这是我个人挑战「***Measure, Integration & Real Analysis, by Sheldon Axler***」的学习笔记，包括课文补注和部分习题。我先从 ***Supplement*** 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节本来不太要命，但超出课文的补助却让我折戟沉沙——的确，它们不需要硬性知识门槛，可以用集合和数理逻辑来推导 $\mathcal{D}$ 的一切。或许是缺乏 Dedekind cut 的系统学习，我推导这一切时感到我在亲手缔造一个数学分支；我不是自傲的意思，只是说，**这非常艰难**。但我还是坚持下来了；在此过程中我肉眼可见我在数理逻辑上的提升。

Abbreviation Table

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B **Note:** C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.
  Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q}\backslash\mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
- Define $-D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \iff a > b,\ b \notin D\}$.
  $-(-D) = -\{a : a < -b,\ b \notin D\} = \{c : c < -a,\ a \geqslant -b,\ \forall b \notin D\} = \{c : c < b,\ \forall b \notin D\} = D$.
  The last equa is becs (a) $d \notin D \Rightarrow \exists\, b \notin D,\ d \geqslant b$, and (b) $d \in D \Rightarrow$ if $\exists\, b \notin D$ suth $d \geqslant b$, then $b \in D$, ctradic.

- **Tips:** *Prove* $\forall \varepsilon > 0,\ \exists\, b \notin D$ *suth* $b - \varepsilon \in D$.

Solus: Asum $\exists\, \varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \iff \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.
Now $\forall d \in D,\ \exists\, n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

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1 *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

Solus: (a) $\forall d \in D,\ \exists\, \varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.
(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.
$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By Tips, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

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3 *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

Solus: (a) $C \subsetneq D \Rightarrow \exists\, x \in D\backslash C \Rightarrow \exists\, y \in D,\ y > x \Rightarrow \exists\, d \in D,\ d > y \iff 0 < d - y \in D - C$.
(b) $0 \in D - C \Rightarrow \exists\, y > x \notin C,\ y \in D \Rightarrow \forall c \in C,\ c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. □

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5 *Prove* (a) D *posi* $\Rightarrow -D$ *not posi,* (b) *non0* $-D$ *not posi* $\Rightarrow D$ *posi.*

Solus: (a) $0 \notin \{a : a < -b,\ b \notin D\} \iff \nexists\, b \notin D,\ 0 < -b \iff \forall b \notin D,\ b \geqslant 0 \Longleftarrow 0 \in D$.
(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.
Or. $\exists\, a < 0,\ a \notin -D = \{a : -a > b,\ b \notin D\} \iff \nexists\, b \notin D,\ -a > b \iff \forall b \notin D,\ 0 < -a \leqslant b$. □

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subsetneq D \iff 0 \in D \iff D$ posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q}\backslash D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \mathbf{Q}\backslash(D \cup \mathbf{Q}^+)$.
  (a) $D^- = \{0\} \Longleftarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leqslant 0\} = \{0\} \Longrightarrow \mathbf{Q}\backslash D = \mathbf{Q}^*$.
  (b) $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. **Coro:** D not posi $\iff 0 \in D^-$.
  (c) $(D^-)^- = \{r \in D : r \leqslant 0\} = \mathbf{Q}\backslash D^+$. **Coro:** D not posi $\iff (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \notin D \iff b \in D^-\backslash\{0\}\}$.
  $(-D)^- = (\mathbf{Q}\backslash -D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that '$a \leqslant cd$' here is equiv to '$a < cd$'.

- For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  $CD = \{0 < cd < (-r)(-s) : r \in C^-\backslash\{0\}, s \in D^-\backslash\{0\}\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.
  If $C, -C$ not posi $\Rightarrow C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlr for $D, -D$ not posi.
- The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.

- For D posi, define $D^{-1} = \{a : a < 1/b,\ b \notin D\} \Rightarrow DD^{-1} = \{a : a \leqslant d/b < 1,\ b \notin D,\ d \in D^+\} = \tilde{1}$.
  The last equa holds becs $\forall a \in \tilde{1} \cap \mathbf{Q}^+$, $\exists\, d \in D^+$ suth $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$.
- For non0 D not posi, define $D^{-1} = \{a : a < 1/b,\ b \in D^-\} \Rightarrow (D^{-1})^- = \{s : 0 \geqslant s \geqslant 1/b,\ \forall b \in D^-\}$.
  Then $DD^{-1} = \{a : a < rs,\ r \in D^-,\ s \in (D^{-1})^-\} = \{a : a < rs,\ \exists\, r \in D^-,\ 0 \leqslant rs \leqslant r/b,\ \forall b \in D^-\}$.
  Asum $\exists\, a \in \tilde{1} \cap \mathbf{Q}^+$, $\forall r \in D^-$, $s \in (D^{-1})^-$, $a \geqslant rs \Longrightarrow a \geqslant r/b \Longleftrightarrow r/a \leqslant b$, $\forall b \in D^-$. Let $r = 0$.

- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
  $CD = -C(-D) = -\{a : a < rt,\ r \in C^-,\ t \in (-D)^-\} = \{-a : a > b,\ b \geqslant rt,\ \forall r \in C^-,\ t \in (-D)^-\}$
  $= \{-a : a > rt,\ \forall r \in C^-,\ 0 \geqslant t \geqslant -s,\ \forall s \notin D\} = \{a : a < ru,\ \forall r \in C^-,\ 0 \leqslant u \leqslant s,\ \forall s \notin D\}$.
  $(r \leqslant 0 < s,\ rs \leqslant ru = -rt \leqslant 0 \leqslant rt \leqslant -rs.)$
- Note the '$0 \leqslant u$'. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$, for if not, $\exists\, u > 0$, and $\exists\, r \in C^-\backslash\{0\}$, such that $\exists\, a < ru < 0$. Hence '$0 \leqslant u$' is actually '$0 < u$'.
- '$u \leqslant s$' cannot be abbreviated as in $\{-a : a > -rs,\ \forall s \notin D,\ r \in C^-\} = \{a : a < rs,\ \forall s \notin D,\ r \in C^-\}$.
  '$u \leqslant s$' cannot be '$u < s$', becs here $rs < ru \Rightarrow \exists\, a = rs$. Simlr for '$a < ru$' to be '$a \leqslant ru$' with '$u < s$'.
- Note that $\{u : 0 < u \leqslant s,\ \forall s \notin D\} \supsetneq D^+$. We show "$0 < u \leqslant s,\ \forall s \notin D$" is equiv to "$\forall u \in D^+$".
  Immed, $LHS \subseteq RHS$. For $RHS \subseteq LHS$, if we cannot fix '$\min \mathbf{Q}\backslash D$' by a certain scalar s_M, then done.
  Othws, becs $s_M = \max D$, now "$a < rs_M,\ \forall r \in C^-$".
  Note that if $0 < d_1 < \cdots < d_n < \cdots < d_\infty \in D^+$ $[d_n < s_M]$ and $0 > r_1 > \cdots > r_m > \cdots > r_\infty \in C^-$,
  NOTICE that d_∞ is an ordered pair (d_n, d_{n+1}) with $d_n < d_{n+1}$. Simlr, (r_m, r_{m+1}) with $r_{m+1} < r_m$.
  We show "$a < r_m\, d_n$" is equiv to "$a < r_m\, s_M$". Becs $r_{m+1}\, s_M < r_m\, s_M < r_m\, d_{n+1} < r_m\, d_n > r_{m+1}\, d_{n+1}$.
  Asum $r_m\, s_M \leqslant t < r_m\, d_n$, $\forall m, n \in \mathbf{N}^+$. Let $u = t/r_m \Rightarrow s_M \geqslant u > d_n$. Ctradic. $\square$

- Consider $-C$ and D both posi. Omitting $C = \tilde{0}$.
  $CD = -[(-C)D] = -\{a : a \leqslant cd,\ c \in (-C)^+,\ d \in D^+\} = \{-a : a > b,\ b > cd,\ \forall c \in (-C)^+,\ d \in D^+\}$
  $= \{a : a < -cd,\ \forall d \in D^+,\ \forall c \text{ suth } \underline{0 < c < -r,\ \exists\, r \in C^-}\}$ $\qquad (rd < \underline{-cd < 0})$

- Let $LHS = \{a : a < ru,\ \forall r \in C^-, 0 < u \leqslant s,\ \forall s \notin D\}$, $RHS = \{a : a \leqslant rd,\ \forall r \in C^-, \forall d \in D^+\}$.
  Where $\tilde{0} \neq C$ is not posi, D posi. We show 'rd' < 'ru', so that $LHS = RHS$.
  Seems equiv to 'd' > 'u' ⩾ 's', while $d \in D$, $s \notin D$, thus ctradic. NOTICE that 'r, d, u' are **not certain**.
  Supp $a < ru,\ \forall r, u$. Asum $a > r'd,\ \exists r', d$. Now $r'd < a < ru,\ \forall r, u$. Let $r = r' \Rightarrow d > u$, ctradic.
  Supp $a \leqslant rd,\ \forall r, d$. Asum $a \geqslant r'u,\ \exists r', u$. $(r' \neq 0)$ Now $r'u \leqslant a \leqslant rd,\ \forall r, d \Rightarrow u \geqslant d,\ \forall d \in D^+$.
  In fact, $u > d,\ \forall d \in D^+ \Rightarrow u \notin D$.
  If $\exists$ certain smallest elem s_M in $\mathbf{Q} \backslash D$, and if $u = s_M$, then $LHS = \{a : a < rs_M,\ \forall r \in C^-\}$.
  Othws, $\exists d \in D^+$, $0 < u < d < s$, $\forall s \notin D$, ctradic.

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4 *Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD)$, $B(C + D) = BC + BD$.*

SOLUS: We discuss in cases.

$\backslash$	1	2	3	4	5	6	7	8
B	+	+	+	+	-	-	-	-
C	+	+	-	-	-	+	-	+
D	+	-	+	-	-	-	+	+

(1) $\big((BC)D\big)^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = \big(B(CD)\big)^+$.
(4) $(BC)D = \{a : a < rs,\ r \in (BC)^-, s \in D^-\}$.

$\big(B(C + D)\big)^+ = \{bs : b \in B^+, s \in (C + D)^+\} = \{bc + bd : b \in B^+, -bc < bd, c \in C, d \in D\} = P$.
$\{bc : b \in B^+, c \in C^+\} + \{b'd : b' \in B^+, d \in D^+\} = \{bc + b'd : b, b' \in B^+, c \in C^+, d \in D^+\} = Q$.

ENDED

0·C

5 *Supp* $a_1, a_2, \dots$ *is a seq in* $\mathbf{Q}$, *and* $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.
Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ *for all* $n \in \mathbf{N}^+$.

Solus: Becs the sup not in seq $\Rightarrow$ infily many disti elem.
$\forall a_i,\ \exists\, a_j,\ a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.
After at most $(n+1)$ steps, we have a_m with $m > n$. Thus $\forall a_{n+i},\ \exists\, a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. $\square$

- *Supp nonempty* $A \subseteq \mathbf{R}$.
- **Tips 1:** *Define* $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. *Prove* $\sup(-A) = -\inf A$.

Solus: $-b$ is an upper bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a lower bound of A.
Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. $\square$

- **Tips 2:** *Show if* $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists\, a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists\, a \in A,\ a < x$.

Solus: (a) $\nexists\, a > x \iff \forall a \in A,\ a \leqslant x$. Then by def of sup.
Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). $\square$

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6 *Supp* $A, B \subseteq \mathbf{R}$ *has infily many disti elem, so has* $A + B = \{a + b : a \in A, b \in B\}$.
Prove $\sup(A + B) = \sup A + \sup B$, *and* $\inf(A + B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B,\ \inf A + \inf B \leqslant \inf(A+B)$.
$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$
$\iff \exists\, a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists\, a + b > \sup(A+B)$. Ctradic.
Simlr for $\inf(A+B) \in A + B$. Or. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. $\square$

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16 *Supp* $\mathbf{R}_1, \mathbf{R}_2$ *are complete ordered fields. Define* φ_1, φ_2 *as in* [0.11].
Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. *Define* $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.
(a) *Show* $\psi = \psi_1 : \mathbf{R}_1 \to \mathbf{R}_2$ *is one-to-one.* (b) *Show* $\psi(0) = 0,\ \psi(1) = 1$.
(c) *Show* $\psi(a \pm b) = \psi(a) \pm (b)$, *and* $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.
(d) *Supp* $a \in \mathbf{R}_1$. *Show* $a > 0 \iff \psi(a) > 0$.

Solus: (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.
Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.
Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.
Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.
(b) Note that $\varphi(q) < 0 \iff q < 0$, and $\varphi(0) = 0$.
$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leqslant 1\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+),\ \varphi_2(q) \leqslant 1$.
(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leqslant a \pm b\}$.
$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leqslant ab^{-1}\}$.
(d) $a > 0 \iff \exists\, n \in \mathbf{N}^+,\ 1/n < a \iff \psi_1(a) > 0$. $\square$

Ended