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## ABBREVIATION TABLE

### A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

### E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existsns	existence
expo	exponent
expr	expression

### L

liney	linear.ly
linity	linearity
len	length
low-	lower-

### R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

### C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	contrapositive

### D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

### I

id	identity
immed	immediately
induc	induct(ion)(ive)
infil	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

### F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

### M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

### S

seq	sequence
simlr	similar.ly
soluts	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

### O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quot	quotient

### T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

## 0·B

**NOTE:**  $C, D$  are Dedekind cuts. Numbers used here are always rational.

- Define  $\tilde{q} = \{a : a < q\}$ , and  $-\tilde{q} = \tilde{-q} = \{a : a < -q\}$ .  
Then  $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leq 0\} = \tilde{0}$ .
- Define  $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$ .  
 $-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$ .

- **TIPS:** Prove  $\forall \varepsilon > 0, \exists b \notin D$  suth  $b - \varepsilon \in D$ .

**SOLUS:** Asum  $\exists \varepsilon > 0$  suth  $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$ .

Then  $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$  for any  $n \in \mathbf{N}^+$ .

Now  $\forall d \in D, \exists n \in \mathbf{N}^+$  suth  $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$ , ctradic.  $\square$

**1** Prove (a)  $D + \tilde{0} = D$ , (b)  $-D$  is Dedekind cut, and  $D + (-D) = \tilde{0}$ .

**SOLUS:** (a)  $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$ .

(b) Asum  $x \in -D$  is the largest elem of  $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$ .

Let  $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$ .

Thus by def,  $x + \delta \in -D$ , ctradic the max of  $x \in -D$ . Hence  $-D$  is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$ .

Supp  $a \in \tilde{0} \Rightarrow -a > 0$ . By TIPS,  $\exists b \notin D$  suth  $b + a \in D$ .

Note that  $b < b - a \notin D \Rightarrow -b > -b + a \in -D$ . Then  $(-b + a) + (b + a) = 2a < 0$ .

Thus  $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$ .  $\square$

**CORO:**  $\{a_1 + a_2 : a_1, a_2 \in D\} = D + D = \tilde{0} = \{a : a < 0\} \Rightarrow \forall a \in D, a + a < 0 \Leftrightarrow a < 0$ .

**5** Prove  $D$  is posi  $\Leftrightarrow -D$  is not posi  $\Leftrightarrow 0 \in D$ .

**SOLUS:**  $0 \notin -D = \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, b < 0 \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D$ .  $\square$

- Define  $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$ . Then  $D^+ = \emptyset \Leftrightarrow D \subseteq \mathbf{Q} \setminus \mathbf{Q}^+ \Leftrightarrow 0 \notin D \Leftrightarrow D$  not posi.  
Define  $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$ .  
(a)  $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi.   **CORO:**  $D$  not posi  $\Leftrightarrow 0 \in D^-$ .  
(b)  $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$ .   **CORO:**  $D$  not posi  $\Leftrightarrow (-D)^- = D$ .
- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$ .  
 $(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$ .
- For  $C, D$  posi, define  $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .  
 $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$ . Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.
- For  $-C, -D$  posi, define  $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .  
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$ .  
If  $C, D$  not posi while  $-C, -D$  not posi  $\Rightarrow C = D = \tilde{0}$ , then with the asum  $\tilde{0}\tilde{0} = \tilde{0}$ , it still holds.
- The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.

- For  $C$  not posi and  $D$  posi, we expect that  $CD$  not posi. Consider  $C$  and  $-D$  both not posi.

$$\begin{aligned} CD = -C(-D) &= -\{a : a < rt, r \in C^-, t \in (-D)^-\} \\ &= \{-a : a > rt, \forall r \in C^-, 0 \geq t \geq -s, \forall s \notin D\} \quad (r \leq 0, -s \leq t \leq 0 \Rightarrow -rs \geq rt \geq 0) \\ &= \{a : a < ru, \forall r \in C^-, 0 \leq u \leq s, \forall s \notin D\}. \end{aligned}$$

- Note the ' $0 \leq u'$ . Becs  $C^- \neq \emptyset \Rightarrow 0 \in C^-$ . If it is to be exactly  $a < 0$  in LHS ( $= \tilde{0}$ ), then  $C^- = \{0\}$ , for if not,  $\exists u = s > 0$ , and  $\exists r \in C^- \setminus \{0\}$ , such that  $\exists a < ru < 0$ . Hence ' $0 \leq u'$  is actually ' $0 < u'$ .
- ' $u \leq s'$  cannot be abbreviated as in  $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$ . ' $a < ru$ ' cannot be changed to ' $a \leq ru$ ' with " $0 < u < s, \forall s \notin D$ "?????
- Let  ${}^+D = \{u : 0 < u \leq s, \forall s \notin D\} \supsetneq D^+$ .

- Consider  $-C$  and  $D$  both posi. Omitting  $C = \tilde{0}$ , becs  $C$  not posi  $\not\Rightarrow -C$  posi.

$$\begin{aligned} CD = -[(-C)D] &= -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \\ &= \{a : a < -b < -cd, \forall d \in D^+, \forall c \text{ suth } 0 < c < -c', \exists c' \notin C\} \quad (0 < cd < -c'd) \end{aligned}$$

To make those comfortable, notice the following:

$$(a) A = \{a : a < M\}, B = \{b : b < a, \forall a \in A \Rightarrow b < a < M\} \subseteq \{b : b < M\} = A. \quad \text{又 } \forall b < M, \exists \epsilon > 0, b + \epsilon < M \Rightarrow b \in B.$$

$$(b) A = \{a : M < a\}, B = \{b : b < a, \forall a \in A\} \supseteq \{b : b \leq M\} \quad \text{又 Supp } b \text{ suth } \forall a \in A, b < a. \text{ If } b > M, \text{ then } b \in A \Rightarrow \exists b' \in A, b' < b.$$

$$\begin{aligned} CD &= \{a : a < -cd, \forall d \in D^+, \forall c \text{ suth } 0 < c < -c', \exists c' \notin C\} \quad (c'd < -cd < 0) \\ &= \{a : a \leq c'd, \forall c' \in C^-, \forall d \in D^+\} = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}. \end{aligned}$$

- Let  $LHS = \{cs : c \in C, s \notin D\}$ ,  $RHS = \{a : a \leq rd, \forall r \in C^-, \forall d \in D^+\}$ .

$\text{Supp } a \leq rd$ . Let  $b \notin D \Rightarrow d < b$ . Let  $q = b/d$ .

$$r' = rq \Rightarrow r'd = rb.$$

$\text{Supp } c, s \text{ suth } cs \in LHS$ . Then  $cs < rs, \forall r \in C^-$ .

- For a posi  $D$ ,  $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$ .

Now  $(\tilde{1}D)^+ \subseteq D^+$ . 又  $\forall d \in D^+, \exists \epsilon > 0, d + \epsilon \in D^+ \Rightarrow d = (d + \epsilon) \frac{d}{d + \epsilon} \in \tilde{1}D$ .

- 3** Show  $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$  posi.

**SOLUS:** (a)  $\text{Supp } C \subsetneq D$ . Let  $d \in D \setminus C \subseteq \mathbb{Q} \setminus C$ . For any  $a \in \tilde{0}$ , let  $y = d - a > d \Rightarrow a = d - y \in D - C$ .

(b)  $\text{Supp } D - C$  posi. Becs  $d = y$  for some  $d \in D, y > x \notin C$ .

Note that  $x \notin C \Rightarrow \forall c \in C, c < x < y = d \Rightarrow C \subseteq D$ . □

#### 4 Prove

**SOLUS:**

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# 0.C

**5** *Supp  $a_1, a_2, \dots$  is a seq in  $\mathbf{Q}$ , and  $\sup\{a_1, a_2, \dots\} = \sqrt{2}$ .*

*Prove  $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$  for all  $n \in \mathbf{N}^+$ .*

**SOLUS:** Becs the sup not in seq  $\Rightarrow$  inflly many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$ . For  $a_{n+k}$ , choose  $a_i > a_{n+k}$ . If  $i \in \{1, \dots, n\}$ , then choose  $a_j > a_i$ .

After at most  $(n+1)$  steps, we have  $a_m$  with  $m > n$ . Thus  $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$ .  $\square$

• *Supp nonempty  $A \subseteq \mathbf{R}$ .*

• **TIPS 1:** Define  $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$ . Prove  $\sup(-A) = -\inf A$ .

**SOLUS:**  $-b$  is an upper bound of  $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$  is a lower bound of  $A$ .

Thus  $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$ .  $\square$

• **TIPS 2:** Show if  $x \in \mathbf{R}$ , (a)  $\sup A > x \Rightarrow \exists a \in A, a > x$ , (b)  $\inf A < x \Rightarrow \exists a \in A, a < x$ .

**SOLUS:** (a)  $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x$ . Then by def of sup.

Or. By (b),  $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$ .

Simlr for (b).  $\square$

**6** *Supp  $A, B \subseteq \mathbf{R}$  has inflly many disti elem, so has  $A + B = \{a + b : a \in A, b \in B\}$ .*

*Prove  $\sup(A + B) = \sup A + \sup B$ , and  $\inf(A + B) = \inf A + \inf B$ .*

**SOLUS:**  $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$ ,  $\inf A + \inf B \leq \inf(A + B)$ .

$\sup A + \sup B > \sup(A + B) \Leftrightarrow \sup A > \sup(A + B) - \sup B$

$\Leftrightarrow \exists a + \sup B > \sup(A + B) \Leftrightarrow \sup B > \sup(A + B) - a \Leftrightarrow \exists a + b > \sup(A + B)$ . Ctradic.

Simlr for  $\inf(A + B) \in A + B$ . Or. Apply to  $-A - B$ , becs  $\sup(-A) = -\inf A$ .  $\square$

**16** *Supp  $\mathbf{R}_1, \mathbf{R}_2$  are complete ordered fields. Define  $\varphi_1, \varphi_2$  as in [0.11].*

Define  $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Define  $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$ .

(a) Show  $\psi = \psi_1 : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  is one-to-one. (b) Show  $\psi(0) = 0, \psi(1) = 1$ .

(c) Show  $\psi(a \pm b) = \psi(a) \pm \psi(b)$ , and  $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$ .

(d) Supp  $a \in \mathbf{R}_1$ . Show  $a > 0 \Leftrightarrow \psi(a) > 0$ .

**SOLUS:** (a) Define  $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$ .

Define  $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_2$ . Then  $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  well-defined.

Note that  $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$ .

Now  $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_2 = a$ . Rev the roles of  $\mathbf{R}_1, \mathbf{R}_2$ .

(b) Note that  $\varphi(q) < 0 \Leftrightarrow q < 0$ , and  $\varphi(0) = 0$ .

$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leq 1\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ ,  $\varphi_2(q) \leq 1$ .

(c)  $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leq a \pm b\}$ .

$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(p) \cdot \varphi_1(q)^{-1} \leq ab^{-1}\}$ .

(d)  $a > 0 \Leftrightarrow \exists n \in \mathbf{N}^+, 1/n < a \Leftrightarrow \psi_1(a) > 0$ .  $\square$

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