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ABBREVIATION TABLE

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

T U V W X Y Z

uniq	unique
uniques	uniqueness
val	value
-wd	-ward
-ws	-wise
wрто	with respect to

0.B NOTE: C, D are Dedekind cuts. Numbers used here are always rational.

• Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.

Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -0 \leq 0\} = \tilde{0}$.

• Define $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$.

$-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$.

• **TIPS:** Prove $\forall \varepsilon > 0, \exists b \notin D$ suth $b - \varepsilon \in D$.

SOLUS: Asum $\exists \varepsilon > 0$ suth $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$.

Then $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.

Now $\forall d \in D, \exists n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

1 Prove (a) $D + \tilde{0} = D$, (b) $-D$ is Dedekind cut, and $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$.

Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.

Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$.

Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists b \notin D$ suth $b + a \in D$.

Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.

Thus $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

CORO: $\{a_1 + a_2 : a_1, a_2 \in D\} = D + D = \tilde{0} = \{a : a < 0\} \Rightarrow \forall a \in D, a + a < 0 \Leftrightarrow a < 0$.

5 Prove D is posi $\Leftrightarrow -D$ is not posi $\Leftrightarrow 0 \in D$.

SOLUS: $0 \notin -D = \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, b < 0 \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \not\subseteq D \Leftrightarrow 0 \in D$. □

• Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ = \emptyset \Leftrightarrow D \subsetneq \mathbf{Q} \setminus \mathbf{Q}^+ \Leftrightarrow 0 \notin D \Leftrightarrow D$ not posi.

Define $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$.

(a) $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$ posi. **CORO:** D not posi $\Leftrightarrow 0 \in D^-$.

(b) $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$. **CORO:** D not posi $\Leftrightarrow (-D)^- = D$.

• $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$.

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$.

• For C, D posi, define $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.

• For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.

$CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.

If C, D not posi while $-C, -D$ not posi $\Rightarrow C = D = \tilde{0}$, then with the asum $\tilde{0}\tilde{0} = \tilde{0}$, it still holds.

• The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.

- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.

$$\begin{aligned} CD &= -C(-D) = -\{a : a < rt, r \in C^-, t \in (-D)^-\} \\ &= \{-a : a > rt, \forall r \in C^-, 0 \geq t \geq -s, \forall s \notin D\} & (r \leq 0, -s \leq t \leq 0 \Rightarrow -rs \geq rt \geq 0) \\ &= \{a : a < ru, \forall r \in C^-, 0 \leq u \leq s, \forall s \notin D\}. \end{aligned}$$

- Note the ' $0 \leq u$ '. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $a < 0$ in $LHS (= \tilde{0})$, then $C^- = \{0\}$, for if not, $\exists u = s > 0$, and $\exists r \in C^- \setminus \{0\}$, such that $\exists a < ru < 0$. Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
- ' $u \leq s$ ' cannot be abbreviated as in $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$. ' $a < ru$ ' cannot be changed to ' $a \leq ru$ ' with " $0 < u < s, \forall s \notin D$ "????
- Let ${}^+D = \{u : 0 < u \leq s, \forall s \notin D\} \supsetneq D^+$.

- Consider $-C$ and D both posi. Omitting $C = \tilde{0}$, becs C not posi $\nRightarrow -C$ posi.

$$\begin{aligned} CD &= -[(-C)D] = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \\ &= \{a : \underline{a < -b} < -cd, \forall d \in D^+, \forall c \text{ suth } \underline{0 < c < -c'}, \exists c' \notin C\} & (0 < \underline{cd} < -c'd) \end{aligned}$$

To make those comfortable, notice the following:

$$(a) A = \{a : a < M\}, B = \{b : b < a, \forall a \in A \Rightarrow b < a < M\} \subseteq \{b : b < M\} = A.$$

$$\text{又 } \forall b < M, \exists \epsilon > 0, b + \epsilon < M \Rightarrow b \in B.$$

$$(b) A = \{a : M < a\}, B = \{b : b < a, \forall a \in A\} \supseteq \{b : b \leq M\}$$

$$\text{又 Supp } b \text{ suth } \forall a \in A, b < a. \text{ If } b > M, \text{ then } b \in A \Rightarrow \exists b' \in A, b' < b.$$

$$\begin{aligned} CD &= \{a : a < -cd, \forall d \in D^+, \forall c \text{ suth } 0 < c < -c', \exists c' \notin C\} & (c'd < -cd < 0) \\ &= \{a : a \leq c'd, \underline{\forall c' \in C^-}, \forall d \in D^+\} = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}. \end{aligned}$$

- Let $LHS = \{cs : c \in C, s \notin D\}$, $RHS = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}$.

$$\text{Supp } a \leq rd. \text{ Let } b \notin D \Rightarrow d < b. \text{ Let } q = b/d.$$

$$r' = rq \Rightarrow r'd = rb.$$

$$\text{Supp } c, s \text{ suth } cs \in LHS. \text{ Then } cs < rs, \forall r \in C^-.$$

- For a posi D , $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$.

$$\text{Now } (\tilde{1}D)^+ \subseteq D^+. \text{ 又 } \forall d \in D^+, \exists \epsilon > 0, d + \epsilon \in D^+ \Rightarrow d = (d + \epsilon) \frac{d}{d + \epsilon} \in \tilde{1}D.$$

3 Show $C \subsetneq D \iff D - C = \{d - y : d \in D, y > x, x \notin C\}$ posi.

SOLUS: (a) Supp $C \subsetneq D$. Let $d \in D \setminus C \subseteq \mathbf{Q} \setminus C$. For any $a \in \tilde{0}$, let $y = d - a > d \Rightarrow a = d - y \in D - C$.

(b) Supp $D - C$ posi. Becs $d = y$ for some $d \in D, y > x \notin C$.

$$\text{Note that } x \notin C \Rightarrow \forall c \in C, c < x < y = d \Rightarrow C \subseteq D. \quad \square$$

4 Prove

SOLUS:

ENDED

0.C

5 Supp a_1, a_2, \dots is a seq in \mathbf{Q} , and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.

Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.

SOLUS: Becs the sup not in seq \Rightarrow infily many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$. For a_{n+k} , choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most $(n+1)$ steps, we have a_m with $m > n$. Thus $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$. \square

• Supp nonempty $A \subseteq \mathbf{R}$.

• **TIPS 1:** Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.

SOLUS: $-b$ is an upper bound of $-A \iff \forall a \in A, -a \leq -b \iff a \geq b \iff b$ is a lower bound of A .

Thus $-b_M = \sup(-A) \iff -b_M \leq -b \iff b_M \geq b \iff b_M = \inf A$. \square

• **TIPS 2:** Show if $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A, a > x$, (b) $\inf A < x \Rightarrow \exists a \in A, a < x$.

SOLUS: (a) $\nexists a > x \iff \forall a \in A, a \leq x$. Then by def of sup.

OR. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$. Simlr for (b). \square

6 Supp $A, B \subseteq \mathbf{R}$ has infily many disti elem, so has $A + B = \{a + b : a \in A, b \in B\}$.

Prove $\sup(A + B) = \sup A + \sup B$, and $\inf(A + B) = \inf A + \inf B$.

SOLUS: $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B, \inf A + \inf B \leq \inf(A + B)$.

$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$

$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B)$. Ctradic.

Simlr for $\inf(A + B) \in A + B$. OR. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. \square

16 Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in [0.11].

Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Define $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.

(a) Show $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$ is one-to-one. (b) Show $\psi(0) = 0, \psi(1) = 1$.

(c) Show $\psi(a \pm b) = \psi(a) \pm \psi(b)$, and $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.

(d) Supp $a \in \mathbf{R}_1$. Show $a > 0 \iff \psi(a) > 0$.

SOLUS: (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Note that $\varphi(q) < 0 \iff q < 0$, and $\varphi(0) = 0$.

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+), \varphi_2(q) \leq 1$.

(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$.

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$.

(d) $a > 0 \iff \exists n \in \mathbf{N}^+, 1/n < a \iff \psi_1(a) > 0$. \square