这是我个人挑战自学「***Measure, Integration & Real Analysis, by Sheldon Axler***」的学习笔记，包括课文补注和部分习题。
我先从 ***Supplement*** 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。
0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不训。
的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导 $\mathcal{D}$ 的一切。如你所见，字越少，事越大。

Abbreviation Table

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
continu	countinu(ous)(ity)
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
countexa	counterexample
ctradic	contradict(s)(ion)
ctrapos	constrapositive

D

Ddkd	Dedekind
decr	decreasing
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disj	disjoint
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B **NOTE:** C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.
  Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q}\backslash\mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
- Define $-D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \iff a > b,\ b \notin D\}$.
  $-(-D) = -\{a : a < -b,\ b \notin D\} = \{c : c < -a,\ a \geqslant -b,\ \forall b \notin D\} = \{c : c < b,\ \forall b \notin D\} = D$.
  The last equa is becs (a) $d \notin D \Rightarrow \exists\, b \notin D,\ d \geqslant b$, and (b) $d \in D \Rightarrow$ if $\exists\, b \notin D$ suth $d \geqslant b$, then $b \in D$, ctradic.

- **TIPS:** *Prove* $\forall \varepsilon > 0,\ \exists\, b \notin D$ *suth* $b - \varepsilon \in D$.

SOLUS: Asum $\exists\, \varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \iff \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.
Now $\forall d \in D,\ \exists\, n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

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1 *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D,\ \exists\, \varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.
(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.
$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

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3 *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

SOLUS: (a) $C \subsetneq D \Rightarrow \exists\, x \in D\backslash C \Rightarrow \exists\, y \in D,\ y > x \Rightarrow \exists\, d \in D,\ d > y \iff 0 < d - y \in D - C$.
(b) $0 \in D - C \Rightarrow \exists\, y > x \notin C,\ y \in D \Rightarrow \forall c \in C,\ c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. □

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5 *Prove* (a) D *posi* $\Rightarrow -D$ *not posi,* (b) *non0* $-D$ *not posi* $\Rightarrow D$ *posi.*

SOLUS: (a) $0 \notin \{a : a < -b,\ b \notin D\} \iff \nexists\, b \notin D,\ 0 < -b \iff \forall b \notin D,\ b \geqslant 0 \Longleftarrow 0 \in D$.
(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.
OR. $\exists\, a < 0,\ a \notin -D = \{a : -a > b,\ b \notin D\} \iff \nexists\, b \notin D,\ -a > b \iff \forall b \notin D,\ 0 < -a \leqslant b$. □

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subsetneq D \iff 0 \in D \iff D$ posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q}\backslash D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \mathbf{Q}\backslash(D \cup \mathbf{Q}^+)$.
  (a) $D^- = \{0\} \Longleftarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leqslant 0\} = \{0\} \Longrightarrow \mathbf{Q}\backslash D = \mathbf{Q}^*$.
  (b) $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q}\backslash\mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. **CORO:** D not posi $\iff 0 \in D^-$.
  (c) $(D^-)^- = \{r \in D : r \leqslant 0\} = \mathbf{Q}\backslash D^+$. **CORO:** D not posi $\iff (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \notin D \iff b \in D^-\backslash\{0\}\}$.
  $(-D)^- = (\mathbf{Q}\backslash -D) \cap (\mathbf{Q}\backslash\mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q}\backslash\mathbf{Q}^+)$.
  $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that '$a \leqslant cd$' here is equiv to '$a < cd$'.

• For $-C,-D$ posi, define $CD=(-C)(-D)=\{cd: c\in(-C)^+, d\in(-D)^+\}\cup(\mathbf{Q}\backslash\mathbf{Q}^+)$.
$CD=\{0<cd<(-r)(-s): r\in C^-\backslash\{0\},\ s\in D^-\backslash\{0\}\}\cup(\mathbf{Q}\backslash\mathbf{Q}^+)=\{a: a<rs,\ r\in C^-,\ s\in D^-\}$.
If $C,-C$ not posi $\Rightarrow C=\tilde{0}$, then with the asum $\tilde{0}D=\tilde{0}$, it still holds. Simlr for D.

• For D posi, define $D^{-1}=\{a: a<1/b,\ b\notin D\}\Rightarrow DD^{-1}=\{a: a\leqslant d/b<1,\ b\notin D,\ d\in D^+\}=\tilde{1}$.
The last equa holds becs $\forall a\in\tilde{1}\cap\mathbf{Q}^+$, $\exists\, d\in D^+$ suth $b=d/a\notin D\Rightarrow d/b=a\in DD^{-1}$.

• For non0 D not posi, define $D^{-1}=-(-D)^{-1}=-\{a: a<1/b,\ \exists\, b\notin -D\iff \exists\, b\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<-x,\ \exists\, x\geqslant 1/b,\ \forall b$ suth $b\geqslant -s,\ \forall s\notin D\}=\{a: a<-1/b,\ \forall b$ suth $b\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<1/b,\ \forall b$ suth $b\leqslant s,\ \forall s\notin D\}\neq\{a: a<1/s,\ \forall s\notin D\}$.
Let $b_1<\cdots<b_m<\cdots\leqslant s,\ \forall s\notin D\iff\forall s\in D^-\Rightarrow$ each $s_j,b_k<0\Rightarrow 1/s_j\leqslant\cdots<1/b_m<\cdots<1/b_1$.
Thus '$a<1/b$' is equiv to '$a<1/s,\ \exists\, s\in D^-$'. Hence $D^{-1}=\{a: a<1/b,\ b\in D^-\}$.
$DD^{-1}=\{a: a<rs,\ \exists\, r\in D^-,\ \exists\, s$ suth $0\geqslant s\geqslant 1/b,\ \forall b\in D^-\}\subseteq\tilde{1}$.
Asum $\exists\, x$ suth $rs\leqslant x<1,\ \forall r,s$. Let $D\not\ni\cdots\leqslant b_m<\cdots<b_1\leqslant 0$, and $D\not\ni\cdots\leqslant r_m<\cdots<r_1\leqslant 0$.
$1/b_1<\cdots<1/b_m\leqslant\cdots\leqslant\cdots\leqslant s_n<\cdots<s_1\leqslant 0$, and $r_m/b_m\geqslant\cdots\geqslant\cdots\geqslant r_m s_n>\cdots>r_j s_k>\cdots>r_1 s_1$.
Let $r_m=b_m$. Ctradic. OR. $DD^{-1}=D[(-\tilde{1})(-D)^{-1}]=[D(-\tilde{1})](-D)^{-1}=(-D)(-D)^{-1}=\tilde{1}$.

• For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
$CD=-[C(-D)]=-\{a: a<rt,\ r\in C^-,\ t\in(-D)^-\}=\{-a: a>b,\ b\geqslant rt,\ \forall r\in C^-,\ t\in(-D)^-\}$
$=\{-a: a>rt,\ \forall r\in C^-,\ \forall t$ suth $0\geqslant t\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<ru,\ \forall r\in C^-,\ \forall u$ suth $0\leqslant u\leqslant s,\ \forall s\notin D\}$. $(r\leqslant 0<s,\ rs\leqslant ru=-rt\leqslant 0\leqslant rt\leqslant -rs.)$

• Note the '$0\leqslant u$'. Becs $C^-\neq\emptyset\Rightarrow 0\in C^-$. If it is to be exactly $CD=\{a: a<0\}$, then $C^-=\{0\}$, for if not, $\exists\, u>0$, and $\exists\, r\in C^-\backslash\{0\}$, suth $\exists\, a<ru<0$. Hence '$0\leqslant u$' is actually '$0<u$'.

• '$u\leqslant s$' cannot be abbreviated as in $\{-a: a>-rs,\ \forall s\notin D,\ r\in C^-\}=\{a: a<rs,\ \forall s\notin D,\ r\in C^-\}$.
'$u\leqslant s$' cannot be '$u<s$', becs here $rs<ru\Rightarrow\exists\, a=rs$.

• Note that $\{u: 0<u\leqslant s,\ \forall s\notin D\}=\begin{cases} D^+\cup\{\min\mathbf{Q}\backslash D\}, & \text{if it exis,}\\ D^+, & \text{othws.}\end{cases}$ Denote it by $D^{\oplus}=D^{\otimes}\backslash\{0\}$.

• For C not posi and D posi. If $C=\tilde{0}$, then $CD=-[C(-D)]=-\tilde{0}$. Now consider $-C$ and D both posi.
But $CD=-(-C)D=-\{a: a\leqslant cd,\ c\in(-C)^+,\ d\in D^+\}\neq\{a: a<-cd,\ \forall c\in(-C)^+,\ d\in D^+\}$.
Altho "$a\leqslant cd$" is equiv to "$a<cd$" so that $b\notin(-C)D\Rightarrow b\geqslant cd$, which is actually $b>cd,\ \forall c,d$.
And $a<-b<-cd,\ \forall c,d\Rightarrow\forall a,\ \exists\, x$ suth $a<x<-cd,\ \forall c,d$. While a can be the 'boundary' in RHS.

• $LHS=\{a: a<ru,\ \forall r\in C^-,\ \forall u\in D^{\oplus}\},\ \{cs: c\in C,\ s\notin D\}=RHS$.
Becs $cs\leqslant cu<ru$. We show $LHS\subseteq RHS$. Let $c_1<\cdots<c_n<\cdots\in C$, and $s_1>\cdots>s_m\geqslant\cdots\notin D$.
Then $\underline{c_1s_1<\cdots<c_ns_m<\cdots<ru,\ \forall r,u \text{ as in } LHS.\ \text{Thus } a\in LHS\Rightarrow\exists\, a<c_js_k.}$ □
OR. Note that in LHS, '$a<ru$' is equiv to '$a<rs,\ \exists\, s\notin D$.' Now $LHS=\{a: a/s\in C,\ \exists\, s\notin D\}$. □

• For C posi and D not posi. If $D=\tilde{0}$, then $CD=-[(-C)D]=-\tilde{0}$.
$CD=(-C)(-D)=\{a: a<ru,\ \forall r\in(-C)^-,\ \forall u\in(-D)^{\oplus}\}$ $\qquad\mathbf{Q}\backslash -D=\{s: s\geqslant -y,\ \forall y\notin D\}$
$=\{a: a<ru,\ \forall r$ suth $\forall x\notin C,\ 0\geqslant r\geqslant -x,\ \forall u$ suth $0<u\leqslant s,\ \forall s\geqslant -y,\ \forall y\notin D\}$
$=\{a: a<(-r)(-u),\ \forall r$ suth $\forall x\notin C,\ 0\leqslant -r\leqslant x,\ \forall u$ suth $y\leqslant -u<0,\ \forall y\notin D\}$
$=\{a: a<ru,\ \forall r$ suth $\forall x\notin C,\ 0\leqslant r\leqslant x,\ \forall u\in D^-\}=\{a: a<ru,\ \forall r\in C^{\oplus},\ \forall u\in D^-\}$, simlr.

• We show $-D=\{a:a<-b,\ b\notin D\}=(-\tilde{1})D$.
For D posi, $RHS=\{a:a<ru,\ \forall r \text{ suth } -1\leqslant r\leqslant 0,\ \forall u\in D^{\oplus}\}=\{a:a<-u,\ \forall u\in D^{\oplus}\}\supseteq -D$.
Supp x suth $-b\leqslant x<-u,\ \forall b\notin D,\ \forall u \text{ suth } -s\leqslant -u<0,\ \forall s\notin D\Longrightarrow -s\leqslant x<-u$. Let $-u=x$.
For D not posi, $RHS=\{a:a<rb,\ \exists r \text{ suth } -1\leqslant r\leqslant 0,\ b\in D^{-}\}=\{a:a<-b,\ 0\geqslant b\notin D\}=-D$.

• We show $\tilde{1}D=D$. For D not posi, immed. Othws, $\tilde{1}D=\{a:a\leqslant ij<j,\ 0<i<1,\ j\in D^{+}\}\subseteq D$.
Now $(\tilde{1}D)^{+}\subseteq D^{+}$. 又 $\forall d\in D^{+},\ \exists\,\varepsilon>0,\ d+\varepsilon\in D^{+}\Rightarrow d=(d+\varepsilon)\frac{d}{d+\varepsilon}\in(\tilde{1}D)^{+}$.

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4 *Supp B,C,D non0 Dedekind cuts. Show $(BC)D=B(CD)$, $B(C+D)=BC+BD$.*

Solus: We discuss in cases.

\	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

(1) $[(BC)D]^{+}=\{(bc)d:bc\in(BC)^{+},\ d\in D^{+}\}=\{b(cd):b\in B^{+},\ cd\in(CD)^{+}\}=[B(CD)]^{+}$.
$B(C+D)=\{a:a\leqslant bc+bd,\ \exists\, b\in B^{+},\ 0<c+d\in C+D\}$.
$BC+BD=\{x:x\leqslant uc,\ u\in B^{+},\ c\in C^{+}\}+\{y:y\leqslant vd,\ v\in B^{+},\ d\in D^{+}\}$
$=\{a:a\leqslant uc+vd,\ \exists\, u,v\in B^{+},\ c\in C^{+},\ d\in D^{+}\}$. Done.

(3) $(BC)^{-}=\{r:0\geqslant r\geqslant bc,\ \exists\, c\in C^{-},\ \exists\, b\in B^{\oplus}\}$.
$(BC)D=\{a:a<rs,\ r\in(BC)^{-},\ s\in D^{-}\}$
$=\{a:a<bcs,\ \exists\, s\in D^{-},\ c\in C^{-},\ \exists\, b\in B^{\oplus}\}$.
$B(CD)=\{a:a\leqslant bx<bcs,\ \exists\, s\in D^{-},\ c\in C^{-},\ \exists\, b\in B^{+},\ \text{and } cs>x\in(CD)^{+}\}$
$=\{a:a<bcs,\ \exists\, s\in D^{-},\ c\in C^{-},\ \exists\, b\in B^{+}\}$.
Note that $\{q:q<b,\ b\in B^{+}\}=\{q:q<b,\ \exists\, b\in B^{\oplus}\}$. Done.

$B(C+D)=\{a:a<ru,\ \forall r \text{ suth } \forall x\notin B,\ 0<r\leqslant x,\ \forall u \text{ suth } 0\geqslant u>c+d,\ \forall c\in C,\ d\in D\}$
$=\{a:a<ru,\ \forall r\in B^{\oplus},\ \forall u \text{ suth } 0\geqslant u\geqslant c+d,\ \exists\, c\in C^{-},\ d\in D^{-}\}$
$=\{a:a<r(c+d),\ \forall r\in B^{\oplus},\ \forall c\in C^{-},\ d\in D^{-}\}$.
$BC+BD=\{x:x<pc,\ \forall p\in B^{\oplus},\ \forall c\in C^{-}\}+\{y:y<qd,\ \forall q\in B^{\oplus},\ \forall d\in D^{-}\}$
$=\{a:a<pc+qd,\ \forall p,q\in B^{\oplus},\ \forall c\in C^{-},\ d\in D^{-}\}$. Done immed.

(5) $(BC)^{-}=\{r:0\geqslant r\geqslant bc,\ \exists\, b\in B^{-},\ \exists\, c\in C^{\oplus}\}$. $(CD)^{-}=\{r:0\geqslant r\geqslant cd,\ \exists\, d\in D^{-},\ \exists\, c\in C^{\oplus}\}$.
$(BC)D=\{a:a<rd\leqslant bcd,\ \exists\, d\in D^{-},\ b\in B^{-},\ \exists\, c\in C^{\oplus}\}$.
$B(CD)=\{a:a<br\leqslant bcd,\ \exists\, b\in B^{-},\ d\in D^{-},\ \exists\, c\in C^{\oplus}\}$. Done.
Or. By commu and (3), $(BC)D=(CB)D=C(BD)=C(DB)=(CD)B=B(CD)$.

$BC+BD=\{x:x<bc,\ \forall b\in B^{-},\ \forall c\in C^{\oplus}\}+\{y:y<bd,\ \exists\, b\in B^{-},\ d\in D^{-}\}$
$=\{a:a<pc+qd,\ \forall p\in B^{-},\ \forall c\in C^{\oplus},\ \text{and } \exists\, q\in B^{-},\ d\in D^{-}\Rightarrow q\geqslant p\}$
$=\{a:a<pc+y,\ \forall p\in B^{-},\ \forall c\in C^{\oplus},\ \text{and } \forall y \text{ suth } y\geqslant qd,\ \forall q\in B^{-},\ d\in D^{-}\}$
$=\{a:a<p(c+d'),\ \forall p\in B^{-},\ \forall c\in C^{\oplus},\ \text{and } \forall d' \text{ suth } d'\leqslant d,\ \forall d\in D^{-}\}$.

(I) If $C+D$ not posi. Then $B(C+D)=\{a:a<bx,\ \exists\, b\in B^{-},\ 0\geqslant x>c+d,\ \forall (c,d)\in C\times D\}$.
Rewrite as $\{a:a<t,\ \forall t \text{ suth } t\geqslant bx,\ \forall b\in B^{-},\ x\in(C+D)^{-}\}$. Done.

(II) If $C+D$ posi. Then $B(C+D)=\{a:a<bx,\ \forall b\in B^{-},\ x\in(C+D)^{\oplus}\}$.
If $(C+D)^{\oplus}=(C+D)^{+}$. Then $B(C+D)=\{a:a<bc+bd,\ \forall b\in B^{-},\ \forall (c,d)\in C^{\oplus}\times D^{\oplus}\}$.
Othws, $\exists\, s=\min \mathbf{Q}\backslash(C+D)\Rightarrow B(C+D)=\{a:a<bs,\ \forall b\in B^{-}\}$. Done.

$$
\begin{aligned}
(4)\ (BC)D &= \{xd : d \in D,\ x \geqslant bc,\ \forall b \in B^-,\ c \in C^-\} \\
&= \{a : a < bcd,\ \forall d \in D^-,\ \forall b \in B^-,\ c \in C^-\} \\
&= \{by : b \in B,\ y \geqslant cd,\ \forall c \in C^-,\ d \in D^-\} = B(CD).
\end{aligned}
$$

$$
\begin{aligned}
B(C+D) &= \{a : a < t,\ \forall t \text{ suth } t \geqslant b(c+d),\ \forall b \in B^-,\ (c,d) \in C^- \times D^-\} \\
&= \{a : a < t_1,\ \forall t_1 \text{ suth } t_1 \geqslant bc,\ \forall (b,c) \in B^- \times C^-\} \\
&\quad + \{a : a < t_2,\ \forall t_2 \text{ suth } t_2 \geqslant bd,\ \forall (b,d) \in B^- \times D^-\} = BC + BD. \text{ Done.}
\end{aligned}
$$

Note: Supp for any B posi, C posi, D not posi, assoc holds.

Supp B posi, C not posi, D posi. Then $(\overset{+}{B}\overset{-}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{B}\overset{-}{C}) = (\overset{+}{D}\overset{+}{B})\overset{-}{C} = \overset{+}{B}(\overset{+}{D}\overset{-}{C})$. Convly true.

Simlr for the case B not posi, C posi, D not posi, equiv to the case B not posi, C not posi, D posi.

(2) holds $\Rightarrow (\overset{-}{B}\overset{+}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{C}\overset{-}{B}) = (\overset{+}{C}\overset{+}{D})\overset{-}{B}$, (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds $\Rightarrow \overset{+}{B}(\overset{+}{C} + \overset{-}{D}) = (-\overset{-}{B})\big[-(\overset{+}{C} + \overset{-}{D})\big] \overset{(5)}{=\!=} (-\overset{-}{B})\big[(-\overset{-}{C}) + (-\overset{+}{D})\big] \overset{(5)}{=\!=} \overset{+}{B}\overset{+}{C} + \overset{+}{B}\overset{-}{D}$, by def of multi.

Thus (5) $\Rightarrow$ (2) in distr. Convly as well.

(6) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists\, b \in B^-,\ c \in C^{\oplus}\}$.

$(BC)D = \{a : a < ru,\ \forall r \in (BC)^-,\ u \in D^{\oplus}\} = \{a : a < bcd,\ \forall b \in B^-,\ c \in C^{\oplus},\ d \in D^{\oplus}\}$.

$(CD)^{\oplus} = \{u : 0 < u \leqslant s,\ \forall s > cd,\ \forall c \in C^+,\ d \in D^+\}$.

$B(CD) = \{a : a < ru,\ \forall r \in B^-,\ u \in (CD)^{\oplus}\}$. Done.

$B(C+D) = \{a : a < b(c+d),\ \forall b \in B^-,\ \forall (c,d) \in C^{\oplus} \times D^{\oplus}\}$.

$BC + BD = \{a : a < bc,\ \forall b \in B^-,\ c \in C^{\oplus}\} + \{a : a < bd,\ \forall b \in B^-,\ d \in D^{\oplus}\}$. Done.

Or. (2) instead of (6) ? Notice that the distr in (6) cannot be shown without (6). ☐

Ended

0·C

2 *Supp nonempty $U \subseteq V \subseteq \mathbf{R}$. Show $\sup U \leqslant \sup V$.*

Solus: Asum $\sup U > \sup V \Rightarrow \exists\, t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$, ctradic. ☐

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5 *Supp $a_1, a_2, \cdots \in \mathbf{Q}$, and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.*
Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.

Solus: Becs the sup not in seq $\Rightarrow$ infily many disti elem.

$\forall a_i,\ \exists\, a_j,\ a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most n steps, we must have a_m with $m > n$. Thus $\forall a_{n+i},\ \exists\, a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. ☐

- *Supp nonempty $A \subseteq \mathbf{R}$.*
- **Tips 1:** *Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.*

Solus: $-b$ is an upper bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a lower bound of A.

Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. ☐

- **Tips 2:** *Show if $x \in \mathbf{R}$,* (a) $\sup A > x \Rightarrow \exists\, a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists\, a \in A,\ a < x$.

Solus: (a) $\nexists\, a > x \iff \forall a \in A,\ a \leqslant x \Longrightarrow \sup A \leqslant x$.

Or. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). ☐

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6 *Supp nonempty $A, B \subseteq \mathbf{R}$. Prove* $\sup(A+B) = \sup A + \sup B$, *and* $\inf(A+B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B$, $\inf A + \inf B \leqslant \inf(A+B)$.

$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$

$\iff \exists\, a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists\, a + b > \sup(A+B)$. Ctradic.

Simlr for $\inf(A+B) \in A+B$. Or. Apply to $-A-B$, becs $\sup(-A) = -\inf A$. $\square$

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16 *Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in* [0.11].

Define $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ by $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant a\}$. Show

(a) *$\psi : \mathbf{R}_1 \to \mathbf{R}_2$ is one-to-one.* (b) $\psi(0) = 0$, $\psi(1) = 1$.

(c) $\psi(a+b) = \psi(a) + \psi(b)$. (d) $\psi(ab) = \psi(a)\psi(b)$.

Solus: (a) Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)}\varphi_1 = a$. Let $\psi_1(a) = \sup_{\mathcal{R}_1(a)}\varphi_2 = \psi(a)$.

Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)}\varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)}\varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)}\varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)}\varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Becs $\varphi_1(q) \leqslant \varphi_1(0) \iff q \leqslant 0 \iff \varphi_2(q) \leqslant \varphi_2(0)$. Simlr for $\psi(1) = 1$.

(c) $S = \{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant a+b\} \supseteq \{\varphi_2(p+q) : p, q \in \mathbf{Q},\ \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b\} = T$

$\Rightarrow \sup S \geqslant \sup T$. Asum it is '$>$'. Now $\exists\, t \in \mathbf{Q}$ suth $\sup S \geqslant \varphi_2(t) > \sup T = \psi(a) + \psi(b)$.

Which means $\varphi_2(t) > \varphi_2(p+q)$, $\forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leqslant a$ and $\varphi_1(q) \leqslant b$.

Now $a + b \geqslant \varphi_1(t) > \varphi_1\big[(t+p+q)/2\big] > \varphi_1(p+q)$, $\forall p, q$. Ctradic.

(d) We show it for (I) $a, b > 0$, (II) $a > 0 > b$.

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant ab\}$, $\sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant b\} = RHS$.

(I) $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q},\ 0 < \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b \Longrightarrow \varphi_1(pq) \leqslant ab\}$.

(II) $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q},\ \varphi_1(q) \leqslant b$, and $s \geqslant p, \forall p$ suth $0 < \varphi_1(p) \leqslant a\}$.

Note that $\varphi_1(s) \geqslant \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geqslant a$, for if not, $\exists\, p' \in \mathbf{Q}$ suth $\varphi_1(s) < \varphi_1(p') \leqslant a$.

So that $\varphi_1(sq) \leqslant a\varphi_1(q) \leqslant ab$.

Now $LHS \geqslant RHS$. Asum it is '$>$'. Then $\exists\, t \in \mathbf{Q}$ suth $LHS \geqslant \varphi_2(t) > RHS$.

Which means $\varphi_2(t) > \psi(a)\varphi_2(q)$, $\forall q \in \mathbf{Q}$ suth $\varphi_1(q) \leqslant b$.

(I) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $0 < \varphi_1(s) \leqslant a$.

(II) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $\varphi_2(s) \geqslant \varphi_2(p), \forall p$ suth $0 < \varphi_1(p) \leqslant a$.

Thus $ab \geqslant \varphi_1(t) > \varphi_1\big[(t+sq)/2\big] > \varphi_1(sq)$, $\forall s, q$. Ctradic. $\square$

Ended

0·D

15 *Supp F is a closed subset of* $\mathbf{R}$. *Prove $S = \{a^2 : a \in F\}$ is closed.*

SOLUS: Supp S not closed $\Rightarrow \exists\, a_1^2, a_2^2, \dots \in S$ convg to L^2 suth $\pm L \notin F$. Let $\varepsilon_1 > \varepsilon_2 > \dots \in \mathbf{R}^+$.
Becs $\forall \varepsilon_p, \exists\, m \in \mathbf{N}^+, \forall k \geqslant m,\ \|a_k^2 - L^2\|_\infty = |a_k - L| \cdot |a_k + L| < \varepsilon_p$,
if $|a_k - L| < \varepsilon_p$, then let $b_p = a_k$, if $|a_k + L| < \varepsilon_p$, then let $c_p = a_k$.
Then we have at least one seq in F with lim $\pm L \notin F \Rightarrow F$ not closed. $\square$

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17 *Supp $F \subseteq \mathbf{R}$, and $\forall n \in \mathbf{N}^+$, $F \cap [-n, n]$ is closed. Prove F is closed.*

SOLUS: Becs $G = (\mathbf{R} \backslash F) \cup (-\infty, -n) \cup (n, \infty)$ is open for all $n \in \mathbf{N}^+$. Let $n > \sup\{|a| : a \in \mathbf{R} \backslash F\}$.
By [0.59], $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \cdots$ (disj) $\Longrightarrow I_1 \cup I_2 \cup \cdots = \mathbf{R} \backslash F$. $\square$

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20 *Prove $\forall b \in \mathbf{R}^n$, $\delta > 0$, $\{a \in \mathbf{R}^n : \|a - b\|_\infty > \delta\}, \{a \in \mathbf{R}^n : \|a - b\| > \delta\}$ are open.*

SOLUS: Asum $\exists\, b \in \mathbf{R}^n, \delta > 0$ suth $\{a \in \mathbf{R}^n : \|a - b\|_\infty \leqslant \delta\}$ is not closed.

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21 *Supp X is an open subset of* $\mathbf{R}$. *Prove $\exists\, a_k, b_k \in \mathbf{R}$ suth $X = \bigcup_{k=1}^{\infty} [a_k, b_k]$.*

SOLUS: Let $X = \bigcup_{k=1}^{\infty} (c_k, d_k)$, where each $d_k < c_{k+1}$. Let $I_k = (c_k, d_k)$.
Let $a_{1,k}, a_{2,k}, \dots$ be convg of lim c_k. Simlr, let $b_{1,k}, b_{2,k}, \dots$ of lim d_k. Let $E_{j,k} = [a_{j,k}, b_{j,k}]$.
Now $\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{j,k} = \bigcup_{k=1}^{\infty} I_k = X$. Rearrange the order of $E_{j,k}$'s. $\square$

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23 *Supp F_1, F_2 are disj closed subsets of* $\mathbf{R}$ *suth $U = F_1 \cup F_2$ is interval. Prove F_1 or F_2 open.*

SOLUS: If U open, then $F_1 = \emptyset$, $F_2 = \emptyset$ or $\mathbf{R}$. Now supp U not open $\Rightarrow F_1$ or F_2 not open.
We show F_1 open $\Longleftrightarrow F_2$ not open. Note that F_1 open $\Longleftrightarrow F_1 = \emptyset$. Simlr for F_2.
WLOG, asum F_1, F_2 not open, and $x \in F_1, y \in F_2$ with $x < y \Rightarrow x, y \in U \supseteq [x, y]$.
NOTICE that $T = [x, y] \cap F_1$ is a closed subset that has infily many elem. 又 $\sup T < y$.
(I) If $\sup T \notin F_1$. Then $\exists$ a convg seq in T with lim $\sup T$.
(II) Othws, $\forall t \in (\sup T, y]$, $t \notin F_1 \Longleftrightarrow t \in F_2$. Now $\exists$ a convg seq in F_2 with lim $\sup T \notin F_2$.
Ctradic the asum $\Rightarrow$ at least one of F_1, F_2 is empty. $\square$

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25 *Give an exa of inv $f : \mathbf{R} \to \mathbf{R} \backslash \mathbf{Q}$.*

SOLUS: Consider a seq disti $a_1, b_1, a_2, b_2, \dots \in \mathbf{R}$ where $\{a_1, a_2, \dots\} \in \mathbf{Q}$ and each $b_i \in \mathbf{R} \backslash \mathbf{Q}$.
Define $\varphi(a_j) = b_{2j-1}$, $\varphi(b_k) = b_{2k} \Rightarrow \varphi^{-1}(b_i) = \begin{cases} a_{(i+1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$
Let $B = \{b_1, b_2, \dots\}$, $U = \mathbf{Q} \cup B$, $K = \mathbf{R} \backslash U$. Extend $\varphi \in B^U$ to $\psi \in (K \cup B)^{K \cup U}$ by $\psi|_K = I$. $\square$

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26 *Supp $E, G \subseteq \mathbf{R}^n$, and G is open. Prove $E + G = \{x + y : x \in E, y \in G\}$ is open.*

SOLUS: Asum $E + G$ not open $\Longleftrightarrow \mathbf{R}^n \backslash (E + G)$ not closed.
Then $\exists\, a = x + y \in E + G$ suth $\forall \delta > 0, \exists\, b \notin E + G$ suth $\|a - b\|_\infty < \delta$.
Let $z = b - x \notin G \Rightarrow \|y - z\|_\infty < \delta \Rightarrow z \in B(y, \delta) \subseteq G$, $\exists\, \delta > 0$. $\square$
OR. $\exists\, a_1, a_2, \dots \notin E + G$ while its lim $L = e + g \in E + G$. 又 $\forall x \in E$, $a_k - x \notin G$
$\Rightarrow \lim_{k \to \infty}(a_k - e) = L - e = g \in G$. Thus $\mathbf{R}^n \backslash G$ not closed $\Longleftrightarrow G$ not open. $\square$

ENDED

0·E

- *Supp $b \in A \subseteq \mathbf{R}^m$, $f : A \to \mathbf{R}^n$. Prove*
  $[P]$ *f is continu at b* $\Longleftrightarrow$ $\forall b_1, b_2, \cdots \in A$ *suth* $\lim_{k\to\infty} b_k = b$, $\lim_{k\to\infty} f(b_k) = f(b)$. $[Q]$

Solus: $Q \Rightarrow P:$ Supp $\varepsilon > 0$ suth $\forall \delta > 0, \exists\, a \in A$ with $\|a - b\|_\infty < \delta$ and $\left\| f(a) - f(b) \right\|_\infty \geqslant \varepsilon$.
Fix a δ. Define $\delta_k = \delta / k \Rightarrow \exists\, a_k$ for each. Now $\lim_{k\to\infty} a_k = b$.
Thus $\forall m, \forall k \geqslant m$, $\left\| f(a_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic Q.

$P \Rightarrow Q:$ Supp $b_1, b_2, \cdots \in A$ suth $\forall \delta > 0$, $\exists\, m, \forall k \geqslant m$, $\|b_k - b\|_\infty < \delta$.
Asum $\varepsilon > 0$ suth $\forall m, \forall k \geqslant m$, $\left\| f(b_k) - f(b) \right\|_\infty \geqslant \varepsilon$. Ctradic P. □

Ended