



这是我个人挑战自学「*Measure, Integration & Real Analysis, by Sheldon Axler*」的学习笔记，包括课文补注和部分习题。我先从 *Supplement* 即第 0 章开始。我当时并没有学过数学分析，以为学完 Axler 的这个 *Supplement* 就能具备所有必要的知识基础。0.B 节是我碰到的第一个挫折。本来课文非常温顺，但习题做起来却让我感到知识的桀骜不驯。的确，它们不需要硬性知识门槛，可以用初中学过的不等式和高中学过的集合与量词来推导 \mathcal{D} 的一切。如你所见，字越少，事越大。

ABBREVIATION TABLE

A B

abs	absolute
add	addi(tion)(tive)
adj	adjoint
algo	algorithm
arb	arbitrary
assoc	associa(tive)(tivity)
asum	assum(e)(ption)
becs	because

E

-ec	-ec(t)(tor)(tion)(tive)
elem	element(s)
ent	entr(y)(ies)
equa	equality
equiv	equivalen(t)(ce)
exa	example
exe	exercise
exis	exist(s)(ing)
existns	existence
expo	exponent
expr	expression

L

liney	linear(ly)
linity	linearity
len	length
low-	lower-

R

recurly	recursively
repeti	repetition(s)
repres	represent(s)(ation(s))
req	require(s)(d)/requiring
respectly	respectively
restr	restrict(ion)(ive)(ing)
rev	revers(e(s))(ed)(ing)

C

clod	closed under
coeff	coefficient
combina	combination
commu	commut(es)(ing)(ativity)
cond	condition
contin	countinu(ous)(ity)
corres	correspond(s)(ing)
conveni	convenience
convly	conversely
countexa	counterexample
ctradic	contradict(s)(ion)
ctrapos	constrapositive

F G H

factoriz	factorizaion
fini	finite
finide	finite-dimensional
homo	homogeneity
hypo	hypothesis

M N

max	maxi(mal(ity))(mum)
min	mini(mal(ity))(mum)
multi	multipl(e)(icati-on/ve)
non0	nonzero
nonC	nonconst
notat	notation(al)

S

seq	sequence
simlr	similar(ly)
solus	solution
sp	space
stmt	statement
std	standard
supp	suppose
surj	surjectiv(e)(ity)
suth	such that

D

Ddkd	Dedekind
def	definition
deg	degree
deri	derivative(s)
diff	differentia(l)(ting)(tion)
dim	dimension(al)
disj	disjoint
disti	distinct
distr	distributive propert(ies)(ty)
div	div(ide)(ision)

I

id	identity
immed	immediately
induc	induct(ion)(ive)
infily	infinitely
inje	injectiv(e)(ity)
inv	inver(se)(tib-le/ility)
iso	isomorph(ism)(ic)

O P Q

othws	otherwise
orthog	orthogonal
orthon	orthonormal
poly	polynomial
posi	positive
prod	product
quad	quadratic
quotient	quot

T U V W X Y Z

uniq	unique
uniques	uniqueness
val	value
-wd	-ward
-ws	-wise
wrto	with respect to

0.B NOTE: C, D are Dedekind cuts. Numbers used here are always rational.

• Define $\tilde{q} = \{a : a < q\}$, and $-\tilde{q} = \widetilde{-q} = \{a : a < -q\}$.

Then $\tilde{0} = \{a : a < 0\} = \mathbf{Q} \setminus \mathbf{Q}^* \Rightarrow -\tilde{0} = \{a : a < -0 \leq 0\} = \tilde{0}$.

• Define $-D = \{a : a < -b, b \notin D\} = \{-a : -a < -b \Leftrightarrow a > b, b \notin D\}$.

$-(-D) = -\{a : a < -b, b \notin D\} = \{c : c < -a, a \geq -b, \forall b \notin D\} = \{c : c < b, \forall b \notin D\} = D$.

The last equa is becs (a) $d \notin D \Rightarrow \exists b \notin D, d \geq b$, and (b) $d \in D \Rightarrow$ if $\exists b \notin D$ suth $d \geq b$, then $b \in D$, ctradic.

• **TIPS:** Prove $\forall \varepsilon > 0, \exists b \notin D$ suth $b - \varepsilon \in D$.

SOLUS: Asum $\exists \varepsilon > 0$ suth $\nexists b \notin D, b - \varepsilon \in D \Leftrightarrow \forall b \notin D, b - \varepsilon \notin D$.

Then $(b - \varepsilon) - \dots - \varepsilon = b - n \cdot \varepsilon \notin D$ for any $n \in \mathbf{N}^+$.

Now $\forall d \in D, \exists n \in \mathbf{N}^+$ suth $b - n \cdot \varepsilon < d \Rightarrow b - n \cdot \varepsilon \in D$, ctradic. □

1 Prove (a) $D + \tilde{0} = D$, (b) $-D$ is Dedekind cut, and $D + (-D) = \tilde{0}$.

SOLUS: (a) $\forall d \in D, \exists \varepsilon > 0, d + \varepsilon \in D \Rightarrow (d + \varepsilon) + (-\varepsilon) \in D + \tilde{0}$.

(b) Asum $x \in -D$ is the largest elem of $-D \Rightarrow \exists b \notin D, x < -b \Rightarrow 0 < -b - x$.

Let $\delta = (-b - x)/2 \Rightarrow 0 < \delta < -b - x \Rightarrow x < x + \delta < -b$.

Thus by def, $x + \delta \in -D$, ctradic the max of $x \in -D$. Hence $-D$ is Ddkd cut.

$D + (-D) = \{x + y : x + y < x - b, x \in D, b \notin D\}$.

Supp $a \in \tilde{0} \Rightarrow -a > 0$. By TIPS, $\exists b \notin D$ suth $b + a \in D$.

Note that $b < b - a \notin D \Rightarrow -b > -b + a \in -D$. Then $(-b + a) + (b + a) = 2a < 0$.

Thus $\forall a \in \tilde{0}, \exists b \notin D, d = b + \frac{1}{2}a \in D, c = -b + \frac{1}{2}a \in -D \Rightarrow c + d = a \in D + (-D)$. □

3 Show $C \subsetneq D \Leftrightarrow D - C = \{d - y : d \in D, y > x, x \notin C\}$ posi.

SOLUS: (a) $C \subsetneq D \Rightarrow \exists x \in D \setminus C \Rightarrow \exists y \in D, y > x \Rightarrow \exists d \in D, d > y \Leftrightarrow 0 < d - y \in D - C$.

(b) $0 \in D - C \Rightarrow \exists y > x \notin C, y \in D \Rightarrow \forall c \in C, c < x < y \in D \Rightarrow C \subseteq D$. 又 $D - C \neq \tilde{0}$. □

5 Prove (a) D posi $\Rightarrow -D$ not posi, (b) non0 $-D$ not posi $\Rightarrow D$ posi.

SOLUS: (a) $0 \notin \{a : a < -b, b \notin D\} \Leftrightarrow \nexists b \notin D, 0 < -b \Leftrightarrow \forall b \notin D, b \geq 0 \Leftrightarrow 0 \in D$.

(b) Becs $\tilde{0}$ is the largest non posi cuts. Thus $-D \neq \tilde{0} \Rightarrow -D \subsetneq \tilde{0} \Rightarrow \tilde{0} - (-D) = D$ posi.

OR. $\exists a < 0, a \notin -D = \{a : -a > b, b \notin D\} \Leftrightarrow \nexists b \notin D, -a > b \Leftrightarrow \forall b \notin D, 0 < -a \leq b$. □

• Define $D^+ = \{d \in D : d > 0\} = D \cap \mathbf{Q}^+$. Then $D^+ \neq \emptyset \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subsetneq D \Leftrightarrow 0 \in D \Leftrightarrow D$ posi.

Define $D^- = \{r \notin D : r \leq 0\} = (\mathbf{Q} \setminus D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \mathbf{Q} \setminus (D \cup \mathbf{Q}^+)$.

(a) $D^- = \{0\} \Leftrightarrow D = \tilde{0}$. Convly, $\{r \notin D : r \leq 0\} = \{0\} \Rightarrow \mathbf{Q} \setminus D = \mathbf{Q}^*$.

(b) $D^- = \emptyset \Leftrightarrow D \cup \mathbf{Q}^+ = \mathbf{Q} \Leftrightarrow \mathbf{Q} \setminus \mathbf{Q}^+ \subseteq D \Leftrightarrow 0 \in D \Leftrightarrow D$ posi. **CORO:** D not posi $\Leftrightarrow 0 \in D^-$.

(c) $(D^-)^- = \{r \in D : r \leq 0\} = \mathbf{Q} \setminus D^+$. **CORO:** D not posi $\Leftrightarrow (D^-)^- = D$.

• $(-D)^+ = (-D) \cap \mathbf{Q}^+ = \{a : 0 < a < -b, b \notin D \Leftrightarrow b \in D^- \setminus \{0\}\}$.

$(-D)^- = (\mathbf{Q} \setminus -D) \cap (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : 0 \geq a \geq -b, \forall b \notin D\}$.

• For C, D posi, define $CD = \{a : a \leq cd, c \in C^+, d \in D^+\} = \{cd : c \in C^+, d \in D^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.

$\{cd : c \in C^+, d \in D^+\} = CD \cap \mathbf{Q}^+ = (CD)^+$. Note that ' $a \leq cd$ ' here is equiv to ' $a < cd$ '.

- For $-C, -D$ posi, define $CD = (-C)(-D) = \{cd : c \in (-C)^+, d \in (-D)^+\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+)$.
 $CD = \{0 < cd < (-r)(-s) : r \in C^- \setminus \{0\}, s \in D^- \setminus \{0\}\} \cup (\mathbf{Q} \setminus \mathbf{Q}^+) = \{a : a < rs, r \in C^-, s \in D^-\}$.
 If $C, -C$ not posi $\Rightarrow C = \tilde{0}$, then with the asum $\tilde{0}D = \tilde{0}$, it still holds. Simlrr for D .

- For D posi, define $D^{-1} = \{a : a < 1/b, b \notin D\} \Rightarrow DD^{-1} = \{a : a \leq d/b < 1, b \notin D, d \in D^+\} = \tilde{1}$.
 The last equa holds becs $\forall a \in \tilde{1} \cap \mathbf{Q}^+, \exists d \in D^+$ suth $b = d/a \notin D \Rightarrow d/b = a \in DD^{-1}$.
- For non0 D not posi, define $D^{-1} = -(-D)^{-1} = -\{a : a < 1/b, \exists b \notin -D \Leftrightarrow \exists b \geq -s, \forall s \notin D\}$
 $= \{a : a < -x, \exists x \geq 1/b, \forall b$ suth $b \geq -s, \forall s \notin D\} = \{a : a < -1/b, \forall b$ suth $b \geq -s, \forall s \notin D\}$
 $= \{a : a < 1/b, \forall b$ suth $b \leq s, \forall s \notin D\} \neq \{a : a < 1/s, \forall s \notin D\}$.
 Let $b_1 < \dots < b_m < \dots \leq s, \forall s \notin D \Leftrightarrow \forall s \in D^- \Rightarrow$ each $s_j, b_k < 0 \Rightarrow 1/s_j \leq \dots < 1/b_m < \dots < 1/b_1$.
 Thus ' $a < 1/b$ ' is equiv to ' $a < 1/s, \exists s \in D^-$ '. Hence $D^{-1} = \{a : a < 1/b, b \in D^-\}$.
 $DD^{-1} = \{a : a < rs, \exists r \in D^-, \exists s$ suth $0 \geq s \geq 1/b, \forall b \in D^-\} \subseteq \tilde{1}$.
 Asum $\exists x$ suth $rs \leq x < 1, \forall r, s$. Let $D \not\supseteq \dots \leq b_m < \dots < b_1 \leq 0$, and $D \not\supseteq \dots \leq r_m < \dots < r_1 \leq 0$.
 $1/b_1 < \dots < 1/b_m \leq \dots \leq \dots \leq s_n < \dots < s_1 \leq 0$, and $r_m/b_m \geq \dots \geq \dots \geq r_m s_n > \dots > r_j s_k > \dots > r_1 s_1$.
 Let $r_m = b_m$. Ctradic. OR. $DD^{-1} = D[(-\tilde{1})(-D)^{-1}] = [D(-\tilde{1})](-D)^{-1} = (-D)(-D)^{-1} = \tilde{1}$.

- For C not posi and D posi, we expect that CD not posi. Consider C and $-D$ both not posi.
 $CD = -[C(-D)] = -\{a : a < rt, r \in C^-, t \in (-D)^-\} = \{-a : a > b, b \geq rt, \forall r \in C^-, t \in (-D)^-\}$
 $= \{-a : a > rt, \forall r \in C^-, \forall t$ suth $0 \geq t \geq -s, \forall s \notin D\}$
 $= \{a : a < ru, \forall r \in C^-, \forall u$ suth $0 \leq u \leq s, \forall s \notin D\}$. ($r \leq 0 < s, rs \leq ru = -rt \leq 0 \leq rt \leq -rs$.)
- Note the ' $0 \leq u$ '. Becs $C^- \neq \emptyset \Rightarrow 0 \in C^-$. If it is to be exactly $CD = \{a : a < 0\}$, then $C^- = \{0\}$,
 for if not, $\exists u > 0$, and $\exists r \in C^- \setminus \{0\}$, suth $\exists a < ru < 0$. Hence ' $0 \leq u$ ' is actually ' $0 < u$ '.
- ' $u \leq s$ ' cannot be abbreviated as in $\{-a : a > -rs, \forall s \notin D, r \in C^-\} = \{a : a < rs, \forall s \notin D, r \in C^-\}$.
 ' $u \leq s$ ' cannot be ' $u < s$ ', becs here $rs < ru \Rightarrow \exists a = rs$.
- Note that $\{u : 0 < u \leq s, \forall s \notin D\} = \begin{cases} D^+ \cup \{\min \mathbf{Q} \setminus D\}, & \text{if it exists,} \\ D^+, & \text{othws.} \end{cases}$ Denote it by $D^\oplus = D^\otimes \setminus \{0\}$.

- For C not posi and D posi. If $C = \tilde{0}$, then $CD = -[C(-D)] = -\tilde{0}$. Now consider $-C$ and D both posi.
 But $CD = -(-C)D = -\{a : a \leq cd, c \in (-C)^+, d \in D^+\} \neq \{a : a < -cd, \forall c \in (-C)^+, d \in D^+\}$.
 Altho " $a \leq cd$ " is equiv to " $a < cd$ " so that $b \notin (-C)D \Rightarrow b \geq cd$, which is actually $b > cd, \forall c, d$.
 And $a < -b < -cd, \forall c, d \Rightarrow \forall a, \exists x$ suth $a < x < -cd, \forall c, d$. While a can be the 'boundary' in RHS.

- $LHS = \{a : a < ru, \forall r \in C^-, \forall u \in D^\oplus\}, \{cs : c \in C, s \notin D\} = RHS$.
 Becs $cs \leq cu < ru$. We show $LHS \subseteq RHS$. Let $c_1 < \dots < c_n < \dots \in C$, and $s_1 > \dots > s_m \geq \dots \notin D$.
 Then $c_1 s_1 < \dots < c_n s_m < \dots < ru, \forall r, u$ as in LHS. Thus $a \in LHS \Rightarrow \exists a < c_j s_k$. □
 OR. Note that in LHS, ' $a < ru$ ' is equiv to ' $a < rs, \exists s \notin D$.' Now $LHS = \{a : a/s \in C, \exists s \notin D\}$. □

- For C posi and D not posi. If $D = \tilde{0}$, then $CD = -[(-C)D] = -\tilde{0}$.
 $CD = (-C)(-D) = \{a : a < ru, \forall r \in (-C)^-, \forall u \in (-D)^\oplus\} \quad \mathbf{Q} \setminus -D = \{s : s \geq -y, \forall y \notin D\}$
 $= \{a : a < ru, \forall r$ suth $\forall x \notin C, 0 \geq r \geq -x, \forall u$ suth $0 < u \leq s, \forall s \geq -y, \forall y \notin D\}$
 $= \{a : a < (-r)(-u), \forall r$ suth $\forall x \notin C, 0 \leq -r \leq x, \forall u$ suth $y \leq -u < 0, \forall y \notin D\}$
 $= \{a : a < ru, \forall r$ suth $\forall x \notin C, 0 \leq r \leq x, \forall u \in D^-\} = \{a : a < ru, \forall r \in C^\oplus, \forall u \in D^-\}$, simlrr.

- We show $-D = \{a : a < -b, b \notin D\} = (-\tilde{1})D$.

For D posi, $RHS = \{a : a < ru, \forall r \text{ suth } -1 \leq r \leq 0, \forall u \in D^\oplus\} = \{a : a < -u, \forall u \in D^\oplus\} \supseteq -D$.

Supp x suth $-b \leq x < -u, \forall b \notin D, \forall u \text{ suth } -s \leq -u < 0, \forall s \notin D \Rightarrow -s \leq x < -u$. Let $-u = x$.

For D not posi, $RHS = \{a : a < rb, \exists r \text{ suth } -1 \leq r \leq 0, b \in D^-\} = \{a : a < -b, 0 \geq b \notin D\} = -D$.

- We show $\tilde{1}D = D$. For D not posi, immed. Othws, $\tilde{1}D = \{a : a \leq ij < j, 0 < i < 1, j \in D^+\} \subseteq D$.

Now $(\tilde{1}D)^+ \subseteq D^+$. 又 $\forall d \in D^+, \exists \varepsilon > 0, d + \varepsilon \in D^+ \Rightarrow d = (d + \varepsilon) \frac{d}{d + \varepsilon} \in (\tilde{1}D)^+$.

4 Supp B, C, D non0 Dedekind cuts. Show $(BC)D = B(CD), B(C + D) = BC + BD$.

SOLUS: We discuss in cases.

\backslash	1	2	3	4	5	6
B	+	+	+	-	-	-
C	+	+	-	-	+	+
D	+	-	-	-	-	+

$$(1) [(BC)D]^+ = \{(bc)d : bc \in (BC)^+, d \in D^+\} = \{b(cd) : b \in B^+, cd \in (CD)^+\} = [B(CD)]^+.$$

$$B(C + D) = \{a : a \leq bc + bd, \exists b \in B^+, 0 < c + d \in C + D\}.$$

$$BC + BD = \{x : x \leq uc, u \in B^+, c \in C^+\} + \{y : y \leq vd, v \in B^+, d \in D^+\} \\ = \{a : a \leq uc + vd, \exists u, v \in B^+, c \in C^+, d \in D^+\}. \text{ Done.}$$

$$(3) (BC)^- = \{r : 0 \geq r \geq bc, \exists c \in C^-, \exists b \in B^\oplus\}.$$

$$(BC)D = \{a : a < rs, r \in (BC)^-, s \in D^-\} \\ = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^\oplus\}.$$

$$B(CD) = \{a : a \leq bx < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+, \text{ and } cs > x \in (CD)^+\} \\ = \{a : a < bcs, \exists s \in D^-, c \in C^-, \exists b \in B^+\}.$$

Note that $\{q : q < b, b \in B^+\} = \{q : q < b, \exists b \in B^\oplus\}$. Done.

$$B(C + D) = \{a : a < ru, \forall r \text{ suth } \forall x \notin B, 0 < r \leq x, \forall u \text{ suth } 0 \geq u > c + d, \forall c \in C, d \in D\} \\ = \{a : a < ru, \forall r \in B^\oplus, \forall u \text{ suth } 0 \geq u \geq c + d, \exists c \in C^-, d \in D^-\} \\ = \{a : a < r(c + d), \forall r \in B^\oplus, \forall c \in C^-, d \in D^-\}.$$

$$BC + BD = \{x : x < pc, \forall p \in B^\oplus, \forall c \in C^-\} + \{y : y < qd, \forall q \in B^\oplus, \forall d \in D^-\} \\ = \{a : a < pc + qd, \forall p, q \in B^\oplus, \forall c \in C^-, d \in D^-\}. \text{ Done immed.}$$

$$(5) (BC)^- = \{r : 0 \geq r \geq bc, \exists b \in B^-, \exists c \in C^\oplus\}. (CD)^- = \{r : 0 \geq r \geq cd, \exists d \in D^-, \exists c \in C^\oplus\}.$$

$$(BC)D = \{a : a < rd \leq bcd, \exists d \in D^-, b \in B^-, \exists c \in C^\oplus\}.$$

$$B(CD) = \{a : a < br \leq bcd, \exists b \in B^-, d \in D^-, \exists c \in C^\oplus\}. \text{ Done.}$$

OR. By commu and (3), $(BC)D = (CB)D = C(BD) = C(DB) = (CD)B = B(CD)$.

$$BC + BD = \{x : x < bc, \forall b \in B^-, \forall c \in C^\oplus\} + \{y : y < bd, \exists b \in B^-, d \in D^-\} \\ = \{a : a < pc + qd, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \exists q \in B^-, d \in D^-, \Rightarrow q \geq p\} \\ = \{a : a < pc + y, \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall y \text{ suth } y \geq qd, \forall q \in B^-, d \in D^-\} \\ = \{a : a < p(c + d'), \forall p \in B^-, \forall c \in C^\oplus, \text{ and } \forall d' \text{ suth } d' \leq d, \forall d \in D^-\}.$$

$$(I) \text{ If } C + D \text{ not posi. Then } B(C + D) = \{a : a < bx, \exists b \in B^-, 0 \geq x > c + d, \forall (c, d) \in C \times D\}.$$

Rewrite as $\{a : a < t, \forall t \text{ suth } t \geq bx, \forall b \in B^-, x \in (C + D)^-\}$. Done.

$$(II) \text{ If } C + D \text{ posi. Then } B(C + D) = \{a : a < bx, \forall b \in B^-, x \in (C + D)^\oplus\}.$$

If $(C + D)^\oplus = (C + D)^+$. Then $B(C + D) = \{a : a < bc + bd, \forall b \in B^-, \forall (c, d) \in C^\oplus \times D^\oplus\}$.

Othws, $\exists s = \min \mathbf{Q} \setminus (C + D) \Rightarrow B(C + D) = \{a : a < bs, \forall b \in B^-\}$. Done.

$$\begin{aligned}
(4) \quad (BC)D &= \{xd : d \in D, x \geq bc, \forall b \in B^-, c \in C^-\} \\
&= \{a : a < bcd, \forall d \in D^-, \forall b \in B^-, c \in C^-\} \\
&= \{by : b \in B, y \geq cd, \forall c \in C^-, d \in D^-\} = B(CD). \\
B(C + D) &= \{a : a < t, \forall t \text{ suth } t \geq b(c + d), \forall b \in B^-, (c, d) \in C^- \times D^-\} \\
&= \{a : a < t_1, \forall t_1 \text{ suth } t_1 \geq bc, \forall (b, c) \in B^- \times C^-\} \\
&\quad + \{a : a < t_2, \forall t_2 \text{ suth } t_2 \geq bd, \forall (b, d) \in B^- \times D^-\} = BC + BD. \text{ Done.}
\end{aligned}$$

NOTE: Supp for any B posi, C posi, D not posi, assoc holds.

Supp B posi, C not posi, D posi. Then $(\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{B}\bar{C}) = (\bar{D}\bar{B})\bar{C} = \bar{B}(\bar{D}\bar{C})$. Convly true.

Simlr for the case B not posi, C posi, D not posi, equiv to the case B not posi, C not posi, D posi.

(2) holds $\Rightarrow (\bar{B}\bar{C})\bar{D} = \bar{D}(\bar{B}\bar{C}) = (\bar{C}\bar{D})\bar{B}$, (6) holds. Convly true. Simlr to (3) with (5) in assoc.

(5) holds $\Rightarrow \bar{B}(\bar{C} + \bar{D}) = (-B)[- (\bar{C} + \bar{D})] \stackrel{(5)}{=} (-B)[(-C) + (-D)] \stackrel{(5)}{=} \bar{B}\bar{C} + \bar{B}\bar{D}$, by def of multi.

Thus (5) \Rightarrow (2) in distr. Convly as well.

$$\begin{aligned}
(6) \quad (BC)^- &= \{r : 0 \geq r \geq bc, \exists b \in B^-, c \in C^\oplus\}. \\
(BC)D &= \{a : a < ru, \forall r \in (BC)^-, u \in D^\oplus\} = \{a : a < bcd, \forall b \in B^-, c \in C^\oplus, d \in D^\oplus\}. \\
(CD)^\oplus &= \{u : 0 < u \leq s, \forall s > cd, \forall c \in C^+, d \in D^+\}. \\
B(CD) &= \{a : a < ru, \forall r \in B^-, u \in (CD)^\oplus\}. \text{ Done.} \\
B(C + D) &= \{a : a < b(c + d), \forall b \in B^-, \forall (c, d) \in C^\oplus \times D^\oplus\}. \\
BC + BD &= \{a : a < bc, \forall b \in B^-, c \in C^\oplus\} + \{a : a < bd, \forall b \in B^-, d \in D^\oplus\}. \text{ Done.}
\end{aligned}$$

OR. (2) instead of (6) ? NOTICE that the distr in (6) cannot be shown without (6). □

ENDED

0.C

2 Supp nonempty $U \subseteq V \subseteq \mathbf{R}$. Show $\sup U \leq \sup V$.

SOLUS: Asum $\sup U > \sup V \Rightarrow \exists t \in U \cap (\sup V, \sup U] \Rightarrow \sup V < t \in V$, ctrad. □

5 Supp $a_1, a_2, \dots \in \mathbf{Q}$, and $\sup\{a_1, a_2, \dots\} = \sqrt{2}$.

Prove $\sup\{a_n, a_{n+1}, \dots\} = \sqrt{2}$ for all $n \in \mathbf{N}^+$.

SOLUS: Becs the sup not in seq \Rightarrow infily many disti elem.

$\forall a_i, \exists a_j, a_i < a_j < \sqrt{2}$. For a_{n+k} , choose $a_i > a_{n+k}$. If $i \in \{1, \dots, n\}$, then choose $a_j > a_i$.

After at most n steps, we must have a_m with $m > n$. Thus $\forall a_{n+i}, \exists a_{n+j}, a_{n+i} < a_{n+j} < \sqrt{2}$. □

• Supp nonempty $A \subseteq \mathbf{R}$.

• **TIPS 1:** Define $-A = \{-a : a \in A\} \Rightarrow -(-A) = A$. Prove $\sup(-A) = -\inf A$.

SOLUS: $-b$ is an upper bound of $-A \Leftrightarrow \forall a \in A, -a \leq -b \Leftrightarrow a \geq b \Leftrightarrow b$ is a lower bound of A .

Thus $-b_M = \sup(-A) \Leftrightarrow -b_M \leq -b \Leftrightarrow b_M \geq b \Leftrightarrow b_M = \inf A$. □

• **TIPS 2:** Show if $x \in \mathbf{R}$, (a) $\sup A > x \Rightarrow \exists a \in A, a > x$, (b) $\inf A < x \Rightarrow \exists a \in A, a < x$.

SOLUS: (a) $\nexists a > x \Leftrightarrow \forall a \in A, a \leq x \Rightarrow \sup A \leq x$.

OR. By (b), $\inf(-A) = -\sup A < -x \Rightarrow \exists -a \in A, -a < -x$.

Simlr for (b). □

6 Supp nonempty $A, B \subseteq \mathbf{R}$. Prove $\sup(A + B) = \sup A + \sup B$, and $\inf(A + B) = \inf A + \inf B$.

SOLUS: $\inf A + \inf B \leq a + b \leq \sup A + \sup B \Rightarrow \sup(A + B) \leq \sup A + \sup B$, $\inf A + \inf B \leq \inf(A + B)$.

$$\sup A + \sup B > \sup(A + B) \iff \sup A > \sup(A + B) - \sup B$$

$$\iff \exists a + \sup B > \sup(A + B) \iff \sup B > \sup(A + B) - a \iff \exists a + b > \sup(A + B). \text{ Ctradic.}$$

Simlr for $\inf(A + B) \in A + B$. OR. Apply to $-A - B$, becs $\sup(-A) = -\inf A$. \square

16 Supp $\mathbf{R}_1, \mathbf{R}_2$ are complete ordered fields. Define φ_1, φ_2 as in [0.11].

Define $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$ by $\psi(a) = \sup\{\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq a\}$. Show

(a) $\psi : \mathbf{R}_1 \rightarrow \mathbf{R}_2$ is one-to-one. (b) $\psi(0) = 0, \psi(1) = 1$.

(c) $\psi(a + b) = \psi(a) + \psi(b)$. (d) $\psi(ab) = \psi(a)\psi(b)$.

SOLUS: (a) Define $\mathcal{R}_1(a) = \{q \in \mathbf{Q} : \varphi_1(q) \leq a\} \Rightarrow \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Let $\psi_1(a) = \sup_{\mathcal{R}_1(a)} \varphi_2 = \psi(a)$.

Define $\mathcal{R}_2(c) = \{q \in \mathbf{Q} : \varphi_2(q) \leq c\} \Rightarrow \sup_{\mathcal{R}_2(c)} \varphi_2 = c$.

Define $\psi_2(c) = \sup_{\mathcal{R}_2(c)} \varphi_1$. Then $\psi_2 : \mathbf{R}_2 \rightarrow \mathbf{R}_1$ well-defined.

Note that $\mathcal{R}_2(\psi_1(a)) = \{q \in \mathbf{Q} : \varphi_2(q) \leq \sup_{\mathcal{R}_1(a)} \varphi_2\} = \mathcal{R}_1(a)$.

Now $\psi_2(\psi_1(a)) = \sup_{\mathcal{R}_1(a)} \varphi_1 = a$. Rev the roles of $\mathbf{R}_1, \mathbf{R}_2$.

(b) Becs $\varphi_1(q) \leq \varphi_1(0) \iff q \leq 0 \iff \varphi_2(q) \leq \varphi_2(0)$. Simlr for $\psi(1) = 1$.

(c) $S = \{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq a + b\} \supseteq \{\varphi_2(p + q) : p, q \in \mathbf{Q}, \varphi_1(p) \leq a, \varphi_1(q) \leq b\} = T$
 $\Rightarrow \sup S \geq \sup T$. Asum it is ' $>$ '. Now $\exists t \in \mathbf{Q}$ suth $\sup S \geq \varphi_2(t) > \sup T = \psi(a) + \psi(b)$.

Which means $\varphi_2(t) > \varphi_2(p + q), \forall p, q \in \mathbf{Q}$ suth $\varphi_1(p) \leq a$ and $\varphi_1(q) \leq b$.

Now $a + b \geq \varphi_1(t) > \varphi_1[(t + p + q)/2] > \varphi_1(p + q), \forall p, q$. Ctradic.

(d) We show it for (I) $a, b > 0$, (II) $a > 0 > b$.

$LHS = \sup\{\varphi_2(t) : t \in \mathbf{Q}, \varphi_1(t) \leq ab\}, \sup\{\psi(a)\varphi_2(q) : q \in \mathbf{Q}, \varphi_1(q) \leq b\} = RHS$.

(I) $RHS = \sup\{\varphi_2(p)\varphi_2(q) : p, q \in \mathbf{Q}, 0 < \varphi_1(p) \leq a, \varphi_1(q) \leq b \implies \varphi_1(pq) \leq ab\}$.

(II) $RHS = \sup\{\varphi_2(s)\varphi_2(q) : s, q \in \mathbf{Q}, \varphi_1(q) \leq b, \text{ and } s \geq p, \forall p \text{ suth } 0 < \varphi_1(p) \leq a\}$.

Note that $\varphi_1(s) \geq \varphi_1(p), \forall p \Rightarrow \varphi_1(s) \geq a$, for if not, $\exists p' \in \mathbf{Q}$ suth $\varphi_1(s) < \varphi_1(p') \leq a$.

So that $\varphi_1(sq) \leq a\varphi_1(q) \leq ab$.

Now $LHS \geq RHS$. Asum it is ' $>$ '. Then $\exists t \in \mathbf{Q}$ suth $LHS \geq \varphi_2(t) > RHS$.

Which means $\varphi_2(t) > \psi(a)\varphi_2(q), \forall q \in \mathbf{Q}$ suth $\varphi_1(q) \leq b$.

(I) $\implies \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q$ and $\forall s \in \mathbf{Q}$ suth $0 < \varphi_1(s) \leq a$.

(II) $\implies \varphi_2(t) > \varphi_2(s)\varphi_2(q), \forall q$ and $\forall s \in \mathbf{Q}$ suth $\varphi_2(s) \geq \varphi_2(p), \forall p$ suth $0 < \varphi_1(p) \leq a$.

Thus $ab \geq \varphi_1(t) > \varphi_1[(t + sq)/2] > \varphi_1(sq), \forall s, q$. Ctradic. \square

ENDED

0.D

15 Supp F is a closed subset of \mathbf{R} . Prove $S = \{a^2 : a \in F\}$ is closed.

SOLUS: Let $\mathbf{R} \setminus F = \bigcup_{k=1}^{\infty} I_k$ (disj)

17 Supp $F \subseteq \mathbf{R}$, and $\forall n \in \mathbf{N}^+$, $F \cap [-n, n]$ is closed. Prove F is closed.

SOLUS: Becs $G = (\mathbf{R} \setminus F) \cup (-\infty, -n) \cup (n, \infty)$ is open for all $n \in \mathbf{N}^+$. Let $n > \sup\{|a| : a \in \mathbf{R} \setminus F\}$.

By [0.59], $G = (-\infty, -n) \cup (n, \infty) \cup I_1 \cup I_2 \cup \dots$ (disj) $\Rightarrow I_1 \cup I_2 \cup \dots = \mathbf{R} \setminus F$. □

20 Prove $\forall b \in \mathbf{R}^n$, $\delta > 0$, $\{a \in \mathbf{R}^n : \|a - b\|_{\infty} > \delta\}$, $\{a \in \mathbf{R}^n : \|a - b\| > \delta\}$ are open.

SOLUS: Asum $\exists b \in \mathbf{R}^n, \delta > 0$ suth $\{a \in \mathbf{R}^n : \|a - b\|_{\infty} \leq \delta\}$ is not closed.

21 Supp X is an open subset of \mathbf{R} . Prove $\exists a_k, b_k \in \mathbf{R}$ suth $X = \bigcup_{k=1}^{\infty} [a_k, b_k]$.

SOLUS: Let $X = \bigcup_{k=1}^{\infty} (c_k, d_k)$, where each $d_k < c_{k+1}$. Let $I_k = (c_k, d_k)$.

Let $a_{1,k}, a_{2,k}, \dots$ be convg of $\lim c_k$. Simlr, let $b_{1,k}, b_{2,k}, \dots$ of $\lim d_k$. Let $E_{j,k} = [a_{j,k}, b_{j,k}]$.

Now $\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{j,k} = \bigcup_{k=1}^{\infty} I_k = X$. Rearrange the order of $E_{j,k}$'s. □

23 Supp F_1, F_2 are disj closed subsets of \mathbf{R} suth $U = F_1 \cup F_2$ is interval. Prove F_1 or F_2 open.

SOLUS: If U open, then $F_1 = \emptyset$, $F_2 = \emptyset$ or \mathbf{R} . Now supp U not open $\Rightarrow F_1$ or F_2 not open.

We show F_1 open $\Leftrightarrow F_2$ not open. Note that F_1 open $\Leftrightarrow F_1 = \emptyset$. Simlr for F_2 .

WLOG, asum F_1, F_2 not open, and $x \in F_1, y \in F_2$ with $x < y \Rightarrow x, y \in U \supseteq [x, y]$.

NOTICE that $T = [x, y] \cap F_1$ is a closed subset that has infily many elem. $\nexists \sup T < y$.

(I) If $\sup T \notin F_1$. Then \exists a convg seq in T with $\lim \sup T$.

(II) Othws, $\forall t \in (\sup T, y], t \notin F_1 \Leftrightarrow t \in F_2$. Now \exists a convg seq in F_2 with $\lim \sup T \notin F_2$.

Ctradic the asum \Rightarrow at least one of F_1, F_2 is empty. □

25 Give an exa of inv $f : \mathbf{R} \rightarrow \mathbf{R} \setminus \mathbf{Q}$.

SOLUS: Consider a seq disti $a_1, b_1, a_2, b_2, \dots \in \mathbf{R}$ where each $a_j \in \mathbf{R} \setminus \mathbf{Q}$ and $\{b_1, b_2, \dots\} = \mathbf{Q}$.

Define $f(a_j) = b_{2j+1}$, $f(b_k) = b_{2k} \Rightarrow f^{-1}(b_i) = \begin{cases} a_{(i-1)/2}, & \text{if } i \text{ is odd,} \\ b_{i/2}, & \text{if } i \text{ is even.} \end{cases}$ □

NOTE: \nexists a seq of all irrational numbers, but whenever use f , the input in $\mathbf{R} \setminus \mathbf{Q}$ are always countable.

26 Supp $E, G \subseteq \mathbf{R}^n$, and G is open. Prove $E + G = \{x + y : x \in E, y \in G\}$ is open.

SOLUS: Asum $E + G$ not open $\Leftrightarrow \mathbf{R}^n \setminus (E + G)$ not closed.

Then $\exists a = x + y \in E + G$ suth $\forall \delta > 0, \exists b \notin E + G$ suth $\|a - b\|_{\infty} < \delta$.

Let $z = b - x \notin G \Rightarrow \|y - z\|_{\infty} < \delta \Rightarrow z \in B(y, \delta) \subseteq G, \exists \delta > 0$. □

OR. $\exists a_1, a_2, \dots \notin E + G$ while its $\lim L = e + g \in E + G$. $\nexists \forall x \in E, a_k - x \notin G$

$\Rightarrow \lim_{k \rightarrow \infty} (a_k - e) = L - e = g \in G$. Thus $\mathbf{R}^n \setminus G$ not closed $\Leftrightarrow G$ not open. □

ENDED

0.E

• Supp $b \in A \subseteq \mathbf{R}^m, f : A \rightarrow \mathbf{R}^n$. Prove

$[P]$ f is continu at $b \iff \forall b_1, b_2, \dots \in A$ suth $\lim_{k \rightarrow \infty} b_k = b, \lim_{k \rightarrow \infty} f(b_k) = f(b)$. $[Q]$

SOLUS: $Q \Rightarrow P$: Supp $\varepsilon > 0$ suth $\forall \delta > 0, \exists a \in A$ with $\|a - b\|_\infty < \delta$ and $\|f(a) - f(b)\|_\infty \geq \varepsilon$.

Fix a δ . Define $\delta_k = \delta/k \Rightarrow \exists a_k$ for each. Now $\lim_{k \rightarrow \infty} a_k = b$.

Thus $\forall m, \forall k \geq m, \|f(a_k) - f(b)\|_\infty \geq \varepsilon$. Ctradic Q .

$P \Rightarrow Q$: Supp $b_1, b_2, \dots \in A$ suth $\forall \delta > 0, \exists m, \forall k \geq m, \|b_k - b\|_\infty < \delta$.

Asum $\varepsilon > 0$ suth $\forall m, \forall k \geq m, \|f(b_k) - f(b)\|_\infty \geq \varepsilon$. Ctradic P . □

ENDED