# ABBREVIATION TABLE

## A B

| | |
|---|---|
| abs | absolute |
| add | addi(tion)(tive) |
| adj | adjoint |
| algo | algorithm |
| arb | arbitrary |
| assoc | associa(tive)(tivity) |
| asum | assum(e)(ption) |
| becs | because |

## C

| | |
|---|---|
| closd | closed under |
| coeff | coefficient |
| combina | combination |
| commu | commut(es)(ing)(ativity) |
| cond | condition |
| corres | correspond(s)(ing) |
| conveni | convenience |
| convly | conversely |
| count- | counter- |
| ctradic | contradict(s)(ion) |
| ctrapos | constrapositive |

## D

| | |
|---|---|
| Ddkd | Dedekind |
| def | definition |
| deg | degree |
| deri | derivative(s) |
| diff | differentia(l)(ting)(tion) |
| dim | dimension(al) |
| disti | distinct |
| distr | distributive propert(ies)(ty) |
| div | div(ide)(ision) |

## E

| | |
|---|---|
| -ec | -ec(t)(tor)(tion)(tive) |
| elem | element(s) |
| ent | entr(y)(ies) |
| equa | equality |
| equiv | equivalen(t)(ce) |
| exa | example |
| exe | exercise |
| exis | exist(s)(ing) |
| existns | existence |
| expo | exponent |
| expr | expression |

## F G H

| | |
|---|---|
| factoriz | factorizaion |
| fini | finite |
| finide | finite-dimensional |
| homo | homogeneity |
| hypo | hypothesis |

## I

| | |
|---|---|
| id | identity |
| immed | immediately |
| induc | induct(ion)(ive) |
| infily | infinitely |
| inje | injectiv(e)(ity) |
| inv | inver(se)(tib-le/ility) |
| iso | isomorph(ism)(ic) |

## L

| | |
|---|---|
| liney | linear(ly) |
| linity | linearity |
| len | length |
| low- | lower- |

## M N

| | |
|---|---|
| max | maxi(mal(ity))(mum) |
| min | mini(mal(ity))(mum) |
| multi | multipl(e)(icati-on/ve) |
| non0 | nonzero |
| nonC | nonconst |
| notat | notation(al) |

## O P Q

| | |
|---|---|
| othws | otherwise |
| orthog | orthogonal |
| orthon | orthonormal |
| poly | polynomial |
| posi | positive |
| prod | product |
| quad | quadratic |
| quotient | quot |

## R

| | |
|---|---|
| recurly | recursively |
| repeti | repetition(s) |
| repres | represent(s)(ation(s)) |
| req | require(s)(d)/requiring |
| respectly | respectively |
| restr | restrict(ion)(ive)(ing) |
| rev | revers(e(s))(ed)(ing) |

## S

| | |
|---|---|
| seq | sequence |
| simlr | similar(ly) |
| solus | solution |
| sp | space |
| stmt | statement |
| std | standard |
| supp | suppose |
| surj | surjectiv(e)(ity) |
| suth | such that |

## T U V W X Y Z

| | |
|---|---|
| uniq | unique |
| uniqnes | uniqueness |
| val | value |
| -wd | -ward |
| -ws | -wise |
| wrto | with respect to |

## 0·B **Note:** $C, D$ are Dedekind cuts. Numbers used here are always rational.

- $\tilde{0} = \{a : a < 0\} = \mathbf{Q}\backslash\mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
  $D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \iff a > b,\ b \notin D\}$.

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- **Tips:** *Prove* $\forall\,\varepsilon > 0,\ \exists\, b \notin D$ *suth* $b - \varepsilon \in D$.

**Solus:** Asum $\exists\,\varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \iff \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.
For any $d \in D$, $\exists\, n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

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**1** *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

**Solus:** (a) $\forall d \in D,\ \exists\,\varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.
$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By Tips, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

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**5** *Prove* $D$ *is posi* $\iff -D$ *is not posi* $\iff 0 \in D$.

**Solus:** $0 \notin -D = \{a : a < -b,\ b \notin D\} \iff \nexists\, b \notin D,\ b < 0 \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subsetneq D \iff 0 \in D$. □

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- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ = \emptyset \iff D \subsetneq \mathbf{Q}\backslash\mathbf{Q}^+ \iff 0 \notin D \iff D$ not posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q}\backslash D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \mathbf{Q}\backslash(D \cup \mathbf{Q}^+)$.
  Then $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. And $\tilde{0}^- = \{0\}$.
  $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \in (\mathbf{Q}\backslash D) \cap (\mathbf{Q}\backslash\mathbf{Q}^*) \subseteq D^-\}$.
  $(-D)^- = (\mathbf{Q}\backslash -D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.
  Define $\tilde{1} = \{a : a < 1\} \Rightarrow \tilde{1}^+ = \{a : 0 < a < 1\},\ \tilde{1}^- = \{a : a \leqslant 0\} = \tilde{0} \cup \{0\}$.
  Then $-\tilde{1} = \{a : -a > 1 \iff a < -1\}$ not posi.

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- For $C, D$ posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  Note that $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$.

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- For $C, D$ posi, define $(-C)(-D) = CD$.

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- For a posi $D$, $\tilde{1}D = \{a : a \leqslant ij < j,\ 0 < i < 1,\ j \in D^+\} \subseteq D$.
  Now $(\tilde{1}D)^+ \subseteq D^+$. 又 $\forall d \in D^+,\ \exists\,\varepsilon > 0,\ d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon)\frac{d}{d + \varepsilon} \in \tilde{1}D$.

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- We want to define $CD$ for posi $C$ and non-posi $D$.
  For non-posi $D$, define $-\tilde{1}D = \tilde{1}(-D) = -D$. For posi $C$, define $-\tilde{1}C = -C$.
  $CD = C(-\tilde{1}-D)$

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**3** *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

**Solus:** (a) Supp $C \subsetneq D$. Let $d \in D \backslash C \subseteq \mathbf{Q} \backslash C$. For any $a \in \tilde{0}$, let $y = d - a > d \Rightarrow a = d - y \in D - C$.

(b) Supp $D - C$ posi. Becs $d = y$ for some $d \in D$, $y > x \notin C$.

Note that $x \notin C \Rightarrow \forall c \in C$, $c < x < y = d \Rightarrow C \subseteq D$. $\square$

**Ended**

# 0·C

**5** *Supp* $a_1, a_2, \ldots$ *is a seq in* $\mathbf{Q}$, *and* $\sup\{a_1, a_2, \ldots\} = \sqrt{2}$.
*Prove* $\sup\{a_n, a_{n+1}, \ldots\} = \sqrt{2}$ *for all* $n \in \mathbf{N}^+$.

**Solus:** Becs the sup not in seq $\Rightarrow$ infily many disti elem.
$\forall a_i,\ \exists a_j,\ a_i < a_j < \sqrt{2}$. For $a_{n+k}$, choose $a_i > a_{n+k}$. If $i \in \{1, \ldots, n\}$, then choose $a_j > a_i$.
After at most $(n+1)$ steps, we have $a_m$ with $m > n$. Thus $\forall a_{n+i},\ \exists a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. $\square$

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- *Supp nonempty* $A \subseteq \mathbf{R}$.
- **Tips 1:** *Define* $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. *Prove* $\sup(-A) = -\inf A$.

**Solus:** $-b$ is an upper bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a lower bound of $A$.
Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. $\square$

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- **Tips 2:** *Show if* $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists a \in A,\ a < x$.

**Solus:** (a) $\nexists a > x \iff \forall a \in A,\ a \leqslant x$. Then by def of sup.
Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). $\square$

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**6** *Supp* $A, B \subseteq \mathbf{R}$ *has infily many disti elem, so has* $A + B = \{a + b : a \in A, b \in B\}$.
*Prove* $\sup(A+B) = \sup A + \sup B$, *and* $\inf(A+B) = \inf A + \inf B$.

**Solus:** $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B,\ \inf A + \inf B \leqslant \inf(A+B)$.
$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$
$\iff \exists a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists a + b > \sup(A+B)$. Ctradic.
Simlr for $\inf(A+B) \in A + B$. Or. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. $\square$

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**16** *Supp* $\mathbf{R}_1, \mathbf{R}_2$ *are complete ordered fields. Define* $\varphi_1, \varphi_2$ *as in* [0.11].
*Define* $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. *Define* $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.
(a) *Show* $\psi = \psi_1 : \mathbf{R}_1 \to \mathbf{R}_2$ *is one-to-one.* (b) *Show* $\psi(0) = 0,\ \psi(1) = 1$.
(c) *Show* $\psi(a \pm b) = \psi(a) \pm (b)$, *and* $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.
(d) *Supp* $a \in \mathbf{R}_1$. *Show* $a > 0 \iff \psi(a) > 0$.

**Solus:** (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.
Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.
Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.
Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.
(b) Note that $\varphi(q) < 0 \iff q < 0$, and $\varphi(0) = 0$.
$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \cdots + 1)^{-1} \leqslant 1\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+),\ \varphi_2(q) \leqslant 1$.
(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leqslant a \pm b\}$.
$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leqslant ab^{-1}\}$.
(d) $a > 0 \iff \exists n \in \mathbf{N}^+,\ 1/n < a \iff \psi_1(a) > 0$. $\square$

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**Ended**