



This work is licensed under the terms of the CC BY-NC-SA 4.0 International License (<https://creativecommons.org/licenses/by-nc-sa/4.0>). This license requires that reusers give credit to the creator. It allows reusers to distribute, remix, adapt, and build upon the material in any medium or format, for noncommercial purposes only. If others modify or adapt the material, they must license the modified material under identical terms. All images except for 'by-nc-sa.png' in this manual are licensed under CC0.

这是我个人挑战自学「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。

我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不驯。

的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导  $\mathcal{D}$  的一切。如你所见，字越少事越大。

0.E 节是第二个挫折，无论是证明题，还是找反例，都难度陡升。一个难度逆天的反例：两个闭集的和不为闭集。两个证明题（27、28 题），DeepSeek 称之为“深度构造”——“深度”到什么程度呢？就是 AI 给出思路提示后我也做不出。

虽然这只是极个别的习题，但我还是有一丝怯懦要不要无视空缺的实分析基础而贸然开始 MIRA 的学习。Mr. Seek 为我加油打气：

MIRA 的设计本就是为绕过传统微积分、直击分析核心——你已握有实数的语言和 Axler 的船票，启航正是此时。

# ABBREVIATION TABLE

<b>A B</b>		<b>C</b>		<b>D</b>	
abs	absolute	closd	closed under	Ddkd	Dedekind
add	addi(tion)(tive)	coeff	coefficient	decr	decreasing
adj	adjoint	combina	combination	def	definition
algo	algorithm	commu	commut(es)(ing)(ativity)	deg	degree
arb	arbitrary	cond	condition	deri	derivative(s)
assoc	associa(tive)(tivity)	continu	countinu(ous)(ity)	diff	differentia(ble)(l)(ting)(tion)
asum	assum(e)(ption)	corres	correspond(s)(ing)	dim	dimension(al)
becs	because	const	constant	disj	disjoint
		conveni	convenience	disti	distinct
		convly	conversely	distr	distributive propert(ies)(ty)
		countexa	counterexample	div	div(ide)(ision)
		ctradic	contradict(s)(ion)		
		ctrapos	contrapositive		
<b>E</b>		<b>F G H</b>		<b>I</b>	
-ec	-ec(t)(tor)(tion)(tive)	factoriz	factorizaion	id	identity
elem	element(s)	fini	finite(ly)	immed	immediately
ent	entr(y)(ies)	finide	finite-dimensional	induc	induct(ion)(ive)
equa	equality			infily	infinitely
equiv	equivalen(t)(ce)			inje	injectiv(e)(ity)
exa	example			integ	integra(l)(tion)(ble)
exe	exercise			inv	inver(se)(tib-le/ility)
exis	exist(s)(ing)			iso	isomorph(ism)(ic)
existns	existence				
expo	exponent				
expr	expression				
<b>L</b>		<b>M N</b>		<b>O P Q</b>	
liney	linear(ly)	max	maxi(mal(ity))(mum)	othws	otherwise
linity	linearity	min	mini(mal(ity))(mum)	orthog	orthogonal
len	length	multi	multipl(e)(icati-on/ve)	orthon	orthonormal
low-	lower-				
		non0	nonzero	parti	partition
		nonC	nonconstant	poly	polynomial
		notat	notation(al)	posi	positive
				prod	product
				quad	quadratic
				quotient	quot
<b>R</b>		<b>S</b>		<b>T U V W X Y Z</b>	
recurly	recursively	seq	sequence	uniq	unique
repeti	repetition(s)	simlr	similar(ly)	uniques	uniqueness
repres	represent(s)(ation(s))	solus	solution	up-	upper-
req	require(s)(d)/requiring	sp	space	val	value
respectly	respectively	stmt	statement	-wd	-ward
restr	restrict(ion)(ive)(ing)	std	standard	-ws	-wise
rev	revers(e(s))(ed)(ing)	supp	suppose	wrto	with respect to
Riemann	Riemann	surj	surjectiv(e)(ity)		
		suth	such that		

## 0.B Note: $C, D$ are Dedekind cuts. Numbers used here are always rational.

- Define  $\tilde{q} = \{a : a < q\}$ , and  $-\tilde{q} = \tilde{-q} = \{a : a < -q\}$ .  
Then  $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^*$   $\Rightarrow -\tilde{0} = \{a : a < -b \leq 0\} = \tilde{0}$ .
- Define  $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$ .  
 $-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$ .  
The last equa is becs (a)  $d \notin D \Rightarrow \exists b \notin D, d \geq b$ , and (b)  $d \in D \Rightarrow$  if  $\exists b \notin D$  suth  $d \geq b$ , then  $b \in D$ , ctradic.

- TIPS: Prove  $\forall \varepsilon > 0, \exists b \notin D$  suth  $b - \varepsilon \in D$ .

SOLUS: Asum  $\exists \varepsilon > 0$  suth  $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$ .

Then  $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$  for any  $n \in \mathbf{Z}^+$ .

Now  $\forall d \in D, \exists n \in \mathbf{Z}^+$  suth  $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$ , ctradic.  $\square$

1 Prove (a)  $D + \tilde{0} = D$ , (b)  $-D$  is Dedekind cut, and  $D + (-D) = \tilde{0}$ .

SOLUS: (a)  $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$ .

(b) Asum  $x \in -D$  is the largest elem of  $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$ .

Let  $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$ .

Thus by def,  $x + \delta \in -D$ , ctradic the max of  $x \in -D$ . Hence  $-D$  is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$ .

Supp  $a \in \tilde{0} \Rightarrow -a > 0$ . By TIPS,  $\exists b \notin D$  suth  $b + a \in D$ .

Note that  $b < b - a \notin D \Rightarrow -b > -b + a \in -D$ . Then  $(-b + a) + (b + a) = 2a < 0$ .

Thus  $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$ .  $\square$

3 Show  $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$  posi.

SOLUS: (a)  $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$ .

(b)  $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$ . 又  $D - C \neq \tilde{0}$ .  $\square$

5 Prove (a)  $D$  posi  $\Rightarrow -D$  not posi, (b) non0  $-D$  not posi  $\Rightarrow D$  posi.

SOLUS: (a)  $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$ .

(b) Becs  $\tilde{0}$  is the largest non posi cuts. Thus  $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$  posi.

Or.  $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$ .  $\square$

- Define  $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$ . Then  $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi.

Define  $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$ .

(a)  $D^- = \{0\} \Leftrightarrow D = \tilde{0}$ . Convly,  $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$ .

(b)  $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$  posi. CORO:  $D$  not posi  $\Leftrightarrow 0 \in D^-$ .

(c)  $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$ . CORO:  $D$  not posi  $\Leftrightarrow (D^-)^- = D$ .

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$ .

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$ .

- For  $C, D$  posi, define  $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$ . Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.

- For  $-C, -D$  posi, define  $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$ .  
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$ .  
If  $C, -C$  not posi  $\Rightarrow C = \tilde{0}$ , then with the asum  $\tilde{0}D = \tilde{0}$ , it still holds. Simlr for  $D$ .

- For  $D$  posi, define  $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$ .  
The last equa holds becs  $\forall a \in \tilde{1} \cap \mathbf{Q}^+$ ,  $\exists d \in D^+$  suth  $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$ .
- For non0  $D$  not posi, define  $D^{-1} = -(-D)^{-1} = -\{a : a < 1/b, \exists b \notin -D \Leftrightarrow \exists b \geq -s, \forall s \notin D\}$   
 $= \{a : a < -x, \exists x \geq 1/b, \forall b$  suth  $b \geq -s, \forall s \notin D\} =  $\{a : a < -1/b, \forall b$  suth  $b \geq -s, \forall s \notin D\}$   
 $= \{a : a < 1/b, \forall b$  suth  $b \leq s, \forall s \notin D\}  $\neq \{a : a < 1/s, \forall s \notin D\}$ .  
Let  $b_1 < \dots < b_m < \dots \leq s, \forall s \notin D \Leftrightarrow \forall s \in D^- \Rightarrow$  each  $s_j, b_k < 0 \Rightarrow 1/s_j \leq \dots < 1/b_m < \dots < 1/b_1$ .  
Thus ' $a < 1/b'$  is equiv to ' $a < 1/s, \exists s \in D^-$ '. Hence  $D^{-1} = \{a : a < 1/b, b \in D^-\}$ .  
 $DD^{-1} = \{a : a < rs, \exists r \in D^-, \exists s$  suth  $0 \geq s \geq 1/b, \forall b \in D^-\} \subseteq \tilde{1}$ .  
Asum  $\exists x$  suth  $rs \leq x < 1, \forall r, s$ . Let  $D \not\ni \dots \leq b_m < \dots < b_1 \leq 0$ , and  $D \not\ni \dots \leq r_m < \dots < r_1 \leq 0$ .  
 $1/b_1 < \dots < 1/b_m \leq \dots \leq s_n < \dots < s_1 \leq 0$ , and  $r_m/b_m \geq \dots \geq r_m s_n > \dots > r_j s_k > \dots > r_1 s_1$ .  
Let  $r_m = b_m$ . Ctradic.      Or.  $DD^{-1} = D[(-\tilde{1})(-D)^{-1}] = [D(-\tilde{1})](-D)^{-1} = (-D)(-D)^{-1} = \tilde{1}$ .$$

- For  $C$  not posi and  $D$  posi, we expect that  $CD$  not posi. Consider  $C$  and  $-D$  both not posi.  
 $CD = -[C(-D)] = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$   
 $= \{-a : a > rt, \forall r \in C^-, \forall t$  suth  $0 \geq t \geq -s, \forall s \notin D\}$   
 $= \{a : a < ru, \forall r \in C^-, \forall u$  suth  $0 \leq u \leq s, \forall s \notin D\}$ .      ( $r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs$ .)
- Note the ' $0 \leq u$ '. Becs  $C^- \neq \emptyset \Rightarrow 0 \in C^-$ . If it is to be exactly  $CD = \{a : a < 0\}$ , then  $C^- = \{0\}$ ,  
for if not,  $\exists u > 0$ , and  $\exists r \in C^- \setminus \{0\}$ , suth  $\exists a < ru < 0$ . Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
- ' $u \leq s'$  cannot be abbreviated as in  $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$ .  
' $u \leq s'$  cannot be ' $u < s'$ ', becs here  $rs < ru \Rightarrow \exists a = rs$ .
- Note that  $\{u : 0 < u \leq s, \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q} \setminus D\}, & \text{if it exists,} \\ D^+, & \text{othws.} \end{cases}$  Denote it by  $D^\oplus = D^\otimes \setminus \{0\}$ .

- For  $C$  not posi and  $D$  posi. If  $C = \tilde{0}$ , then  $CD = -[C(-D)] = -\tilde{0}$ . Now consider  $-C$  and  $D$  both posi.  
But  $CD = -(-C)D = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \neq \{a : a < -cd, \forall c \in (-C)^+, d \in D^+\}$ .  
Altho " $a \leq cd$ " is equiv to " $a < cd$ " so that  $b \notin (-C)D \Rightarrow b \geq cd$ , which is actually  $b > cd, \forall c, d$ .  
And  $a < -b < -cd, \forall c, d \Rightarrow \forall a, \exists x$  suth  $a < x < -cd, \forall c, d$ . While  $a$  can be the 'boundary' in RHS.
- $LHS = \{a : a < ru, \forall r \in C^-, \forall u \in D^\oplus\}, \{cs : c \in C, s \notin D\} = RHS$ .  
Becs  $cs \leq cu < ru$ . We show  $LHS \subseteq RHS$ . Let  $c_1 < \dots < c_n < \dots \in C$ , and  $s_1 > \dots > s_m \geq \dots \notin D$ .  
Then  $c_1 s_1 < \dots < c_n s_m < \dots < ru, \forall r, u$  as in LHS. Thus  $a \in LHS \Rightarrow \exists a < c_j s_k$ .  $\square$   
Or. Note that in LHS, ' $a < ru$ ' is equiv to ' $a < rs, \exists s \notin D$ '. Now  $LHS = \{a : a/s \in C, \exists s \notin D\}$ .  $\square$

- For  $C$  posi and  $D$  not posi. If  $D = \tilde{0}$ , then  $CD = -[(-C)D] = -\tilde{0}$ .  
 $CD = (-C)(-D) = \{a : a < ru, \forall r \in (-C)^-, \forall u \in (-D)^\oplus\} \quad \mathbf{Q} \setminus -D = \{s : s \geq -y, \forall y \notin D\}$   
 $= \{a : a < ru, \forall r$  suth  $\forall x \notin C, 0 \geq r \geq -x, \forall u$  suth  $0 < u \leq s, \forall s \geq -y, \forall y \notin D\}$   
 $= \{a : a < (-r)(-u), \forall r$  suth  $\forall x \notin C, 0 \leq -r \leq x, \forall u$  suth  $y \leq -u < 0, \forall y \notin D\}$   
 $= \{a : a < ru, \forall r$  suth  $\forall x \notin C, 0 \leq r \leq x, \forall u \in D^-\} = \{a : a < ru, \forall r \in C^\oplus, \forall u \in D^-\}$ , simlr.

- We show  $-D = \{a : a < -b, b \notin D\} = (-\tilde{1})D$ .

For  $D$  posi,  $RHS = \{a : a < ru, \forall r \text{ suth } -1 \leq r \leq 0, \forall u \in D^\oplus\} = \{a : a < -u, \forall u \in D^\oplus\} \supseteq -D$ .

$\text{Supp } x \text{ suth } -b \leq x < -u, \forall b \notin D, \forall u \text{ suth } -s \leq -u < 0, \forall s \notin D \Rightarrow -s \leq x < -u$ . Let  $-u = x$ .

For  $D$  not posi,  $RHS = \{a : a < rb, \exists r \text{ suth } -1 \leq r \leq 0, b \in D^-\} = \{a : a < -b, 0 \geq b \notin D\} = -D$ .

- We show  $\tilde{1}D = D$ . For  $D$  not posi, immed. Othws,  $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$ .

Now  $(\tilde{1}D)^+ \subseteq D^+$ . 又  $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in (\tilde{1}D)^+$ .

#### 4 *Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD)$ , $B(C + D) = BC + BD$ .*

**SOLUS:** We discuss in cases.

\	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

$$(1) [(BC)D]^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = [B(CD)]^+.$$

$$B(C + D) = \{a : a \leq bc + bd, \exists b \in B^+, 0 < c + d \in C + D\}.$$

$$\begin{aligned} BC + BD &= \{x : x \leq uc, u \in B^+, c \in C^+\} + \{y : y \leq vd, v \in B^+, d \in D^+\} \\ &= \{a : a \leq uc + vd, \exists u, v \in B^+, c \in C^+, d \in D^+\}. \text{ Done.} \end{aligned}$$

$$(3) (BC)^- = \{r : 0 \geq r \geq bc, \exists c \in C^-, \exists b \in B^\oplus\}.$$

$$\begin{aligned} (BC)D &= \{a : a < rs, r \in (BC)^-, s \in D^-\} \\ &= \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^\oplus\}. \end{aligned}$$

$$\begin{aligned} B(CD) &= \{a : a \leq bx < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+, \text{ and } cs > x \in (CD)^+\} \\ &= \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+\}. \end{aligned}$$

Note that  $\{q : q < b, b \in B^+\} = \{q : q < b, \exists b \in B^\oplus\}$ . Done.

$$\begin{aligned} B(C + D) &= \{a : a < ru, \forall r \text{ suth } \forall x \notin B, 0 < r \leq x, \forall u \text{ suth } 0 \geq u > c + d, \forall c \in C, d \in D\} \\ &= \{a : a < ru, \forall r \in B^\oplus, \forall u \text{ suth } 0 \geq u \geq c + d, \exists c \in C^-, d \in D^-\} \\ &= \{a : a < r(c + d), \forall r \in B^\oplus, \forall c \in C^-, d \in D^-\}. \end{aligned}$$

$$\begin{aligned} BC + BD &= \{x : x < pc, \forall p \in B^\oplus, \forall c \in C^-\} + \{y : y < qd, \forall q \in B^\oplus, \forall d \in D^-\} \\ &= \{a : a < pc + qd, \forall p, q \in B^\oplus, \forall c \in C^-, d \in D^-\}. \text{ Done immed.} \end{aligned}$$

$$(5) (BC)^- = \{r : 0 \geq r \geq bc, \exists b \in B^-, \exists c \in C^\oplus\}. (CD)^- = \{r : 0 \geq r \geq cd, \exists d \in D^-, \exists c \in C^\oplus\}.$$

$$(BC)D = \{a : a < rd \leq bcd, \exists d \in D^-, b \in B^-, \exists c \in C^\oplus\}.$$

$$B(CD) = \{a : a < br \leq bcd, \exists b \in B^-, d \in D^-, \exists c \in C^\oplus\}. \text{ Done.}$$

OR. By commu and (3),  $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$ .

$$\begin{aligned} BC + BD &= \{x : x < bc, \forall b \in B^-, \forall c \in C^\oplus\} + \{y : y < bd, \exists b \in B^-, d \in D^-\} \\ &= \{a : a < pc + qd, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \exists q \in B^-, d \in D^- \Rightarrow q \geq p\} \\ &= \{a : a < pc + y, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall y \text{ suth } y \geq qd, \forall q \in B^-, d \in D^-\} \\ &= \{a : a < p(c + d'), \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall d' \text{ suth } d' \leq d, \forall d \in D^-\}. \end{aligned}$$

$$(I) \text{ If } C + D \text{ not posi. Then } B(C + D) = \{a : a < bx, \exists b \in B^-, 0 \geq x > c + d, \forall (c, d) \in C \times D\}.$$

Rewrite as  $\{a : a < t, \forall t \text{ suth } t \geq bx, \forall b \in B^-, x \in (C + D)^-\}$ . Done.

$$(II) \text{ If } C + D \text{ posi. Then } B(C + D) = \{a : a < bx, \forall b \in B^-, x \in (C + D)^\oplus\}.$$

If  $(C + D)^\oplus = (C + D)^+$ . Then  $B(C + D) = \{a : a < bc + bd, \forall b \in B^-, \forall (c, d) \in C^\oplus \times D^\oplus\}$ .

Othws,  $\exists s = \min \mathbf{Q} \setminus (C + D) \Rightarrow B(C + D) = \{a : a < bs, \forall b \in B^-\}$ . Done.

$$\begin{aligned}
(4) \quad (BC)D &= \{xd : d \in D, x \geq bc, \forall b \in B^-, c \in C^-\} \\
&= \{a : a < bcd, \forall d \in D^-, \forall b \in B^-, c \in C^-\} \\
&= \{by : b \in B, y \geq cd, \forall c \in C^-, d \in D^-\} = B(CD).
\end{aligned}$$

$$\begin{aligned}
B(C+D) &= \{a : a < t, \forall t \text{ suth } t \geq b(c+d), \forall b \in B^-, (c,d) \in C^- \times D^-\} \\
&= \{a : a < t_1, \forall t_1 \text{ suth } t_1 \geq bc, \forall (b,c) \in B^- \times C^-\} \\
&\quad + \{a : a < t_2, \forall t_2 \text{ suth } t_2 \geq bd, \forall (b,d) \in B^- \times D^-\} = BC + BD. \text{ Done.}
\end{aligned}$$

**NOTE:** Supp for any  $B$  posi,  $C$  posi,  $D$  not posi, assoc holds.

Supp  $B$  posi,  $C$  not posi,  $D$  posi. Then  $(\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{B}\bar{C}) = (\bar{D}\bar{B})\bar{C} = \bar{B}(\bar{D}\bar{C})$ . Convly true.

Simlr for the case  $B$  not posi,  $C$  posi,  $D$  not posi, equiv to the case  $B$  not posi,  $C$  not posi,  $D$  posi.

(2) holds  $\Rightarrow (\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{C}\bar{B}) = (\bar{C}\bar{D})\bar{B}$ , (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds  $\Rightarrow \bar{B}(\bar{C} + \bar{D}) = (-\bar{B})[-(\bar{C} + \bar{D})] \xrightarrow{(5)} (-\bar{B})[(-\bar{C}) + (-\bar{D})] \xrightarrow{(5)} \bar{B}\bar{C} + \bar{B}\bar{D}$ , by def of multi.

Thus (5)  $\Rightarrow$  (2) in distr. Convly as well.

$$(6) \quad (BC)^- = \{r : 0 \geq r \geq bc, \exists b \in B^-, c \in C^\oplus\}.$$

$$(BC)D = \{a : a < ru, \forall r \in (BC)^-, u \in D^\oplus\} = \{a : a < bcd, \forall b \in B^-, c \in C^\oplus, d \in D^\oplus\}.$$

$$(CD)^\oplus = \{u : 0 < u \leq s, \forall s > cd, \forall c \in C^+, d \in D^+\}.$$

$$B(CD) = \{a : a < ru, \forall r \in B^-, u \in (CD)^\oplus\}. \text{ Done.}$$

$$B(C+D) = \{a : a < b(c+d), \forall b \in B^-, \forall (c,d) \in C^\oplus \times D^\oplus\}.$$

$$BC + BD = \{a : a < bc, \forall b \in B^-, c \in C^\oplus\} + \{a : a < bd, \forall b \in B^-, d \in D^\oplus\}. \text{ Done.}$$

OR. (2) instead of (6) ? NOTICE that the distr in (6) cannot be shown without (6). □

ENDED

## 0.C

**2** Supp non- $\emptyset$   $U \subseteq V \subseteq \mathbf{R}$ . Show  $\sup U \leq \sup V$ .

**SOLUS:** Asum  $\sup U > \sup V \Rightarrow \exists t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$ , ctradic. □

**5** Supp  $\sup\{a_1, a_2, \dots\} = \sqrt{2}$  and each  $a_k \in \mathbf{Q}$ . Prove each  $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ .

**SOLUS:** Becs the sup not in seq  $\Rightarrow$  infly many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$ . For  $a_{n+k}$ , choose  $a_i > a_{n+k}$ . If  $i \in \{1, \dots, n\}$ , then choose  $a_j > a_i$ .

After at most  $n$  steps, we must have  $a_m$  with  $m > n$ . Thus  $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$ . □

• Supp non- $\emptyset$   $A \subseteq \mathbf{R}$ .

• **TIPS 1:** Define  $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$ . Prove  $\sup(-A) = -\inf A$ .

**SOLUS:**  $-b$  is an up-bound of  $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$  is a low-bound of  $A$ .

Thus  $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$ . □

• **TIPS 2:** Show if  $x \in \mathbf{R}$ , (a)  $\sup A > x \Rightarrow \exists a \in A, a > x$ , (b)  $\inf A < x \Rightarrow \exists a \in A, a < x$ .

**SOLUS:** (a)  $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x \Rightarrow \sup A \leq x$ .

Or. By (b),  $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$ .

Simlr for (b). □

**6** *Supp non- $\emptyset$   $A, B \subseteq \mathbf{R}$ . Prove  $\sup(A + B) = \sup A + \sup B$ , and  $\inf(A + B) = \inf A + \inf B$ .*

**SOLUS:**  $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$ ,  $\inf A + \inf B \leq \inf(A + B)$ .

$$\sup A + \sup B > \sup(A + B) \Leftrightarrow \sup A > \sup(A + B) - \sup B$$

$$\Leftrightarrow \exists a + \sup B > \sup(A + B) \Leftrightarrow \sup B > \sup(A + B) - a \Leftrightarrow \exists a + b > \sup(A + B). \text{ Ctradic.}$$

Simlr for  $\inf(A + B) \in A + B$ . Or. Apply to  $-A - B$ , becs  $\sup(-A) = -\inf A$ .  $\square$

**16** *Supp  $\mathbf{R}_1, \mathbf{R}_2$  are complete ordered fields. Define  $\varphi_1, \varphi_2$  as in [0.11].*

*Define  $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  by  $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq a\}$ . Show*

(a)  $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  is one-to-one. (b)  $\psi(0) = 0, \psi(1) = 1$ .

(c)  $\psi(a + b) = \psi(a) + \psi(b)$ . (d)  $\psi(ab) = \psi(a)\psi(b)$ .

**SOLUS:** (a) Define  $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Let  $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$ .

Define  $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$ .

Define  $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$ . Then  $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  well-defined.

Note that  $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$ .

Now  $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$ . Rev the roles of  $\mathbf{R}_1, \mathbf{R}_2$ .

(b) Becs  $\varphi_1(q) \leq \varphi_1(0) \Leftrightarrow q \leq 0 \Leftrightarrow \varphi_2(q) \leq \varphi_2(0)$ . Simlr for  $\psi(1) = 1$ .

(c)  $S = \{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq a + b\} \supseteq \{\varphi_2(p + q) : p, q \in \mathbf{Q}, \varphi_1(p) \leq a, \varphi_1(q) \leq b\} = T$   
 $\Rightarrow \sup S \geq \sup T$ . Asum it is ' $>$ '. Now  $\exists t \in \mathbf{Q}$  suth  $\sup S \geq \varphi_2(t) > \sup T = \psi(a) + \psi(b)$ .

Which means  $\varphi_2(t) > \varphi_2(p + q), \forall p, q \in \mathbf{Q}$  suth  $\varphi_1(p) \leq a$  and  $\varphi_1(q) \leq b$ .

Now  $a + b \geq \varphi_1(t) > \varphi_1[(t + p + q)/2] > \varphi_1(p + q), \forall p, q$ . Ctradic.

(d) We show it for (I)  $a, b > 0$ , (II)  $a > 0 > b$ .

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq ab\}, \sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq b\} = RHS$ .

(I)  $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q}, 0 < \varphi_1(p) \leq a, \varphi_1(q) \leq b \Rightarrow \varphi_1(pq) \leq ab\}$ .

(II)  $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q}, \varphi_1(q) \leq b, \text{ and } s \geq p, \forall p \text{ suth } 0 < \varphi_1(p) \leq a\}$ .

Note that  $\varphi_1(s) \geq \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geq a$ , for if not,  $\exists p' \in \mathbf{Q}$  suth  $\varphi_1(s) < \varphi_1(p') \leq a$ .

So that  $\varphi_1(sq) \leq a\varphi_1(q) \leq ab$ .

Now  $LHS \geq RHS$ . Asum it is ' $>$ '. Then  $\exists t \in \mathbf{Q}$  suth  $LHS \geq \varphi_2(t) > RHS$ .

Which means  $\varphi_2(t) > \psi(a)\varphi_2(q), \forall q \in \mathbf{Q}$  suth  $\varphi_1(q) \leq b$ .

(I)  $\Rightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q \text{ and } \forall s \in \mathbf{Q} \text{ suth } 0 < \varphi_1(s) \leq a$ .

(II)  $\Rightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q \text{ and } \forall s \in \mathbf{Q} \text{ suth } \varphi_2(s) \geq \varphi_2(p), \forall p \text{ suth } 0 < \varphi_1(p) \leq a$ .

Thus  $ab \geq \varphi_1(t) > \varphi_1[(t + sq)/2] > \varphi_1(sq), \forall s, q$ . Ctradic.  $\square$

ENDED

## 0.D

• *Supp non- $\emptyset$   $A$  is closed and open subsets of  $\mathbf{R}^n$ . Prove  $A = \mathbf{R}^n$ .*

**SOLUS:** Asum  $A \neq \mathbf{R}^n$ . Let  $a \in A, b \in \mathbf{R}^n \setminus A$ . Define  $f(t) = (1-t)a + tb$ .

If  $f(t_1) = f(t_2) \Rightarrow (t_1 - t_2)(b - a) = 0 \Rightarrow t_1 = t_2$ . Inje.

Let  $\mathcal{A} = \{t \in [0, 1] : f(t) \in A\}$ , and  $\sup \mathcal{A} = t_M \in [0, 1]$ . Let  $c = f(t_M)$ .

(I) If  $c \in A \Rightarrow \exists \delta > 0, B(c, \delta) \subseteq A$ . Let  $t \neq t_M$  be suth  $f(t) \in B(c, \delta) \Rightarrow \|(t - t_M)(b - a)\|_\infty < \delta$ .

Let  $\varepsilon = |t - t_M| > 0 \Rightarrow f(t_M + \varepsilon) \in B(c, \delta) \subseteq A \Rightarrow t_M \geq t_M + \varepsilon$ , ctradic.

(II) If  $c \in \mathbf{R}^n \setminus A$ . Simlr.  $\forall t' \in [t_M - \varepsilon, t_M], t' \notin \mathcal{A}$ . Now  $\forall t \in \mathcal{A}, t < t_M \Rightarrow t < t_M - \varepsilon$ .  $\square$

**15**  $\text{Supp } F$  is a closed subset of  $\mathbf{R}$ . Prove  $S = \{a^2 : a \in F\}$  is closed.

**SOLUS:**  $\text{Supp } S$  not closed  $\Rightarrow \exists a_1^2, a_2^2, \dots \in S$  convg to  $L^2$  suth  $\pm L \notin F$ . Let  $\varepsilon_1 > \varepsilon_2 > \dots \in \mathbf{R}^+$ .

Becs  $\forall \varepsilon_p, \exists m \in \mathbf{Z}^+, \forall k \geq m, \|a_k^2 - L^2\|_\infty = |a_k - L| \cdot |a_k + L| < \varepsilon_p$ ,

if  $|a_k - L| < \varepsilon_p$ , then let  $b_p = a_k$ , if  $|a_k + L| < \varepsilon_p$ , then let  $c_p = a_k$ .

Then we have at least one seq in  $F$  with  $\lim \pm L \notin F \Rightarrow F$  not closed.  $\square$

**17**  $\text{Supp } F \subseteq \mathbf{R}$ , and  $\forall n \in \mathbf{Z}^+, F \cap [-n, n]$  is closed. Prove  $F$  is closed.

**SOLUS:** Becs  $G = (\mathbf{R} \setminus F) \cup (-\infty, -n) \cup (n, \infty)$  is open for all  $n \in \mathbf{Z}^+$ . Let  $n > \sup\{|a| : a \in \mathbf{R} \setminus F\}$ .

By [0.59],  $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \dots$  (disj)  $\Rightarrow I_1 \cup I_2 \cup \dots = \mathbf{R} \setminus F$ .  $\square$

**20** Prove  $\forall b \in \mathbf{R}^n, \delta > 0, B = \{a \in \mathbf{R}^n : \|a - b\| \leq \delta\}$  is closed. *See below [0.46].*

**SOLUS:** Asum  $\exists a_1, a_2, \dots \in B$  convg to  $L$  suth  $\|L - b\| > \delta$ . Then  $\forall \varepsilon > 0, \exists m \in \mathbf{Z}^+, \forall k \geq m,$

$\|a_k - L\| < \varepsilon \Rightarrow \delta < \|L - b\| \leq \|a_k - L\| + \|a_k - b\| < \delta + \varepsilon$ . Now  $0 < \|L - b\| - \delta < \varepsilon$ . Ctradic.  $\square$

**21**  $\text{Supp } X$  is an open subset of  $\mathbf{R}$ . Prove  $\exists a_k, b_k \in \mathbf{R}$  suth  $X = \bigcup_{k=1}^{\infty} [a_k, b_k]$ .

**SOLUS:** Let  $X = \bigcup_{k=1}^{\infty} (c_k, d_k)$ , where each  $d_k < c_{k+1}$ . Let  $I_k = (c_k, d_k)$ .

Let  $a_{1,k}, a_{2,k}, \dots$  be convg of  $\lim c_k$ . Simlr, let  $b_{1,k}, b_{2,k}, \dots$  of  $\lim d_k$ . Let  $E_{j,k} = [a_{j,k}, b_{j,k}]$ .

Now  $\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{j,k} = \bigcup_{k=1}^{\infty} I_k = X$ . Rearrange the order of the  $E$ 's.  $\square$

**23**  $\text{Supp } F_1, F_2$  are disj closed subsets of  $\mathbf{R}$  suth  $U = F_1 \cup F_2$  is interval. Prove  $F_1$  or  $F_2$  open.

**SOLUS:** If  $U$  open, then  $F_1 = \emptyset, F_2 = \emptyset$  or  $\mathbf{R}$ . Now supp  $U$  not open  $\Rightarrow F_1$  or  $F_2$  not open.

We show  $F_1$  open  $\Leftrightarrow F_2$  not open. Note that  $F_1$  open  $\Leftrightarrow F_1 = \emptyset$ . Simlr for  $F_2$ .

WLOG, asum  $F_1, F_2$  both not open, and  $x \in F_1, y \in F_2$  with  $x < y \Rightarrow x, y \in U \supseteq [x, y]$ .

NOTICE that  $T = [x, y] \cap F_1$  is a closed subset that has infly many elem.  $\forall \sup T < y$ .

(I) If  $\sup T \notin F_1$ . Then  $\exists$  a convg seq in  $T$  with  $\lim \sup T$ .

(II) Othws,  $\forall t \in (\sup T, y], t \notin F_1 \Leftrightarrow t \in F_2$ . Now  $\exists$  a convg seq in  $F_2$  with  $\lim \sup T \notin F_2$ .

Ctradic the asum  $\Rightarrow$  at least one of  $F_1, F_2$  is empty.  $\square$

**25** Give an exa of inv  $f : \mathbf{R} \rightarrow \mathbf{R} \setminus \mathbf{Q}$ .

**SOLUS:** Consider a seq disti  $a_1, b_1, a_2, b_2, \dots \in \mathbf{R}$  where  $\{a_1, a_2, \dots\} \in \mathbf{Q}$  and each  $b_i \in \mathbf{R} \setminus \mathbf{Q}$ .

Define  $\varphi(a_j) = b_{2j-1}, \varphi(b_k) = b_{2k} \Rightarrow \varphi^{-1}(b_i) = \begin{cases} a_{(i+1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$

Let  $B = \{b_1, b_2, \dots\}, U = \mathbf{Q} \cup B, K = \mathbf{R} \setminus U$ . Extend  $\varphi \in B^U$  to  $\psi \in (K \cup B)^{K \cup U}$  by  $\psi|_K = I$ .  $\square$

**26**  $\text{Supp } E, G \subseteq \mathbf{R}^n$ , and  $G$  is open. Prove  $E + G = \{x + y : x \in E, y \in G\}$  is open.

**SOLUS:** Asum  $E + G$  not open  $\Leftrightarrow \mathbf{R}^n \setminus (E + G)$  not closed.

Then  $\exists a = x + y \in E + G$  suth  $\forall \delta > 0, \exists b \notin E + G$  suth  $\|a - b\|_\infty < \delta$ .

Let  $z = b - x \notin G \Rightarrow \|y - z\|_\infty < \delta \Rightarrow z \in B(y, \delta) \subseteq G, \exists \delta > 0$ .  $\square$

Or.  $\exists a_1, a_2, \dots \notin E + G$  while its  $\lim L = e + g \in E + G$ .  $\forall x \in E, a_k - x \notin G$

$\Rightarrow \lim_{k \rightarrow \infty} (a_k - e) = L - e = g \in G$ . Thus  $\mathbf{R}^n \setminus G$  not closed  $\Leftrightarrow G$  not open.  $\square$

# 0·E

**1** Prove every convg seq in  $\mathbf{R}^n$  is bounded.

**SOLUS:**  $\text{Supp } a_1, a_2, \dots \in \mathbf{R}^n$  suth  $\forall \varepsilon > 0, \exists m \in \mathbf{Z}^+, \forall k \geq m, |\|a_k\|_\infty - \|L\|_\infty| \leq \|a_k - L\|_\infty < \varepsilon$ .

Thus  $\|a_k\|_\infty \in (\|L\|_\infty - \varepsilon, \|L\|_\infty + \varepsilon)$ . Now asum  $\sup\{\|a_k\|_\infty : k \in \mathbf{Z}^+\} = \infty$ .

Which means  $\forall t \in \mathbf{R}^+, \forall m \in \mathbf{Z}^+, \exists k \geq m, \max\{\|a_1\|_\infty, \dots, \|a_m\|_\infty\} \leq \|a_k\|_\infty > t$ . Ctradic.  $\square$

• **TIPS 1:** By def, a seq  $a_1, a_2, \dots \in \mathbf{R}^n$  convg to  $L \Rightarrow$  every subseq convg to  $L$ .

**3**  $\text{Supp } F \subseteq \mathbf{R}^n$ , every seq in  $F$  has a convg subseq with lim in  $F$ . Prove  $F$  is closed bounded.

**SOLUS:**  $\text{Supp } a_1, a_2, \dots \in F$  convg with lim  $L$ . Becs  $\exists$  subseq with lim in  $F$ . By TIPS (1),  $L \in F$ .

Asum  $\sup\{\|a\|_\infty : a \in F\} = \infty \Rightarrow \forall \Delta > 0, \exists a_1, a_2, \dots \in F$  with each  $\|a_{k+1}\|_\infty \geq \|a_k\|_\infty + \Delta$ .

Thus every subseq is unbounded  $\Rightarrow$  not convg. Ctradic.  $\square$

•  $\text{Supp } b \in A \subseteq \mathbf{R}^m, f : A \rightarrow \mathbf{R}^n$ . Prove

[P]  $f$  is continu at  $b \iff \forall b_1, b_2, \dots \in A$  suth  $\lim_{k \rightarrow \infty} b_k = b, \lim_{k \rightarrow \infty} f(b_k) = f(b)$ . [Q]

**SOLUS:**  $\neg P \Rightarrow \neg Q : \exists \varepsilon > 0, \forall k \in \mathbf{Z}^+, \exists b_k \in A$  suth  $\|b_k - b\|_\infty < 1/k$  and  $\|f(b_k) - f(b)\|_\infty \geq \varepsilon$ .

Now  $\lim_{k \rightarrow \infty} b_k = b$  while  $\exists \varepsilon > 0, \forall m, \exists k \geq m, \|f(b_k) - f(b)\|_\infty \geq \varepsilon$ .

$\neg Q \Rightarrow \neg P : \text{Supp } b_1, b_2, \dots \in A$  suth  $\forall \delta > 0, \exists m, \forall k \geq m, \|b_k - b\|_\infty < \delta$ .

Asum  $\exists \varepsilon > 0, \forall m [\Rightarrow \forall \delta > 0], \exists k \geq m, \|f(b_k) - f(b)\|_\infty \geq \varepsilon$ . Immed.  $\square$

**10**  $\text{Supp bounded } A \subseteq \mathbf{R}^m$  and uniformly continu  $f \in \mathbf{R}^A$ . Prove  $f$  is bounded.

**SOLUS:** Asum  $\forall \Delta > 0, \exists a_1, a_2, \dots \in A$  suth each  $|f(a_{k+1}) - f(a_k)| \geq |f(a_{k+1})| - |f(a_k)| \geq \Delta$ .

$\exists$  subseq  $a_{j_1}, a_{j_2}, \dots$  convg to  $L \Rightarrow \forall \delta > 0, \exists j_m, \forall j_x, j_y \in \{j_m, j_{m+1}, \dots\}, \|a_{j_x} - a_{j_y}\| < \delta$ .

$\forall \varepsilon > 0, \exists \delta > 0, \forall a, b \in A$  suth  $\|a - b\| < \delta$ , we have  $f(a) \in (f(b) - \varepsilon, f(b) + \varepsilon)$ .

Let  $b_{k_i} = a_{j_{k_i}}$ . Becs  $\exists k_1 < k_2 < \dots$  suth  $f(b_{k_1}), f(b_{k_2}), \dots$  is monotone.

If incre, then let  $k_1$  suth  $f(b_{k_1}) > 0 \Rightarrow f(b_{k_{i+1}}) > f(b_{k_i}) + \Delta$ . Ctradic. Simlr for decre.  $\square$

**17**  $\text{Supp uniformly continu } f, g : \mathbf{R}^R$ . Prove  $f \circ g \in \mathbf{R}^R$  is uniformly continu.

**SOLUS:**  $\forall \varepsilon > 0, \exists \delta > 0, |f(t_1) - f(t_2)|, \forall t_1, t_2 \in g(\mathbf{R})$  suth  $|t_1 - t_2| < \delta$ .

Then  $\exists \rho > 0, \forall a, b$  suth  $|a - b| < \rho$ , we have  $|g(a) - g(b)| < \delta$ .

These  $g(a), g(b)$  is contained in the set of all pairs of  $t_1, t_2$ .  $\square$

**18**  $\text{Supp } h : \mathbf{R}^m \rightarrow \mathbf{R}^n$ . Prove  $h$  continu  $\iff h^{-1}(G)$  is open for all open  $G \subseteq \mathbf{R}^n$ .

**SOLUS:**  $\text{Supp } h$  continu and open  $G \subseteq \mathbf{R}^n$ . We show  $\mathbf{R}^m \setminus h^{-1}(G) = \{t \in \mathbf{R}^m : h(t) \notin G\}$  is closed.

Let  $t_1, t_2, \dots \in \mathbf{R}^m \setminus h^{-1}(G)$  convg to  $L \Rightarrow h(t_1), h(t_2), \dots \in \mathbf{R}^n \setminus G$  convg to  $h(L)$ .

$\text{Supp open } G \subseteq \mathbf{R}^n$  suth  $\mathbf{R}^m \setminus h^{-1}(G)$  not closed. Asum  $h$  continu.

$\exists t_1, t_2, \dots \in \mathbf{R}^m \setminus h^{-1}(G)$  convg to  $L \in h^{-1}(G) \Rightarrow h(t_1), h(t_2), \dots \in \mathbf{R}^n \setminus G$  convg to  $h(L) \notin G$ .  $\square$

**22** *Supp decre seq  $F_1 \supsetneq F_2 \supsetneq \dots$  non- $\emptyset$  closed bounded subsets of  $\mathbf{R}^n$ . Prove  $F_\infty = \bigcap_{k=1}^\infty F_k \neq \emptyset$ .*

**SOLUS:** Define  $\|F\|_\infty = \sup\{\|a\|_\infty : a \in F\}$  and  $\|F\|_0 = \inf\{\|a\|_\infty : a \in F\}$  for all bounded  $F \subseteq \mathbf{R}^n$ .

Note that by def of sup and inf,  $\|F\|_\infty = -\infty \iff \|F\|_0 = \infty \iff \|F\|_\infty < \|F\|_0 \iff F = \emptyset$ .

Now  $\|F_1\|_\infty, \|F_2\|_\infty, \dots$  bounded decre, and  $\|F_1\|_0, \|F_2\|_0, \dots$  bounded incre.

Consider  $\|F_\infty\|_\infty = \lim_{k \rightarrow \infty} \|F_k\|_\infty$ , and  $\|F_\infty\|_0 = \lim_{k \rightarrow \infty} \|F_k\|_0$ .

Note that  $\|F_\infty\|_\infty - \|F_\infty\|_0 = \lim_{k \rightarrow \infty} [\|F_k\|_\infty - \|F_k\|_0] \geq 0$ .

又 For a closed bounded  $F$ ,  $\|F\|_\infty = \|a\|$  for some  $a \in F$ , simlr for  $\|F\|_0$ .

Or. Pick  $a_k \in F_k$  for each  $\Rightarrow \exists$  subseq  $a_{j_1}, a_{j_2}, \dots$  convg to  $a \in \bigcap_{k=1}^\infty F_k$ .  $\square$

**26** *Supp  $F \subseteq \mathbf{R}^n$  suth every continu  $f \in \mathbf{R}^F$  attains a max. Prove  $F$  is closed bounded.*

**SOLUS:** Asum  $F$  not bounded. Define  $f(a) = \|a\|_\infty$ . Ctradic.

Asum  $\exists a_1, a_2, \dots \in F$  convg to  $L \notin F$ . Define  $f(a) = 1/\|a - L\|_\infty$ . Ctradic.  $\square$

**27** *Supp  $f \in \mathbf{R}^{\mathbf{R}}$  is incre. Prove  $\exists$  countable  $A \subseteq \mathbf{R}$  suth  $f|_{\mathbf{R} \setminus A}$  is continu.*

**SOLUS:** Asum  $\forall$  countable  $A \subseteq \mathbf{R}$ ,  $f|_{\mathbf{R} \setminus A}$  is not continu at an uncountable set  $B$  of elem in  $\mathbf{R} \setminus A$ .

$\forall b \in B, \exists \varepsilon > 0, \forall k \in \mathbf{Z}^+, \exists a_k \in \mathbf{R} \setminus A$  suth  $|a_k - b| < 1/k, |f(a_k) - f(b)| \geq \varepsilon$ .

Now  $a_1, a_2, \dots$  convg to  $b$ .

**29** *Supp continu  $f : [a, b] \rightarrow \mathbf{R}$ , and  $t$  is between  $f(a), f(b)$ . Prove  $\exists c \in [a, b], f(c) = t$ .*

**SOLUS:** Let  $a_0 = a, b_0 = b$ .

Step 1 Pick  $c_1 \in (a_0, b_0)$ . If  $f(c_1) = t$ , then stop.

If  $f(a_0) < t < f(b_0)$ . Let  $(a_1, b_1) = \begin{cases} (a_0, c_1), & \text{if } t < f(c_1), \\ (c_1, b_0), & \text{if } f(c_1) < t. \end{cases}$

If  $f(b_0) < t < f(a_0)$ . Let  $(a_1, b_1) = \begin{cases} (c_1, b_0), & \text{if } t < f(c_1), \\ (a_0, c_1), & \text{if } f(c_1) < t. \end{cases}$

Step m Pick  $c_m \in (a_{m-1}, b_{m-1})$ . If  $f(c_m) = t$ , then stop.

If  $f(a_{m-1}) < t < f(b_{m-1})$ . Let  $(a_m, b_m) = \begin{cases} (a_{m-1}, c_m), & \text{if } t < f(c_m), \\ (c_m, b_{m-1}), & \text{if } f(c_m) < t. \end{cases}$

If  $f(b_{m-1}) < t < f(a_{m-1})$ . Let  $(a_m, b_m) = \begin{cases} (c_m, b_{m-1}), & \text{if } t < f(c_m), \\ (a_{m-1}, c_m), & \text{if } f(c_m) < t. \end{cases}$

Either we stop at some  $m$  and done or we get a seq  $(a_1, b_1) \supsetneq (a_2, b_2) \supsetneq \dots$

suth  $\forall \varepsilon > 0, \exists m \in \mathbf{Z}^+, \forall k \geq m, b_k - a_k < \varepsilon \Rightarrow \lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} b_k$ , let it be  $c \in (a_0, b_0)$ .

Now  $\forall \varepsilon > 0, \exists m \in \mathbf{Z}^+, \forall k \geq m, |t - f(a_k)| < \varepsilon \Rightarrow \lim_{k \rightarrow \infty} f(a_k) = t = f(c)$ .  $\square$

**L1** *Supp  $p \in \mathbf{R}^{\mathbf{R}}$  is a poly. Prove  $p$  continu.*

**SOLUS:** Write  $p(x) = a_0 + a_1x + \dots + a_mx^m$  with  $a_m \neq 0$ . Supp  $x_0 \in \mathbf{R}$ . Then write:

$\forall \varepsilon > 0, \exists \delta_M > 0, \forall \delta$  suth  $|\delta| < \delta_M, |p(x_0 + \delta) - p(x_0)| = |\delta \cdot q(x_0, \delta)| < \varepsilon \iff |q(x_0, \delta)| < |\varepsilon / \delta|$

where  $q(x_0, \delta) = \sum_{k=1}^m \left( \sum_{i=1}^k a_k C_k^i \delta^{k-i-1} x_0^i \right) \Rightarrow |q(x_0, \delta)| \leq \sum_{k=1}^m \sum_{i=1}^k |\delta|^{k-i-1} |a_k C_k^i x_0^i|$ .

Let  $|\delta| < 1$  suth  $\max\{|a_k C_k^i x_0^i| : i, k \in \mathbf{Z}^+, 1 \leq k \leq m, 1 \leq i \leq k\} \cdot |\delta| \leq |\varepsilon| < |\varepsilon / \delta|$ .  $\square$

**31** *Supp*  $p \in \mathbf{R}^{\mathbf{R}}$  *is a poly with odd deg. Prove*  $\exists b \in \mathbf{R}$  *suth*  $p(b) = 0$ .

**SOLUS:** Write  $p(x) = a_0 + a_1x + \dots + a_mx^m$  with  $a_m \neq 0$ . *Supp*  $a_m > 0$ . [If  $a_m < 0$ , then apply to  $-p$ .]

$$\text{Now } p(x) = x^m \left[ \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \dots + \frac{a_{m-1}}{x} + a_m \right] \text{ for all } x \in \mathbf{R} \setminus \{0\}.$$

$$\text{While } \left| \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \dots + \frac{a_{m-1}}{x} \right| \leq \delta^m |a_0| + \delta^{m-1} |a_1| + \dots + \delta |a_{m-1}|; \text{ let } \delta = 1/|x|.$$

Let  $x$  be suth  $\delta < 1$  and  $\max\{|a_0|, |a_1|, \dots, |a_{m-1}|\} \cdot \delta \leq |a_m|/m$ . Thus  $p(-|x|) < 0 < p(|x|)$ .  $\square$

**33** *Supp*  $a_1, a_2, \dots \in \mathbf{R}^n$  *is Cauchy seq and*  $\exists$  *subseq convg to L. Prove the seq convg to L.*

**SOLUS:** *Supp* subseq  $a_{j_1}, a_{j_2}, \dots$  *convg to L. Then*  $\forall \varepsilon > 0, \exists j_m, \forall j_k \in \{j_m, j_{m+1}, \dots\}, \|a_{j_k} - L\| < \varepsilon$ ,

$$\text{and } \exists n \in \mathbf{Z}^+, \forall i \geq n, \|a_i - L\| = \|(a_i - a_{j_k}) + (a_{j_k} - L)\| \leq \|a_i - a_{j_k}\| + \|a_{j_k} - L\| < 2\varepsilon.$$

**34** *Supp closed*  $F_1 \subseteq \mathbf{R}^n$  *and closed bounded*  $F_2 \subseteq \mathbf{R}^n$ . *Prove*  $F_1 + F_2$  *is closed.*

**SOLUS:** Asum  $\exists$  convg  $a_1, a_2, \dots \in F_1 + F_2$  with  $\lim L \notin F_1 + F_2$ . Let each  $a_k = x_k + y_k$ , and  $z_k = L - x_k$ .

$$\forall \varepsilon > 0, \exists m \in \mathbf{Z}^+, \forall k \geq m, \|a_k - L\| = \|x_k + y_k - L\| = \|y_k - z_k\| < \varepsilon.$$

*Supp subseq*  $y_{j_1}, y_{j_2}, \dots$  *convg to*  $L_y \in F_2$ . Then  $\forall \varepsilon > 0, \exists j_m, \forall j_k \in \{j_m, j_{m+1}, \dots\}$ ,

$$\|z_{j_k} - L_y\| = \|(z_{j_k} - y_{j_k}) + (y_{j_k} - L_y)\| < \varepsilon. \text{ Now } x_{j_1}, x_{j_2}, \dots \in F_1 \text{ convg to } L - L_y \notin F_1.$$

ENDED

# 1

• **TIPS 1:** *Supp*  $I = [a, b]$  and  $f, g : I \rightarrow \mathbf{R}$  is bounded and  $P$  is a parti. Then

$$(a) \inf_I f + \inf_I g \leq \inf_I(f + g) \leq \sup_I(f + g) \leq \sup_I f + \sup_I g$$

$$(1) L(f, P) + L(g, P) \leq L(f + g, P) \leq L(f + g) \Rightarrow L(f) + L(g) \leq L(f + g).$$

$$(2) U(f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P) \Rightarrow U(f + g) \leq U(f) + U(g).$$

$$(b) -L(f, P) = U(-f, P), \inf(-A) = -\sup A, \forall -(-f) = f \text{ and } -(-A) = A.$$

$$\text{Let } U_f = \{U(f, P) = -L(-f, P) : P \text{ is a parti}\} = -L_{-f}$$

$$\Rightarrow U(f) = \inf U_f = -\sup L_{-f} = -L(-f).$$

$$(c) \text{ Let } P, P' \text{ be parti. Let } P'' = P \cup P'. \text{ Then } U(f, P) - L(f, P') \geq U(f, P'') - L(f, P'').$$

$$\text{Thus } \inf\{U(f, P) - L(f, P') : P, P'\} = \inf\{U(f, P) - L(f, P) : P\}.$$

$$\text{COMMENT: LHS} = \inf\{U(f, P) - L(f, P') : P, P'\} = U(f) - L(f).$$

**A.3** *Supp bounded*  $f : [a, b] \rightarrow \mathbf{R}$ .

*Prove*  $[P] f$  *Rieman integ*  $\iff \forall k \in \mathbf{Z}^+, \exists$  parti  $P_k$  *suth*  $U(f, P_k) - L(f, P_k) < 1/k$ . [Q]

**SOLUS:**  $Q \Rightarrow P : 0 = \inf\{U(f, P) - L(f, P') : P, P'\} = \inf_P U(f, P) + \inf_P -L(f, P) = U(f) - L(f)$ .

$\neg Q \Rightarrow \neg P : \exists k \in \mathbf{Z}^+, \forall P, \inf\{U(f, P) - L(f, P) : P\} \geq 1/k$ . By TIPS (1)(c).  $\square$

**A.6** *Supp Rieman integ*  $f : [a, b] \rightarrow \mathbf{R}$ . *Supp*  $g : [a, b] \rightarrow \mathbf{R}$  *suth*  $g(x) = f(x)$

*for all*  $x \in [a, b] \setminus \{r_1, \dots, r_n\}$ . *Prove*  $g$  *is Rieman integ and*  $\int_a^b g = \int_a^b f$ .

**SOLUS:**

**A.7** *Supp bounded*  $f : [a, b] \rightarrow \mathbf{R}$ . *Let*  $P_n$  *be the parti that divides*  $[a, b]$  *into*  $2^n$  *intervals of equal size.*

*Prove*  $L(f) = \lim_{n \rightarrow \infty} L(f, P_n)$  *and*  $U(f) = \lim_{n \rightarrow \infty} U(f, P_n)$ .

**SOLUS:**

**A.8** Supp Rieman integ  $f : [a, b] \rightarrow \mathbf{R}$ . Prove  $\int_a^b f = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{j=1}^n f\left(a + \frac{j(b-a)}{n}\right)$ .

SOLUS:

---

SOLUS:

---

SOLUS:

---

SOLUS:

---

SOLUS:

---

ENDED