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这是我个人挑战「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节本来不太要命，但超出课文的补助却让我折戟沉沙——的确，它们不需要硬性知识门槛，可以用集合和数理逻辑来推导 \mathcal{D} 的一切。或许是缺乏 Dedekind cut 的系统学习，我推导这一切时感到我在亲手缔造一个数学分支；我不是自傲的意思，只是说，**这非常艰难**。但我还是坚持下来了；在此过程中我肉眼可见我在数理逻辑上的提升。

ABBREVIATION TABLE

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

T U V W X Y Z

uniq	unique
uniques	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0.B NOTE: C, D are Dedekind cuts. Numbers used here are always rational.

• Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.

Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -0 \leq 0\} = \tilde{0}$.

• Define $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$.

$-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$.

The last equa is becs (a) $d \notin D \Rightarrow \exists b \notin D, d \geq b$, and (b) $d \in D \Rightarrow$ if $\exists b \notin D$ suth $d \geq b$, then $b \in D$, ctradic.

• TIPS: Prove $\forall \varepsilon > 0, \exists b \notin D$ suth $b - \varepsilon \in D$.

SOLUS: Asum $\exists \varepsilon > 0$ suth $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$.

Then $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.

Now $\forall d \in D, \exists n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

1 Prove (a) $D + \tilde{0} = D$, (b) $-D$ is Dedekind cut, and $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$.

Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.

Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$.

Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists b \notin D$ suth $b + a \in D$.

Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.

Thus $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

CORO: $-(-D) + (-D) = \tilde{0} \Rightarrow -(-D) = D$, by the unques.

3 Show $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$ posi.

SOLUS: (a) $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists y' \in D, y' > y \Leftrightarrow 0 < y' - y \in D - C$.

(b) $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. □

5 Prove (a) D posi $\Rightarrow -D$ not posi, (b) non0 $-D$ not posi $\Rightarrow D$ posi.

SOLUS: (a) $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftarrow 0 \in D$.

(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.

OR. $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$. □

4 (a) Supp $D \neq \tilde{0}$. Find a formula for D^{-1} suth D^{-1} is Dedekind cut and $DD^{-1} = \tilde{1}$.

(b) Show assoc for multi and distr holds in \mathcal{D} .

SOLUS: Let $D^{-1} = \{a : a < 1/d, \forall d \in D\} \Rightarrow DD^{-1} = \{a : a < rs, r \in D^\pm, s \in (D^{-1})^\pm\}$.

$(D^{-1})^+ = \{a : 0 < a < 1/d, \forall d \in D^+\}, (D^{-1})^- = \{a : 0 \geq a \geq 1/d, \exists d \in D\}$.

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \iff \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \iff 0 \in D \iff D$ posi.
Define $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$.
(a) $D^- = \{0\} \iff D = \tilde{0}$. Convly, $\{r \notin D : r \leq 0\} = \{0\} \implies \mathbf{Q} \setminus D = \mathbf{Q}^*$.
(b) $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. **CORO:** D not posi $\iff 0 \in D^-$.
(c) $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$. **CORO:** D not posi $\iff (-D)^- = D$.
-

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \iff b \in D^- \setminus \{0\}\}$.
 $(-D)^- = (\mathbf{Q} \setminus (-D)) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$.
-

- For C, D posi, define $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.
 $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.
 - For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.
If $C, -C$ not posi $\implies C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlr for $D, -D$ not posi.
 - The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.
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- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
 $CD = -C(-D) = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$
 $= \{-a : a > rt, \forall r \in C^-, 0 \geq t \geq -s, \forall s \notin D\} = \{a : a < ru, \forall r \in C^-, 0 \leq u \leq s, \forall s \notin D\}$.
($r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs$.)
 - Note the ' $0 \leq u$ '. Becs $C^- \neq \emptyset \implies 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$, for if not, $\exists u > 0$, and $\exists r \in C^- \setminus \{0\}$, such that $\exists a < ru < 0$. Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
 - ' $u \leq s$ ' cannot be abbreviated as in $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$.
' $u \leq s$ ' cannot be ' $u < s$ ', becs here $rs < ru \implies \exists a = rs$. Simlr for ' $a < ru$ ' to be ' $a \leq ru$ ' with ' $u < s$ '.
 - Note that $\{u : 0 < u \leq s, \forall s \notin D\} \not\supseteq D^+$.
-

- Consider $-C$ and D both posi. Omitting $C = \tilde{0}$.
 $CD = -[(-C)D] = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} = \{-a : a > b, b > cd, \forall c \in (-C)^+, d \in D^+\}$
 $= \{a : a < -cd, \forall d \in D^+, \forall c$ suth $\underline{0 < c < -r, \exists r \in C^-}$ ($rd < -cd < 0$)
 $\stackrel{*}{=} \{a : a < rd, \forall r \in C^-, \forall d \in D^+\}$.

Intuitively, $0 < d_1 < \dots < d_n < \sup D$, and $\sup C \leq r_m < \dots < r_1 < 0$. Becs $a < r_j d_k, \forall j, k$.

又 $r_{j+1} < r_j d_k < r_j d_{k-1} \implies a < r_m d_n < r_j d_k$. Thus ' $a < rd$ ' cannot be ' $a \leq rd$ '.

(*a) Supp $a < -cd < 0, \forall c \in (-C)^+, d \in D^+$. Asum $-cd > a \geq rd', \exists r \in C^-, d' \in D^+$.

Let $d = d' \implies r \leq -c \iff c \leq -r$. Now $r < 0$.

If \exists min upper bound in \mathbf{Q} , then $r = \sup C$,

othws $r > \sup C \implies -r < -\sup C = \sup(-C) \implies \exists c > -r, c < \sup(-C)$, simlr if $\sup C \notin \mathbf{Q}$.

(*b) Supp $a \leq c'd, \forall c', d$ ($\implies a < 0$). Asum $a \geq -cd, \exists c', d'$.

- Let $LHS = \{a : a < ru, \forall r \in C^-, 0 < u \leq s, \forall s \notin D\}$, $RHS = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}$.
Where $\tilde{0} \neq C$ is not posi, D posi. We show ' $rd' < 'ru'$ ', so that $LHS = RHS$.
Seems equiv to ' $d' > 'u' \geq 's'$ ', while $d \in D, s \notin D$, thus ctrad. NOTICE that ' r, d, u ' are **not certain**.

Supp $a < ru, \forall r, u$. Asum $a > r'd, \exists r', d$. Now $r'd < a < ru, \forall r, u$. Let $r = r' \Rightarrow d > u$, ctrad. \square

Supp $a \leq rd, \forall r, d$. Asum $a \geq r'u, \exists r', u$. ($r' \neq 0$) Now $r'u \leq a \leq rd, \forall r, d \Rightarrow u \geq d, \forall d \in D^+$.

In fact, $u > d, \forall d \in D^+ \Rightarrow u \notin D$.

If \exists certain smallest elem s_M in $\mathbf{Q} \setminus D$, and if $u = s_M$, then $LHS = \{a : a < rs_M, \forall r \in C^-\}$.

Othws, $\exists d \in D^+, 0 < u < d < s, \forall s \notin D$, ctrad. \square

- For a posi D , $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$.

Now $(\tilde{1}D)^+ \subseteq D^+$. $\text{又 } \forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in \tilde{1}D$.

ENDED

0.C

5 Supp a_1, a_2, \dots is a seq in \mathbf{Q} , and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.

Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.

SOLUS: Becs the sup not in seq \Rightarrow infily many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$. For a_{n+k} , choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most $(n+1)$ steps, we have a_m with $m > n$. Thus $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$. \square

• Supp nonempty $A \subseteq \mathbf{R}$.

• **TIPS 1:** Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.

SOLUS: $-b$ is an upper bound of $-A \iff \forall a \in A, -a \leq -b \iff a \geq b \iff b$ is a lower bound of A .

Thus $-b_M = \sup(-A) \iff -b_M \leq -b \iff b_M \geq b \iff b_M = \inf A$. \square

• **TIPS 2:** Show if $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A, a > x$, (b) $\inf A < x \Rightarrow \exists a \in A, a < x$.

SOLUS: (a) $\nexists a > x \iff \forall a \in A, a \leq x$. Then by def of sup.

OR. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$. Simlr for (b). \square

6 Supp $A, B \subseteq \mathbf{R}$ has infily many disti elem, so has $A + B = \{a + b : a \in A, b \in B\}$.

Prove $\sup(A + B) = \sup A + \sup B$, and $\inf(A + B) = \inf A + \inf B$.

SOLUS: $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B, \inf A + \inf B \leq \inf(A + B)$.

$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$

$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B)$. Ctradic.

Simlr for $\inf(A + B) \in A + B$. OR. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. \square

16 Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in [0.11].

Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Define $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.

(a) Show $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$ is one-to-one. (b) Show $\psi(0) = 0, \psi(1) = 1$.

(c) Show $\psi(a \pm b) = \psi(a) \pm \psi(b)$, and $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.

(d) Supp $a \in \mathbf{R}_1$. Show $a > 0 \iff \psi(a) > 0$.

SOLUS: (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Note that $\varphi(q) < 0 \iff q < 0$, and $\varphi(0) = 0$.

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+), \varphi_2(q) \leq 1$.

(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$.

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$.

(d) $a > 0 \iff \exists n \in \mathbf{N}^+, 1/n < a \iff \psi_1(a) > 0$. \square