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这是我个人挑战自学「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 *Supplement* 就能具备所有必要的知识基础。0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不驯。的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导  $\mathcal{D}$  的一切。如你所见，字越少，事越大。

# ABBREVIATION TABLE

## A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

## E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

## L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

## R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

## C

clod	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
contin	countinu(ous)(ity)
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
countexa	counterexample
ctradic	contradict(s)(ion)
ctrapos	constrapositive

## F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

## M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

## S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

## D

Ddkd	Dedekind
decr	decreasing
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disj	disjoint
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

## I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

## O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

## T U V W X Y Z

uniq	unique
uniques	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

**0.B** NOTE:  $C, D$  are Dedekind cuts. Numbers used here are always rational.

• Define  $\tilde{q} = \{a : a < q\}$ , and  $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$ .

Then  $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -0 \leq 0\} = \tilde{0}$ .

• Define  $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$ .

$-( -D ) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$ .

The last equa is becs (a)  $d \notin D \Rightarrow \exists b \notin D, d \geq b$ , and (b)  $d \in D \Rightarrow$  if  $\exists b \notin D$  suth  $d \geq b$ , then  $b \in D$ , ctradic.

• **TIPS:** Prove  $\forall \varepsilon > 0, \exists b \notin D$  suth  $b - \varepsilon \in D$ .

**SOLUS:** Asum  $\exists \varepsilon > 0$  suth  $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$ .

Then  $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$  for any  $n \in \mathbf{N}^+$ .

Now  $\forall d \in D, \exists n \in \mathbf{N}^+$  suth  $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$ , ctradic. □

**1** Prove (a)  $D + \tilde{0} = D$ , (b)  $-D$  is Dedekind cut, and  $D + (-D) = \tilde{0}$ .

**SOLUS:** (a)  $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$ .

(b) Asum  $x \in -D$  is the largest elem of  $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$ .

Let  $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$ .

Thus by def,  $x + \delta \in -D$ , ctradic the max of  $x \in -D$ . Hence  $-D$  is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$ .

Supp  $a \in \tilde{0} \Rightarrow -a > 0$ . By TIPS,  $\exists b \notin D$  suth  $b + a \in D$ .

Note that  $b < b - a \notin D \Rightarrow -b > -b + a \in -D$ . Then  $(-b + a) + (b + a) = 2a < 0$ .

Thus  $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$ . □

**3** Show  $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$  posi.

**SOLUS:** (a)  $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$ .

(b)  $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$ . 又  $D - C \neq \tilde{0}$ . □

**5** Prove (a)  $D$  posi  $\Rightarrow -D$  not posi, (b) non0  $-D$  not posi  $\Rightarrow D$  posi.

**SOLUS:** (a)  $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$ .

(b) Becs  $\tilde{0}$  is the largest non posi cuts. Thus  $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$  posi.

OR.  $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$ . □

• Define  $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$ . Then  $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi.

Define  $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$ .

(a)  $D^- = \{0\} \Leftrightarrow D = \tilde{0}$ . Convly,  $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$ .

(b)  $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi. **CORO:**  $D$  not posi  $\Leftrightarrow 0 \in D^-$ .

(c)  $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$ . **CORO:**  $D$  not posi  $\Leftrightarrow (D^-)^- = D$ .

•  $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$ .

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$ .

• For  $C, D$  posi, define  $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$ . Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.

- For  $-C, -D$  posi, define  $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .  
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$ .  
 If  $C, -C$  not posi  $\Rightarrow C = \tilde{0}$ , then with the asum  $\tilde{0}D = \tilde{0}$ , it still holds. Simlir for  $D$ .

- For  $D$  posi, define  $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$ .  
 The last equa holds becs  $\forall a \in \tilde{1} \cap \mathbf{Q}^+, \exists d \in D^+$  suth  $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$ .
- For non0  $D$  not posi, define  $D^{-1} = -(-D)^{-1} = -\{a : a < 1/b, \exists b \notin -D \Leftrightarrow \exists b \geq -s, \forall s \notin D\}$   
 $= \{a : a < -x, \exists x \geq 1/b, \forall b$  suth  $b \geq -s, \forall s \notin D\} = \{a : a < -1/b, \forall b$  suth  $b \geq -s, \forall s \notin D\}$   
 $= \{a : a < 1/b, \forall b$  suth  $b \leq s, \forall s \notin D\} \neq \{a : a < 1/s, \forall s \notin D\}$ .  
 Let  $b_1 < \dots < b_m < \dots \leq s, \forall s \notin D \Leftrightarrow \forall s \in D^- \Rightarrow$  each  $s_j, b_k < 0 \Rightarrow 1/s_j \leq \dots < 1/b_m < \dots < 1/b_1$ .  
 Thus ' $a < 1/b$ ' is equiv to ' $a < 1/s, \exists s \in D^-$ '. Hence  $D^{-1} = \{a : a < 1/b, b \in D^-\}$ .  
 $DD^{-1} = \{a : a < rs, \exists r \in D^-, \exists s$  suth  $0 \geq s \geq 1/b, \forall b \in D^-\} \subseteq \tilde{1}$ .  
 Asum  $\exists x$  suth  $rs \leq x < 1, \forall r, s$ . Let  $D \not\ni \dots \leq b_m < \dots < b_1 \leq 0$ , and  $D \not\ni \dots \leq r_m < \dots < r_1 \leq 0$ .  
 $1/b_1 < \dots < 1/b_m \leq \dots \leq \dots \leq s_n < \dots < s_1 \leq 0$ , and  $r_m/b_m \geq \dots \geq \dots \geq r_m s_n > \dots > r_j s_k > \dots > r_1 s_1$ .  
 Let  $r_m = b_m$ . Ctradic. OR.  $DD^{-1} = D[(-\tilde{1})(-D)^{-1}] = [D(-\tilde{1})](-D)^{-1} = (-D)(-D)^{-1} = \tilde{1}$ .

- For  $C$  not posi and  $D$  posi, we expect that  $CD$  not posi. Consider  $C$  and  $-D$  both not posi.  
 $CD = -[C(-D)] = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$   
 $= \{-a : a > rt, \forall r \in C^-, \forall t$  suth  $0 \geq t \geq -s, \forall s \notin D\}$   
 $= \{a : a < ru, \forall r \in C^-, \forall u$  suth  $0 \leq u \leq s, \forall s \notin D\}$ . ( $r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs$ .)
- Note the ' $0 \leq u$ '. Becs  $C^- \neq \emptyset \Rightarrow 0 \in C^-$ . If it is to be exactly  $CD = \{a : a < 0\}$ , then  $C^- = \{0\}$ ,  
 for if not,  $\exists u > 0$ , and  $\exists r \in C^- \setminus \{0\}$ , suth  $\exists a < ru < 0$ . Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
- ' $u \leq s$ ' cannot be abbreviated as in  $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$ .  
 ' $u \leq s$ ' cannot be ' $u < s$ ', becs here  $rs < ru \Rightarrow \exists a = rs$ .
- Note that  $\{u : 0 < u \leq s, \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q} \setminus D\}, & \text{if it exists,} \\ D^+, & \text{othws.} \end{cases}$  Denote it by  $D^\oplus = D^\otimes \setminus \{0\}$ .

- For  $C$  not posi and  $D$  posi. If  $C = \tilde{0}$ , then  $CD = -[C(-D)] = -\tilde{0}$ . Now consider  $-C$  and  $D$  both posi.  
 But  $CD = -(-C)D = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \neq \{a : a < -cd, \forall c \in (-C)^+, d \in D^+\}$ .  
 Altho " $a \leq cd$ " is equiv to " $a < cd$ " so that  $b \notin (-C)D \Rightarrow b \geq cd$ , which is actually  $b > cd, \forall c, d$ .  
 And  $a < -b < -cd, \forall c, d \Rightarrow \forall a, \exists x$  suth  $a < x < -cd, \forall c, d$ . While  $a$  can be the 'boundary' in RHS.

- $LHS = \{a : a < ru, \forall r \in C^-, \forall u \in D^\oplus\}, \{cs : c \in C, s \notin D\} = RHS$ .  
 Becs  $cs \leq cu < ru$ . We show  $LHS \subseteq RHS$ . Let  $c_1 < \dots < c_n < \dots \in C$ , and  $s_1 > \dots > s_m \geq \dots \notin D$ .  
 Then  $c_1 s_1 < \dots < c_n s_m < \dots < ru, \forall r, u$  as in LHS. Thus  $a \in LHS \Rightarrow \exists a < c_j s_k$ . □  
 OR. Note that in LHS, ' $a < ru$ ' is equiv to ' $a < rs, \exists s \notin D$ .' Now  $LHS = \{a : a/s \in C, \exists s \notin D\}$ . □

- For  $C$  posi and  $D$  not posi. If  $D = \tilde{0}$ , then  $CD = -[(-C)D] = -\tilde{0}$ .  
 $CD = (-C)(-D) = \{a : a < ru, \forall r \in (-C)^-, \forall u \in (-D)^\oplus\} \quad \mathbf{Q} \setminus -D = \{s : s \geq -y, \forall y \notin D\}$   
 $= \{a : a < ru, \forall r$  suth  $\forall x \notin C, 0 \geq r \geq -x, \forall u$  suth  $0 < u \leq s, \forall s \geq -y, \forall y \notin D\}$   
 $= \{a : a < (-r)(-u), \forall r$  suth  $\forall x \notin C, 0 \leq -r \leq x, \forall u$  suth  $y \leq -u < 0, \forall y \notin D\}$   
 $= \{a : a < ru, \forall r$  suth  $\forall x \notin C, 0 \leq r \leq x, \forall u \in D^-\} = \{a : a < ru, \forall r \in C^\oplus, \forall u \in D^-\}$ , simlir.

- We show  $-D = \{a : a < -b, b \notin D\} = (-\tilde{1})D$ .

For  $D$  posi,  $RHS = \{a : a < ru, \forall r \text{ suth } -1 \leq r \leq 0, \forall u \in D^\oplus\} = \{a : a < -u, \forall u \in D^\oplus\} \supseteq -D$ .

Supp  $x \text{ suth } -b \leq x < -u, \forall b \notin D, \forall u \text{ suth } -s \leq -u < 0, \forall s \notin D \Rightarrow -s \leq x < -u$ . Let  $-u = x$ .

For  $D$  not posi,  $RHS = \{a : a < rb, \exists r \text{ suth } -1 \leq r \leq 0, b \in D^-\} = \{a : a < -b, 0 \geq b \notin D\} = -D$ .

- We show  $\tilde{1}D = D$ . For  $D$  not posi, immed. Othws,  $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$ .

Now  $(\tilde{1}D)^+ \subseteq D^+$ . 又  $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in (\tilde{1}D)^+$ .

**4 Supp  $B, C, D$  non0 Dedekind cuts. Show  $(BC)D = B(CD), B(C + D) = BC + BD$ .**

**SOLUS:** We discuss in cases.

$\backslash$	1	2	3	4	5	6
$B$	+	+	+	-	-	-
$C$	+	+	-	-	+	+
$D$	+	-	-	-	-	+

$$(1) [(BC)D]^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = [B(CD)]^+.$$

$$B(C + D) = \{a : a \leq bc + bd, \exists b \in B^+, 0 < c + d \in C + D\}.$$

$$BC + BD = \{x : x \leq uc, u \in B^+, c \in C^+\} + \{y : y \leq vd, v \in B^+, d \in D^+\} \\ = \{a : a \leq uc + vd, \exists u, v \in B^+, c \in C^+, d \in D^+\}. \text{ Done.}$$

$$(3) (BC)^- = \{r : 0 \geq r \geq bc, \exists c \in C^-, \exists b \in B^\oplus\}.$$

$$(BC)D = \{a : a < rs, r \in (BC)^-, s \in D^-\} \\ = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^\oplus\}.$$

$$B(CD) = \{a : a \leq bx < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+, \text{ and } cs > x \in (CD)^+\} \\ = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+\}.$$

Note that  $\{q : q < b, b \in B^+\} = \{q : q < b, \exists b \in B^\oplus\}$ . Done.

$$B(C + D) = \{a : a < ru, \forall r \text{ suth } \forall x \notin B, 0 < r \leq x, \forall u \text{ suth } 0 \geq u > c + d, \forall c \in C, d \in D\} \\ = \{a : a < ru, \forall r \in B^\oplus, \forall u \text{ suth } 0 \geq u \geq c + d, \exists c \in C^-, d \in D^-\} \\ = \{a : a < r(c + d), \forall r \in B^\oplus, \forall c \in C^-, d \in D^-\}.$$

$$BC + BD = \{x : x < pc, \forall p \in B^\oplus, \forall c \in C^-\} + \{y : y < qd, \forall q \in B^\oplus, \forall d \in D^-\} \\ = \{a : a < pc + qd, \forall p, q \in B^\oplus, \forall c \in C^-, d \in D^-\}. \text{ Done immed.}$$

$$(5) (BC)^- = \{r : 0 \geq r \geq bc, \exists b \in B^-, \exists c \in C^\oplus\}. (CD)^- = \{r : 0 \geq r \geq cd, \exists d \in D^-, \exists c \in C^\oplus\}.$$

$$(BC)D = \{a : a < rd \leq bcd, \exists d \in D^-, b \in B^-, \exists c \in C^\oplus\}.$$

$$B(CD) = \{a : a < br \leq bcd, \exists b \in B^-, d \in D^-, \exists c \in C^\oplus\}. \text{ Done.}$$

OR. By commu and (3),  $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$ .

$$BC + BD = \{x : x < bc, \forall b \in B^-, \forall c \in C^\oplus\} + \{y : y < bd, \exists b \in B^-, d \in D^-\} \\ = \{a : a < pc + qd, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \exists q \in B^-, d \in D^-, \Rightarrow q \geq p\} \\ = \{a : a < pc + y, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall y \text{ suth } y \geq qd, \forall q \in B^-, d \in D^-\} \\ = \{a : a < p(c + d'), \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall d' \text{ suth } d' \leq d, \forall d \in D^-\}.$$

$$(I) \text{ If } C + D \text{ not posi. Then } B(C + D) = \{a : a < bx, \exists b \in B^-, 0 \geq x > c + d, \forall (c, d) \in C \times D\}.$$

Rewrite as  $\{a : a < t, \forall t \text{ suth } t \geq bx, \forall b \in B^-, x \in (C + D)^-\}$ . Done.

$$(II) \text{ If } C + D \text{ posi. Then } B(C + D) = \{a : a < bx, \forall b \in B^-, x \in (C + D)^\oplus\}.$$

If  $(C + D)^\oplus = (C + D)^+$ . Then  $B(C + D) = \{a : a < bc + bd, \forall b \in B^-, \forall (c, d) \in C^\oplus \times D^\oplus\}$ .

Othws,  $\exists s = \min \mathbf{Q} \setminus (C + D) \Rightarrow B(C + D) = \{a : a < bs, \forall b \in B^-\}$ . Done.

$$\begin{aligned}
(4) \quad (BC)D &= \{xd : d \in D, x \geq bc, \forall b \in B^-, c \in C^-\} \\
&= \{a : a < bcd, \forall d \in D^-, \forall b \in B^-, c \in C^-\} \\
&= \{by : b \in B, y \geq cd, \forall c \in C^-, d \in D^-\} = B(CD). \\
B(C + D) &= \{a : a < t, \forall t \text{ suth } t \geq b(c + d), \forall b \in B^-, (c, d) \in C^- \times D^-\} \\
&= \{a : a < t_1, \forall t_1 \text{ suth } t_1 \geq bc, \forall (b, c) \in B^- \times C^-\} \\
&\quad + \{a : a < t_2, \forall t_2 \text{ suth } t_2 \geq bd, \forall (b, d) \in B^- \times D^-\} = BC + BD. \text{ Done.}
\end{aligned}$$

NOTE: Supp for any  $B$  posi,  $C$  posi,  $D$  not posi, assoc holds.

Supp  $B$  posi,  $C$  not posi,  $D$  posi. Then  $(\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{B}\bar{C}) = (\bar{D}\bar{B})\bar{C} = \bar{B}(\bar{D}\bar{C})$ . Convly true.

Simlr for the case  $B$  not posi,  $C$  posi,  $D$  not posi, equiv to the case  $B$  not posi,  $C$  not posi,  $D$  posi.

(2) holds  $\Rightarrow (\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{B}\bar{C}) = (\bar{C}\bar{D})\bar{B}$ , (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds  $\Rightarrow \bar{B}(\bar{C} + \bar{D}) = (-B)[- (\bar{C} + \bar{D})] \stackrel{(5)}{=} (-B)[(-C) + (-D)] \stackrel{(5)}{=} \bar{B}\bar{C} + \bar{B}\bar{D}$ , by def of multi.

Thus (5)  $\Rightarrow$  (2) in distr. Convly as well.

$$\begin{aligned}
(6) \quad (BC)^- &= \{r : 0 \geq r \geq bc, \exists b \in B^-, c \in C^\oplus\}. \\
(BC)D &= \{a : a < ru, \forall r \in (BC)^-, u \in D^\oplus\} = \{a : a < bcd, \forall b \in B^-, c \in C^\oplus, d \in D^\oplus\}. \\
(CD)^\oplus &= \{u : 0 < u \leq s, \forall s > cd, \forall c \in C^+, d \in D^+\}. \\
B(CD) &= \{a : a < ru, \forall r \in B^-, u \in (CD)^\oplus\}. \text{ Done.} \\
B(C + D) &= \{a : a < b(c + d), \forall b \in B^-, \forall (c, d) \in C^\oplus \times D^\oplus\}. \\
BC + BD &= \{a : a < bc, \forall b \in B^-, c \in C^\oplus\} + \{a : a < bd, \forall b \in B^-, d \in D^\oplus\}. \text{ Done.}
\end{aligned}$$

OR. (2) instead of (6) ? NOTICE that the distr in (6) cannot be shown without (6). □

**ENDED**

## 0.C

**2** Supp nonempty  $U \subseteq V \subseteq \mathbf{R}$ . Show  $\sup U \leq \sup V$ .

SOLUS: Asum  $\sup U > \sup V \Rightarrow \exists t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$ , ctrad. □

**5** Supp  $a_1, a_2, \dots \in \mathbf{Q}$ , and  $\sup\{a_1, a_2, \dots\} = \sqrt{2}$ .

Prove  $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$  for all  $n \in \mathbf{N}^+$ .

SOLUS: Becs the sup not in seq  $\Rightarrow$  infily many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$ . For  $a_{n+k}$ , choose  $a_i > a_{n+k}$ . If  $i \in \{1, \dots, n\}$ , then choose  $a_j > a_i$ .

After at most  $n$  steps, we must have  $a_m$  with  $m > n$ . Thus  $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$ . □

• Supp nonempty  $A \subseteq \mathbf{R}$ .

• **TIPS 1:** Define  $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$ . Prove  $\sup(-A) = -\inf A$ .

SOLUS:  $-b$  is an upper bound of  $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$  is a lower bound of  $A$ .

Thus  $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$ . □

• **TIPS 2:** Show if  $x \in \mathbf{R}$ , (a)  $\sup A > x \Rightarrow \exists a \in A, a > x$ , (b)  $\inf A < x \Rightarrow \exists a \in A, a < x$ .

SOLUS: (a)  $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x \Rightarrow \sup A \leq x$ .

OR. By (b),  $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$ .

Simlr for (b). □

**6** Supp nonempty  $A, B \subseteq \mathbf{R}$ . Prove  $\sup(A + B) = \sup A + \sup B$ , and  $\inf(A + B) = \inf A + \inf B$ .

**SOLUS:**  $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$ ,  $\inf A + \inf B \leq \inf(A + B)$ .

$$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$$

$$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B). \text{ Ctradic.}$$

Simlr for  $\inf(A + B) \in A + B$ . OR. Apply to  $-A - B$ , becs  $\sup(-A) = -\inf A$ .  $\square$

**16** Supp  $\mathbf{R}_1, \mathbf{R}_2$  are complete ordered fields. Define  $\varphi_1, \varphi_2$  as in [0.11].

Define  $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  by  $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq a\}$ . Show

(a)  $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  is one-to-one. (b)  $\psi(0) = 0, \psi(1) = 1$ .

(c)  $\psi(a + b) = \psi(a) + \psi(b)$ . (d)  $\psi(ab) = \psi(a)\psi(b)$ .

**SOLUS:** (a) Define  $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Let  $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$ .

Define  $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$ .

Define  $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$ . Then  $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  well-defined.

Note that  $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$ .

Now  $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Rev the roles of  $\mathbf{R}_1, \mathbf{R}_2$ .

(b) Becs  $\varphi_1(q) \leq \varphi_1(0) \iff q \leq 0 \iff \varphi_2(q) \leq \varphi_2(0)$ . Simlr for  $\psi(1) = 1$ .

(c)  $S = \{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq a + b\} \supseteq \{\varphi_2(p + q) : p, q \in \mathbf{Q}, \varphi_1(p) \leq a, \varphi_1(q) \leq b\} = T$   
 $\Rightarrow \sup S \geq \sup T$ . Asum it is ' $>$ '. Now  $\exists t \in \mathbf{Q}$  suth  $\sup S \geq \varphi_2(t) > \sup T = \psi(a) + \psi(b)$ .

Which means  $\varphi_2(t) > \varphi_2(p + q), \forall p, q \in \mathbf{Q}$  suth  $\varphi_1(p) \leq a$  and  $\varphi_1(q) \leq b$ .

Now  $a + b \geq \varphi_1(t) > \varphi_1[(t + p + q)/2] > \varphi_1(p + q), \forall p, q$ . Ctradic.

(d) We show it for (I)  $a, b > 0$ , (II)  $a > 0 > b$ .

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq ab\}, \sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq b\} = RHS$ .

(I)  $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q}, 0 < \varphi_1(p) \leq a, \varphi_1(q) \leq b \implies \varphi_1(pq) \leq ab\}$ .

(II)  $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q}, \varphi_1(q) \leq b, \text{ and } s \geq p, \forall p \text{ suth } 0 < \varphi_1(p) \leq a\}$ .

Note that  $\varphi_1(s) \geq \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geq a$ , for if not,  $\exists p' \in \mathbf{Q}$  suth  $\varphi_1(s) < \varphi_1(p') \leq a$ .

So that  $\varphi_1(sq) \leq a\varphi_1(q) \leq ab$ .

Now  $LHS \geq RHS$ . Asum it is ' $>$ '. Then  $\exists t \in \mathbf{Q}$  suth  $LHS \geq \varphi_2(t) > RHS$ .

Which means  $\varphi_2(t) > \psi(a)\varphi_2(q), \forall q \in \mathbf{Q}$  suth  $\varphi_1(q) \leq b$ .

(I)  $\implies \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q$  and  $\forall s \in \mathbf{Q}$  suth  $0 < \varphi_1(s) \leq a$ .

(II)  $\implies \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q$  and  $\forall s \in \mathbf{Q}$  suth  $\varphi_2(s) \geq \varphi_2(p), \forall p$  suth  $0 < \varphi_1(p) \leq a$ .

Thus  $ab \geq \varphi_1(t) > \varphi_1[(t + sq)/2] > \varphi_1(sq), \forall s, q$ . Ctradic.  $\square$

**ENDED**

## 0.D

• Supp nonempty  $A$  is closed and open subsets of  $\mathbf{R}^n$ . Prove  $A = \mathbf{R}^n$ .

**SOLUS:** Asum  $A \neq \mathbf{R}^n$ . Let  $a \in A, b \in \mathbf{R}^n \setminus A$ . Define  $f(t) = (1 - t)a + tb$ .

If  $f(t_1) = f(t_2) \Rightarrow (t_1 - t_2)(b - a) = 0 \Rightarrow t_1 = t_2$ . Inje.

Let  $\mathcal{A} = \{t \in [0, 1] : f(t) \in A\}$ , and  $\sup \mathcal{A} = t_M \in [0, 1]$ . Let  $c = f(t_M)$ .

(I) If  $c \in A \Rightarrow \exists \delta > 0, B(c, \delta) \subseteq A$ . Let  $t \neq t_M$  be suth  $f(t) \in B(c, \delta) \Rightarrow \|(t - t_M)(b - a)\|_\infty < \delta$ .

Let  $\varepsilon = |t - t_M| > 0 \Rightarrow f(t_M + \varepsilon) \in B(c, \delta) \subseteq A \Rightarrow t_M \geq t_M + \varepsilon$ , ctradic.

(II) If  $c \in \mathbf{R}^n \setminus A$ . Simlr.  $\forall t' \in [t_M - \varepsilon, t_M), t' \notin \mathcal{A}$ . Now  $\forall t \in \mathcal{A}, t < t_M \Rightarrow t < t_M - \varepsilon$ .  $\square$



**15** Supp  $F$  is a closed subset of  $\mathbf{R}$ . Prove  $S = \{a^2 : a \in F\}$  is closed.

**SOLUS:** Supp  $S$  not closed  $\Rightarrow \exists a_1^2, a_2^2, \dots \in S$  convg to  $L^2$  suth  $\pm L \notin F$ . Let  $\varepsilon_1 > \varepsilon_2 > \dots \in \mathbf{R}^+$ .

Becs  $\forall \varepsilon_p, \exists m \in \mathbf{N}^+, \forall k \geq m, \|a_k^2 - L^2\|_\infty = |a_k - L| \cdot |a_k + L| < \varepsilon_p$ ,

if  $|a_k - L| < \varepsilon_p$ , then let  $b_p = a_k$ , if  $|a_k + L| < \varepsilon_p$ , then let  $c_p = a_k$ .

Then we have at least one seq in  $F$  with  $\lim \pm L \notin F \Rightarrow F$  not closed.  $\square$

**17** Supp  $F \subseteq \mathbf{R}$ , and  $\forall n \in \mathbf{N}^+, F \cap [-n, n]$  is closed. Prove  $F$  is closed.

**SOLUS:** Becs  $G = (\mathbf{R} \setminus F) \cup (-\infty, -n) \cup (n, \infty)$  is open for all  $n \in \mathbf{N}^+$ . Let  $n > \sup\{|a| : a \in \mathbf{R} \setminus F\}$ .

By [0.59],  $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \dots$  (disj)  $\Rightarrow I_1 \cup I_2 \cup \dots = \mathbf{R} \setminus F$ .  $\square$

**20** Prove  $\forall b \in \mathbf{R}^n, \delta > 0, B = \{a \in \mathbf{R}^n : \|a - b\| \leq \delta\}$  are open.

**SOLUS:** Asum  $\exists a_1, a_2, \dots \in B$  convg to  $L$  suth  $\|L - b\| > \delta$ . Then  $\forall \varepsilon > 0, \exists m \in \mathbf{N}^+, \forall k \geq m$ ,

$\|a_k - L\| < \varepsilon \Rightarrow \delta < \|L - b\| \leq \|a_k - L\| + \|a_k - b\| < \delta + \varepsilon$ . Let  $\varepsilon = \|L - b\|$ . Ctradic.  $\square$

**21** Supp  $X$  is an open subset of  $\mathbf{R}$ . Prove  $\exists a_k, b_k \in \mathbf{R}$  suth  $X = \bigcup_{k=1}^{\infty} [a_k, b_k]$ .

**SOLUS:** Let  $X = \bigcup_{k=1}^{\infty} (c_k, d_k)$ , where each  $d_k < c_{k+1}$ . Let  $I_k = (c_k, d_k)$ .

Let  $a_{1,k}, a_{2,k}, \dots$  be convg of  $\lim c_k$ . Simlr, let  $b_{1,k}, b_{2,k}, \dots$  of  $\lim d_k$ . Let  $E_{j,k} = [a_{j,k}, b_{j,k}]$ .

Now  $\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{j,k} = \bigcup_{k=1}^{\infty} I_k = X$ . Rearrange the order of  $E_{j,k}$ 's.  $\square$

**23** Supp  $F_1, F_2$  are disj closed subsets of  $\mathbf{R}$  suth  $U = F_1 \cup F_2$  is interval. Prove  $F_1$  or  $F_2$  open.

**SOLUS:** If  $U$  open, then  $F_1 = \emptyset, F_2 = \emptyset$  or  $\mathbf{R}$ . Now supp  $U$  not open  $\Rightarrow F_1$  or  $F_2$  not open.

We show  $F_1$  open  $\Leftrightarrow F_2$  not open. Note that  $F_1$  open  $\Leftrightarrow F_1 = \emptyset$ . Simlr for  $F_2$ .

WLOG, asum  $F_1, F_2$  both not open, and  $x \in F_1, y \in F_2$  with  $x < y \Rightarrow x, y \in U \supseteq [x, y]$ .

NOTICE that  $T = [x, y] \cap F_1$  is a closed subset that has infily many elem.  $\forall \sup T < y$ .

(I) If  $\sup T \notin F_1$ . Then  $\exists$  a convg seq in  $T$  with  $\lim \sup T$ .

(II) Othws,  $\forall t \in (\sup T, y], t \notin F_1 \Leftrightarrow t \in F_2$ . Now  $\exists$  a convg seq in  $F_2$  with  $\lim \sup T \notin F_2$ .

Ctradic the asum  $\Rightarrow$  at least one of  $F_1, F_2$  is empty.  $\square$

**25** Give an exa of inv  $f : \mathbf{R} \rightarrow \mathbf{R} \setminus \mathbf{Q}$ .

**SOLUS:** Consider a seq disti  $a_1, b_1, a_2, b_2, \dots \in \mathbf{R}$  where  $\{a_1, a_2, \dots\} \in \mathbf{Q}$  and each  $b_i \in \mathbf{R} \setminus \mathbf{Q}$ .

Define  $\varphi(a_j) = b_{2j-1}, \varphi(b_k) = b_{2k} \Rightarrow \varphi^{-1}(b_i) = \begin{cases} a_{(i+1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$

Let  $B = \{b_1, b_2, \dots\}, U = \mathbf{Q} \cup B, K = \mathbf{R} \setminus U$ . Extend  $\varphi \in B^U$  to  $\psi \in (K \cup B)^{K \cup U}$  by  $\psi|_K = I$ .  $\square$

**26** Supp  $E, G \subseteq \mathbf{R}^n$ , and  $G$  is open. Prove  $E + G = \{x + y : x \in E, y \in G\}$  is open.

**SOLUS:** Asum  $E + G$  not open  $\Leftrightarrow \mathbf{R}^n \setminus (E + G)$  not closed.

Then  $\exists a = x + y \in E + G$  suth  $\forall \delta > 0, \exists b \notin E + G$  suth  $\|a - b\|_\infty < \delta$ .

Let  $z = b - x \notin G \Rightarrow \|y - z\|_\infty < \delta \Rightarrow z \in B(y, \delta) \subseteq G, \exists \delta > 0$ .  $\square$

OR.  $\exists a_1, a_2, \dots \notin E + G$  while its  $\lim L = e + g \in E + G$ .  $\forall x \in E, a_k - x \notin G$

$\Rightarrow \lim_{k \rightarrow \infty} (a_k - e) = L - e = g \in G$ . Thus  $\mathbf{R}^n \setminus G$  not closed  $\Leftrightarrow G$  not open.  $\square$



## 0.E

• Supp  $b \in A \subseteq \mathbf{R}^m, f : A \rightarrow \mathbf{R}^n$ . Prove

[P]  $f$  is continu at  $b \iff \forall b_1, b_2, \dots \in A$  suth  $\lim_{k \rightarrow \infty} b_k = b, \lim_{k \rightarrow \infty} f(b_k) = f(b)$ . [Q]

SOLUS:  $Q \Rightarrow P$ : Supp  $\varepsilon > 0$  suth  $\forall \delta > 0, \exists a \in A$  with  $\|a - b\|_\infty < \delta$  and  $\|f(a) - f(b)\|_\infty \geq \varepsilon$ .

Fix a  $\delta$ . Define  $\delta_k = \delta/k \Rightarrow \exists a_k$  for each. Now  $\lim_{k \rightarrow \infty} a_k = b$ .

Thus  $\forall m, \forall k \geq m, \|f(a_k) - f(b)\|_\infty \geq \varepsilon$ . Ctradic  $Q$ .

$P \Rightarrow Q$ : Supp  $b_1, b_2, \dots \in A$  suth  $\forall \delta > 0, \exists m, \forall k \geq m, \|b_k - b\|_\infty < \delta$ .

Asum  $\varepsilon > 0$  suth  $\forall m, \forall k \geq m, \|f(b_k) - f(b)\|_\infty \geq \varepsilon$ . Ctradic  $P$ . □

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1 Prove every convg seq in  $\mathbf{R}^n$  is bounded.

SOLUS: Supp  $a_1, a_2, \dots \in \mathbf{R}^n$  suth  $\forall \varepsilon > 0, \exists m \in \mathbf{N}^+, \forall k \geq m, \left| \|a_k\|_\infty - \|L\|_\infty \right| \leq \|a_k - L\|_\infty < \varepsilon$ .

Thus  $\|a_k\|_\infty \in \left( \left| \varepsilon - \|L\|_\infty \right|, \varepsilon + \|L\|_\infty \right)$ . Now asum  $\sup\{\|a_k\|_\infty : k \in \mathbf{N}^+\} = \infty$ .

Which means  $\forall T \in \mathbf{R}^+, \forall m \in \mathbf{N}^+, \exists k > m, \max\{\|a_1\|_\infty, \dots, \|a_m\|_\infty\} < \|a_k\|_\infty > T$ . Ctradic. □

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3 Supp  $F \subseteq \mathbf{R}^n$  suth every seq in  $F$  has a subseq that convg in  $F$ . Prove  $F$  is closed bounded.

SOLUS: Supp  $a_1, a_2, \dots \in F$  and  $\exists k_1, k_2, \dots$  incre in  $\mathbf{N}^+$  suth  $a_{k_1}, a_{k_2}, \dots$  convg to  $L \in F$ .

Asum  $\sup F = \infty \Rightarrow \forall a \in F, \exists b \in F, b > a$ .

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