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这是我个人挑战自学「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 *Supplement* 就能具备所有必要的知识基础。0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不驯。的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导  $\mathcal{D}$  的一切。

# ABBREVIATION TABLE

## A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

## E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

## L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

## R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

## C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

## F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

## M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

## S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

## D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

## I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

## O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

## T U V W X Y Z

uniq	unique
uniques	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

**0.B** NOTE:  $C, D$  are Dedekind cuts. Numbers used here are always rational.

• Define  $\tilde{q} = \{a : a < q\}$ , and  $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$ .

Then  $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -0 \leq 0\} = \tilde{0}$ .

• Define  $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$ .

$-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$ .

The last equa is becs (a)  $d \notin D \Rightarrow \exists b \notin D, d \geq b$ , and (b)  $d \in D \Rightarrow$  if  $\exists b \notin D$  suth  $d \geq b$ , then  $b \in D$ , ctradic.

• **TIPS:** Prove  $\forall \varepsilon > 0, \exists b \notin D$  suth  $b - \varepsilon \in D$ .

**SOLUS:** Asum  $\exists \varepsilon > 0$  suth  $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$ .

Then  $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$  for any  $n \in \mathbf{N}^+$ .

Now  $\forall d \in D, \exists n \in \mathbf{N}^+$  suth  $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$ , ctradic. □

**1** Prove (a)  $D + \tilde{0} = D$ , (b)  $-D$  is Dedekind cut, and  $D + (-D) = \tilde{0}$ .

**SOLUS:** (a)  $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$ .

(b) Asum  $x \in -D$  is the largest elem of  $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$ .

Let  $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$ .

Thus by def,  $x + \delta \in -D$ , ctradic the max of  $x \in -D$ . Hence  $-D$  is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$ .

Supp  $a \in \tilde{0} \Rightarrow -a > 0$ . By TIPS,  $\exists b \notin D$  suth  $b + a \in D$ .

Note that  $b < b - a \notin D \Rightarrow -b > -b + a \in -D$ . Then  $(-b + a) + (b + a) = 2a < 0$ .

Thus  $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$ . □

**3** Show  $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$  posi.

**SOLUS:** (a)  $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$ .

(b)  $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$ . 又  $D - C \neq \tilde{0}$ . □

**5** Prove (a)  $D$  posi  $\Rightarrow -D$  not posi, (b) non0  $-D$  not posi  $\Rightarrow D$  posi.

**SOLUS:** (a)  $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$ .

(b) Becs  $\tilde{0}$  is the largest non posi cuts. Thus  $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$  posi.

OR.  $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$ . □

• Define  $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$ . Then  $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi.

Define  $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$ .

(a)  $D^- = \{0\} \Leftrightarrow D = \tilde{0}$ . Convly,  $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$ .

(b)  $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi. **CORO:**  $D$  not posi  $\Leftrightarrow 0 \in D^-$ .

(c)  $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$ . **CORO:**  $D$  not posi  $\Leftrightarrow (D^-)^- = D$ .

•  $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$ .

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$ .

• For  $C, D$  posi, define  $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$ . Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.

- For  $-C, -D$  posi, define  $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .  
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$ .  
 If  $C, -C$  not posi  $\Rightarrow C = \tilde{0}$ , then with the asum  $\tilde{0}D = \tilde{0}$ , it still holds. Simlr for  $D, -D$  not posi.

- For  $D$  posi, define  $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$ .  
 The last equa holds becs  $\forall a \in \tilde{1} \cap \mathbf{Q}^+, \exists d \in D^+$  suth  $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$ .
- For non0  $D$  not posi, define  $D^{-1} = \{a : a < 1/b, b \in D^-\} \Rightarrow (D^{-1})^- = \{s : 0 \geq s \geq 1/b, \forall b \in D^-\}$ .  
 Then  $DD^{-1} = \{a : a < rs, r \in D^-, s \in (D^{-1})^-\} = \{a : a < rs, \exists r \in D^-, 0 \leq rs \leq r/b, \forall b \in D^-\}$ .  
 Asum  $\exists a \in \tilde{1} \cap \mathbf{Q}^+, \forall r \in D^-, s \in (D^{-1})^-, a \geq rs \Rightarrow a \geq r/b \Leftrightarrow r/a \leq b, \forall b \in D^-$ . Let  $r = 0$ .

- For  $C$  not posi and  $D$  posi, we expect that  $CD$  not posi. Consider  $C$  and  $-D$  both not posi.  
 $CD = -C(-D) = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$   
 $= \{-a : a > rt, \forall r \in C^-, \forall t \text{ suth } 0 \geq t \geq -s, \forall s \notin D\}$   
 $= \{a : a < ru, \forall r \in C^-, \forall u \text{ suth } 0 \leq u \leq s, \forall s \notin D\}. \quad (r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs.)$
- Note the ' $0 \leq u$ '. Becs  $C^- \neq \emptyset \Rightarrow 0 \in C^-$ . If it is to be exactly  $CD = \{a : a < 0\}$ , then  $C^- = \{0\}$ ,  
 for if not,  $\exists u > 0$ , and  $\exists r \in C^- \setminus \{0\}$ , suth  $\exists a < ru < 0$ . Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
- ' $u \leq s$ ' cannot be abbreviated as in  $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$ .  
 ' $u \leq s$ ' cannot be ' $u < s$ ', becs here  $rs < ru \Rightarrow \exists a = rs$ . Simlr for ' $a < ru$ ' to be ' $a \leq ru$ ' with ' $u < s$ '.
- Note that  $\{u : 0 < u \leq s, \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q} \setminus D\}, & \text{if it exists,} \\ D^+, & \text{othws.} \end{cases}$  Denote it by  $D^\oplus = D^\otimes \setminus \{0\}$ .

- For  $C$  not posi and  $D$  posi. If  $C = \tilde{0}$ , then  $CD = -C(-D) = -\tilde{0}$ . Now consider  $-C$  and  $D$  both posi.  
 But  $CD = -(-C)D = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \neq \{a : a < -cd, \forall c \in (-C)^+, d \in D^+\}$ .  
 Altho " $a \leq cd$ " is equiv to " $a < cd$ " so that  $b \notin (-C)D \Rightarrow b \geq cd$ , which is actually  $b > cd, \forall c, d$ .  
 And  $a < -b < -cd, \forall c, d \Rightarrow \forall a, \exists x \text{ suth } a < x < -cd, \forall c, d$ . While  $a$  can be the 'boundary' in RHS.

- $LHS = \{a : a < ru, \forall r \in C^-, \forall u \in D^\oplus\}, \{cs : c \in C, s \notin D\} = RHS$ .  
 Becs  $cs \leq cu < ru$ . We show  $LHS \subseteq RHS$ . Let  $c_1 < \dots < c_n < \dots \in C$ , and  $s_1 > \dots > s_m > \dots \notin D$ .  
 Then  $c_1 s_1 < \dots < c_n s_m < \dots < ru, \forall r, u$  as in LHS. Thus  $a \in LHS \Rightarrow \exists a < c_j s_k$ . □

- For  $C$  posi and  $D$  not posi. If  $D = \tilde{0}$ , then  $CD = -(-C)D = -\tilde{0}$ .  
 Now consider  $-C$  not posi and  $-D$  posi.  $\mathbf{Q} \setminus -D = \{s : s \geq -y, \forall y \notin D\}$ .  
 $CD = (-C)(-D) = \{a : a < ru, \forall r \in (-C)^-, \forall u \in (-D)^\oplus\}$   
 $= \{a : a < ru, \forall r \text{ suth } \forall x \notin C, 0 \geq r \geq -x, \forall u \text{ suth } 0 < u \leq s, \forall s \geq -y, \forall y \notin D\}$   
 $= \{a : a < (-r)(-u), \forall r \text{ suth } \forall x \notin C, 0 \leq -r \leq x, \forall u \text{ suth } y \leq -u < 0, \forall y \notin D\}$   
 $= \{a : a < ru, \forall r \text{ suth } \forall x \notin C, 0 \leq r \leq x, \forall u \in D^-\}$ .
- Note the ' $0 \leq r$ '. Becs  $D^- \neq \emptyset \Rightarrow 0 \in D^-$ . If it is to be exactly  $CD = \{a : a < 0\}$ , then  $D^- = \{0\}$ ,  
 for if not,  $\exists r > 0$ , and  $\exists u \in D^- \setminus \{0\}$ , suth  $\exists a < ru < 0$ . Hence ' $0 \leq r$ ' is actually ' $0 < r$ '.

- We show  $-D = \{a : a < -b, b \notin D\} = (-\tilde{1})D$ .  
 For  $D$  posi,  $RHS = \{a : a < ru, \forall r \text{ suth } -1 \leq r \leq 0, \forall u \in D^\oplus\} = \{a : a < -u, \forall u \in D^\oplus\} \supseteq -D$ .  
 Supp  $x \text{ suth } -b \leq x < -u, \forall b \notin D, \forall u \text{ suth } -s \leq -u < 0, \forall s \notin D \Rightarrow -b \leq x < -s, \forall b, s \notin D$ .  
 For  $D$  not posi,  $RHS = \{a : a < rb, \exists r \text{ suth } -1 \leq r \leq 0, b \in D^-\} = \{a : a < -b, 0 \geq b \notin D\} = -D$ .

- We show  $\tilde{1}D = D$ . For  $D$  not posi, immed. Othws,  $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$ .  
Now  $(\tilde{1}D)^+ \subseteq D^+$ . 又  $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in \tilde{1}D$ .

**4** Supp  $B, C, D$  non0 Dedekind cuts. Show  $(BC)D = B(CD), B(C + D) = BC + BD$ .

**SOLUS:** We discuss in cases.

$\backslash$	1	2	3	4	5	6	7	8
B	+	+	+	+	-	-	-	-
C	+	+	-	-	-	+	-	+
D	+	-	+	-	-	-	+	+

$$(4) (BC)^- = \{r : 0 \geq r \geq bc, \exists c \in C^-, \exists b \in B^\oplus\}.$$

$$(BC)D = \{a : a < rs, r \in (BC)^-, s \in D^-\}$$

$$= \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^\oplus\}.$$

$$B(CD) = \{a : a \leq bx < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+, \text{ and } cs > x \in (CD)^+\}$$

$$= \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+\}.$$

Note that  $\{q : q < b, b \in B^+\} = \{q : q < b, \exists b \in B^\oplus\}$ . Done.

$$B(C + D) = \{a : a < ru, \forall r \text{ suth } \forall x \notin B, 0 < r \leq x, \forall u \text{ suth } 0 \geq u > c + d, \forall c \in C, d \in D\}$$

$$= \{a : a < ru, \forall r \in B^\oplus, \forall u \text{ suth } 0 \geq u \geq c + d, \exists c \in C^-, d \in D^-\}$$

$$= \{a : a < r(c + d), \forall r \in B^\oplus, \forall c \in C^-, d \in D^-\}.$$

$$BC + BD = \{x : x < pc, \forall p \in B^\oplus, \forall c \in C^-\} + \{y : y < qd, \forall q \in B^\oplus, \forall d \in D^-\}$$

$$= \{a : a < pc + qd, \forall p, q \in B^\oplus, \forall c \in C^-, d \in D^-\}.$$

Done immed.

$$(6) (BC)^- = \{r : 0 \geq r \geq bc, \exists b \in B^-, \exists c \in C^\oplus\}. (CD)^- = \{r : 0 \geq r \geq cd, \exists d \in D^-, \exists c \in C^\oplus\}.$$

$$(BC)D = \{a : a < rd \leq bcd, \exists d \in D^-, b \in B^-, \exists c \in C^\oplus\}.$$

$$B(CD) = \{a : a < br \leq bcd, \exists b \in B^-, d \in D^-, \exists c \in C^\oplus\}. \text{ Done.}$$

$$\text{OR. } (BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD).$$

$$BC + BD = \{x : x < bc, \forall b \in B^-, \forall c \in C^\oplus\} + \{y : y < bd, \exists b \in B^-, d \in D^-\}$$

$$= \{a : a < pc + qd, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \exists q \in B^-, d \in D^- \Rightarrow q \geq p\}.$$

$$\text{OR. } = \{bc : b \in B, c \notin C\} + \{a : a < bd, \exists b \in B^-, d \in D^-\}$$

$$= \{pc + a : a < qd, \exists p \in B, q \in B^-, c \notin C, d \in D^-\}$$

$$= \{x = pc + (x - pc) : x - pc < qd, \exists p \in B, q \in B^-, c \notin C, d \in D^-\}$$

$$= \{a : a < pc + qd, \exists c \notin C, d \in D^-, p \in B, q \in B^- \Rightarrow p < q \leq 0, \text{ and } c \neq 0\}.$$

$$(I) \text{ If } C = -D. \text{ Then } B(C + D) = \tilde{0}. \text{ Note that } c \in \mathbf{Q} \setminus C = \mathbf{Q} \setminus -D = \{c : c \geq -d, \forall d \in D^-\}.$$

Thus  $pc + qd = pc + (qd - pd) + pd \leq 0$ , becs  $qd - pd \leq 0$  and  $pc + pd \leq 0$ . Now immed done.

$$(II) \text{ If non0 } C + D \text{ not posi. Then } B(C + D) = \{a : a < bx, \exists b \in B^-, x > c + d, \forall (c, d) \in C \times D\}.$$

$$(III) \text{ If } C + D \text{ posi. Then } B(C + D) = \{a : a < bx, \forall b \in B^-, x \in (C + D)^\oplus\}.$$

$$\text{If } (C + D)^\oplus = (C + D)^+. \text{ Then } B(C + D) = \{a : a < bc + bd, \forall b \in B^-, \forall (c, d) \in C^+ \times D^+\}.$$

$$\text{Othws, } \exists s = \min \mathbf{Q} \setminus (C + D) \Rightarrow B(C + D) = \{a : a < bs, \forall b \in B^-\}.$$

## 0.C

**5** Supp  $a_1, a_2, \dots$  is a seq in  $\mathbf{Q}$ , and  $\sup\{a_1, a_2, \dots\} = \sqrt{2}$ .

Prove  $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$  for all  $n \in \mathbf{N}^+$ .

**SOLUS:** Becs the sup not in seq  $\Rightarrow$  infily many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$ . For  $a_{n+k}$ , choose  $a_i > a_{n+k}$ . If  $i \in \{1, \dots, n\}$ , then choose  $a_j > a_i$ .

After at most  $(n+1)$  steps, we have  $a_m$  with  $m > n$ . Thus  $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$ .  $\square$

• Supp nonempty  $A \subseteq \mathbf{R}$ .

• **TIPS 1:** Define  $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$ . Prove  $\sup(-A) = -\inf A$ .

**SOLUS:**  $-b$  is an upper bound of  $-A \iff \forall a \in A, -a \leq -b \iff a \geq b \iff b$  is a lower bound of  $A$ .

Thus  $-b_M = \sup(-A) \iff -b_M \leq -b \iff b_M \geq b \iff b_M = \inf A$ .  $\square$

• **TIPS 2:** Show if  $x \in \mathbf{R}$ , (a)  $\sup A > x \Rightarrow \exists a \in A, a > x$ , (b)  $\inf A < x \Rightarrow \exists a \in A, a < x$ .

**SOLUS:** (a)  $\nexists a > x \iff \forall a \in A, a \leq x$ . Then by def of sup.

OR. By (b),  $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$ .

Simlr for (b).  $\square$

**6** Supp  $A, B \subseteq \mathbf{R}$  has infily many disti elem, so has  $A + B = \{a + b : a \in A, b \in B\}$ .

Prove  $\sup(A + B) = \sup A + \sup B$ , and  $\inf(A + B) = \inf A + \inf B$ .

**SOLUS:**  $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B, \inf A + \inf B \leq \inf(A + B)$ .

$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$

$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B)$ . Ctradic.

Simlr for  $\inf(A + B) \in A + B$ . OR. Apply to  $-A - B$ , becs  $\sup(-A) = -\inf A$ .  $\square$

**16** Supp  $\mathbf{R}_1, \mathbf{R}_2$  are complete ordered fields. Define  $\varphi_1, \varphi_2$  as in [0.11].

Define  $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Define  $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$ .

(a) Show  $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  is one-to-one. (b) Show  $\psi(0) = 0, \psi(1) = 1$ .

(c) Show  $\psi(a \pm b) = \psi(a) \pm \psi(b)$ , and  $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$ .

(d) Supp  $a \in \mathbf{R}_1$ . Show  $a > 0 \iff \psi(a) > 0$ .

**SOLUS:** (a) Define  $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$ .

Define  $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$ . Then  $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  well-defined.

Note that  $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$ .

Now  $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Rev the roles of  $\mathbf{R}_1, \mathbf{R}_2$ .

(b) Note that  $\varphi(q) < 0 \iff q < 0$ , and  $\varphi(0) = 0$ .

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+), \varphi_2(q) \leq 1$ .

(c)  $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$ .

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$ .

(d)  $a > 0 \iff \exists n \in \mathbf{N}^+, 1/n < a \iff \psi_1(a) > 0$ .  $\square$

ENDED