Abbreviation Table

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
count-	counter-
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B **Note:** C, D are Dedekind cuts. Numbers used here are always rational.

- $\tilde{0} = \{a : a < 0\} = \mathbf{Q}\backslash\mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
  $D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \Longleftrightarrow a > b,\ b \notin D\}$.

- **Tips:** *Prove* $\forall\,\varepsilon > 0,\ \exists\, b \notin D$ *suth* $b - \varepsilon \in D$.

Solus: Asum $\exists\,\varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \Longleftrightarrow \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n\cdot\varepsilon \notin D$ for any $n \in \mathbf{N}^+$.
For any $d \in D$, $\exists\, n \in \mathbf{N}^+$ suth $b - n\cdot\varepsilon < d \Rightarrow b - n\cdot\varepsilon \in D$, ctradic. □

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1 *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

Solus: (a) $\forall d \in D,\ \exists\,\varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By Tips, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

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5 *Prove* D *is posi* $\Longleftrightarrow -D$ *is not posi* $\Longleftrightarrow 0 \in D$.

Solus: $0 \notin -D = \{a : a < -b,\ b \notin D\} \Longleftrightarrow \nexists\, b \notin D,\ b < 0 \Longleftrightarrow \mathbf{Q}\backslash\mathbf{Q}^+ \subsetneq D \Longleftrightarrow 0 \in D$. □

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ = \emptyset \Longleftrightarrow D \subsetneq \mathbf{Q}\backslash\mathbf{Q}^+ \Longleftrightarrow 0 \notin D \Longleftrightarrow D$ not posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q}\backslash D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \mathbf{Q}\backslash(D \cup \mathbf{Q}^+)$.
  Then $D^- = \emptyset \Longleftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Longleftrightarrow \mathbf{Q}\backslash\mathbf{Q}^+ \subseteq D \Longleftrightarrow 0 \in D \Longleftrightarrow D$ posi. And D not posi $\Longleftrightarrow 0 \in D^-$.
  If D not posi, then $(D^-)^- = \{r \in D : r \leqslant 0\} = D = \mathbf{Q}\backslash D^+$.
- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \notin D \Longleftrightarrow b \in D^-\backslash\{0\}\}$.
  $(-D)^- = (\mathbf{Q}\backslash -D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.
  Define $\tilde{1} = \{a : a < 1\} \Rightarrow \tilde{1}^+ = \{a : 0 < a < 1\}$, $\tilde{1}^- = \{a : a \leqslant 0\} = \tilde{0} \cup \{0\}$.
  Then $-\tilde{1} = \{a : -a > 1 \Longleftrightarrow a < -1\}$ not posi.

- For C, D posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  Note that $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$.
- For C, D not posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  $CD = \{0 < cd < (-r)(-s) : r \in C^-\backslash\{0\},\ s \in D^-\backslash\{0\}\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : a < rs,\ r \in C^-,\ s \in D^-\}$.
- The intuitive key point is that the prod of cuts is the cut with the endpoint being the prod of endpoints of cuts.
- For C not posi and D posi, we expect that CD not posi.
  $CD = -C(-D) = -\{a : a < rt,\ r \in C^-,\ t \in (-D)^-\}$
  $\quad = \{-a : a \geqslant rt,\ \forall r \in C^-,\ t \in (-D)^- \Rightarrow 0 \geqslant r \notin C,\ 0 \geqslant t \geqslant -s,\ \forall s \notin D\}$
  Becs $r \leqslant 0,\ -s \leqslant t \leqslant 0 \Rightarrow -rs \geqslant rt \geqslant 0$. Now $CD = \{a : a \leqslant rs \Longleftrightarrow -a \geqslant -rs,\ \forall s \notin D,\ 0 \geqslant r \notin C\}$.
  Thus defining CD. Now we show this is equiv to $CD = \{cs : c \in C,\ s \notin D\}$.

• For C posi and D not posi, define $CD = \{rd : r \notin C,\ d \in D\} \Rightarrow CD$ not posi.

• Thus $CD = DC$ in all cases.

• For a posi D, $\tilde{1}D = \{a : a \leqslant ij < j,\ 0 < i < 1,\ j \in D^+\} \subseteq D$.
Now $(\tilde{1}D)^+ \subseteq D^+$. 又 $\forall d \in D^+, \exists\, \varepsilon > 0,\ d + \varepsilon \in D^+ \Rightarrow d = (d+\varepsilon)\frac{d}{d+\varepsilon} \in \tilde{1}D$.

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3 *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

Solus: (a) Supp $C \subsetneq D$. Let $d \in D \backslash C \subseteq \mathbf{Q} \backslash C$. For any $a \in \tilde{0}$, let $y = d - a > d \Rightarrow a = d - y \in D - C$.

(b) Supp $D - C$ posi. Becs $d = y$ for some $d \in D,\ y > x \notin C$.
Note that $x \notin C \Rightarrow \forall c \in C,\ c < x < y = d \Rightarrow C \subseteq D$. □

Ended

0·C

5 *Supp* $a_1, a_2, \dots$ *is a seq in* $\mathbf{Q}$, *and* $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.
Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ *for all* $n \in \mathbf{N}^+$.

Solus: Becs the sup not in seq $\Rightarrow$ infily many disti elem.
$\forall a_i,\ \exists a_j,\ a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.
After at most $(n+1)$ steps, we have a_m with $m > n$. Thus $\forall a_{n+i},\ \exists a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. $\square$

- *Supp nonempty* $A \subseteq \mathbf{R}$.
- **Tips 1:** *Define* $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. *Prove* $\sup(-A) = -\inf A$.

Solus: $-b$ is an upper bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a lower bound of A.
Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. $\square$

- **Tips 2:** *Show if* $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists a \in A,\ a < x$.

Solus: (a) $\nexists a > x \iff \forall a \in A,\ a \leqslant x$. Then by def of sup.
Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). $\square$

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6 *Supp* $A, B \subseteq \mathbf{R}$ *has infily many disti elem, so has* $A + B = \{a + b : a \in A, b \in B\}$.
Prove $\sup(A+B) = \sup A + \sup B$, *and* $\inf(A+B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B,\ \inf A + \inf B \leqslant \inf(A+B)$.
$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$
$\iff \exists a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists a + b > \sup(A+B)$. Ctradic.
Simlr for $\inf(A+B) \in A + B$. Or. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. $\square$

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16 *Supp* $\mathbf{R}_1, \mathbf{R}_2$ *are complete ordered fields. Define* φ_1, φ_2 *as in* [0.11].
Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. *Define* $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2$.
(a) *Show* $\psi = \psi_1 : \mathbf{R}_1 \to \mathbf{R}_2$ *is one-to-one.* (b) *Show* $\psi(0) = 0,\ \psi(1) = 1$.
(c) *Show* $\psi(a \pm b) = \psi(a) \pm (b)$, *and* $\psi(ab^{-1}) = \psi(a)\psi(b)^{-1}$.
(d) *Supp* $a \in \mathbf{R}_1$. *Show* $a > 0 \iff \psi(a) > 0$.

Solus: (a) Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.
Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.
Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.
Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.
(b) Note that $\varphi(q) < 0 \iff q < 0$, and $\varphi(0) = 0$.
$\forall q \in \mathcal{R}_1(1) = \{1/m \in \mathbf{Q} : m \in \mathbf{N}^+, \varphi_1(1/m) = (1 + \dots + 1)^{-1} \leqslant 1\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+),\ \varphi_2(q) \leqslant 1$.
(c) $\mathcal{R}_1(a \pm b) = \{p \pm q \in \mathbf{Q} : \varphi_1(p) \pm \varphi_1(q) \leqslant a \pm b\}$.
$\mathcal{R}_1(ab^{-1}) = \{pq^{-1} \in \mathbf{Q} : \varphi_1(q) \cdot \varphi_1(q)^{-1} \leqslant ab^{-1}\}$.
(d) $a > 0 \iff \exists n \in \mathbf{N}^+,\ 1/n < a \iff \psi_1(a) > 0$. $\square$

Ended