AMS 597 Spring 2021
Homework 6 Due 04/20/2021 10:00 AM

Instruction: Submit your homework via Blackboard. If you have difficulty submitting via Blackboard, email the TA directly. No late homework will be accepted, based on the time the email is sent out. If you are submitting via email (not recommended), put this as the subject of your email

AMS597 2021: Homework/Exam #6. ID: XXXXXXXXX. Name: XXX XXX

1. Compute a Monte Carlo estimate of

$$\int_0^{\pi/3} sinx\mathrm{d}x.$$

   Compare your estimate with the exact value of the integral.

2. We will estimate ω of

$$\omega = \int_0^{0.5} e^{-x}\mathrm{d}x$$

   using two different approaches:

   (a) Compute a Monte Carlo estimate ($\hat{\omega}$) of ω by sampling from Uniform(0, 0.5), and estimate the variance of $\hat{\omega}$.

   (b) Compute a Monte Carlo estimate (ω^*) of ω by sampling from the exponential distribution, and estimate the variance of ω^*.

   (c) Compare the two variances. Which one is smaller?

3. Write a function to compute a Monte Carlo estimate of the $Beta(a, b)$ cdf, $F(x)$

   (a) by sampling from Uniform$(0, x)$ (name this function `my.pbeta1`)

   (b) by sampling from $U \sim Gamma(a, 1)$ and $V \sim Gamma(b, 1)$ and using the result that $Y = U/(U + V) \sim Beta(a, b)$ (name this function `my.pbeta2`)

   (c) Use `my.pbeta1` and `my.pbeta2` to estimate $F(x)$ of $Beta(3, 3)$ for $x = 0.1, 0.2, \ldots, 0.9$. Compare the estimates with the values returned by the `pbeta` function in R.

4. (a) Generate $X_1, \ldots, X_{20}$ from $N(0, 1)$. Consider testing $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$. Compute the p-value from (1) one sample t-test and (2) exact wilcoxon signed rank test. Repeat this process 1000 times. Estimate the empirical Type I error for both tests at $\alpha = 0.05$. (Hint: Empirical Type I error is the proportion of wrongly rejected null hypothesis).

   (b) Now generate $X_1, \ldots, X_{20}$ from $N(0.5, 1)$. Consider testing $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$. Compute the p-value from (1) one sample t-test and (2) exact wilcoxon signed rank test. Repeat this process 1000 times. Estimate the empirical power for both tests at $\alpha = 0.05$.

5. (a) Generate $X_1, \ldots, X_n$ from $N(0,1)$ and $Y_1, \ldots, Y_n$ from $N(0.5, 1.5)$. Consider testing $H_0 : \mu_X - \mu_Y = 0$ vs $H_a : \mu_X - \mu_Y \neq 0$. Compute the p-value from two sample t-test. Repeat this process 1000 times. Estimate the empirical power for this test at $\alpha = 0.05$ for $n = 10, 20, 30, ..., 100$. Based on your plot, what is the minimum sample size to achieve power $> 80\%$.

   (b) Edit: You do not need to show how the formula is derived. An approximate sample size formula for comparing two population means using z-test for the hypothesis in (a) is

$$n = \frac{(z_{\alpha/2} + z_\beta)^2(\sigma_X^2 + \sigma_Y^2)}{(\mu_X - \mu_Y)^2}$$

   Compare your results in (a) to this sample size formula.