Newton's method in infinite dimensions and Galerkin approximations

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Introduction

Newton's method in $\mathbb{R}$

Consider the root finding problem $F(x) = 0$ where $F \in C^1(\mathbb{R};\mathbb{R})$.

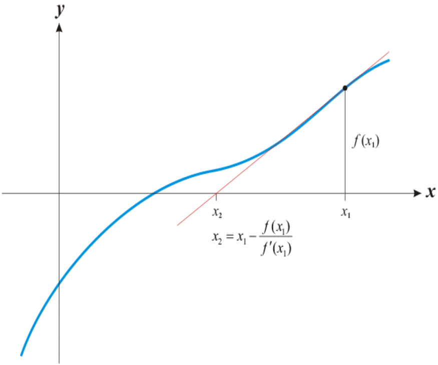

Figure: Image from [3]

Newton's method in $\mathbb{R}$

Let x^* be the root of F. By Taylor's theorem,

$$\begin{aligned} 0 = F(x^*) &= F(x^k + (x^* - x^k)) \\ &= F(x^k) + F'(x^k)(x^* - x^k) + o(x^* - x^k) \end{aligned}$$

Find the root δx^k of the local linear approximation :

$$F'(x^k)\delta x^k + F(x^k) = 0.$$

Then δx^k approximates $x^* - x^k$, giving the update [2] [3]

$$x^{k+1} = x^k + \delta x^k.$$

Newton's method in $\mathbb{R}^n$

Consider $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. The local linear approximation is given by [2]

$$F(x^k + \delta x^k) = F(x^k) + F'(x^k)\delta x^k + o(\|\delta x^k\|).$$

Then we solve for δx^k in

$$F'(x^k)\delta x^k + F(x^k) = 0$$

and obtain the update

$$x^{k+1} = x^k + \delta x^k.$$

Question

- Let V, W be Banach spaces and $F : V \to W$ be an nonlinear operator. We are interested in the problem:

  $$\text{Find a function } u \in V \text{ s.t. } F(u) = 0.$$

  Can we apply Newton's method to this problem in infinite dimensional spaces? [1]

Question

- Let V, W be Banach spaces and $F : V \to W$ be an nonlinear operator. We are interested in the problem:
$$\text{Find a function } u \in V \text{ s.t. } F(u) = 0.$$
Can we apply Newton's method to this problem in infinite dimensional spaces? [1]
- Answer: Yes!

Newton's method in infinite dimensions

Definition (Fréchet differentiability [2])

Let V, W be normed vector spaces. $F : V \to W$ is **Fréchet differentiable** at u if there exists a bounded linear operator $F'(u) : V \to W$ that satisfies

$$\lim_{v \to 0} \frac{\|F(u+v) - F(u) - F'(u)(v)\|_W}{\|v\|_V} = 0 \quad \text{for all } v \in V.$$

Newton's method in infinite dimensions

Suppose $F : V \to W$ is Fréchet differentiable in V. Then

$$F(u^k + \delta u^k) = F(u^k) + F'(u^k)(\delta u^k) + o(\|\delta u^k\|).$$

Now we have the problem: find $\delta u^k \in V$ s.t.

$$F'(u^k)(\delta u^k) + F(u^k) = 0.$$

Then we have the Newton update

$$u^{k+1} = u^k + \delta u^k.$$

Applications to PDEs

Abstract weak form

Let $G : V \times V \to \mathbb{R}$ be **nonlinear** in $u \in V$ and **linear** in v. We have the following abstract nonlinear weak form of a PDE: Find $u \in V$ s.t.

$$G(u; v) = 0 \quad \forall\, v \in V. \tag{1}$$

Define $F : V \to V^*, F(u)(v) = G(u; v)$. Then (1) is equivalent to

$$F(u) = 0$$

where 0 is the zero functional on V.

Newton's method for PDEs

Recall the linearized equation

$$F'(u^k)(\delta u^k) + F(u^k) = 0.$$

This is equivalent to

$$F'(u^k)(\delta u^k)(v) + F(u^k)(v) = 0 \quad \forall\, v \in V.$$

Newton's method for PDEs

Then we have the following Newton's iterations for a nonlinear PDE: Find $\delta u^k \in V$ s.t.

$$F'(u^k)(\delta u^k)(v) + F(u^k)(v) = 0 \quad \forall\, v \in V$$

and then update

$$u^{k+1} = u^k + \delta u^k.$$

Galerkin approximations

Galerkin approximations

A conforming Galerkin approximation of the problem,

find $\delta u^k \in V$ s.t.

$$F'(u^k)(\delta u^k)(v) + F(u^k)(v) = 0 \quad \forall\, v \in V,$$

is given by

find $\delta u_h^k \in V_h \subset V$ s.t.

$$F'(u_h^k)(\delta u_h^k)(v_h) + F(u_h^k)(v_h) = 0 \quad \forall\, v_h \in V_h.$$

This is the so-called "linearize, then discretize" strategy.

SVK hyperelasticity equation

The St. Venant-Kirchhoff (SVK) model is a simple model of a hyperelastic material. Its weak form is given by

$$
\begin{aligned}
&\text{Find } u \text{ s.t. } u - u_0 \in W^{1,4}_{\Gamma_1}(\Omega) \text{ s.t.} \\
&\int_\Omega \left(\mu(\mathbb{F}(u)^\top \mathbb{F}(u) - \mathbb{I}) + \lambda \left(\nabla \cdot u + \frac{|\nabla u|^2}{2} \right) \mathbb{F}(u) \right) : \nabla v \; dx \\
&= \int_\Omega f \cdot v \; dx + \int_{\Gamma_2} g \cdot v \; ds \; \forall \; v \in W^{1,4}_{\Gamma_1}(\Omega),
\end{aligned}
$$

where $\mathbb{F}(u) := \mathbb{I} + \nabla u$ is the deformation gradient.

SVK hyperelasticity equation

The abstract form is given by

$$\text{Find } u \text{ s.t. } u - u_0 \in W^{1,4}_{\Gamma_1}(\Omega) \text{ s.t. } a(u; v) = L(v) \quad \forall\, v \in W^{1,4}_{\Gamma_1}(\Omega)$$

where

$$L(v) = \int_\Omega f \cdot v \; dx + \int_{\Gamma_2} g \cdot v \; ds.$$

Here $a(u; v)$ is nonlinear in u and linear in v.

Numerical Experiments

- Classical Newton's method (without linesearch)
- Finite element method using polynomials of degree $p = 4$

N	L^4 error	ROC in L^4	$W^{1,4}$ error	ROC in $W^{1,4}$
2	3.48935e-06		0.000146509	
4	1.04745e-07	5.058	8.98981e-06	4.02655
8	3.22015e-09	5.02361	5.60221e-07	4.00422
16	1.00153e-10	5.00685	3.50223e-08	3.99965
32	3.1259e-12	5.00179	2.1901e-09	3.9992

Table: Errors of Newton's and finite element method

Globalization of Newton's method

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- If the forces f, g are large, how do we make sure the iteration converges?
- Answer: globalization through parameter continuation

Parameter continuations

Motivations and assumptions

- Prior knowledge of solution when $f = g = 0$
- The solution u depends continuously on the forcing term f, g

Parameter continuations

- Consider a partition of $[0, 1]$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 1$.
- Find the approximate discrete solution $u_h^{(i)}$ of $u^{(i)}$ to the equation $a(u^{(i)}; v) = \alpha_i L(v) = \int_\Omega (\alpha_i f) \cdot v \, dx + \int_{\Gamma_2} (\alpha_i g) \cdot v \, ds$
- use $u_h^{(i)}$ as the initial guess for Newton's method applied to $a(u^{(i+1)}; v) = \alpha_{i+1} L(v)$ [1].

Summary

- Newton's method in infinite dimensional normed spaces
- Galerkin approximations
- Applications to SVK hyperelasticity equation
- Globalization using parameter continuations

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Questions?

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