这是我个人挑战自学「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。

我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不训。

的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导 $\mathcal{D}$ 的一切。如你所见，字越少，事越大。

ABBREVIATION TABLE

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
continu	countinu(ous)(ity)
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
countexa	counterexample
ctradic	contradict(s)(ion)
ctrapos	constrapositive

F G H

factoriz	factorizaion
fini	finite(ly)
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

D

Ddkd	Dedekind
decr	decreasing
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disj	disjoint
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
up-	upper-
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B   Note: C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.
  Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \backslash \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.

- Define $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Longleftrightarrow a > b, b \notin D\}$.
  $-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geqslant -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$.
  The last equa is becs (a) $d \notin D \Rightarrow \exists b \notin D, d \geqslant b$, and (b) $d \in D \Rightarrow$ if $\exists b \notin D$ suth $d \geqslant b$, then $b \in D$, ctradic.

- Tips: *Prove $\forall \varepsilon > 0, \exists b \notin D$ suth $b - \varepsilon \in D$.*

  Solus: Asum $\exists \varepsilon > 0$ suth $\nexists b \notin D, b - \varepsilon \in D \Longleftrightarrow \forall b \notin D, b - \varepsilon \notin D$.

  Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.

  Now $\forall d \in D, \exists n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic.   □

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1 *Prove* (a) $D + \tilde{0} = D,$ (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

Solus: (a) $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$.

Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.

Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$.

Supp $a \in \tilde{0} \Rightarrow -a > 0$. By Tips, $\exists b \notin D$ suth $b + a \in D$.

Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.

Thus $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$.   □

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3 *Show* $C \subsetneq D \Longleftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$ *posi.*

Solus: (a) $C \subsetneq D \Rightarrow \exists x \in D \backslash C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Longleftrightarrow 0 < d - y \in D - C$.

(b) $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$. $\not\!\!\curlyvee D - C \neq \tilde{0}$.   □

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5 *Prove* (a) D *posi* $\Rightarrow -D$ *not posi,* (b) *non0* $-D$ *not posi* $\Rightarrow D$ *posi.*

Solus: (a) $0 \notin \{a : a < -b, b \notin D\} \Longleftrightarrow \nexists b \notin D, 0 < -b \Longleftrightarrow \forall b \notin D, b \geqslant 0 \Longleftarrow 0 \in D$.

(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.

Or. $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Longleftrightarrow \nexists b \notin D, -a > b \Longleftrightarrow \forall b \notin D, 0 < -a \leqslant b$.□

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \Longleftrightarrow \mathbf{Q} \backslash \mathbf{Q}^+ \subsetneq D \Longleftrightarrow 0 \in D \Longleftrightarrow D$ posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q} \backslash D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \mathbf{Q} \backslash (D \cup \mathbf{Q}^+)$.
  (a) $D^- = \{0\} \Longleftarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leqslant 0\} = \{0\} \Longrightarrow \mathbf{Q} \backslash D = \mathbf{Q}^*$.
  (b) $D^- = \emptyset \Longleftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Longleftrightarrow \mathbf{Q} \backslash \mathbf{Q}^+ \subseteq D \Longleftrightarrow 0 \in D \Longleftrightarrow D$ posi.   Coro: D not posi $\Longleftrightarrow 0 \in D^-$.
  (c) $(D^-)^- = \{r \in D : r \leqslant 0\} = \mathbf{Q} \backslash D^+$.   Coro: D not posi $\Longleftrightarrow (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Longleftrightarrow b \in D^- \backslash \{0\}\}$.
  $(-D)^- = (\mathbf{Q} \backslash -D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b, \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leqslant cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+)$.
  $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that '$a \leqslant cd$' here is equiv to '$a < cd$'.

- For $-C, -D$ posi, define $CD = (-\tilde{C})(-\tilde{D}) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  $CD = \{0 < cd < (-r)(-s) : r \in C^-\backslash\{0\}, s \in D^-\backslash\{0\}\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.
  If $C, -C$ not posi $\Rightarrow C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlr for D.

- For D posi, define $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leqslant d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$.
  The last equa holds becs $\forall a \in \tilde{1} \cap \mathbf{Q}^+, \exists d \in D^+$ suth $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$.

- For non0 D not posi, define $D^{-1} = -(-D)^{-1} = -\{a : a < 1/b, \exists b \notin -D \Longleftrightarrow \exists b \geqslant -s, \forall s \notin D\}$
  $= \{a : a < -x, \exists x \geqslant 1/b, \forall b \text{ suth } b \geqslant -s, \forall s \notin D\} = \{a : a < -1/b, \forall b \text{ suth } b \geqslant -s, \forall s \notin D\}$
  $= \{a : a < 1/b, \forall b \text{ suth } b \leqslant s, \forall s \notin D\} \neq \{a : a < 1/s, \forall s \notin D\}$.
  Let $b_1 < \cdots < b_m < \cdots \leqslant s, \forall s \notin D \Longleftrightarrow \forall s \in D^- \Rightarrow$ each $s_j, b_k < 0 \Rightarrow 1/s_j \leqslant \cdots < 1/b_m < \cdots < 1/b_1$.
  Thus '$a < 1/b$' is equiv to '$a < 1/s, \exists s \in D^-$'. Hence $D^{-1} = \{a : a < 1/b, b \in D^-\}$.

  $DD^{-1} = \{a : a < rs, \exists r \in D^-, \exists s \text{ suth } 0 \geqslant s \geqslant 1/b, \forall b \in D^-\} \subseteq \tilde{1}$.
  Asum $\exists x$ suth $rs \leqslant x < 1, \forall r, s$. Let $D \not\ni \cdots \leqslant b_m < \cdots < b_1 \leqslant 0$, and $D \not\ni \cdots \leqslant r_m < \cdots < r_1 \leqslant 0$.
  $1/b_1 < \cdots < 1/b_m \leqslant \cdots \leqslant \cdots \leqslant s_n < \cdots < s_1 \leqslant 0$, and $r_m/b_m \geqslant \cdots \geqslant \cdots \geqslant r_m s_n > \cdots > r_j s_k > \cdots > r_1 s_1$.
  Let $r_m = b_m$. Ctradic.   OR. $DD^{-1} = D[(-\tilde{1})(-D)^{-1}] = [D(-\tilde{1})](-D)^{-1} = (-D)(-D)^{-1} = \tilde{1}$.

- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
  $CD = -[C(-D)] = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geqslant rt, \forall r \in C^-, t \in (-D)^-\}$
  $= \{-a : a > rt, \forall r \in C^-, \forall t \text{ suth } 0 \geqslant t \geqslant -s, \forall s \notin D\}$
  $= \{a : a < ru, \forall r \in C^-, \forall u \text{ suth } 0 \leqslant u \leqslant s, \forall s \notin D\}$. $(r \leqslant 0 < s, rs \leqslant ru = -rt \leqslant 0 \leqslant rt \leqslant -rs.)$

- Note the '$0 \leqslant u$'. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$,
  for if not, $\exists u > 0$, and $\exists r \in C^-\backslash\{0\}$, suth $\exists a < ru < 0$. Hence '$0 \leqslant u$' is actually '$0 < u$'.

- '$u \leqslant s$' cannot be abbreviated as in $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$.
  '$u \leqslant s$' cannot be '$u < s$', becs here $rs < ru \Rightarrow \exists a = rs$.

- Note that $\{u : 0 < u \leqslant s, \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q}\backslash D\}, & \text{if it exis,} \\ D^+, & \text{othws.} \end{cases}$ Denote it by $D^\oplus = D^\otimes\backslash\{0\}$.

- For C not posi and D posi. If $C = \tilde{0}$, then $CD = -[C(-D)] = -\tilde{0}$. Now consider $-C$ and D both posi.
  But $CD = -(-C)D = -\{a : a \leqslant cd, c \in (-C)^+, d \in D^+\} \neq \{a : a < -cd, \forall c \in (-C)^+, d \in D^+\}$.
  Altho "$a \leqslant cd$" is equiv to "$a < cd$" so that $b \notin (-C)D \Rightarrow b \geqslant cd$, which is actually $b > cd, \forall c, d$.
  And $a < -b < -cd, \forall c, d \Rightarrow \forall a, \exists x$ suth $a < x < -cd, \forall c, d$. While a can be the 'boundary' in RHS.

- $LHS = \{a : a < ru, \forall r \in C^-, \forall u \in D^\oplus\}, \{cs : c \in C, s \notin D\} = RHS$.
  Becs $cs \leqslant cu < ru$. We show $LHS \subseteq RHS$. Let $c_1 < \cdots < c_n < \cdots \in C$, and $s_1 > \cdots > s_m \geqslant \cdots \notin D$.
  Then $\underline{c_1 s_1 < \cdots < c_n s_m < \cdots < ru, \forall r, u \text{ as in } LHS. \text{ Thus } a \in LHS \Rightarrow \exists a < c_j s_k.}$ □
  OR. Note that in LHS, '$a < ru$' is equiv to '$a < rs, \exists s \notin D$.' Now $LHS = \{a : a/s \in C, \exists s \notin D\}$.   □

- For C posi and D not posi. If $D = \tilde{0}$, then $CD = -[(-C)D] = -\tilde{0}$.
  $CD = (-\tilde{C})(-\tilde{D}) = \{a : a < ru, \forall r \in (-C)^-, \forall u \in (-D)^\oplus\}$ $\mathbf{Q}\backslash -D = \{s : s \geqslant -y, \forall y \notin D\}$
  $= \{a : a < ru, \forall r \text{ suth } \forall x \notin C, 0 \geqslant r \geqslant -x, \forall u \text{ suth } 0 < u \leqslant s, \forall s \geqslant -y, \forall y \notin D\}$
  $= \{a : a < (-r)(-u), \forall r \text{ suth } \forall x \notin C, 0 \leqslant -r \leqslant x, \forall u \text{ suth } y \leqslant -u < 0, \forall y \notin D\}$
  $= \{a : a < ru, \forall r \text{ suth } \forall x \notin C, 0 \leqslant r \leqslant x, \forall u \in D^-\} = \{a : a < ru, \forall r \in C^\oplus, \forall u \in D^-\}$, simlr.

- We show $-D = \{a : a < -b, b \notin D\} = (-\tilde{1})D$.

  For D posi, $RHS = \{a : a < ru, \forall r \text{ suth } -1 \leqslant r \leqslant 0, \forall u \in D^{\oplus}\} = \{a : a < -u, \forall u \in D^{\oplus}\} \supseteq -D$.

  Supp x suth $-b \leqslant x < -u, \forall b \notin D, \forall u \text{ suth } -s \leqslant -u < 0, \forall s \notin D \Longrightarrow -s \leqslant x < -u$. Let $-u = x$.

  For D not posi, $RHS = \{a : a < rb, \exists r \text{ suth } -1 \leqslant r \leqslant 0, b \in D^-\} = \{a : a < -b, 0 \geqslant b \notin D\} = -D$.

- We show $\tilde{1}D = D$. For D not posi, immed. Othws, $\tilde{1}D = \{a : a \leqslant ij < j, 0 < i < 1, j \in D^+\} \subseteq D$.

  Now $(\tilde{1}D)^+ \subseteq D^+$. $\not\supseteq$ $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon)\frac{d}{d+\varepsilon} \in (\tilde{1}D)^+$.

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4 *Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD), B(C + D) = BC + BD$.*

SOLUS: We discuss in cases.

\	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

(1) $[(BC)D]^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = [B(CD)]^+$.

   $B(C + D) = \{a : a \leqslant bc + bd, \exists b \in B^+, 0 < c + d \in C + D\}$.

   $BC + BD = \{x : x \leqslant uc, u \in B^+, c \in C^+\} + \{y : y \leqslant vd, v \in B^+, d \in D^+\}$

   $\qquad = \{a : a \leqslant uc + vd, \exists u, v \in B^+, c \in C^+, d \in D^+\}$. Done.

(3) $(BC)^- = \{r : 0 \geqslant r \geqslant bc, \exists c \in C^-, \exists b \in B^{\oplus}\}$.

   $(BC)D = \{a : a < rs, r \in (BC)^-, s \in D^-\}$

   $\qquad = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^{\oplus}\}$.

   $B(CD) = \{a : a \leqslant bx < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+, \text{ and } cs > x \in (CD)^+\}$

   $\qquad = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+\}$.

   Note that $\{q : q < b, b \in B^+\} = \{q : q < b, \exists b \in B^{\oplus}\}$. Done.

   $B(C + D) = \{a : a < ru, \forall r \text{ suth } \forall x \notin B, 0 < r \leqslant x, \forall u \text{ suth } 0 \geqslant u > c + d, \forall c \in C, d \in D\}$

   $\qquad = \{a : a < ru, \forall r \in B^{\oplus}, \forall u \text{ suth } 0 \geqslant u \geqslant c + d, \exists c \in C^-, d \in D^-\}$

   $\qquad = \{a : a < r(c + d), \forall r \in B^{\oplus}, \forall c \in C^-, d \in D^-\}$.

   $BC + BD = \{x : x < pc, \forall p \in B^{\oplus}, \forall c \in C^-\} + \{y : y < qd, \forall q \in B^{\oplus}, \forall d \in D^-\}$

   $\qquad = \{a : a < pc + qd, \forall p, q \in B^{\oplus}, \forall c \in C^-, d \in D^-\}$. $\qquad\qquad$ Done immed.

(5) $(BC)^- = \{r : 0 \geqslant r \geqslant bc, \exists b \in B^-, \exists c \in C^{\oplus}\}$. $(CD)^- = \{r : 0 \geqslant r \geqslant cd, \exists d \in D^-, \exists c \in C^{\oplus}\}$.

   $(BC)D = \{a : a < rd \leqslant bcd, \exists d \in D^-, b \in B^-, \exists c \in C^{\oplus}\}$.

   $B(CD) = \{a : a < br \leqslant bcd, \exists b \in B^-, d \in D^-, \exists c \in C^{\oplus}\}$. Done.

   OR. By commu and (3), $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$.

   $BC + BD = \{x : x < bc, \forall b \in B^-, \forall c \in C^{\oplus}\} + \{y : y < bd, \exists b \in B^-, d \in D^-\}$

   $\qquad = \{a : a < pc + qd, \forall p \in B^-, \forall c \in C^{\oplus}, \text{ and } \exists q \in B^-, d \in D^- \Rightarrow q \geqslant p\}$

   $\qquad = \{a : a < pc + y, \forall p \in B^-, \forall c \in C^{\oplus}, \text{ and } \forall y \text{ suth } y \geqslant qd, \forall q \in B^-, d \in D^-\}$

   $\qquad = \{a : a < p(c + d'), \forall p \in B^-, \forall c \in C^{\oplus}, \text{ and } \forall d' \text{ suth } d' \leqslant d, \forall d \in D^-\}$.

   (I) If $C + D$ not posi. Then $B(C + D) = \{a : a < bx, \exists b \in B^-, 0 \geqslant x > c + d, \forall (c, d) \in C \times D\}$.

   $\qquad$ Rewrite as $\{a : a < t, \forall t \text{ suth } t \geqslant bx, \forall b \in B^-, x \in (C + D)^-\}$. Done.

   (II) If $C + D$ posi. Then $B(C + D) = \{a : a < bx, \forall b \in B^-, x \in (C + D)^{\oplus}\}$.

   $\qquad$ If $(C + D)^{\oplus} = (C + D)^+$. Then $B(C + D) = \{a : a < bc + bd, \forall b \in B^-, \forall (c, d) \in C^{\oplus} \times D^{\oplus}\}$.

   $\qquad$ Othws, $\exists s = \min \mathbf{Q} \backslash (C + D) \Rightarrow B(C + D) = \{a : a < bs, \forall b \in B^-\}$. Done.

(4) $(BC)D = \{xd : d \in D, x \geqslant bc, \forall b \in B^-, c \in C^-\}$
$\qquad = \{a : a < bcd, \forall d \in D^-, \forall b \in B^-, c \in C^-\}$
$\qquad = \{by : b \in B, y \geqslant cd, \forall c \in C^-, d \in D^-\} = B(CD)$.
$\quad B(C+D) = \{a : a < t, \forall t \text{ suth } t \geqslant b(c+d), \forall b \in B^-, (c,d) \in C^- \times D^-\}$
$\qquad = \{a : a < t_1, \forall t_1 \text{ suth } t_1 \geqslant bc, \forall (b,c) \in B^- \times C^-\}$
$\qquad + \{a : a < t_2, \forall t_2 \text{ suth } t_2 \geqslant bd, \forall (b,d) \in B^- \times D^-\} = BC + BD$. Done.

NOTE: Supp for any B posi, C posi, D not posi, assoc holds.

$\qquad$ Supp B posi, C not posi, D posi. Then $(\overset{+}{B}\overset{-}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{B}\overset{-}{C}) = (\overset{+}{D}\overset{+}{B})\overset{-}{C} = \overset{+}{B}(\overset{+}{D}\overset{-}{C})$. Convly true.

$\qquad$ Simlr for the case B not posi, C posi, D not posi, equiv to the case B not posi, C not posi, D posi.

(2) holds $\Rightarrow (\overset{-}{B}\overset{+}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{C}\overset{-}{B}) = (\overset{+}{C}\overset{+}{D})\overset{-}{B}$, (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds $\Rightarrow \overset{+}{B}(\overset{+}{C}+\overset{-}{D}) = (-B)\left[-(\overset{+}{C}+\overset{-}{D})\right] \overset{(5)}{=\!=} (-B)\left[(\overset{-}{C})+(\overset{+}{D})\right] \overset{(5)}{=\!=} \overset{+}{B}\overset{+}{C} + \overset{+}{B}\overset{-}{D}$, by def of multi.

$\qquad$ Thus (5) $\Rightarrow$ (2) in distr. Convly as well.

(6) $(BC)^- = \{r : 0 \geqslant r \geqslant bc, \exists b \in B^-, c \in C^\oplus\}$.
$\qquad (BC)D = \{a : a < ru, \forall r \in (BC)^-, u \in D^\oplus\} = \{a : a < bcd, \forall b \in B^-, c \in C^\oplus, d \in D^\oplus\}$.
$\qquad (CD)^\oplus = \{u : 0 < u \leqslant s, \forall s > cd, \forall c \in C^+, d \in D^+\}$.
$\qquad B(CD) = \{a : a < ru, \forall r \in B^-, u \in (CD)^\oplus\}$. Done.

$\qquad B(C+D) = \{a : a < b(c+d), \forall b \in B^-, \forall (c,d) \in C^\oplus \times D^\oplus\}$.
$\qquad BC + BD = \{a : a < bc, \forall b \in B^-, c \in C^\oplus\} + \{a : a < bd, \forall b \in B^-, d \in D^\oplus\}$. Done.

$\quad$ OR. (2) instead of (6) ? NOTICE that the distr in (6) cannot be shown without (6). $\qquad\square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **ENDED**

0·C

2 *Supp nonempty $U \subseteq V \subseteq \mathbf{R}$. Show $\sup U \leqslant \sup V$.*

SOLUS: Asum $\sup U > \sup V \Rightarrow \exists t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$, ctradic. $\qquad\square$

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5 *Supp $a_1, a_2, \cdots \in \mathbf{Q}$, and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.*
$\quad$ *Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.*

SOLUS: Becs the sup not in seq $\Rightarrow$ infily many disti elem.

$\qquad \forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

$\qquad$ After at most n steps, we must have a_m with $m > n$. Thus $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$. $\qquad\square$

• *Supp nonempty $A \subseteq \mathbf{R}$.*

• **TIPS 1:** *Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.*

SOLUS: $-b$ is an up-bound of $-A \Longleftrightarrow \forall a \in A, -a \leqslant -b \Longleftrightarrow a \geqslant b \Longleftrightarrow b$ is a low-bound of A.

$\qquad$ Thus $-b_M = \sup(-A) \Longleftrightarrow -b_M \leqslant -b \Longleftrightarrow b_M \geqslant b \Longleftrightarrow b_M = \inf A$. $\qquad\square$

• **TIPS 2:** *Show if $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A, a > x$, (b) $\inf A < x \Rightarrow \exists a \in A, a < x$.*

SOLUS: (a) $\nexists a > x \Longleftrightarrow \forall a \in A, a \leqslant x \Longrightarrow \sup A \leqslant x$.

$\qquad$ OR. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$. $\qquad$ Simlr for (b). $\qquad\square$

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6 *Supp nonempty $A, B \subseteq \mathbf{R}$. Prove* $\sup(A + B) = \sup A + \sup B$, *and* $\inf(A + B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A + B) \leqslant \sup A + \sup B$, $\inf A + \inf B \leqslant \inf(A + B)$.

$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$

$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B)$. Ctradic.

Simlr for $\inf(A + B) \in A + B$. Or. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. $\qquad \square$

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16 *Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in* [0.11].

Define $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ by $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leqslant a\}$. Show

(a) $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ *is one-to-one.*   (b) $\psi(0) = 0, \psi(1) = 1$.

(c) $\psi(a + b) = \psi(a) + \psi(b)$.   (d) $\psi(ab) = \psi(a)\psi(b)$.

Solus: (a) Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Let $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$.

Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Becs $\varphi_1(q) \leqslant \varphi_1(0) \iff q \leqslant 0 \iff \varphi_2(q) \leqslant \varphi_2(0)$. Simlr for $\psi(1) = 1$.

(c) $S = \{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leqslant a + b\} \supseteq \{\varphi_2(p + q) : p, q \in \mathbf{Q}, \varphi_1(p) \leqslant a, \varphi_1(q) \leqslant b\} = T$

$\Rightarrow \sup S \geqslant \sup T$. Asum it is '>'. Now $\exists t \in \mathbf{Q}$ suth $\sup S \geqslant \varphi_2(t) > \sup T = \psi(a) + \psi(b)$.

Which means $\varphi_2(t) > \varphi_2(p + q)$, $\forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leqslant a$ and $\varphi_1(q) \leqslant b$.

Now $a + b \geqslant \varphi_1(t) > \varphi_1[(t + p + q)/2] > \varphi_1(p + q)$, $\forall p, q$. Ctradic.

(d) We show it for (I) $a, b > 0$, (II) $a > 0 > b$.

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leqslant ab\}$, $\sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leqslant b\} = RHS$.

(I) $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q}, 0 < \varphi_1(p) \leqslant a, \varphi_1(q) \leqslant b \Rightarrow \varphi_1(pq) \leqslant ab\}$.

(II) $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q}, \varphi_1(q) \leqslant b, \text{and } s \geqslant p, \forall p \text{ suth } 0 < \varphi_1(p) \leqslant a\}$.

Note that $\varphi_1(s) \geqslant \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geqslant a$, for if not, $\exists p' \in \mathbf{Q}$ suth $\varphi_1(s) < \varphi_1(p') \leqslant a$.

So that $\varphi_1(sq) \leqslant a\varphi_1(q) \leqslant ab$.

Now $LHS \geqslant RHS$. Asum it is '>'. Then $\exists t \in \mathbf{Q}$ suth $LHS \geqslant \varphi_2(t) > RHS$.

Which means $\varphi_2(t) > \psi(a)\varphi_2(q)$, $\forall q \in \mathbf{Q}$ suth $\varphi_1(q) \leqslant b$.

(I) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $0 < \varphi_1(s) \leqslant a$.

(II) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $\varphi_2(s) \geqslant \varphi_2(p)$, $\forall p$ suth $0 < \varphi_1(p) \leqslant a$.

Thus $ab \geqslant \varphi_1(t) > \varphi_1[(t + sq)/2] > \varphi_1(sq)$, $\forall s, q$. Ctradic. $\qquad \square$

Ended

0·D

• *Supp nonempty A is closed and open subsets of $\mathbf{R}^n$. Prove $A = \mathbf{R}^n$.*

Solus: Asum $A \neq \mathbf{R}^n$. Let $a \in A$, $b \in \mathbf{R}^n \backslash A$. Define $f(t) = (1 - t)a + tb$.

If $f(t_1) = f(t_2) \Rightarrow (t_1 - t_2)(b - a) = 0 \Rightarrow t_1 = t_2$. Inje.

Let $\mathcal{A} = \{t \in [0, 1) : f(t) \in A\}$, and $\sup \mathcal{A} = t_M \in [0, 1]$. Let $c = f(t_M)$.

(I) If $c \in A \Rightarrow \exists \delta > 0$, $B(c, \delta) \subseteq A$. Let $t \neq t_M$ be suth $f(t) \in B(c, \delta) \Rightarrow \|(t - t_M)(b - a)\|_\infty < \delta$.

Let $\varepsilon = |t - t_M| > 0 \Rightarrow f(t_M + \varepsilon) \in B(c, \delta) \subseteq A \Rightarrow t_M \geqslant t_M + \varepsilon$, ctradic.

(II) If $c \in \mathbf{R}^n \backslash A$. Simlr. $\forall t' \in [t_M - \varepsilon, t_M)$, $t' \notin \mathcal{A}$. Now $\forall t \in \mathcal{A}$, $t < t_M \Rightarrow t < t_M - \varepsilon$. $\qquad \square$

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15 *Supp F is a closed subset of* **R**. *Prove* $S = \{a^2 : a \in F\}$ *is closed.*

Solus: Supp S not closed $\Rightarrow \exists a_1^2, a_2^2, \cdots \in S$ convg to L^2 suth $\pm L \notin F$. Let $\varepsilon_1 > \varepsilon_2 > \cdots \in \mathbf{R}^+$.

Becs $\forall \varepsilon_p, \exists m \in \mathbf{N}^+, \forall k \geqslant m, \|a_k^2 - L^2\|_\infty = |a_k - L| \cdot |a_k + L| < \varepsilon_p,$

if $|a_k - L| < \varepsilon_p$, then let $b_p = a_k$, if $|a_k + L| < \varepsilon_p$, then let $c_p = a_k$.

Then we have at least one seq in F with $\lim \pm L \notin F \Rightarrow F$ not closed. $\square$

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17 *Supp $F \subseteq \mathbf{R}$, and $\forall n \in \mathbf{N}^+$, $F \cap [-n, n]$ is closed. Prove F is closed.*

Solus: Becs $G = (\mathbf{R} \backslash F) \cup (-\infty, -n) \cup (n, \infty)$ is open for all $n \in \mathbf{N}^+$. Let $n > \sup\{|a| : a \in \mathbf{R} \backslash F\}$.

By [0.59], $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \cdots$ (disj) $\Longrightarrow I_1 \cup I_2 \cup \cdots = \mathbf{R} \backslash F$. $\square$

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20 *Prove $\forall b \in \mathbf{R}^n$, $\delta > 0$, $B = \{a \in \mathbf{R}^n : \|a - b\| \leqslant \delta\}$ is closed.*   *See below* [0.46].

Solus: Asum $\exists a_1, a_2, \cdots \in B$ convg to L suth $\|L - b\| > \delta$. Then $\forall \varepsilon > 0, \exists m \in \mathbf{N}^+, \forall k \geqslant m,$

$\underline{\|a_k - L\| < \varepsilon} \Rightarrow \delta < \|L - b\| \leqslant \|a_k - L\| + \|a_k - b\| < \delta + \varepsilon$. Now $0 < \|L - b\| - \delta < \varepsilon$. Ctradic. $\square$

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21 *Supp X is an open subset of* **R**. *Prove $\exists a_k, b_k \in \mathbf{R}$ suth $X = \bigcup_{k=1}^\infty [a_k, b_k]$.*

Solus: Let $X = \bigcup_{k=1}^\infty (c_k, d_k)$, where each $d_k < c_{k+1}$. Let $I_k = (c_k, d_k)$.

Let $a_{1,k}, a_{2,k}, \ldots$ be convg of $\lim c_k$. Simlr, let $b_{1,k}, b_{2,k}, \ldots$ of $\lim d_k$. Let $E_{j,k} = [a_{j,k}, b_{j,k}]$.

Now $\bigcup_{k=1}^\infty \bigcup_{j=1}^\infty E_{j,k} = \bigcup_{k=1}^\infty I_k = X$. Rearrange the order of $E_{j,k}$'s. $\square$

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23 *Supp F_1, F_2 are disj closed subsets of* **R** *suth $U = F_1 \cup F_2$ is interval. Prove F_1 or F_2 open.*

Solus: If U open, then $F_1 = \emptyset$, $F_2 = \emptyset$ or **R**. Now supp U not open $\Rightarrow F_1$ or F_2 not open.

We show F_1 open $\iff F_2$ not open. Note that F_1 open $\iff F_1 = \emptyset$. Simlr for F_2.

WLOG, asum F_1, F_2 both not open, and $x \in F_1, y \in F_2$ with $x < y \Rightarrow x, y \in U \supseteq [x, y]$.

Notice that $T = [x, y] \cap F_1$ is a closed subset that has infily many elem. $\,\nexists\, \sup T < y$.

(I) If $\sup T \notin F_1$. Then $\exists$ a convg seq in T with $\lim \sup T$.

(II) Othws, $\forall t \in (\sup T, y], t \notin F_1 \iff t \in F_2$. Now $\exists$ a convg seq in F_2 with $\lim \sup T \notin F_2$.

Ctradic the asum $\Rightarrow$ at least one of F_1, F_2 is empty. $\square$

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25 *Give an exa of inv $f : \mathbf{R} \to \mathbf{R} \backslash \mathbf{Q}$.*

Solus: Consider a seq disti $a_1, b_1, a_2, b_2, \cdots \in \mathbf{R}$ where $\{a_1, a_2, \ldots\} \in \mathbf{Q}$ and each $b_i \in \mathbf{R} \backslash \mathbf{Q}$.

Define $\varphi(a_j) = b_{2j-1}$, $\varphi(b_k) = b_{2k} \Rightarrow \varphi^{-1}(b_i) = \begin{cases} a_{(i+1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$

Let $B = \{b_1, b_2, \ldots\}$, $U = \mathbf{Q} \cup B$, $K = \mathbf{R} \backslash U$. Extend $\varphi \in B^U$ to $\psi \in (K \cup B)^{K \cup U}$ by $\psi|_K = I$. $\square$

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26 *Supp $E, G \subseteq \mathbf{R}^n$, and G is open. Prove $E + G = \{x + y : x \in E, y \in G\}$ is open.*

Solus: Asum $E + G$ not open $\iff \mathbf{R}^n \backslash (E + G)$ not closed.

Then $\exists a = x + y \in E + G$ suth $\forall \delta > 0, \exists b \notin E + G$ suth $\|a - b\|_\infty < \delta$.

Let $z = b - x \notin G \Rightarrow \|y - z\|_\infty < \delta \Rightarrow z \in B(y, \delta) \subseteq G, \exists \delta > 0$. $\square$

Or. $\exists a_1, a_2, \cdots \notin E + G$ while its $\lim L = e + g \in E + G$. $\,\nexists\, \forall x \in E, a_k - x \notin G$

$\Rightarrow \lim_{k \to \infty} (a_k - e) = L - e = g \in G$. Thus $\mathbf{R}^n \backslash G$ not closed $\iff G$ not open. $\square$

0·E

1 *Prove every convg seq in $\mathbf{R}^n$ is bounded.*

Solus: Supp $a_1, a_2, \cdots \in \mathbf{R}^n$ suth $\forall \varepsilon > 0, \exists m \in \mathbf{N}^+, \forall k \geqslant m, \left| \|a_k\|_\infty - \|L\|_\infty \right| \leqslant \|a_k - L\|_\infty < \varepsilon$.

Thus $\|a_k\|_\infty \in \left(\left| \varepsilon - \|L\|_\infty \right|, \varepsilon + \|L\|_\infty \right)$. Now asum $\sup\{\|a_k\|_\infty : k \in \mathbf{N}^+\} = \infty$.

Which means $\forall t \in \mathbf{R}^+, \forall m \in \mathbf{N}^+, \exists k \geqslant m, \max\{\|a_1\|_\infty, \ldots, \|a_m\|_\infty\} \leqslant \|a_k\|_\infty > t$. Ctradic. $\square$

- **Tips 1:** By def, a seq $a_1, a_2, \cdots \in \mathbf{R}^n$ convg to $L \Longrightarrow$ every subseq convg to L.

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3 *Supp $F \subseteq \mathbf{R}^n$, every seq in F has a convg subseq with lim in F. Prove F is closed bounded.*

Solus: Supp $a_1, a_2, \cdots \in F$ convg with lim L. Becs $\exists$ subseq with lim in F. By Tips (1), $L \in F$.

Asum $\sup\{\|a\|_\infty : a \in F\} = \infty \Rightarrow \forall \Delta > 0, \exists a_1, a_2, \cdots \in F$ with each $\|a_{k+1}\|_\infty \geqslant \|a_k\|_\infty + \Delta$.

Thus every subseq is unbounded $\Rightarrow$ not convg. Ctradic. $\square$

- *Supp $b \in A \subseteq \mathbf{R}^m, f : A \to \mathbf{R}^n$. Prove*
  $[P]$ f is continu at $b \iff \forall b_1, b_2, \cdots \in A$ suth $\lim_{k \to \infty} b_k = b, \lim_{k \to \infty} f(b_k) = f(b)$. $[Q]$

Solus: $Q \Rightarrow P$: Supp $\varepsilon > 0$ suth $\forall \delta > 0, \exists a \in A$ with $\|a - b\|_\infty < \delta$ and $\left\| f(a) - f(b) \right\|_\infty \geqslant \varepsilon$.

Fix a δ. Define $\delta_k = \delta/k \Rightarrow \exists a_k$ for each. Now $\lim_{k \to \infty} a_k = b$.

Thus $\forall m, \forall k \geqslant m, \left\| f(a_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic Q.

$P \Rightarrow Q$: Supp $b_1, b_2, \cdots \in A$ suth $\forall \delta > 0, \exists m, \forall k \geqslant m, \|b_k - b\|_\infty < \delta$.

Asum $\varepsilon > 0$ suth $\forall m, \forall k \geqslant m, \left\| f(b_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic P. $\square$

Solus:

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10 *Supp bounded $A \subseteq \mathbf{R}^m$ and uniformly continu $f \in \mathbf{R}^A$. Prove f is bounded.*

Solus: Asum $\forall a \in A, \exists b \in A, f(b) > f(a)$. Let .

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11 *Supp $f \in \mathbf{R}^\mathbf{R}$ is diff and f' is bounded. Prove f is uniformly continu.*

Solus:

Solus:

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18 *Supp $h : \mathbf{R}^m \to \mathbf{R}^n$. Prove h continu $\iff h^{-1}(G)$ is open for all open $G \subseteq \mathbf{R}^n$.*

Solus: Supp h continu and open $G \subseteq \mathbf{R}^n$. We show $\mathbf{R}^m \backslash h^{-1}(G) = \{t \in \mathbf{R}^m : h(t) \notin G\}$ is closed.

Let $t_1, t_2, \cdots \in \mathbf{R}^m \backslash h^{-1}(G)$ convg to $L \Rightarrow h(t_1), h(t_2), \cdots \in \mathbf{R}^n \backslash G$ convg to $h(L)$.

Supp open $G \subseteq \mathbf{R}^n$ suth $\mathbf{R}^m \backslash h^{-1}(G)$ not closed. Asum show h continu.

$\exists t_1, t_2, \cdots \in \mathbf{R}^m \backslash h^{-1}(G)$ convg to $L \in h^{-1}(G) \Longrightarrow h(t_1), h(t_2), \cdots \in \mathbf{R}^n \backslash G$ convg to $h(L) \notin G$. $\square$

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22 *Supp $\mathbf{R}^n \supseteq F_1 \supseteq F_2 \supseteq \cdots$ is a decre seq of non-$\emptyset$ subsets. Prove $\bigcap_{k=1}^\infty F_k \neq \emptyset$.*

Solus:

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23 *Supp closed $F \subseteq \mathbf{R}$, continu $g \in \mathbf{R}^F$. Prove $\exists$ continu $h \in \mathbf{R}^{\mathbf{R}}$ suth $\forall x \in F$, $g(x) = h(x)$.*

SOLUS:

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26 *Supp $F \subseteq \mathbf{R}^n$ suth every continu $f \in \mathbf{R}^F$ attains a max. Prove F is closed bounded.*

SOLUS:

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27 *Supp $f \in \mathbf{R}^{\mathbf{R}}$ is incre. Prove $\exists$ countable $A \subseteq \mathbf{R}$ suth $f|_{\mathbf{R} \setminus A}$ continu.*

SOLUS:

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29 *Supp continu $f : [a,b] \to \mathbf{R}$, and t is between $f(a), f(b)$. Prove $\exists c \in [a,b]$, $f(c) = t$.*

SOLUS:

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33 *Supp $a_1, a_2, \cdots \in \mathbf{R}^n$ is Cauchy seq and $\exists$ subseq convg to L. Prove the seq convg to L.*

SOLUS: Supp subseq $a_{j_1}, a_{j_2}, \cdots$ convg to L. Then $\forall \varepsilon > 0, \exists j_m, \forall j_k \in \{j_m, j_{m+1}, \cdots\}$, $\|a_{j_k} - L\| < \varepsilon$,

and $\exists n \in \mathbf{N}^+, \forall i \geqslant n$, $\|a_i - L\| = \|(a_i - a_{j_k}) + (a_{j_k} - L)\| \leqslant \|a_i - a_{j_k}\| + \|a_{j_k} - L\| < 2\varepsilon.$ □

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34 *Supp closed $F_1 \subseteq \mathbf{R}^n$ and closed bounded $F_2 \subseteq \mathbf{R}^n$. Prove $F_1 + F_2$ is closed.*

SOLUS:

ENDED