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这是我个人挑战「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节本来不太要命，但超出课文的补助却让我折戟沉沙——的确，它们不需要硬性知识门槛，可以用集合和数理逻辑来推导  $\mathcal{D}$  的一切。或许是缺乏 Dedekind cut 的系统学习，我推导这一切时感到我在亲手缔造一个数学分支；我不是自傲的意思，只是说，这非常艰难。但我还是坚持下来了；在此过程中我肉眼可见我在数理逻辑上的提升。

## ABBREVIATION TABLE

### A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

### E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existsns	existence
expo	exponent
expr	expression

### L

liney	linear.ly
linity	linearity
len	length
low-	lower-

### R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

### C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	contrapositive

### D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

### I

id	identity
immed	immediately
induc	induct(ion)(ive)
infil	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

### F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

### M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

### S

seq	sequence
simlr	similar.ly
soluts	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

### O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quot	quotient

### T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

## 0.B Note: $C, D$ are Dedekind cuts. Numbers used here are always rational.

- Define  $\tilde{q} = \{a : a < q\}$ , and  $-\tilde{q} = \tilde{-q} = \{a : a < -q\}$ .  
Then  $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^*$   $\Rightarrow -\tilde{0} = \{a : a < -b \leq 0\} = \tilde{0}$ .
- Define  $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$ .  
 $-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$ .  
The last equa is becs (a)  $d \notin D \Rightarrow \exists b \notin D, d \geq b$ , and (b)  $d \in D \Rightarrow$  if  $\exists b \notin D$  suth  $d \geq b$ , then  $b \in D$ , ctradic.

- TIPS: Prove  $\forall \varepsilon > 0, \exists b \notin D$  suth  $b - \varepsilon \in D$ .

SOLUS: Asum  $\exists \varepsilon > 0$  suth  $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$ .

Then  $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$  for any  $n \in \mathbf{N}^+$ .

Now  $\forall d \in D, \exists n \in \mathbf{N}^+$  suth  $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$ , ctradic.  $\square$

1 Prove (a)  $D + \tilde{0} = D$ , (b)  $-D$  is Dedekind cut, and  $D + (-D) = \tilde{0}$ .

SOLUS: (a)  $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$ .

(b) Asum  $x \in -D$  is the largest elem of  $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$ .

Let  $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$ .

Thus by def,  $x + \delta \in -D$ , ctradic the max of  $x \in -D$ . Hence  $-D$  is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$ .

Supp  $a \in \tilde{0} \Rightarrow -a > 0$ . By TIPS,  $\exists b \notin D$  suth  $b + a \in D$ .

Note that  $b < b - a \notin D \Rightarrow -b > -b + a \in -D$ . Then  $(-b + a) + (b + a) = 2a < 0$ .

Thus  $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$ .  $\square$

3 Show  $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$  posi.

SOLUS: (a)  $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$ .

(b)  $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$ . 又  $D - C \neq \tilde{0}$ .  $\square$

5 Prove (a)  $D$  posi  $\Rightarrow -D$  not posi, (b) non0  $-D$  not posi  $\Rightarrow D$  posi.

SOLUS: (a)  $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$ .

(b) Becs  $\tilde{0}$  is the largest non posi cuts. Thus  $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$  posi.

Or.  $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$ .  $\square$

- Define  $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$ . Then  $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi.

Define  $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$ .

(a)  $D^- = \{0\} \Leftrightarrow D = \tilde{0}$ . Convly,  $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$ .

(b)  $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi. CORO:  $D$  not posi  $\Leftrightarrow 0 \in D^-$ .

(c)  $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$ . CORO:  $D$  not posi  $\Leftrightarrow (-D)^- = D$ .

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$ .

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$ .

- For  $C, D$  posi, define  $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$ . Note that ' $a \leq cd$ ' here is equiv to ' $a < cd'$ .

- For  $-C, -D$  posi, define  $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)\$ .  
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$ .  
If  $C, -C$  not posi  $\Rightarrow C = \tilde{0}$ , then with the asum  $\tilde{0}D = \tilde{0}$ , it still holds. Simlr for  $D, -D$  not posi.
  - The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.
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- For  $D$  posi, define  $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$ .  
The last equa holds becs  $\forall a \in \tilde{1} \cap \mathbf{Q}^+$ ,  $\exists d \in D^+$  suth  $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$ .
  - For non0  $D$  not posi, define  $D^{-1} = \{a : a < 1/b, b \in D^-\} \Rightarrow (D^{-1})^- = \{s : 0 \geq s \geq 1/b, \forall b \in D^-\}$ .  
Then  $DD^{-1} = \{a : a < rs, r \in D^-, s \in (D^{-1})^-\} = \{a : a < rs, \exists r \in D^-, 0 \leq rs \leq r/b, \forall b \in D^-\}$ .  
Asum  $\exists a \in \tilde{1} \cap \mathbf{Q}^+$ ,  $\forall r \in D^-, s \in (D^{-1})^-, a \geq rs \Rightarrow a \geq r/b \Leftrightarrow r/a \leq b, \forall b \in D^-$ . Let  $r = 0$ .
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- For  $C$  not posi and  $D$  posi, we expect that  $CD$  not posi. Consider  $C$  and  $-D$  both not posi.  
 $CD = -C(-D) = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$   
 $= \{-a : a > rt, \forall r \in C^-, 0 \geq t \geq -s, \forall s \notin D\} = \{a : a < ru, \forall r \in C^-, 0 \leq u \leq s, \forall s \notin D\}$ .  
 $(r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs)$
  - Note the ' $0 \leq u'$ . Becs  $C^- \neq \emptyset \Rightarrow 0 \in C^-$ . If it is to be exactly  $CD = \{a : a < 0\}$ , then  $C^- = \{0\}$ ,  
for if not,  $\exists u > 0$ , and  $\exists r \in C^- \setminus \{0\}$ , such that  $\exists a < ru < 0$ . Hence ' $0 \leq u'$  is actually ' $0 < u'$ .
  - ' $u \leq s'$  cannot be abbreviated as in  $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$ .  
' $u \leq s'$  cannot be ' $u < s'$ ', becs here  $rs < ru \Rightarrow \exists a = rs$ . Simlr for ' $a < ru$ ' to be ' $a \leq ru$ ' with ' $u < s'$ '.
  - Note that  $\{u : 0 < u \leq s, \forall s \notin D\} \supsetneq D^+$ .
- 

- Consider  $-C$  and  $D$  both posi. Omitting  $C = \tilde{0}$ .  
 $CD = -[(-C)D] = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} = \{-a : a > b, b > cd, \forall c \in (-C)^+, d \in D^+\}$   
 $= \{a : a < -cd, \forall d \in D^+, \forall c \text{ suth } 0 < c < -r, \exists r \in C^-\} \quad (rd < -cd < 0)$   
 $\stackrel{*}{=} \{a : a < rd, \forall r \in C^-, \forall d \in D^+\}$ .

Intuitively,  $0 < d_1 < \dots < d_n < \sup D$ , and  $\sup C \leq r_m < \dots < r_1 < 0$ . Becs  $a < r_j d_k, \forall j, k$ ,  
 $\forall r_{j+1} < r_j d_k < r_j d_{k-1} \Rightarrow a < r_m d_n < r_j d_k$ . Thus ' $a < rd'$  cannot be ' $a \leq rd'$ .

(\*.a) Supp  $a < -cd < 0, \forall c \in (-C)^+, d \in D^+$ . Asum  $-cd > a \geq rd', \exists r \in C^-, d' \in D^+$ .

Let  $d = d' \Rightarrow r \leq -c \Leftrightarrow c \leq -r$ . Now  $r < 0$ .

If  $\exists$  min upper bound in  $\mathbf{Q}$ , then  $r = \sup C$ ,

othws  $r > \sup C \Rightarrow -r < -\sup C = \sup(-C) \Rightarrow \exists c > -r, c < \sup(-C)$ , simlr if  $\sup C \notin \mathbf{Q}$ .

(\*.b) Supp  $a \leq c'd, \forall c', d$  ( $\Rightarrow a < 0$ ). Asum  $a \geq -cd, \exists c', d'$ .

- Let  $LHS = \{a : a < ru, \forall r \in C^-, 0 < u \leq s, \forall s \notin D\}$ ,  $RHS = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}$ . Where  $\tilde{0} \neq C$  is not posi,  $D$  posi. We show ' $rd' < 'ru'$ ', so that  $LHS = RHS$ .  
**Seems** equiv to ' $d' > 'u' \geq 's'$ ', while  $d \in D, s \notin D$ , thus contradic. NOTICE that ' $r, d, u$ ' are **not certain**.  
 $\text{Supp } a < ru, \forall r, u$ . Asum  $a > r'd, \exists r', d$ . Now  $r'd < a < ru, \forall r, u$ . Let  $r = r' \Rightarrow d > u$ , contradic.  
 $\text{Supp } a \leq rd, \forall r, d$ . Asum  $a \geq r'u, \exists r', u$ . ( $r' \neq 0$ ) Now  $r'u \leq a \leq rd, \forall r, d \Rightarrow u \geq d, \forall d \in D^+$ . In fact,  $u > d, \forall d \in D^+ \Rightarrow u \notin D$ .  
If  $\exists$  certain smallest elem  $s_M$  in  $\mathbf{Q} \setminus D$ , and if  $u = s_M$ , then  $LHS = \{a : a < rs_M, \forall r \in C^-\}$ . Othws,  $\exists d \in D^+, 0 < u < d < s, \forall s \notin D$ , contradic.
- 

- For a posi  $D$ ,  $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$ .

Now  $(\tilde{1}D)^+ \subseteq D^+$ . 又  $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in \tilde{1}D$ .

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ENDED

# 0.C

**5** *Supp  $a_1, a_2, \dots$  is a seq in  $\mathbf{Q}$ , and  $\sup\{a_1, a_2, \dots\} = \sqrt{2}$ .*

*Prove  $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$  for all  $n \in \mathbf{N}^+$ .*

**SOLUS:** Becs the sup not in seq  $\Rightarrow$  inflly many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$ . For  $a_{n+k}$ , choose  $a_i > a_{n+k}$ . If  $i \in \{1, \dots, n\}$ , then choose  $a_j > a_i$ .

After at most  $(n+1)$  steps, we have  $a_m$  with  $m > n$ . Thus  $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$ .  $\square$

• *Supp nonempty  $A \subseteq \mathbf{R}$ .*

• **TIPS 1:** Define  $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$ . Prove  $\sup(-A) = -\inf A$ .

**SOLUS:**  $-b$  is an upper bound of  $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$  is a lower bound of  $A$ .

Thus  $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$ .  $\square$

• **TIPS 2:** Show if  $x \in \mathbf{R}$ , (a)  $\sup A > x \Rightarrow \exists a \in A, a > x$ , (b)  $\inf A < x \Rightarrow \exists a \in A, a < x$ .

**SOLUS:** (a)  $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x$ . Then by def of sup.

Or. By (b),  $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$ .

Simlr for (b).  $\square$

**6** *Supp  $A, B \subseteq \mathbf{R}$  has inflly many disti elem, so has  $A + B = \{a + b : a \in A, b \in B\}$ .*

*Prove  $\sup(A + B) = \sup A + \sup B$ , and  $\inf(A + B) = \inf A + \inf B$ .*

**SOLUS:**  $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$ ,  $\inf A + \inf B \leq \inf(A + B)$ .

$\sup A + \sup B > \sup(A + B) \Leftrightarrow \sup A > \sup(A + B) - \sup B$

$\Leftrightarrow \exists a + \sup B > \sup(A + B) \Leftrightarrow \sup B > \sup(A + B) - a \Leftrightarrow \exists a + b > \sup(A + B)$ . Ctradic.

Simlr for  $\inf(A + B) \in A + B$ . Or. Apply to  $-A - B$ , becs  $\sup(-A) = -\inf A$ .  $\square$

**16** *Supp  $\mathbf{R}_1, \mathbf{R}_2$  are complete ordered fields. Define  $\varphi_1, \varphi_2$  as in [0.11].*

Define  $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Define  $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$ .

(a) Show  $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  is one-to-one. (b) Show  $\psi(0) = 0, \psi(1) = 1$ .

(c) Show  $\psi(a \pm b) = \psi(a) \pm \psi(b)$ , and  $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$ .

(d) Supp  $a \in \mathbf{R}_1$ . Show  $a > 0 \Leftrightarrow \psi(a) > 0$ .

**SOLUS:** (a) Define  $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$ .

Define  $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_2$ . Then  $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  well-defined.

Note that  $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$ .

Now  $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_2 = a$ . Rev the roles of  $\mathbf{R}_1, \mathbf{R}_2$ .

(b) Note that  $\varphi(q) < 0 \Leftrightarrow q < 0$ , and  $\varphi(0) = 0$ .

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ ,  $\varphi_2(q) \leq 1$ .

(c)  $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$ .

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(p) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$ .

(d)  $a > 0 \Leftrightarrow \exists n \in \mathbf{N}^+, 1/n < a \Leftrightarrow \psi_1(a) > 0$ .  $\square$

ENDED