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这是我个人挑战自学「***Measure, Integration & Real Analysis, by Sheldon Axler***」的学习笔记，包括课文补注和部分习题。

我先从 ***Supplement*** 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 Supplement 就能具备所有必要的知识基础。

0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不训。

的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导 $\mathcal{D}$ 的一切。如你所见，字越少事越大。

0.E 节是第二个挫折，无论是证明题，还是找反例，都难度陡升。一个难度逆天的反例：两个闭集的和不为闭集。两个证明题（27、28 题），DeepSeek 称之为 “深度构造” —— “深度” 到什么程度呢？就是 AI 给出思路提示后我也做不出。

虽然这只是极个别的习题，但我还是有一丝怯懦要不要无视空缺的实分析基础而贸然开始 MIRA 的学习。Mr. Seek 为我加油打气：

MIRA的设计本就是为绕过传统微积分、直击分析核心——你已握有实数的语言和 Axler的船票，启航正是此时。

接下来，我不再要求自己独立做出一切，而是在全力思考后学习（照搬）别人的解答，如 Samy Lahlou Kamal。

我没有任何前置的分析学训练（就连中学微积分都不会算的那种），所以很多题目自然难以给出正确解答；即便我给出解答后自认为没有问题。我想这个情况也是原书作者 Axler 教授始料未及的。来自网络的习题解答能让苦思冥想而不得的我学到很多。

Abbreviation Table

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

C

closd	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
continu	countinu(ous)(ity)
corres	correspond(s)(ing)
const	constant
conveni	convenience
convly	conversely
countexa	counterexample
ctradic	contradict(s)(ion)
ctrapos	contrapositive

D

Ddkd	Dedekind
decr	decreasing
def	definition
deg	degree
deri	derivative(s)
diff	differentia(ble)(l)(ting)(tion)
dim	dimension(al)
disj	disjoint
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

F G H

factoriz	factorizaion
fini	finite(ly)
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
integ	integra(l)(tion)(ble)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconstant
notat	notation(al)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
parti	partition
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)
Rieman	Riemann

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

T U V W X Y Z

uniq	unique
uniqnes	uniqueness
up-	upper-
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0·B **Note:** C, D are Dedekind cuts. Numbers used here are always rational.

- Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.
  Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \backslash \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -b \leqslant 0\} = \tilde{0}$.
- Define $-D = \{a : a < -b,\ b \notin D\} = \{-a : -a < -b \iff a > b,\ b \notin D\}$.
  $-(-D) = -\{a : a < -b,\ b \notin D\} = \{c : c < -a,\ a \geqslant -b,\ \forall b \notin D\} = \{c : c < b,\ \forall b \notin D\} = D$.
  The last equa is becs (a) $d \notin D \Rightarrow \exists\, b \notin D,\ d \geqslant b$, and (b) $d \in D \Rightarrow$ if $\exists\, b \notin D$ suth $d \geqslant b$, then $b \in D$, ctradic.

- **Tips:** *Prove* $\forall \varepsilon > 0,\ \exists\, b \notin D$ *suth* $b - \varepsilon \in D$.

Solus: Asum $\exists\, \varepsilon > 0$ suth $\nexists\, b \notin D,\ b - \varepsilon \in D \iff \forall b \notin D,\ b - \varepsilon \notin D$.
Then $(b - \varepsilon) - \cdots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{Z}^+$.
Now $\forall d \in D,\ \exists\, n \in \mathbf{Z}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. $\square$

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1 *Prove* (a) $D + \tilde{0} = D$, (b) $-D$ *is Dedekind cut, and* $D + (-D) = \tilde{0}$.

Solus: (a) $\forall d \in D,\ \exists\, \varepsilon > 0,\ d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.
(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists\, b \notin D,\ x < -b \Rightarrow 0 < -b - x$.
Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.
Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.
$D + (-D) = \{x + y : x + y < x - b,\ x \in D,\ b \notin D\}$.
Supp $a \in \tilde{0} \Rightarrow -a > 0$. By Tips, $\exists\, b \notin D$ suth $b + a \in D$.
Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.
Thus $\forall a \in \tilde{0},\ \exists\, b \notin D,\ d = b + \frac{1}{2}a \in D,\ c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. $\square$

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3 *Show* $C \subsetneq D \iff D - C = \{d - y : d \in D,\ y > x,\ x \notin C\}$ *posi.*

Solus: (a) $C \subsetneq D \Rightarrow \exists\, x \in D \backslash C \Rightarrow \exists\, y \in D,\ y > x \Rightarrow \exists\, d \in D,\ d > y \iff 0 < d - y \in D - C$.
(b) $0 \in D - C \Rightarrow \exists\, y > x \notin C,\ y \in D \Rightarrow \forall c \in C,\ c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. $\square$

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5 *Prove* (a) D *posi* $\Rightarrow -D$ *not posi,* (b) *non0* $-D$ *not posi* $\Rightarrow D$ *posi.*

Solus: (a) $0 \notin \{a : a < -b,\ b \notin D\} \iff \nexists\, b \notin D,\ 0 < -b \iff \forall b \notin D,\ b \geqslant 0 \Longleftarrow 0 \in D$.
(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.
Or. $\exists\, a < 0,\ a \notin -D = \{a : -a > b,\ b \notin D\} \iff \nexists\, b \notin D,\ -a > b \iff \forall b \notin D,\ 0 < -a \leqslant b.$ $\square$

- Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \iff \mathbf{Q} \backslash \mathbf{Q}^+ \subsetneq D \iff 0 \in D \iff D$ posi.
  Define $D^- = \{r \notin D : r \leqslant 0\} = (\mathbf{Q} \backslash D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \mathbf{Q} \backslash (D \cup \mathbf{Q}^+)$.
  (a) $D^- = \{0\} \Longleftarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leqslant 0\} = \{0\} \Longrightarrow \mathbf{Q} \backslash D = \mathbf{Q}^*$.
  (b) $D^- = \emptyset \iff D \cup \mathbf{Q}^+ = \mathbf{Q} \iff \mathbf{Q} \backslash \mathbf{Q}^+ \subseteq D \iff 0 \in D \iff D$ posi. **Coro:** D not posi $\iff 0 \in D^-$.
  (c) $(D^-)^- = \{r \in D : r \leqslant 0\} = \mathbf{Q} \backslash D^+$. **Coro:** D not posi $\iff (D^-)^- = D$.

- $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b,\ b \notin D \iff b \in D^- \backslash \{0\}\}$.
  $(-D)^- = (\mathbf{Q} \backslash -D) \cap (\mathbf{Q} \backslash \mathbf{Q}^+) = \{a : 0 \geqslant a \geqslant -b,\ \forall b \notin D\}$.

- For C, D posi, define $CD = \{a : a \leqslant cd,\ c \in C^+,\ d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \backslash \mathbf{Q}^+)$.
  $\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that '$a \leqslant cd$' here is equiv to '$a < cd$'.

• For $-C,-D$ posi, define $CD=(-C)(-D)=\{cd: c\in(-C)^+, d\in(-D)^+\}\cup(\mathbf{Q}\backslash\mathbf{Q}^+)$.
$CD=\{0<cd<(-r)(-s): r\in C^-\backslash\{0\}, s\in D^-\backslash\{0\}\}\cup(\mathbf{Q}\backslash\mathbf{Q}^+)=\{a: a<rs, r\in C^-, s\in D^-\}$.
If $C,-C$ not posi $\Rightarrow C=\tilde{0}$, then with the asum $\tilde{0}D=\tilde{0}$, it still holds. Simlr for D.

• For D posi, define $D^{-1}=\{a: a<1/b,\ b\notin D\}\Rightarrow DD^{-1}=\{a: a\leqslant d/b<1,\ b\notin D, d\in D^+\}=\tilde{1}$.
The last equa holds becs $\forall a\in\tilde{1}\cap\mathbf{Q}^+$, $\exists d\in D^+$ suth $b=d/a\notin D\Rightarrow d/b=a\in DD^{-1}$.

• For non0 D not posi, define $D^{-1}=-(-D)^{-1}=-\{a: a<1/b,\ \exists b\notin -D\iff\exists b\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<-x,\ \exists x\geqslant 1/b,\ \forall b$ suth $b\geqslant -s,\ \forall s\notin D\}=\{a: a<-1/b,\ \forall b$ suth $b\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<1/b,\ \forall b$ suth $b\leqslant s,\ \forall s\notin D\}\neq\{a: a<1/s,\ \forall s\notin D\}$.
Let $b_1<\cdots<b_m<\cdots\leqslant s,\ \forall s\notin D\iff\forall s\in D^-\Rightarrow$ each $s_j, b_k<0\Rightarrow 1/s_j\leqslant\cdots<1/b_m<\cdots<1/b_1$.
Thus '$a<1/b$' is equiv to '$a<1/s,\ \exists s\in D^-$'. Hence $D^{-1}=\{a: a<1/b,\ b\in D^-\}$.
$DD^{-1}=\{a: a<rs,\ \exists r\in D^-, \exists s$ suth $0\geqslant s\geqslant 1/b,\ \forall b\in D^-\}\subseteq\tilde{1}$.
Asum $\exists x$ suth $rs\leqslant x<1,\ \forall r,s$. Let $D\not\ni\cdots\leqslant b_m<\cdots<b_1\leqslant 0$, and $D\not\ni\cdots\leqslant r_m<\cdots<r_1\leqslant 0$.
$1/b_1<\cdots<1/b_m\leqslant\cdots\leqslant\cdots\leqslant s_n<\cdots<s_1\leqslant 0$, and $r_m/b_m\geqslant\cdots\geqslant\cdots\geqslant r_m s_n>\cdots>r_j s_k>\cdots>r_1 s_1$.
Let $r_m=b_m$. Ctradic. OR. $DD^{-1}=D[(-\tilde{1})(-D)^{-1}]=[D(-\tilde{1})](-D)^{-1}=(-D)(-D)^{-1}=\tilde{1}$.

• For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
$CD=-[C(-D)]=-\{a: a<rt,\ r\in C^-, t\in(-D)^-\}=\{-a: a>b,\ b\geqslant rt,\ \forall r\in C^-, t\in(-D)^-\}$
$=\{-a: a>rt,\ \forall r\in C^-, \forall t$ suth $0\geqslant t\geqslant -s,\ \forall s\notin D\}$
$=\{a: a<ru,\ \forall r\in C^-, \forall u$ suth $0\leqslant u\leqslant s,\ \forall s\notin D\}$. $(r\leqslant 0<s,\ rs\leqslant ru=-rt\leqslant 0\leqslant rt\leqslant -rs.)$

• Note the '$0\leqslant u$'. Becs $C^-\neq\emptyset\Rightarrow 0\in C^-$. If it is to be exactly $CD=\{a: a<0\}$, then $C^-=\{0\}$,
for if not, $\exists u>0$, and $\exists r\in C^-\backslash\{0\}$, suth $\exists a<ru<0$. Hence '$0\leqslant u$' is actually '$0<u$'.

• '$u\leqslant s$' cannot be abbreviated as in $\{-a: a>-rs,\ \forall s\notin D, r\in C^-\}=\{a: a<rs,\ \forall s\notin D, r\in C^-\}$.
'$u\leqslant s$' cannot be '$u<s$', becs here $rs<ru\Rightarrow\exists a=rs$.

• Note that $\{u: 0<u\leqslant s,\ \forall s\notin D\}=\begin{cases} D^+\cup\{\min\mathbf{Q}\backslash D\}, & \text{if it exis,}\\ D^+, & \text{othws.}\end{cases}$ Denote it by $D^{\oplus}=D^{\otimes}\backslash\{0\}$.

• For C not posi and D posi. If $C=\tilde{0}$, then $CD=-[C(-D)]=-\tilde{0}$. Now consider $-C$ and D both posi.
But $CD=-(-C)D=-\{a: a\leqslant cd,\ c\in(-C)^+, d\in D^+\}\neq\{a: a<-cd,\ \forall c\in(-C)^+, d\in D^+\}$.
Altho "$a\leqslant cd$" is equiv to "$a<cd$" so that $b\notin(-C)D\Rightarrow b\geqslant cd$, which is actually $b>cd,\ \forall c,d$.
And $a<-b<-cd,\ \forall c,d\Rightarrow\forall a, \exists x$ suth $a<x<-cd,\ \forall c,d$. While a can be the 'boundary' in RHS.

• $LHS=\{a: a<ru,\ \forall r\in C^-,\ \forall u\in D^{\oplus}\}$, $\{cs: c\in C, s\notin D\}=RHS$.
Becs $cs\leqslant cu<ru$. We show $LHS\subseteq RHS$. Let $c_1<\cdots<c_n<\cdots\in C$, and $s_1>\cdots>s_m\geqslant\cdots\notin D$.
Then $\underline{c_1s_1<\cdots<c_ns_m<\cdots<ru,\ \forall r,u \text{ as in } LHS. \text{ Thus } a\in LHS\Rightarrow\exists a<c_js_k.}$ □
OR. Note that in LHS, '$a<ru$' is equiv to '$a<rs,\ \exists s\notin D$.' Now $LHS=\{a: a/s\in C,\ \exists s\notin D\}$. □

• For C posi and D not posi. If $D=\tilde{0}$, then $CD=-[(-C)D]=-\tilde{0}$.
$CD=(-C)(-D)=\{a: a<ru,\ \forall r\in(-C)^-,\ \forall u\in(-D)^{\oplus}\}$ $\quad\mathbf{Q}\backslash -D=\{s: s\geqslant -y,\ \forall y\notin D\}$
$=\{a: a<ru,\ \forall r$ suth $\forall x\notin C, 0\geqslant r\geqslant -x,\ \forall u$ suth $0<u\leqslant s,\ \forall s\geqslant -y,\ \forall y\notin D\}$
$=\{a: a<(-r)(-u),\ \forall r$ suth $\forall x\notin C, 0\leqslant -r\leqslant x,\ \forall u$ suth $y\leqslant -u<0,\ \forall y\notin D\}$
$=\{a: a<ru,\ \forall r$ suth $\forall x\notin C, 0\leqslant r\leqslant x,\ \forall u\in D^-\}=\{a: a<ru,\ \forall r\in C^{\oplus},\ \forall u\in D^-\}$, simlr.

- We show $-D = \{a : a < -b,\ b \notin D\} = (-\tilde{1})D$.
  For D posi, $RHS = \{a : a < ru,\ \forall r \text{ suth } -1 \leqslant r \leqslant 0,\ \forall u \in D^{\oplus}\} = \{a : a < -u,\ \forall u \in D^{\oplus}\} \supseteq -D$.
  Supp x suth $-b \leqslant x < -u,\ \forall b \notin D,\ \forall u \text{ suth } -s \leqslant -u < 0,\ \forall s \notin D \Longrightarrow -s \leqslant x < -u$. Let $-u = x$.
  For D not posi, $RHS = \{a : a < rb,\ \exists r \text{ suth } -1 \leqslant r \leqslant 0,\ b \in D^-\} = \{a : a < -b,\ 0 \geqslant b \notin D\} = -D$.

- We show $\tilde{1}D = D$. For D not posi, immed. Othws, $\tilde{1}D = \{a : a \leqslant ij < j,\ 0 < i < 1,\ j \in D^+\} \subseteq D$.
  Now $(\tilde{1}D)^+ \subseteq D^+$. 又 $\forall d \in D^+,\ \exists \varepsilon > 0,\ d + \varepsilon \in D^+ \Rightarrow d = (d+\varepsilon)\frac{d}{d+\varepsilon} \in (\tilde{1}D)^+$.

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4 *Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD)$, $B(C+D) = BC + BD$.*

Solus: We discuss in cases.

\	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

(1) $[(BC)D]^+ = \{(bc)d : bc \in (BC)^+,\ d \in D^+\} = \{b(cd) : b \in B^+,\ cd \in (CD)^+\} = [B(CD)]^+$.
$B(C+D) = \{a : a \leqslant bc + bd,\ \exists b \in B^+,\ 0 < c + d \in C + D\}$.
$$BC + BD = \{x : x \leqslant uc,\ u \in B^+,\ c \in C^+\} + \{y : y \leqslant vd,\ v \in B^+,\ d \in D^+\}$$
$$= \{a : a \leqslant uc + vd,\ \exists u, v \in B^+,\ c \in C^+,\ d \in D^+\}.$$ Done.

(3) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists c \in C^-,\ \exists b \in B^{\oplus}\}$.
$$(BC)D = \{a : a < rs,\ r \in (BC)^-,\ s \in D^-\}$$
$$= \{a : a < bcs,\ \exists s \in D^-,\ c \in C^-,\ \exists b \in B^{\oplus}\}.$$
$$B(CD) = \{a : a \leqslant bx < bcs,\ \exists s \in D^-,\ c \in C^-,\ \exists b \in B^+,\ \text{and } cs > x \in (CD)^+\}$$
$$= \{a : a < bcs,\ \exists s \in D^-,\ c \in C^-,\ \exists b \in B^+\}.$$
Note that $\{q : q < b,\ b \in B^+\} = \{q : q < b,\ \exists b \in B^{\oplus}\}$. Done.

$$B(C+D) = \{a : a < ru,\ \forall r \text{ suth } \forall x \notin B,\ 0 < r \leqslant x,\ \forall u \text{ suth } 0 \geqslant u > c + d,\ \forall c \in C,\ d \in D\}$$
$$= \{a : a < ru,\ \forall r \in B^{\oplus},\ \forall u \text{ suth } 0 \geqslant u \geqslant c + d,\ \exists c \in C^-,\ d \in D^-\}$$
$$= \{a : a < r(c+d),\ \forall r \in B^{\oplus},\ \forall c \in C^-,\ d \in D^-\}.$$
$$BC + BD = \{x : x < pc,\ \forall p \in B^{\oplus},\ \forall c \in C^-\} + \{y : y < qd,\ \forall q \in B^{\oplus},\ \forall d \in D^-\}$$
$$= \{a : a < pc + qd,\ \forall p, q \in B^{\oplus},\ \forall c \in C^-,\ d \in D^-\}.$$ Done immed.

(5) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists b \in B^-,\ \exists c \in C^{\oplus}\}$. $(CD)^- = \{r : 0 \geqslant r \geqslant cd,\ \exists d \in D^-,\ \exists c \in C^{\oplus}\}$.
$(BC)D = \{a : a < rd \leqslant bcd,\ \exists d \in D^-,\ b \in B^-,\ \exists c \in C^{\oplus}\}$.
$B(CD) = \{a : a < br \leqslant bcd,\ \exists b \in B^-,\ d \in D^-,\ \exists c \in C^{\oplus}\}$. Done.
Or. By commu and (3), $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$.

$$BC + BD = \{x : x < bc,\ \forall b \in B^-,\ \forall c \in C^{\oplus}\} + \{y : y < bd,\ \exists b \in B^-,\ d \in D^-\}$$
$$= \{a : a < pc + qd,\ \forall p \in B^-,\ \forall c \in C^{\oplus},\ \text{and } \exists q \in B^-,\ d \in D^- \Rightarrow q \geqslant p\}$$
$$= \{a : a < pc + y,\ \forall p \in B^-,\ \forall c \in C^{\oplus},\ \text{and } \forall y \text{ suth } y \geqslant qd,\ \forall q \in B^-,\ d \in D^-\}$$
$$= \{a : a < p(c + d'),\ \forall p \in B^-,\ \forall c \in C^{\oplus},\ \text{and } \forall d' \text{ suth } d' \leqslant d,\ \forall d \in D^-\}.$$

(I) If $C + D$ not posi. Then $B(C+D) = \{a : a < bx,\ \exists b \in B^-,\ 0 \geqslant x > c + d,\ \forall (c,d) \in C \times D\}$.
Rewrite as $\{a : a < t,\ \forall t \text{ suth } t \geqslant bx,\ \forall b \in B^-,\ x \in (C+D)^-\}$. Done.

(II) If $C + D$ posi. Then $B(C+D) = \{a : a < bx,\ \forall b \in B^-,\ x \in (C+D)^{\oplus}\}$.
If $(C+D)^{\oplus} = (C+D)^+$. Then $B(C+D) = \{a : a < bc + bd,\ \forall b \in B^-,\ \forall (c,d) \in C^{\oplus} \times D^{\oplus}\}$.
Othws, $\exists s = \min \mathbf{Q} \backslash (C+D) \Rightarrow B(C+D) = \{a : a < bs,\ \forall b \in B^-\}$. Done.

$$\begin{aligned}
(4)\ (BC)D &= \{xd : d \in D,\ x \geqslant bc,\ \forall b \in B^-,\ c \in C^-\} \\
&= \{a : a < bcd,\ \forall d \in D^-,\ \forall b \in B^-,\ c \in C^-\} \\
&= \{by : b \in B,\ y \geqslant cd,\ \forall c \in C^-,\ d \in D^-\} = B(CD).
\end{aligned}$$

$$\begin{aligned}
B(C+D) &= \{a : a < t,\ \forall t \text{ suth } t \geqslant b(c+d),\ \forall b \in B^-,\ (c,d) \in C^- \times D^-\} \\
&= \{a : a < t_1,\ \forall t_1 \text{ suth } t_1 \geqslant bc,\ \forall (b,c) \in B^- \times C^-\} \\
&\quad + \{a : a < t_2,\ \forall t_2 \text{ suth } t_2 \geqslant bd,\ \forall (b,d) \in B^- \times D^-\} = BC + BD. \text{ Done.}
\end{aligned}$$

NOTE: Supp for any B posi, C posi, D not posi, assoc holds.

Supp B posi, C not posi, D posi. Then $(\overset{+}{B}\overset{-}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{B}\overset{-}{C}) = (\overset{+}{D}\overset{+}{B})\overset{-}{C} = \overset{+}{B}(\overset{+}{D}\overset{-}{C})$. Convly true.

Simlr for the case B not posi, C posi, D not posi, equiv to the case B not posi, C not posi, D posi.

(2) holds $\Rightarrow (\overset{-}{B}\overset{+}{C})\overset{+}{D} = \overset{+}{D}(\overset{+}{C}\overset{-}{B}) = (\overset{+}{C}\overset{+}{D})\overset{-}{B}$, (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds $\Rightarrow \overset{+}{B}(\overset{+}{C} + \overset{-}{D}) = (-\overset{-}{B})[-(\overset{+}{C} + \overset{-}{D})] \overset{(5)}{=\!=} (-\overset{-}{B})[(-\overset{-}{C}) + (-\overset{+}{D})] \overset{(5)}{=\!=} \overset{+}{B}\overset{+}{C} + \overset{+}{B}\overset{-}{D}$, by def of multi.

Thus (5) $\Rightarrow$ (2) in distr. Convly as well.

(6) $(BC)^- = \{r : 0 \geqslant r \geqslant bc,\ \exists\, b \in B^-,\ c \in C^{\oplus}\}$.

$(BC)D = \{a : a < ru,\ \forall r \in (BC)^-,\ u \in D^{\oplus}\} = \{a : a < bcd,\ \forall b \in B^-,\ c \in C^{\oplus},\ d \in D^{\oplus}\}$.

$(CD)^{\oplus} = \{u : 0 < u \leqslant s,\ \forall s > cd,\ \forall c \in C^+,\ d \in D^+\}$.

$B(CD) = \{a : a < ru,\ \forall r \in B^-,\ u \in (CD)^{\oplus}\}$. Done.

$B(C+D) = \{a : a < b(c+d),\ \forall b \in B^-,\ \forall (c,d) \in C^{\oplus} \times D^{\oplus}\}$.

$BC + BD = \{a : a < bc,\ \forall b \in B^-,\ c \in C^{\oplus}\} + \{a : a < bd,\ \forall b \in B^-,\ d \in D^{\oplus}\}$. Done.

OR. (2) instead of (6) ? NOTICE that the distr in (6) cannot be shown without (6). $\square$

ENDED

0·C

2 *Supp non-$\emptyset$ $U \subseteq V \subseteq \mathbf{R}$. Show* $\sup U \leqslant \sup V$.

SOLUS: Asum $\sup U > \sup V \Rightarrow \exists\, t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$, ctradic. $\square$

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5 *Supp* $\sup\{a_1, a_2, \dots\} = \sqrt{2}$ *and each* $a_k \in \mathbf{Q}$. *Prove each* $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$.

SOLUS: Becs the sup not in seq $\Rightarrow$ infily many disti elem.

$\forall a_i,\ \exists\, a_j,\ a_i < a_j < \sqrt{2}$. For a_{n+k}, choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most n steps, we must have a_m with $m > n$. Thus $\forall a_{n+i},\ \exists\, a_{n+j},\ a_{n+i} < a_{n+j} < \sqrt{2}$. $\square$

• *Supp non-$\emptyset$ $A \subseteq \mathbf{R}$.*

• **TIPS 1:** *Define* $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. *Prove* $\sup(-A) = -\inf A$.

SOLUS: $-b$ is an up-bound of $-A \iff \forall a \in A,\ -a \leqslant -b \iff a \geqslant b \iff b$ is a low-bound of A.

Thus $-b_M = \sup(-A) \iff -b_M \leqslant -b \iff b_M \geqslant b \iff b_M = \inf A$. $\square$

• **TIPS 2:** *Show if* $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists\, a \in A,\ a > x$, (b) $\inf A < x \Rightarrow \exists\, a \in A,\ a < x$.

SOLUS: (a) $\nexists\, a > x \iff \forall a \in A,\ a \leqslant x \Longrightarrow \sup A \leqslant x$.

OR. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A,\ -a < -x$. Simlr for (b). $\square$

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6 *Supp non-$\emptyset$ $A, B \subseteq \mathbf{R}$. Prove* $\sup(A+B) = \sup A + \sup B$, *and* $\inf(A+B) = \inf A + \inf B$.

Solus: $\inf A + \inf B \leqslant a + b \leqslant \sup A + \sup B \Rightarrow \sup(A+B) \leqslant \sup A + \sup B$, $\inf A + \inf B \leqslant \inf(A+B)$.

$\sup A + \sup B > \sup(A+B) \iff \sup A > \sup(A+B) - \sup B$

$\iff \exists\, a + \sup B > \sup(A+B) \iff \sup B > \sup(A+B) - a \iff \exists\, a + b > \sup(A+B)$. Ctradic.

Simlr for $\inf(A+B) \in A+B$. Or. Apply to $-A-B$, becs $\sup(-A) = -\inf A$. $\square$

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16 *Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in $[0.11]$.*
Define $\psi : \mathbf{R}_1 \to \mathbf{R}_2$ by $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant a\}$. Show
(a) *$\psi : \mathbf{R}_1 \to \mathbf{R}_2$ is one-to-one.* (b) $\psi(0) = 0$, $\psi(1) = 1$.
(c) $\psi(a+b) = \psi(a) + \psi(b)$. (d) $\psi(ab) = \psi(a)\psi(b)$.

Solus: (a) Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leqslant a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Let $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$.
Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.
Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \to \mathbf{R}_1$ well-defined.
Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leqslant \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.
Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Becs $\varphi_1(q) \leqslant \varphi_1(0) \iff q \leqslant 0 \iff \varphi_2(q) \leqslant \varphi_2(0)$. Simlr for $\psi(1) = 1$.

(c) $S = \{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant a+b\} \supseteq \{\varphi_2(p+q) : p, q \in \mathbf{Q},\ \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b\} = T$
$\Rightarrow \sup S \geqslant \sup T$. Asum it is '$>$'. Now $\exists\, t \in \mathbf{Q}$ suth $\sup S \geqslant \varphi_2(t) > \sup T = \psi(a) + \psi(b)$.
Which means $\varphi_2(t) > \varphi_2(p+q)$, $\forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leqslant a$ and $\varphi_1(q) \leqslant b$.
Now $a + b \geqslant \varphi_1(t) > \varphi_1[(t+p+q)/2] > \varphi_1(p+q)$, $\forall p, q$. Ctradic.

(d) We show it for (I) $a, b > 0$, (II) $a > 0 > b$.
$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q},\ \varphi_1(t) \leqslant ab\}$, $\sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q},\ \varphi_1(q) \leqslant b\} = RHS$.
(I) $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q},\ 0 < \varphi_1(p) \leqslant a,\ \varphi_1(q) \leqslant b \Longrightarrow \varphi_1(pq) \leqslant ab\}$.
(II) $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q},\ \varphi_1(q) \leqslant b$, and $s \geqslant p, \forall p$ suth $0 < \varphi_1(p) \leqslant a\}$.
Note that $\varphi_1(s) \geqslant \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geqslant a$, for if not, $\exists\, p' \in \mathbf{Q}$ suth $\varphi_1(s) < \varphi_1(p') \leqslant a$.
So that $\varphi_1(sq) \leqslant a\varphi_1(q) \leqslant ab$.
Now $LHS \geqslant RHS$. Asum it is '$>$'. Then $\exists\, t \in \mathbf{Q}$ suth $LHS \geqslant \varphi_2(t) > RHS$.
Which means $\varphi_2(t) > \psi(a)\varphi_2(q)$, $\forall q \in \mathbf{Q}$ suth $\varphi_1(q) \leqslant b$.
(I) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $0 < \varphi_1(s) \leqslant a$.
(II) $\Longrightarrow \varphi_2(t) > \varphi_2(s)\varphi_2(q)$, $\forall q$ and $\forall s \in \mathbf{Q}$ suth $\varphi_2(s) \geqslant \varphi_2(p), \forall p$ suth $0 < \varphi_1(p) \leqslant a$.
Thus $ab \geqslant \varphi_1(t) > \varphi_1[(t+sq)/2] > \varphi_1(sq)$, $\forall s, q$. Ctradic. $\square$

Ended

0·D

• *Supp non-$\emptyset$ A is closed and open subsets of $\mathbf{R}^n$. Prove $A = \mathbf{R}^n$.*

Solus: Asum $A \neq \mathbf{R}^n$. Let $a \in A$, $b \in \mathbf{R}^n \backslash A$. Define $f(t) = (1-t)a + tb$.
If $f(t_1) = f(t_2) \Rightarrow (t_1 - t_2)(b - a) = 0 \Rightarrow t_1 = t_2$. Inje.
Let $\mathcal{A} = \{t \in [0,1) : f(t) \in A\}$, and $\sup \mathcal{A} = t_M \in [0,1]$. Let $c = f(t_M)$.
(I) If $c \in A \Rightarrow \exists\, \delta > 0$, $B(c,\delta) \subseteq A$. Let $t \neq t_M$ be suth $f(t) \in B(c,\delta) \Rightarrow \|(t - t_M)(b-a)\|_\infty < \delta$.
Let $\varepsilon = |t - t_M| > 0 \Rightarrow f(t_M + \varepsilon) \in B(c,\delta) \subseteq A \Rightarrow t_M \geqslant t_M + \varepsilon$, ctradic.
(II) If $c \in \mathbf{R}^n \backslash A$. Simlr. $\forall t' \in [t_M - \varepsilon,\ t_M)$, $t' \notin \mathcal{A}$. Now $\forall t \in \mathcal{A}$, $t < t_M \Rightarrow t < t_M - \varepsilon$. $\square$

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15 *Supp F is a closed subset of* $\mathbf{R}$. *Prove* $S = \{a^2 : a \in F\}$ *is closed.*

Solus: Supp S not closed $\Rightarrow \exists\, a_1^2, a_2^2, \cdots \in S$ convg to L^2 suth $\pm L \notin F$. Let $\varepsilon_1 > \varepsilon_2 > \cdots \in \mathbf{R}^+$.
Becs $\forall \varepsilon_p, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m, \|a_k^2 - L^2\|_\infty = |a_k - L| \cdot |a_k + L| < \varepsilon_p$,
if $|a_k - L| < \varepsilon_p$, then let $b_p = a_k$, if $|a_k + L| < \varepsilon_p$, then let $c_p = a_k$.
Then we have at least one seq in F with lim $\pm L \notin F \Rightarrow F$ not closed. □

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17 *Supp* $F \subseteq \mathbf{R}$, *and* $\forall n \in \mathbf{Z}^+$, $F \cap [-n, n]$ *is closed. Prove F is closed.*

Solus: Becs $G = (\mathbf{R}\backslash F) \cup (-\infty, -n) \cup (n, \infty)$ is open for all $n \in \mathbf{Z}^+$. Let $n > \sup\{|a| : a \in \mathbf{R}\backslash F\}$.
By [0.59], $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \cdots$ (disj) $\Longrightarrow I_1 \cup I_2 \cup \cdots = \mathbf{R}\backslash F$. □

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20 *Prove* $\forall b \in \mathbf{R}^n$, $\delta > 0$, $B = \{a \in \mathbf{R}^n : \|a - b\| \leqslant \delta\}$ *is closed.* *See below [0.46].*

Solus: Asum $\exists\, a_1, a_2, \cdots \in B$ convg to L suth $\|L - b\| > \delta$. Then $\forall \varepsilon > 0, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m$,
$\underline{\|a_k - L\| < \varepsilon} \Rightarrow \delta < \|L - b\| \leqslant \|a_k - L\| + \|a_k - b\| < \delta + \varepsilon$. Now $0 < \|L - b\| - \delta < \varepsilon$. Ctradic. □

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21 *Supp X is an open subset of* $\mathbf{R}$. *Prove* $\exists\, a_k, b_k \in \mathbf{R}$ *suth* $X = \bigcup_{k=1}^{\infty} [a_k, b_k]$.

Solus: Let $X = \bigcup_{k=1}^{\infty} (c_k, d_k)$, where each $d_k < c_{k+1}$. Let $I_k = (c_k, d_k)$.
Let $a_{1,k}, a_{2,k}, \ldots$ be convg of lim c_k. Simlr, let $b_{1,k}, b_{2,k}, \ldots$ of lim d_k. Let $E_{j,k} = [a_{j,k}, b_{j,k}]$.
Now $\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{j,k} = \bigcup_{k=1}^{\infty} I_k = X$. Rearrange the order of the E's. □

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23 *Supp F_1, F_2 are disj closed subsets of* $\mathbf{R}$ *suth $U = F_1 \cup F_2$ is interval. Prove F_1 or F_2 open.*

Solus: If U open, then $F_1 = \emptyset$, $F_2 = \emptyset$ or $\mathbf{R}$. Now supp U not open $\Rightarrow F_1$ or F_2 not open.
We show F_1 open $\Longleftrightarrow F_2$ not open. Note that F_1 open $\Longleftrightarrow F_1 = \emptyset$. Simlr for F_2.
WLOG, asum F_1, F_2 both not open, and $x \in F_1, y \in F_2$ with $x < y \Rightarrow x, y \in U \supseteq [x, y]$.
Notice that $T = [x, y] \cap F_1$ is a closed subset that has infily many elem. 又 $\sup T < y$.
(I) If $\sup T \notin F_1$. Then $\exists$ a convg seq in T with lim $\sup T$.
(II) Othws, $\forall t \in (\sup T, y]$, $t \notin F_1 \Longleftrightarrow t \in F_2$. Now $\exists$ a convg seq in F_2 with lim $\sup T \notin F_2$.
Ctradic the asum $\Rightarrow$ at least one of F_1, F_2 is empty. □

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25 *Give an exa of inv* $f : \mathbf{R} \to \mathbf{R}\backslash\mathbf{Q}$.

Solus: Consider a seq disti $a_1, b_1, a_2, b_2, \cdots \in \mathbf{R}$ where $\{a_1, a_2, \ldots\} \in \mathbf{Q}$ and each $b_i \in \mathbf{R}\backslash\mathbf{Q}$.
Define $\varphi(a_j) = b_{2j-1}$, $\varphi(b_k) = b_{2k} \Rightarrow \varphi^{-1}(b_i) = \begin{cases} a_{(i+1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$
Let $B = \{b_1, b_2, \ldots\}$, $U = \mathbf{Q} \cup B$, $K = \mathbf{R}\backslash U$. Extend $\varphi \in B^U$ to $\psi \in (K \cup B)^{K \cup U}$ by $\psi|_K = I$. □

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26 *Supp* $E, G \subseteq \mathbf{R}^n$, *and G is open. Prove* $E + G = \{x + y : x \in E, y \in G\}$ *is open.*

Solus: Asum $E + G$ not open $\Longleftrightarrow \mathbf{R}^n \backslash (E + G)$ not closed.
Then $\exists\, a = x + y \in E + G$ suth $\forall \delta > 0, \exists\, b \notin E + G$ suth $\|a - b\|_\infty < \delta$.
Let $z = b - x \notin G \Rightarrow \|y - z\|_\infty < \delta \Rightarrow z \in B(y, \delta) \subseteq G, \exists\, \delta > 0$. □
Or. $\exists\, a_1, a_2, \cdots \notin E + G$ while its lim $L = e + g \in E + G$. 又 $\forall x \in E$, $a_k - x \notin G$
$\Rightarrow \lim_{k \to \infty} (a_k - e) = L - e = g \in G$. Thus $\mathbf{R}^n \backslash G$ not closed $\Longleftrightarrow G$ not open. □

Ended

0·E

1 *Prove every convg seq in* $\mathbf{R}^n$ *is bounded.*

Solus: Supp $a_1, a_2, \cdots \in \mathbf{R}^n$ suth $\forall \varepsilon > 0, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m, \big|\|a_k\|_\infty - \|L\|_\infty\big| \leqslant \|a_k - L\|_\infty < \varepsilon.$
Thus $\|a_k\|_\infty \in \big(\|L\|_\infty - \varepsilon,\ \|L\|_\infty + \varepsilon\big)$. Now asum $\sup\big\{\|a_k\|_\infty : k \in \mathbf{Z}^+\big\} = \infty$.
Which means $\forall t \in \mathbf{R}^+, \forall m \in \mathbf{Z}^+, \exists\, k \geqslant m, \max\big\{\|a_1\|_\infty, \ldots, \|a_m\|_\infty\big\} \leqslant \|a_k\|_\infty > t$. Ctradic. $\square$

• **Tips 1:** By def, a seq $a_1, a_2, \cdots \in \mathbf{R}^n$ convg to $L \Longrightarrow$ every subseq convg to L.

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3 *Supp* $F \subseteq \mathbf{R}^n$*, every seq in* F *has a convg subseq with lim in* F*. Prove* F *is closed bounded.*

Solus: Supp $a_1, a_2, \cdots \in F$ convg with lim L. Becs $\exists$ subseq with lim in F. By Tips (1), $L \in F$.
Asum $\sup\big\{\|a\|_\infty : a \in F\big\} = \infty \Rightarrow \forall \Delta > 0, \exists\, a_1, a_2, \cdots \in F$ with each $\|a_{k+1}\|_\infty \geqslant \|a_k\|_\infty + \Delta$.
Thus every subseq is unbounded $\Rightarrow$ not convg. Ctradic. $\square$

• *Supp* $b \in A \subseteq \mathbf{R}^m, f : A \to \mathbf{R}^n$*. Prove*
$[P]$ f *is continu at* $b \iff \forall b_1, b_2, \cdots \in A$ *suth* $\lim_{k\to\infty} b_k = b$, $\lim_{k\to\infty} f(b_k) = f(b)$. $[Q]$

Solus: $\neg P \Rightarrow \neg Q$: $\exists\, \varepsilon > 0, \forall k \in \mathbf{Z}^+, \exists\, b_k \in A$ suth $\|b_k - b\|_\infty < 1/k$ and $\big\|f(b_k) - f(b)\big\|_\infty \geqslant \varepsilon$.
Now $\lim_{k\to\infty} b_k = b$ while $\exists\, \varepsilon > 0, \forall m, \exists\, k \geqslant m, \big\|f(b_k) - f(b)\big\|_\infty \geqslant \varepsilon$.

$\neg Q \Rightarrow \neg P$: Supp $b_1, b_2, \cdots \in A$ suth $\forall \delta > 0, \exists\, m, \forall k \geqslant m, \|b_k - b\|_\infty < \delta$.
Asum $\exists\, \varepsilon > 0, \forall m\, [\Rightarrow \forall \delta > 0], \exists\, k \geqslant m, \big\|f(b_k) - f(b)\big\|_\infty \geqslant \varepsilon$. Immed. $\square$

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10 *Supp bounded* $A \subseteq \mathbf{R}^m$ *and uniformly continu* $f \in \mathbf{R}^A$*. Prove* f *is bounded.*

Solus: Asum $\forall \Delta > 0, \exists\, a_1, a_2, \cdots \in A$ suth each $\big|f(a_{k+1}) - f(a_k)\big| \geqslant \big|f(a_{k+1})\big| - \big|f(a_k)\big| \geqslant \Delta$.
$\exists$ subseq $a_{j_1}, a_{j_2}, \cdots$ convg to $L \Rightarrow \forall \delta > 0, \exists\, j_m, \forall j_x, j_y \in \big\{j_m, j_{m+1}, \cdots\big\}, \|a_{j_x} - a_{j_y}\| < \delta$.
$\forall \varepsilon > 0, \exists\, \delta > 0, \forall a, b \in A$ suth $\|a - b\| < \delta$, we have $f(a) \in \big(f(b) - \varepsilon,\ f(b) + \varepsilon\big)$.
Let $b_{k_i} = a_{j_{k_i}}$. Becs $\exists\, k_1 < k_2 < \cdots$ suth $f(b_{k_1}), f(b_{k_2}), \cdots$ is monotone.
If incre, then let k_1 suth $f(b_{k_1}) > 0 \Rightarrow f(b_{k_{i+1}}) > f(b_{k_i}) + \Delta$. Ctradic. Simlr for decre. $\square$

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17 *Supp uniformly continu* $f, g : \mathbf{R}^{\mathbf{R}}$*. Prove* $f \circ g \in \mathbf{R}^{\mathbf{R}}$ *is uniformly continu.*

Solus: $\forall \varepsilon > 0, \exists\, \delta > 0, \big|f(t_1) - f(t_2)\big|, \forall t_1, t_2 \in g(\mathbf{R})$ suth $\big|t_1 - t_2\big| < \delta$.
Then $\exists\, \rho > 0, \forall a, b$ suth $|a - b| < \rho$, we have $\big|g(a) - g(b)\big| < \delta$.
These $g(a), g(b)$ is contained in the set of all pairs of t_1, t_2. $\square$

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18 *Supp* $h : \mathbf{R}^m \to \mathbf{R}^n$*. Prove* h *continu* $\iff h^{-1}(G)$ *is open for all open* $G \subseteq \mathbf{R}^n$*.*

Solus: Supp h continu and open $G \subseteq \mathbf{R}^n$. We show $\mathbf{R}^m \backslash h^{-1}(G) = \big\{t \in \mathbf{R}^m : h(t) \notin G\big\}$ is closed.
Let $t_1, t_2, \cdots \in \mathbf{R}^m \backslash h^{-1}(G)$ convg to $L \Rightarrow h(t_1), h(t_2), \cdots \in \mathbf{R}^n \backslash G$ convg to $h(L)$.
Supp open $G \subseteq \mathbf{R}^n$ suth $\mathbf{R}^m \backslash h^{-1}(G)$ not closed. Asum h continu.
$\exists\, t_1, t_2, \cdots \in \mathbf{R}^m \backslash h^{-1}(G)$ convg to $L \in h^{-1}(G) \Longrightarrow h(t_1), h(t_2), \cdots \in \mathbf{R}^n \backslash G$ convg to $h(L) \notin G$. $\square$

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22 *Supp decre seq $F_1 \supsetneq F_2 \supsetneq \cdots$ non-$\emptyset$ closed bounded subsets of $\mathbf{R}^n$. Prove $F_\infty = \bigcap_{k=1}^\infty F_k \neq \emptyset$.*

SOLUS: Define $\|F\|_\infty = \sup\{\|a\|_\infty : a \in F\}$ and $\|F\|_0 = \inf\{\|a\|_\infty : a \in F\}$ for all bounded $F \subseteq \mathbf{R}^n$.

Note that by def of sup and inf, $\|F\|_\infty = -\infty \iff \|F\|_0 = \infty \iff \|F\|_\infty < \|F\|_0 \iff F = \emptyset$.

Now $\|F_1\|_\infty, \|F_2\|_\infty, \cdots$ bounded decre, and $\|F_1\|_0, \|F_2\|_0, \cdots$ bounded incre.

Consider $\|F_\infty\|_\infty = \lim_{k\to\infty} \|F_k\|_\infty$, and $\|F_\infty\|_0 = \lim_{k\to\infty} \|F_k\|_0$.

Note that $\|F_\infty\|_\infty - \|F_\infty\|_0 = \lim_{k\to\infty}\big[\|F_k\|_\infty - \|F_k\|_0\big] \geqslant 0$.

又 For a closed bounded F, $\|F\|_\infty = \|a\|$ for some $a \in F$, simlr for $\|F\|_0$. ☐

OR. Pick $a_k \in F_k$ for each $\Rightarrow \exists$ subseq $a_{j_1}, a_{j_2}, \cdots$ convg to $a \in \bigcap_{k=1}^\infty F_k$. ☐

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26 *Supp $F \subseteq \mathbf{R}^n$ suth every continu $f \in \mathbf{R}^F$ attains a max. Prove F is closed bounded.*

SOLUS: Asum F not bounded. Define $f(a) = \|a\|_\infty$. Ctradic.

Asum $\exists\, a_1, a_2, \cdots \in F$ convg to $L \notin F$. Define $f(a) = 1/\|a - L\|_\infty$. Ctradic. ☐

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27 *Supp $f \in \mathbf{R}^{\mathbf{R}}$ is incre. Prove $\exists$ countable $A \subseteq \mathbf{R}$ suth $f|_{\mathbf{R}\setminus A}$ is continu.*

SOLUS: Asum $\forall$ countable $A \subseteq \mathbf{R}$, $f|_{\mathbf{R}\setminus A}$ is not continu at an uncountable set B of elem in $\mathbf{R}\setminus A$.

$\forall b \in B, \exists\, \varepsilon > 0, \forall k \in \mathbf{Z}^+, \exists\, a_k \in \mathbf{R}\setminus A$ suth $|a_k - b| < 1/k$, $\big|f(a_k) - f(b)\big| \geqslant \varepsilon$.

Now $a_1, a_2, \cdots$ convg to b.

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29 *Supp continu $f : [a,b] \to \mathbf{R}$, and t is between $f(a), f(b)$. Prove $\exists\, c \in [a,b]$, $f(c) = t$.*

SOLUS: Let $a_0 = a$, $b_0 = b$.

Step 1 Pick $c_1 \in (a_0, b_0)$. If $f(c_1) = t$, then stop.

If $f(a_0) < t < f(b_0)$. Let $(a_1, b_1) = \begin{cases} (a_0, c_1), & \text{if } t < f(c_1), \\ (c_1, b_0), & \text{if } f(c_1) < t. \end{cases}$

If $f(b_0) < t < f(a_0)$. Let $(a_1, b_1) = \begin{cases} (c_1, b_0), & \text{if } t < f(c_1), \\ (a_0, c_1), & \text{if } f(c_1) < t. \end{cases}$

Step m Pick $c_m \in (a_{m-1}, b_{m-1})$. If $f(c_m) = t$, then stop.

If $f(a_{m-1}) < t < f(b_{m-1})$. Let $(a_m, b_m) = \begin{cases} (a_{m-1}, c_m), & \text{if } t < f(c_m), \\ (c_m, b_{m-1}), & \text{if } f(c_m) < t. \end{cases}$

If $f(b_{m-1}) < t < f(a_{m-1})$. Let $(a_m, b_m) = \begin{cases} (c_m, b_{m-1}), & \text{if } t < f(c_m), \\ (a_{m-1}, c_m), & \text{if } f(c_m) < t. \end{cases}$

Either we stop at some m and done or we get a seq $(a_1, b_1) \supsetneq (a_2, b_2) \supsetneq \cdots$

suth $\forall \varepsilon > 0, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m, b_k - a_k < \varepsilon \Longrightarrow \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$, let it be $c \in (a_0, b_0)$.

Now $\forall \varepsilon > 0, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m, \big|t - f(a_k)\big| < \varepsilon \Longrightarrow \lim_{k\to\infty} f(a_k) = t = f(c)$. ☐

L1 *Supp $p \in \mathbf{R}^{\mathbf{R}}$ is a poly. Prove p continu.*

SOLUS: Write $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ with $a_m \neq 0$. Supp $x_0 \in \mathbf{R}$. Then write:

$\forall \varepsilon > 0, \exists\, \delta_M > 0, \forall \delta$ suth $|\delta| < \delta_M$, $\big|p(x_0 + \delta) - p(x_0)\big| = \big|\delta \cdot q(x_0, \delta)\big| < \varepsilon \iff \big|q(x_0, \delta)\big| < |\varepsilon/\delta|$

where $q(x_0, \delta) = \sum_{k=1}^m \Big(\sum_{i=1}^k a_k C_k^i \delta^{k-i-1} x_0^i\Big) \Rightarrow \big|q(x_0, \delta)\big| \leqslant \sum_{k=1}^m \sum_{i=1}^k |\delta|^{k-i-1} \big|a_k C_k^i x_0^i\big|$.

Let $|\delta| < 1$ suth $\max\{|a_k C_k^i x_0^i| : i, k \in \mathbf{Z}^+, 1 \leqslant k \leqslant m, 1 \leqslant i \leqslant k\} \cdot |\delta| \leqslant |\varepsilon| < |\varepsilon/\delta|$. ☐

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31 *Supp $p \in \mathbf{R}^{\mathbf{R}}$ is a poly with odd deg. Prove $\exists\, b \in \mathbf{R}$ suth $p(b) = 0$.*

Solus: Write $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ with $a_m \neq 0$. Supp $a_m > 0$. $\big[$If $a_m < 0$, then apply to $-p$.$\big]$
Now $p(x) = x^m \left[\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \right]$ for all $x \in \mathbf{R} \backslash \{0\}$.
While $\left| \frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} \right| \leqslant \delta^m |a_0| + \delta^{m-1} |a_1| + \cdots + \delta |a_{m-1}|$; let $\delta = 1/|x|$.
Let x be suth $\delta < 1$ and $\max\big\{|a_0|, |a_1|, \ldots, |a_{m-1}|\big\} \cdot \delta \leqslant |a_m|/m$. Thus $p(-|x|) < 0 < p(|x|)$. □

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33 *Supp $a_1, a_2, \cdots \in \mathbf{R}^n$ is Cauchy seq and $\exists$ subseq convg to L. Prove the seq convg to L.*

Solus: Supp subseq $a_{j_1}, a_{j_2}, \cdots$ convg to L. Then $\forall \varepsilon > 0, \underline{\exists\, j_m}, \forall j_k \in \{ j_m, j_{m+1}, \cdots \}$, $\|a_{j_k} - L\| < \varepsilon$,
and $\underline{\underline{\exists\, n \in \mathbf{Z}^+}}, \forall i \geqslant n$, $\|a_i - L\| = \big\| (a_i - a_{j_k}) + (a_{j_k} - L) \big\| \leqslant \|a_i - a_{j_k}\| + \|a_{j_k} - L\| < 2\varepsilon$. □

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34 *Supp closed $F_1 \subseteq \mathbf{R}^n$ and closed bounded $F_2 \subseteq \mathbf{R}^n$. Prove $F_1 + F_2$ is closed.*

Solus: Asum $\exists$ convg $a_1, a_2, \cdots \in F_1 + F_2$ with $\lim L \notin F_1 + F_2$. Let each $a_k = x_k + y_k$, and $z_k = L - x_k$.
$\forall \varepsilon > 0, \exists\, m \in \mathbf{Z}^+, \forall k \geqslant m$, $\|a_k - L\| = \|x_k + y_k - L\| = \|y_k - z_k\| < \varepsilon$.
Supp subseq $y_{j_1}, y_{j_2}, \cdots$ convg to $L_y \in F_2$. Then $\forall \varepsilon > 0, \exists\, j_m, \forall j_k \in \{ j_m, j_{m+1}, \cdots \}$,
$\|z_{j_k} - L_y\| = \big\| (z_{j_k} - y_{j_k}) + (y_{j_k} - L_y) \big\| < \varepsilon$. Now $x_{j_1}, x_{j_2}, \cdots \in F_1$ convg to $L - L_y \notin F_1$. □

Ended

1

A.1 *Supp bounded $f : [a,b] \to \mathbf{R}$ and P is a parti suth $L\big(f, P, [a,b]\big) = U\big(f, P, [a,b]\big)$. Show f is const.*

Solus: Note that $L\big(f, [a,b]\big) \geqslant L\big(f, P, [a,b]\big) = U\big(f, P, [a,b]\big) \geqslant U\big(f, [a,b]\big)$. □

Or. Supp $a < b$ and $P : a = x_0 < x_1 < \cdots < x_{k+1} = b$. We use induc on k.
(i) $k = 0 \Rightarrow P : a = x_0, x_1 = b$. Becs $L\big(f, P, [a,b]\big) = \inf f \leqslant f(x) \leqslant \sup f = U\big(f, P, [a,b]\big)$.
(ii) $k > 0$. Asum if $g : [c,d] \to \mathbf{R}$ with $c < d$, and $Q : c = y_0 < y_1 < \cdots < y_k = d$
suth $L\big(g, Q, [c,d]\big) = U\big(g, Q, [c,d]\big)$, then g is const.
Back to this f and this $P_1 \cup P_2 = P$ with $P_1 : a = x_0 < x_1 < \cdots < x_k$ and $P_2 : x_k < x_{k+1} = b$.
Let $A = [x_0, x_k]$, $B = [x_k, x_{k+1}]$. Then $f|_A$ and $f|_B$ are const by asum. Consider $f(x_k)$. □

A.2 *Supp $a \leqslant s < t \leqslant b$. Define $f : [a,b] \to \mathbf{R}$ by $f(x) = \begin{cases} 1, & s < x < t, \\ 0, & \text{othws.} \end{cases}$ Prove f is Rieman integ on $[a,b]$ and $\int_a^b f = t - s$.*

Solus: Consider $P_\varepsilon : a, s - \varepsilon, s + \varepsilon, t - \varepsilon, t + \varepsilon, b$ with $0 < \varepsilon < \min\{s - a, 2(t - s), b - t\}$.
Now $t - s - 2\varepsilon = L\big(f, P_\varepsilon, [a,b]\big)$, and $U\big(f, P_\varepsilon, [a,b]\big) = t - s + 2\varepsilon$.
Thus $t - s \leqslant L\big(f, [a,b]\big) \leqslant U\big(f, [a,b]\big) \leqslant t - s$. □

• **Tips:** Supp $I = [a,b]$ and $f : I \to \mathbf{R}$ is bounded and P, P' are parti. Let $P'' = P \cup P'$. Then

(a) Supp $g : I \to \mathbf{R}$ is bounded. Now $\inf_I f + \inf_I g \leqslant \inf_I (f+g) \leqslant \sup_I (f+g) \leqslant \sup_I f + \sup_I g$.

(1) $L(f,P) + L(g,P) \leqslant L(f+g,P) \leqslant L(f+g)$, 又 $L(f,P) + L(g,P') \leqslant L(f,P'') + L(g,P'')$.

(2) $U(f+g) \leqslant U(f+g,P) \leqslant U(f,P) + U(g,P)$, 又 $U(f,P'') + U(g,P'') \leqslant U(f,P) + U(g,P')$.

Hence $L(f) + L(g) \leqslant L(f+g) \leqslant U(f+g) \leqslant U(f) + U(g)$.

(b) $-L(f,P) = U(-f,P)$, $\inf(-A) = -\sup A$, 又 $-(-f) = f$ and $-(-A) = A$.

Let $U_f = \{U(f,P) = -L(-f,P) : P \text{ is a parti}\} = -L_{-f}$

$\Longrightarrow U(f) = \inf U_f = -\sup L_{-f} = -L(-f)$.

(c) $U(P) - L(P') \geqslant U(P'') - L(P'') \Longrightarrow U - L = \inf\{U(P) - L(P') : P, P'\} = \inf\{U(P) - L(P) : P\}$.

(d) $U([a,b]) \leqslant U(P \cup P', [a,b]) = U(P, [a,c]) + U(P', [c,b]) \geqslant U([a,c]) + U([c,b])$

$\Longrightarrow U([a,b]) = U([a,c]) + U([c,b])$. Simlr for $L([a,b])$.

• **New Nota:** $\Delta_f([a,b]) = U(f,[a,b]) - L(f,[a,b])$, $\Delta_f(P,[a,b]) = U(f,P,[a,b]) - L(f,P,[a,b])$.

A.3 *Supp bounded $f : [a,b] \to \mathbf{R}$.*

Prove f Rieman integ $\Longleftrightarrow \forall \varepsilon > 0, \exists$ parti P suth $U(f,P) - L(f,P) < \varepsilon$.

Solus: $(\Leftarrow)$ $0 \leqslant U - L \leqslant \inf\{U(P) - L(P') : P, P'\} \leqslant \inf\{U(P) - L(P) : P\} = 0$.

$(\Rightarrow)$ Immed by Tips (c). Or. $\forall \varepsilon > 0, \exists P_1, P_2$ suth $U(P_1) < U + \varepsilon,\ L - \varepsilon < L(P_2)$

$\Longrightarrow \forall \varepsilon > 0, \exists P = P_1 \cup P_2$ suth $U(P) - L(P) \leqslant U(P_1) - L(P_2) < 2\varepsilon$. □

A.12 *Supp Rieman integ $f : [a,b] \to \mathbf{R}$. Prove $|f|$ Rieman integ and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$*

Solus:

A.14 *Supp $f_1, f_2, \cdots$ Rieman integ seq and convg uniformly on $[a,b]$ to $f : [a,b] \to \mathbf{R}$.*

Prove f Rieman integ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Solus:

Solus:

Ended