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这是我个人挑战「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节本来不太要命，但超出课文的补助却让我折戟沉沙——的确，它们不需要硬性知识门槛，可以用集合和数理逻辑来推导 \mathcal{D} 的一切。或许是缺乏 Dedekind cut 的系统学习，我推导这一切时感到我在亲手缔造一个数学分支；我不是自傲的意思，只是说，这非常艰难。但我还是坚持下来了；在此过程中我肉眼可见我在数理逻辑上的提升。

ABBREVIATION TABLE

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existsns	existence
expo	exponent
expr	expression

L

liney	linear.ly
linity	linearity
len	length
low-	lower-

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	contrapositive

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infil	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

S

seq	sequence
simlr	similar.ly
soluts	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quot	quotient

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0.B Note: C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \tilde{-q} = \{a : a < -q\}$.
Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^*$ $\Rightarrow -\tilde{0} = \{a : a < -b \leq 0\} = \tilde{0}$.
- Define $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$.
 $-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$.
The last equa is becs (a) $d \notin D \Rightarrow \exists b \notin D, d \geq b$, and (b) $d \in D \Rightarrow$ if $\exists b \notin D$ suth $d \geq b$, then $b \in D$, ctradic.

- TIPS: Prove $\forall \varepsilon > 0, \exists b \notin D$ suth $b - \varepsilon \in D$.

SOLUS: Asum $\exists \varepsilon > 0$ suth $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$.

Then $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.

Now $\forall d \in D, \exists n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. \square

1 Prove (a) $D + \tilde{0} = D$, (b) $-D$ is Dedekind cut, and $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$.

Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.

Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$.

Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists b \notin D$ suth $b + a \in D$.

Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.

Thus $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. \square

3 Show $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$ posi.

SOLUS: (a) $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$.

(b) $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. \square

5 Prove (a) D posi $\Rightarrow -D$ not posi, (b) non0 $-D$ not posi $\Rightarrow D$ posi.

SOLUS: (a) $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$.

(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.

Or. $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$. \square

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$ posi.

Define $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$.

(a) $D^- = \{0\} \Leftrightarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$.

(b) $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$ posi. CORO: D not posi $\Leftrightarrow 0 \in D^-$.

(c) $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$. CORO: D not posi $\Leftrightarrow (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$.

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that ' $a \leq cd$ ' here is equiv to ' $a < cd'$.

- For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)\$.
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.
If $C, -C$ not posi $\Rightarrow C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlr for $D, -D$ not posi.
 - The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.
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- For D posi, define $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$.
The last equa holds becs $\forall a \in \tilde{1} \cap \mathbf{Q}^+$, $\exists d \in D^+$ suth $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$.
 - For non0 D not posi, define $D^{-1} = \{a : a < 1/b, b \in D^-\} \Rightarrow (D^{-1})^- = \{s : 0 \geq s \geq 1/b, \forall b \in D^-\}$.
Then $DD^{-1} = \{a : a < rs, r \in D^-, s \in (D^{-1})^-\} = \{a : a < rs, \exists r \in D^-, 0 \leq rs \leq r/b, \forall b \in D^-\}$.
Asum $\exists a \in \tilde{1} \cap \mathbf{Q}^+$, $\forall r \in D^-, s \in (D^{-1})^-, a \geq rs \Rightarrow a \geq r/b \Leftrightarrow r/a \leq b, \forall b \in D^-$. Let $r = 0$.
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- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
 $CD = -C(-D) = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$
 $= \{-a : a > rt, \forall r \in C^-, 0 \geq t \geq -s, \forall s \notin D\} = \{a : a < ru, \forall r \in C^-, 0 \leq u \leq s, \forall s \notin D\}$.
 $(r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs)$
 - Note the ' $0 \leq u'$. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$,
for if not, $\exists u > 0$, and $\exists r \in C^- \setminus \{0\}$, such that $\exists a < ru < 0$. Hence ' $0 \leq u'$ is actually ' $0 < u'$.
 - ' $u \leq s'$ cannot be abbreviated as in $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$.
' $u \leq s'$ cannot be ' $u < s'$ ', becs here $rs < ru \Rightarrow \exists a = rs$. Simlr for ' $a < ru$ ' to be ' $a \leq ru$ ' with ' $u < s'$ '.
 - Note that $\{u : 0 < u \leq s, \forall s \notin D\} \supsetneq D^+$. We show " $0 < u \leq s, \forall s \notin D$ " is equiv to " $\forall u \in D^+$ ".
 $LHS = \{a : a < ru, \forall r \in C^-, 0 < u \leq s, \forall s \notin D\}, \{a : a < rd, \forall r \in C^-, \forall d \in D^+\} = RHS$.
Immed, $LHS \subseteq RHS$. For $RHS \subseteq LHS$:
 - (I) If we cannot fix " $\min \mathbf{Q} \setminus D$ " by a certain scalar s_M , then becs ' $u \leq s'$ is actually ' $u < s'$,
 $u \in \mathbf{Q}^+, u \notin D^+ \Leftrightarrow u \geq s, \exists s \notin D \Leftrightarrow u \notin \{u : 0 < u < s, \forall s \notin D\}$.
 - (II) Othws, $s_M = \max D$. Now in LHS , " $a < rs_M, \forall r \in C^-$ ".
Supp $0 < d_1 < \dots < d_n < \dots \in D^+ [d_n < s_M]$ and $0 > r_1 > \dots > r_m > \dots \in C^-$.
Becs $r_{m+1}s_M < r_ms_M < r_md_{n+1} < r_md_n > r_{m+1}d_{n+1}$. Now $r_{m+1}s_M < r_ms_M \notin CD = RHS$.
Asum $\exists t \in RHS \setminus LHS \Rightarrow r_ms_M < t < r_md_n, \forall m, n \in \mathbf{N}^+$. Let $u = t/r_m \Rightarrow s_M > u > d_n$. Ctradic. \square
-

- For C posi and D not posi. Consider C and $-D$ both posi. Omitting $D = \tilde{0}$.
 $CD = -[C(-D)] = -\{a : a \leq cd, c \in C^+, d \in (-D)^+\} = \{-a : a > b, b > cd, \forall c \in C^+, d \in (-D)^+\}$
 $= \{a : a < -cd, \forall c \in C^+, \forall d \text{ suth } 0 < d < -s, \exists s \in D^-\} \quad (sd < -cd < 0)$
 $= \{a : a < ct, \forall c \in C^+, \forall t \text{ suth } s < t < 0, \exists s \in D^-\}$
 $= \{a : a < cs, \forall c \in C^+, \forall s \in D^- \setminus \{0\}\}$.
 - For C not posi and D posi, we show $\{a : a < rd, \forall r \in C^-, \forall d \in D^+\} = \{cs : c \in C, s \notin D\}$.
 $\forall s \notin D, \forall d \in D^+, d < s \Rightarrow \forall c \in C, \forall r \in C^-, rd > cd > cs$, OR. $rd > rs > cs$. Thus $RHS \subseteq LHS$.
Supp $a \in LHS \Rightarrow a/d \in C, \forall d \in D^+$. We show $\exists \varepsilon > 0$ suth $\frac{a}{d+\varepsilon} \in C, \forall d \in D^+$. So that $a \in RHS$.
Asum $\forall \varepsilon > 0, \frac{a}{d'+\varepsilon} \notin C [= r_0 \in C^-], \exists d' \in D^+$. Then $\exists r_1 \in C^-, r_1 > r_0 \Rightarrow a > r_1d' + \varepsilon d'$.
Now $\forall r \in C^-, \forall d \in D^+, rd > r_1d' + \varepsilon d'$. 又 $r_1d' > rd' > rd$ and $\varepsilon d' > 0$. Ctradic.
-

4 Suppose B, C, D non-zero Dedekind cuts. Show $(BC)D = B(CD)$, $B(C + D) = BC + BD$.

SOLUS: We discuss in cases.

\	1	2	3	4	5	6	7	8
B	+	+	+	+	-	-	-	-
C	+	+	-	-	-	+	-	+
D	+	-	+	-	-	-	+	+

$$(1) ((BC)D)^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = (B(CD))^+.$$

$$(4) (BC)D = \{a : a < rs, r \in (BC)^-, s \in D^-\}.$$

$$(B(C + D))^+ = \{bs : b \in B^+, s \in (C + D)^+\} = \{bc + bd : b \in B^+, -bc < bd, c \in C, d \in D\} = P.$$

$$\{bc : b \in B^+, c \in C^+\} + \{b'd : b' \in B^+, d \in D^+\} = \{bc + b'd : b, b' \in B^+, c \in C^+, d \in D^+\} = Q.$$

ENDED

0.C

5 *Supp a_1, a_2, \dots is a seq in \mathbf{Q} , and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.*

Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.

SOLUS: Becs the sup not in seq \Rightarrow inflly many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$. For a_{n+k} , choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most $(n+1)$ steps, we have a_m with $m > n$. Thus $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$. \square

• *Supp nonempty $A \subseteq \mathbf{R}$.*

• **TIPS 1:** Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.

SOLUS: $-b$ is an upper bound of $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$ is a lower bound of A .

Thus $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$. \square

• **TIPS 2:** Show if $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A, a > x$, (b) $\inf A < x \Rightarrow \exists a \in A, a < x$.

SOLUS: (a) $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x$. Then by def of sup.

Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$.

Simlr for (b). \square

6 *Supp $A, B \subseteq \mathbf{R}$ has inflly many disti elem, so has $A + B = \{a + b : a \in A, b \in B\}$.*

Prove $\sup(A + B) = \sup A + \sup B$, and $\inf(A + B) = \inf A + \inf B$.

SOLUS: $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$, $\inf A + \inf B \leq \inf(A + B)$.

$\sup A + \sup B > \sup(A + B) \Leftrightarrow \sup A > \sup(A + B) - \sup B$

$\Leftrightarrow \exists a + \sup B > \sup(A + B) \Leftrightarrow \sup B > \sup(A + B) - a \Leftrightarrow \exists a + b > \sup(A + B)$. Ctradic.

Simlr for $\inf(A + B) \in A + B$. Or. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. \square

16 *Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in [0.11].*

Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Define $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.

(a) Show $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$ is one-to-one. (b) Show $\psi(0) = 0, \psi(1) = 1$.

(c) Show $\psi(a \pm b) = \psi(a) \pm \psi(b)$, and $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.

(d) Supp $a \in \mathbf{R}_1$. Show $a > 0 \Leftrightarrow \psi(a) > 0$.

SOLUS: (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_2$. Then $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_2 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Note that $\varphi(q) < 0 \Leftrightarrow q < 0$, and $\varphi(0) = 0$.

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$, $\varphi_2(q) \leq 1$.

(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$.

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(p) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$.

(d) $a > 0 \Leftrightarrow \exists n \in \mathbf{N}^+, 1/n < a \Leftrightarrow \psi_1(a) > 0$. \square

ENDED