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这是我个人挑战自学「***Measure, Integration & Real Analysis, by Sheldon Axler***」的学习笔记，包括课文补注和部分习题。
我先从 ***Supplement*** 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。
0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不训。
的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导 $\mathcal{D}$ 的一切。如你所见，字越少，事越大。

Abbreviation Table

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B **NOTE:** C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.
  Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \backslash \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
- Define $-D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \iff a > b,\ b \notin D\}$.
  $-(-D) = -\{a : a < -b,\ b \notin D\} = \{c : c < -a,\ a \geqslant -b,\ \forall b \notin D\} = \{c : c < b,\ \forall b \notin D\} = D$.
  The last equa is becs (a) $d \notin D \Rightarrow \exists\, b \notin D,\ d \geqslant b$, and (b) $d \in D \Rightarrow$ if $\exists\, b \notin D$ suth $d \geqslant b$, then $b \in D$, ctradic.

- **TIPS:** *Prove $\forall \varepsilon > 0,\ \exists\, b \notin D$ suth $b - \varepsilon \in D$.*

SOLUS: Asum $\exists\, \varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \iff \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.
Now $\forall d \in D,\ \exists\, n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. $\square$

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1 *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D,\ \exists\, \varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.
(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.
$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. $\square$

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3 *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

SOLUS: (a) $C \subsetneq D \Rightarrow \exists\, x \in D \backslash C \Rightarrow \exists\, y \in D,\ y > x \Rightarrow \exists\, d \in D,\ d > y \iff 0 < d - y \in D - C$.
(b) $0 \in D - C \Rightarrow \exists\, y > x \notin C,\ y \in D \Rightarrow \forall c \in C,\ c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. $\square$

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5 *Prove* (a) D *posi* $\Rightarrow -D$ *not posi,* (b) *non0* $-D$ *not posi* $\Rightarrow D$ *posi.*

SOLUS: (a) $0 \notin \{a : a < -b,\ b \notin D\} \iff \nexists\, b \notin D,\ 0 < -b \iff \forall b \notin D,\ b \geqslant 0 \Longleftarrow 0 \in D$.
(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.
OR. $\exists\, a < 0,\ a \notin -D = \{a : -a > b,\ b \notin D\} \iff \nexists\, b \notin D,\ -a > b \iff \forall b \notin D,\ 0 < -a \leqslant b$. $\square$

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \iff \mathbf{Q} \backslash \mathbf{Q}^+ \subsetneq D \iff 0 \in D \iff D$ posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q} \backslash D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \mathbf{Q} \backslash (D \cup \mathbf{Q}^+)$.
  (a) $D^- = \{0\} \Longleftarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leqslant 0\} = \{0\} \Longrightarrow \mathbf{Q} \backslash D = \mathbf{Q}^*$.
  (b) $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q} \backslash \mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. **CORO:** D not posi $\iff 0 \in D^-$.
  (c) $(D^-)^- = \{r \in D : r \leqslant 0\} = \mathbf{Q} \backslash D^+$. **CORO:** D not posi $\iff (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \notin D \iff b \in D^- \backslash \{0\}\}$.
  $(-D)^- = (\mathbf{Q} \backslash -D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+)$.
  $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that '$a \leqslant cd$' here is equiv to '$a < cd$'.

• For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
$CD = \{0 < cd < (-r)(-s) : r \in C^-\backslash\{0\},\ s \in D^-\backslash\{0\}\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : a < rs,\ r \in C^-,\ s \in D^-\}$.
If $C, -C$ not posi $\Rightarrow C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlr for D.

• For D posi, define $D^{-1} = \{a : a < 1/b,\ b \notin D\} \Rightarrow DD^{-1} = \{a : a \leqslant d/b < 1,\ b \notin D,\ d \in D^+\} = \tilde{1}$.
The last equa holds becs $\forall a \in \tilde{1} \cap \mathbf{Q}^+$, $\exists\, d \in D^+$ suth $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$.

• For non0 D not posi, define $D^{-1} = -(-D)^{-1} = -\{a : a < 1/b,\ \exists\, b \notin -D \iff \exists\, b \geqslant -s,\ \forall s \notin D\}$
$= \{a : a < -x,\ \exists\, x \geqslant 1/b,\ \forall b$ suth $b \geqslant -s,\ \forall s \notin D\} = \{a : a < -1/b,\ \forall b$ suth $b \geqslant -s,\ \forall s \notin D\}$
$= \{a : a < 1/b,\ \forall b$ suth $b \leqslant s,\ \forall s \notin D\} \neq \{a : a < 1/s,\ \forall s \notin D\}$.
Let $b_1 < \cdots < b_m < \cdots \leqslant s,\ \forall s \notin D \iff \forall s \in D^- \Rightarrow$ each $s_j, b_k < 0 \Rightarrow 1/s_j \leqslant \cdots < 1/b_m < \cdots < 1/b_1$.
Thus '$a < 1/b$' is equiv to '$a < 1/s,\ \exists\, s \in D^-$'. Hence $D^{-1} = \{a : a < 1/b,\ b \in D^-\}$.
$DD^{-1} = \{a : a < rs,\ \exists\, r \in D^-,\ \exists\, s$ suth $0 \geqslant s \geqslant 1/b,\ \forall b \in D^-\} \subseteq \tilde{1}$.
Asum $\exists\, x$ suth $rs \leqslant x < 1,\ \forall r, s$. Let $D \not\ni \cdots \leqslant b_m < \cdots < b_1 \leqslant 0$, and $D \not\ni \cdots \leqslant r_m < \cdots < r_1 \leqslant 0$.
$1/b_1 < \cdots < 1/b_m \leqslant \cdots \leqslant \cdots \leqslant s_n < \cdots < s_1 \leqslant 0$, and $r_m/b_m \geqslant \cdots \geqslant \cdots \geqslant r_m s_n > \cdots > r_j s_k > \cdots > r_1 s_1$.
Let $r_m = b_m$. Ctradic. Or. $DD^{-1} = D[(-\tilde{1})(-D)^{-1}] = [D(-\tilde{1})](-D)^{-1} = (-D)(-D)^{-1} = \tilde{1}$.

• For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
$CD = -[C(-D)] = -\{a : a < rt,\ r \in C^-,\ t \in (-D)^-\} = \{-a : a > b,\ b \geqslant rt,\ \forall r \in C^-,\ t \in (-D)^-\}$
$= \{-a : a > rt,\ \forall r \in C^-,\ \forall t$ suth $0 \geqslant t \geqslant -s,\ \forall s \notin D\}$
$= \{a : a < ru,\ \forall r \in C^-,\ \forall u$ suth $0 \leqslant u \leqslant s,\ \forall s \notin D\}$. $(r \leqslant 0 < s,\ rs \leqslant ru = -rt \leqslant 0 \leqslant rt \leqslant -rs.)$

• Note the '$0 \leqslant u$'. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$,
for if not, $\exists\, u > 0$, and $\exists\, r \in C^-\backslash\{0\}$, suth $\exists\, a < ru < 0$. Hence '$0 \leqslant u$' is actually '$0 < u$'.

• '$u \leqslant s$' cannot be abbreviated as in $\{-a : a > -rs,\ \forall s \notin D,\ r \in C^-\} = \{a : a < rs,\ \forall s \notin D,\ r \in C^-\}$.
'$u \leqslant s$' cannot be '$u < s$', becs here $rs < ru \Rightarrow \exists\, a = rs$. Simlr for '$a < ru$' to be '$a \leqslant ru$' with '$u < s$'.

• Note that $\{u : 0 < u \leqslant s,\ \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q}\backslash D\}, & \text{if it exis,} \\ D^+, & \text{othws.} \end{cases}$ Denote it by $D^{\oplus} = D^{\otimes}\backslash\{0\}$.

• For C not posi and D posi. If $C = \tilde{0}$, then $CD = -[C(-D)] = -\tilde{0}$. Now consider $-C$ and D both posi.
But $CD = -(-C)D = -\{a : a \leqslant cd,\ c \in (-C)^+,\ d \in D^+\} \neq \{a : a < -cd,\ \forall c \in (-C)^+,\ d \in D^+\}$.
Altho "$a \leqslant cd$" is equiv to "$a < cd$" so that $b \notin (-C)D \Rightarrow b \geqslant cd$, which is actually $b > cd,\ \forall c, d$.
And $a < -b < -cd,\ \forall c, d \Rightarrow \forall a,\ \exists\, x$ suth $a < x < -cd,\ \forall c, d$. While a can be the 'boundary' in RHS.

• $LHS = \{a : a < ru,\ \forall r \in C^-,\ \forall u \in D^{\oplus}\}$, $\{cs : c \in C,\ s \notin D\} = RHS$.
Becs $cs \leqslant cu < ru$. We show $LHS \subseteq RHS$. Let $c_1 < \cdots < c_n < \cdots \in C$, and $s_1 > \cdots > s_m \geqslant \cdots \notin D$.
Then $\underline{c_1 s_1 < \cdots < c_n s_m < \cdots < ru,\ \forall r, u \text{ as in } LHS.\ \text{Thus } a \in LHS \Rightarrow \exists\, a < c_j s_k.}$ □
Or. Note that in LHS, '$a < ru$' is equiv to '$a < rs,\ \exists\, s \notin D$.' Now $LHS = \{a : a/s \in C,\ \exists\, s \notin D\}$. □

• For C posi and D not posi. If $D = \tilde{0}$, then $CD = -[(-C)D] = -\tilde{0}$.
$CD = (-C)(-D) = \{a : a < ru,\ \forall r \in (-C)^-,\ \forall u \in (-D)^{\oplus}\}$ $\qquad \mathbf{Q}\backslash -D = \{s : s \geqslant -y,\ \forall y \notin D\}$
$= \{a : a < ru,\ \forall r$ suth $\forall x \notin C,\ 0 \geqslant r \geqslant -x,\ \forall u$ suth $0 < u \leqslant s,\ \forall s \geqslant -y,\ \forall y \notin D\}$
$= \{a : a < (-r)(-u),\ \forall r$ suth $\forall x \notin C,\ 0 \leqslant -r \leqslant x,\ \forall u$ suth $y \leqslant -u < 0,\ \forall y \notin D\}$
$= \{a : a < ru,\ \forall r$ suth $\forall x \notin C,\ 0 \leqslant r \leqslant x,\ \forall u \in D^-\} = \{a : a < ru,\ \forall r \in C^{\oplus},\ \forall u \in D^-\}$, simlr.

- We show $-D = \{a : a < -b,\, b \notin D\} = (-\tilde{1})D$.
  For D posi, $RHS = \{a : a < ru,\ \forall r \text{ suth } -1 \leqslant r \leqslant 0,\ \forall u \in D^{\oplus}\} = \{a : a < -u,\ \forall u \in D^{\oplus}\} \supseteq -D$.
  Supp x suth $-b \leqslant x < -u,\ \forall b \notin D,\ \forall u \text{ suth } -s \leqslant -u < 0,\ \forall s \notin D \Longrightarrow -s \leqslant x < -u$. Let $-u = x$.
  For D not posi, $RHS = \{a : a < rb,\ \exists\, r \text{ suth } -1 \leqslant r \leqslant 0,\ b \in D^-\} = \{a : a < -b,\ 0 \geqslant b \notin D\} = -D$.

- We show $\tilde{1}D = D$. For D not posi, immed. Othws, $\tilde{1}D = \{a : a \leqslant ij < j,\ 0 < i < 1,\ j \in D^+\} \subseteq D$.
  Now $(\tilde{1}D)^+ \subseteq D^+$. 又 $\forall d \in D^+, \exists\, \varepsilon > 0,\ d + \varepsilon \in D^+ \Rightarrow d = (d+\varepsilon)\frac{d}{d+\varepsilon} \in (\tilde{1}D)^+$.

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4 *Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD)$, $B(C+D) = BC + BD$.*

Solus: We discuss in cases.

\	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

(1) $[(BC)D]^+ = \{(bc)d : bc \in (BC)^+,\ d \in D^+\} = \{b(cd) : b \in B^+,\ cd \in (CD)^+\} = [B(CD)]^+$.

$$B(C+D) = \{a : a \leqslant bc + bd,\ \exists\, b \in B^+,\ 0 < c + d \in C + D\}.$$

$$\begin{aligned} BC + BD &= \{x : x \leqslant uc,\ u \in B^+,\ c \in C^+\} + \{y : y \leqslant vd,\ v \in B^+,\ d \in D^+\} \\ &= \{a : a \leqslant uc + vd,\ \exists\, u, v \in B^+,\ c \in C^+,\ d \in D^+\}. \text{ Done.} \end{aligned}$$

(3) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists\, c \in C^-,\ \exists\, b \in B^{\oplus}\}$.

$$\begin{aligned} (BC)D &= \{a : a < rs,\ r \in (BC)^-,\ s \in D^-\} \\ &= \{a : a < bcs,\ \exists\, s \in D^-,\ c \in C^-,\ \exists\, b \in B^{\oplus}\}. \\ B(CD) &= \{a : a \leqslant bx < bcs,\ \exists\, s \in D^-,\ c \in C^-,\ \exists\, b \in B^+, \text{ and } cs > x \in (CD)^+\} \\ &= \{a : a < bcs,\ \exists\, s \in D^-,\ c \in C^-,\ \exists\, b \in B^+\}. \end{aligned}$$

Note that $\{q : q < b,\ b \in B^+\} = \{q : q < b,\ \exists\, b \in B^{\oplus}\}$. Done.

$$\begin{aligned} B(C+D) &= \{a : a < ru,\ \forall r \text{ suth } \forall x \notin B,\ 0 < r \leqslant x,\ \forall u \text{ suth } 0 \geqslant u > c + d,\ \forall c \in C,\ d \in D\} \\ &= \{a : a < ru,\ \forall r \in B^{\oplus},\ \forall u \text{ suth } 0 \geqslant u \geqslant c + d,\ \exists\, c \in C^-,\ d \in D^-\} \\ &= \{a : a < r(c+d),\ \forall r \in B^{\oplus},\ \forall c \in C^-,\ d \in D^-\}. \\ BC + BD &= \{x : x < pc,\ \forall p \in B^{\oplus},\ \forall c \in C^-\} + \{y : y < qd,\ \forall q \in B^{\oplus},\ \forall d \in D^-\} \\ &= \{a : a < pc + qd,\ \forall p, q \in B^{\oplus},\ \forall c \in C^-,\ d \in D^-\}. \end{aligned}$$

Done immed.

(5) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists\, b \in B^-,\ \exists\, c \in C^{\oplus}\}$. $(CD)^- = \{r : 0 \geqslant r \geqslant cd,\ \exists\, d \in D^-,\ \exists\, c \in C^{\oplus}\}$.

$$(BC)D = \{a : a < rd \leqslant bcd,\ \exists\, d \in D^-,\ b \in B^-,\ \exists\, c \in C^{\oplus}\}.$$

$$B(CD) = \{a : a < br \leqslant bcd,\ \exists\, b \in B^-,\ d \in D^-,\ \exists\, c \in C^{\oplus}\}. \text{ Done.}$$

Or. By commu and (3), $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$.

$$\begin{aligned} BC + BD &= \{x : x < bc,\ \forall b \in B^-,\ \forall c \in C^{\oplus}\} + \{y : y < bd,\ \exists\, b \in B^-,\ d \in D^-\} \\ &= \{a : a < pc + qd,\ \forall p \in B^-,\ \forall c \in C^{\oplus}, \text{ and } \exists\, q \in B^-,\ d \in D^- \Rightarrow q \geqslant p\} \\ &= \{a : a < pc + y,\ \forall p \in B^-,\ \forall c \in C^{\oplus}, \text{ and } \forall y \text{ suth } y \geqslant qd,\ \forall q \in B^-,\ d \in D^-\} \\ &= \{a : a < p(c + d'),\ \forall p \in B^-,\ \forall c \in C^{\oplus}, \text{ and } \forall d' \text{ suth } d' \leqslant d,\ \forall d \in D^-\}. \end{aligned}$$

(I) If $C + D$ not posi. Then $B(C+D) = \{a : a < bx,\ \exists\, b \in B^-,\ 0 \geqslant x > c + d,\ \forall (c,d) \in C \times D\}$.
Rewrite as $\{a : a < t,\ \forall t \text{ suth } t \geqslant bx,\ \forall b \in B^-,\ x \in (C+D)^-\}$. Done.

(II) If $C + D$ posi. Then $B(C+D) = \{a : a < bx,\ \forall b \in B^-,\ x \in (C+D)^{\oplus}\}$.
If $(C+D)^{\oplus} = (C+D)^+$. Then $B(C+D) = \{a : a < bc + bd,\ \forall b \in B^-,\ \forall (c,d) \in C^{\oplus} \times D^{\oplus}\}$.
Othws, $\exists\, s = \min \mathbf{Q} \backslash (C+D) \Rightarrow B(C+D) = \{a : a < bs,\ \forall b \in B^-\}$. Done.

$$\begin{aligned}(4)\ (BC)D &= \{xd : d \in D,\ x \geqslant bc,\ \forall b \in B^-,\ c \in C^-\}\\ &= \{a : a < bcd,\ \forall d \in D^-,\ \forall b \in B^-,\ c \in C^-\}\\ &= \{by : b \in B,\ y \geqslant cd,\ \forall c \in C^-,\ d \in D^-\} = B(CD).\end{aligned}$$

$$\begin{aligned}B(C+D) &= \{a : a < t,\ \forall t \text{ suth } t \geqslant b(c+d),\ \forall b \in B^-,\ (c,d) \in C^- \times D^-\}\\ &= \{a : a < t_1,\ \forall t_1 \text{ suth } t_1 \geqslant bc,\ \forall (b,c) \in B^- \times C^-\}\\ &\quad + \{a : a < t_2,\ \forall t_2 \text{ suth } t_2 \geqslant bd,\ \forall (b,d) \in B^- \times D^-\} = BC + BD. \text{ Done.}\end{aligned}$$

Note: Supp for any B posi, C posi, D not posi, assoc holds.

Supp B posi, C not posi, D posi. Then $(\overset{+}{B}\overset{-}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{B}\overset{-}{C}) = (\overset{+}{D}\overset{+}{B})\overset{-}{C} = \overset{+}{B}(\overset{+}{D}\overset{-}{C})$. Convly true.

Simlr for the case B not posi, C posi, D not posi, equiv to the case B not posi, C not posi, D posi.

(2) holds $\Rightarrow$ $(\overset{-}{B}\overset{+}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{C}\overset{-}{B}) = (\overset{+}{C}\overset{+}{D})\overset{-}{B}$, (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds $\Rightarrow$ $\overset{+}{B}(\overset{+}{C}+\overset{-}{D}) = (-\overset{-}{B})[-(\overset{+}{C}+\overset{-}{D})] \overset{(5)}{=\!=} (-\overset{-}{B})[(-\overset{-}{C})+(-\overset{+}{D})] \overset{(5)}{=\!=} \overset{+}{B}\overset{+}{C}+\overset{+}{B}\overset{-}{D}$, by def of multi.

Thus (5) $\Rightarrow$ (2) in distr. Convly as well.

(6) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists\, b \in B^-,\ c \in C^{\oplus}\}$.

$(BC)D = \{a : a < ru,\ \forall r \in (BC)^-,\ u \in D^{\oplus}\} = \{a : a < bcd,\ \forall b \in B^-,\ c \in C^{\oplus},\ d \in D^{\oplus}\}$.

$(CD)^{\oplus} = \{u : 0 < u \leqslant s,\ \forall s > cd,\ \forall c \in C^+,\ d \in D^+\}$.

$B(CD) = \{a : a < ru,\ \forall r \in B^-,\ u \in (CD)^{\oplus}\}$. Done.

$B(C+D) = \{a : a < b(c+d),\ \forall b \in B^-,\ \forall (c,d) \in C^{\oplus} \times D^{\oplus}\}$.

$BC + BD = \{a : a < bc,\ \forall b \in B^-,\ c \in C^{\oplus}\} + \{a : a < bd,\ \forall b \in B^-,\ d \in D^{\oplus}\}$. Done.

Or. (2) instead of (6) ? Notice that the distr in (6) cannot be shown without (6). $\square$

Ended

0·C

2 *Supp nonempty $U \subseteq V \subseteq \mathbf{R}$. Show $\sup U \leqslant \sup V$.*

Solus: Asum $\sup U > \sup V \Rightarrow \exists\, t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$, ctradic. $\square$

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5 *Supp $a_1, a_2, \cdots \in \mathbf{Q}$, and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.
Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.*

Solus: Becs the sup not in seq $\Rightarrow$ infily many disti elem.

$\forall a_i,\ \exists\, a_j,\ a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most n steps, we must have a_m with $m > n$. Thus $\forall a_{n+i},\ \exists\, a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. $\square$

- *Supp nonempty $A \subseteq \mathbf{R}$.*
- **Tips 1:** *Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.*

Solus: $-b$ is an upper bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a lower bound of A.

Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. $\square$

- **Tips 2:** *Show if $x \in \mathbf{R}$,* (a) $\sup A > x \Rightarrow \exists\, a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists\, a \in A,\ a < x$.

Solus: (a) $\nexists\, a > x \iff \forall a \in A,\ a \leqslant x \Longrightarrow \sup A \leqslant x$.

Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). $\square$

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6 *Supp nonempty* $A, B \subseteq \mathbf{R}$. *Prove* $\sup(A+B) = \sup A + \sup B$, *and* $\inf(A+B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B$, $\inf A + \inf B \leqslant \inf(A+B)$.

$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$

$\iff \exists\, a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists\, a + b > \sup(A+B)$. Ctradic.

Simlr for $\inf(A+B) \in A+B$. Or. Apply to $-A-B$, becs $\sup(-A) = -\inf A$. $\square$

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16 *Supp* $\mathbf{R}_1, \mathbf{R}_2$ *are complete ordered fields. Define* φ_1, φ_2 *as in* [0.11].

Define $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ *by* $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant a\}$. *Show*

(a) $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ *is one-to-one.* (b) $\psi(0) = 0$, $\psi(1) = 1$.

(c) $\psi(a+b) = \psi(a) + \psi(b)$. (d) $\psi(ab) = \psi(a)\psi(b)$.

Solus: (a) Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Let $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$.

Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Becs $\varphi_1(q) \leqslant \varphi_1(0) \iff q \leqslant 0 \iff \varphi_2(q) \leqslant \varphi_2(0)$. Simlr for $\psi(1) = 1$.

(c) $S = \{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant a+b\} \supseteq \{\varphi_2(p+q) : p, q \in \mathbf{Q},\ \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b\} = T$

$\Rightarrow \sup S \geqslant \sup T$. Asum it is '>'. Now $\exists\, t \in \mathbf{Q}$ suth $\sup S \geqslant \varphi_2(t) > \sup T = \psi(a) + \psi(b)$.

Which means $\varphi_2(t) > \varphi_2(p+q)$, $\forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leqslant a$ and $\varphi_1(q) \leqslant b$.

Now $a + b \geqslant \varphi_1(t) > \varphi_1[(t+p+q)/2] > \varphi_1(p+q)$, $\forall p, q$. Ctradic.

(d) We show it for (I) $a, b > 0$, (II) $a > 0 > b$.

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant ab\}$, $\sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant b\} = RHS$.

(I) $RHS = \sup\{\varphi_2(pq) : p, q \in \mathbf{Q},\ \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b\}$.

(II) $RHS = \sup\{\gamma\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant b, \text{and } \gamma \geqslant \varphi_2(p), \forall p \text{ suth } \varphi_1(p) \leqslant a\}$.

$= \sup\{\varphi_2(pq) : p, q \in \mathbf{Q},\ \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b\}$.

Thus $LHS \geqslant RHS$. Asum it is '>'. Now $\exists\, t \in \mathbf{Q}$ suth $LHS \geqslant \varphi_2(t) > RHS$.

Which means $\varphi_2(t) > \varphi_2(pq)$, $\forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leqslant a$, $\varphi_1(q) \leqslant b$.

Now $ab \geqslant \varphi_1(t) > \varphi_1(p(q+\delta)) > \varphi_1(pq)$, $\forall p, q$. Ctradic. $\square$

Ended

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Ended

0·E

- *Supp* $b \in A \subseteq \mathbf{R}^m, f : A \to \mathbf{R}^n$. *Prove*
  $[P]$ f *is continu at* $b \iff \forall b_1, b_2, \cdots \in A$ *suth* $\lim_{k\to\infty} b_k = b, \lim_{k\to\infty} f(b_k) = f(b)$. $[Q]$

Solus: $Q \Rightarrow P:$ Supp $\varepsilon > 0$ suth $\forall \delta > 0, \exists\, a \in A$ with $\|a - b\|_\infty < \delta$ and $\left\| f(a) - f(b) \right\|_\infty \geqslant \varepsilon$.
Fix a δ. Define $\delta_k = \delta / k \Rightarrow \exists\, a_k$ for each. Now $\lim_{k\to\infty} a_k = b$.
Thus $\forall m, \forall k \geqslant m, \left\| f(a_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic Q.

$P \Rightarrow Q:$ Supp $b_1, b_2, \cdots \in A$ suth $\forall \delta > 0,\ \exists\, m, \forall k \geqslant m,\ \|b_k - b\|_\infty < \delta$.
Asum $\varepsilon > 0$ suth $\forall m, \forall k \geqslant m, \left\| f(b_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic P. $\square$

Ended