

Assignment 2

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Question 1

According to the definition of norm, the equation $\|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{x}_1\|_2$ is the same as:

$$(\mathbf{x} - \mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq (\mathbf{x} - \mathbf{x}_1)^T(\mathbf{x} - \mathbf{x}_1)$$

Which can be further expressed as:

$$\mathbf{x}^T(\mathbf{x}_1 - \mathbf{x}_0) + (\mathbf{x}_1 - \mathbf{x}_0)^T \mathbf{x} \leq \mathbf{x}_1^T \mathbf{x}_1 - \mathbf{x}_0^T \mathbf{x}_0 \quad (1)$$

Since $\mathbf{x}^T(\mathbf{x}_1 - \mathbf{x}_0) = ((\mathbf{x}_1 - \mathbf{x}_0)^T \mathbf{x})^T$, which produce a constant, we have:

$$\mathbf{x}^T(\mathbf{x}_1 - \mathbf{x}_0) = (\mathbf{x}_1 - \mathbf{x}_0)^T \mathbf{x} \quad (2)$$

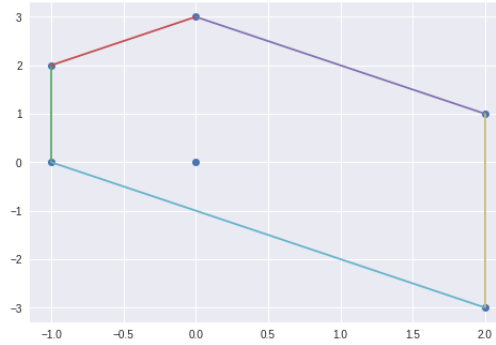
Replace equation 2 to equation 1, there is:

$$2(\mathbf{x}_1 - \mathbf{x}_0)^T \mathbf{x} \leq \mathbf{x}_1^T \mathbf{x}_1 - \mathbf{x}_0^T \mathbf{x}_0$$

Which follow the definition of half-space: $\lambda^T \mathbf{x} \leq \mathbf{b}$, where $\lambda = 2(\mathbf{x}_1 - \mathbf{x}_0)$ and $\mathbf{b} = \mathbf{x}_1^T \mathbf{x}_1 - \mathbf{x}_0^T \mathbf{x}_0$.

Question 2

The polyhedron constructed by the convex hull is the area with the 5 color line as boundary, shown in following figure:



Denote $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the point within \mathcal{R}^2 , the hyper-plane defined by the 5 color line shown in above figure are:

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$$

As a result, the polyhedron constructed by these hyper-plane can be expressed as:

$$\mathcal{A}\mathbf{x} \preceq \mathbf{b}$$

$$\text{Where } \mathcal{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$$

Question 3

(a)

Denote set $\{\mathbf{x} | \alpha \leq \mathbf{a}^T \mathbf{x} \leq \beta\}$ as \mathcal{C} , then, \mathcal{C} is a convex set. The prove is as follow:

Assume $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}$, their convex combination follow:

$$\mathbf{a}^T(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) = \theta \mathbf{a}^T \mathbf{x}_1 + (1 - \theta) \mathbf{a}^T \mathbf{x}_2 \quad (3)$$

Since $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}$, there is:

$$\begin{cases} \mathbf{a}^T \mathbf{x}_1 \geq \alpha \\ \mathbf{a}^T \mathbf{x}_2 \geq \alpha \end{cases} \quad (4)$$

Combine equation 3 and 4, we have:

$$\begin{aligned} \theta \mathbf{a}^T \mathbf{x}_1 + (1 - \theta) \mathbf{a}^T \mathbf{x}_2 &\geq \theta \alpha + (1 - \theta) \alpha \\ &= \alpha \end{aligned}$$

The equation $\theta \mathbf{a}^T \mathbf{x}_1 + (1 - \theta) \mathbf{a}^T \mathbf{x}_2 \leq \beta$ can be proved by using the same method. As a result, when $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}$, $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 \in \mathcal{C}$ hold for any $0 \leq \theta \leq 1$. Thus, \mathcal{C} is a convex set.

(b)

The *rectangle* set \mathcal{G} can be thought as the intersection of different sets that follow:

$$\mathcal{G} = \bigcap_{i=1}^n \mathcal{G}_i$$

Where \mathcal{G}_i is the set: $\{\mathbf{x} | \alpha \leq x_i \leq \beta\}$. It is obvious that \mathcal{G}_i is the *slab* defined in (a) with

$$\mathbf{a} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

In which, the i th element a_i that corresponding to x_i within \mathbf{x} is set to be 1 and the other elements is 0. Therefore, \mathcal{G}_i is convex, which means, \mathcal{G} is the intersection of convex set, which is also convex.

(c)

The set $\{\mathbf{x} | \mathbf{a}_1^T \mathbf{x} \leq b_1, \mathbf{a}_2^T \mathbf{x} \leq b_2\}$ is the same as:

$$\{\mathbf{x} | \mathbf{a}_1^T \mathbf{x} \leq b_1\} \cap \{\mathbf{x} | \mathbf{a}_2^T \mathbf{x} \leq b_2\}$$

For any $i \in \{1, 2\}$, the set $\{\mathbf{x} | \mathbf{a}_i^T \mathbf{x} \leq b_i\}$ define a half-space, which is a convex set. As a result, the set $\{\mathbf{x} | \mathbf{a}_1^T \mathbf{x} \leq b_1, \mathbf{a}_2^T \mathbf{x} \leq b_2\}$ is the intersection of two convex set, which is also convex.

(d)

Assume $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$. The original set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2 \text{ for any } \mathbf{s} \in \mathcal{S}\}$ can be expressed as:

$$\bigcap_{i=1}^n \{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{s}_i\|_2\}$$

The same concept can be expanded to set \mathcal{S} which has infinite element ($n \rightarrow \infty$). According to Question 1, the single set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{s}_i\|_2\}$ define a half-space and is convex. As a result, the intersection of these convex is also convex, which means, $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2 \text{ for any } \mathbf{s} \in \mathcal{S}\}$ is convex.

(e)

We first consider the set:

$$\mathcal{G}_i = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{s}_i\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2 \text{ for all } \mathbf{s} \in \mathcal{S}\} \quad (5)$$

Where $\mathbf{s}_i \in \mathcal{S}$ is a fixed point within \mathcal{S} . This set is consist of points \mathbf{x} that have the minimum distance to \mathcal{S} through \mathbf{s}_i . It is obvious that

$$\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{s}_i\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2\} \cap \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{s}_j\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2\} = \emptyset \quad (6)$$

when $\mathbf{s}_i, \mathbf{s}_j \in \mathcal{S}$ and $\mathbf{s}_i \neq \mathbf{s}_j$ as a point \mathbf{x} can only achieve the minimum distance to \mathcal{S} by hold $\|\mathbf{x} - \mathbf{s}_i\| < \|\mathbf{x} - \mathbf{s}_j\|$ or the inverse. Now assume $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$ and $\mathcal{T} = \{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_k\}$. We define the set:

$$\mathcal{Q}_i = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{s}_i\|_2 \leq \|\mathbf{x} - \mathbf{t}\|_2 \text{ for all } \mathbf{t} \in \mathcal{T}\}$$

Where $\mathbf{s}_i \in \mathcal{S}$. This set describe the point \mathbf{x} that the distance between \mathbf{x} and a point $\mathbf{s}_i \in \mathcal{S}$ is less than the distance between \mathbf{x} and set \mathcal{T} . As a result, the set $\{\mathbf{x} \mid \text{dist}(\mathbf{x}, \mathcal{S}) < \text{dist}(\mathbf{x}, \mathcal{T})\}$ can be then expressed as:

$$\bigcup_{i=1}^n (\mathcal{G}_i \cap \mathcal{Q}_i)$$

According to equation 6, for any $1 \leq i, j \leq n$, $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$. Hence, $(\mathcal{G}_i \cap \mathcal{Q}_i) \cup (\mathcal{G}_j \cap \mathcal{Q}_j) = \emptyset$, which means, the set $\{\mathbf{x} \mid \text{dist}(\mathbf{x}, \mathcal{S}) < \text{dist}(\mathbf{x}, \mathcal{T})\}$ is separated and therefore can not be convex.

(f)

Assume $\mathcal{S}_1 = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$. Then, the set $\mathcal{C} = \{\mathbf{x} \mid \mathbf{x} + \mathcal{S}_1 \subset \mathcal{S}_2\}$ can be replaced as the intersection of different set as:

$$\mathcal{C} = \bigcap_{i=1}^n \mathcal{C}_i$$

Where $\mathcal{C}_i = \{\mathbf{x} \mid \mathbf{x} + \mathbf{s}_i \in \mathcal{S}_2\}$. It can be proved that \mathcal{C}_i is convex: Assume $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}_i$. The equation $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 + \mathbf{s}_i$ can be expressed as:

$$\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 + \mathbf{s}_i = \theta(\mathbf{x}_1 + \mathbf{s}_i) + (1 - \theta)(\mathbf{x}_2 + \mathbf{s}_i)$$

Since $\mathbf{x}_1 + \mathbf{s}_i, \mathbf{x}_2 + \mathbf{s}_i \in \mathcal{S}_2$ and \mathcal{S}_2 is convex, we have $\theta(\mathbf{x}_1 + \mathbf{s}_i) + (1 - \theta)(\mathbf{x}_2 + \mathbf{s}_i) \in \mathcal{S}_2$. Hence, \mathcal{C}_i is convex. As a result, \mathcal{C} is convex as it is the intersection of convex set.

(g)

The equation $\|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2$ is the same as $\|\mathbf{x} - \mathbf{a}\|_2^2 \leq \theta^2 \|\mathbf{x} - \mathbf{b}\|_2^2$. Now, let $\theta^2 = p$, the equation mentioned before can be expanded as:

$$\mathbf{x}^T \mathbf{x} + \frac{2(p\mathbf{b} - \mathbf{a})^T}{1 - p} \mathbf{x} + \frac{\mathbf{a}^T \mathbf{a} - p\mathbf{b}^T \mathbf{b}}{1 - p} \leq 0 \quad (7)$$

Which has the same form as the equation that describe a ball:

$$\|\mathbf{x} - \mathbf{x}_0\|_2^2 = \mathbf{x}^T \mathbf{x} - 2\mathbf{x}_0^T \mathbf{x} + \mathbf{x}_0^T \mathbf{x}_0 \leq r^2 \quad (8)$$

Compare equation 7 with 8, it is easy to figure out that:

$$\mathbf{x}_0 = \frac{\mathbf{a} - p\mathbf{b}}{1 - p} \text{ and } r^2 = \mathbf{x}_0^T \mathbf{x}_0 - \frac{\mathbf{a}^T \mathbf{a} - p\mathbf{b}^T \mathbf{b}}{1 - p}$$

If $r^2 \geq 0$ can be verified, then, equation 7 always describe a ball. Indeed, r^2 can be expanded as:

$$r^2 = \frac{(\mathbf{a} - p\mathbf{b})^T (\mathbf{a} - p\mathbf{b})}{(1 - p)^2} - \frac{\mathbf{a}^T \mathbf{a} - p\mathbf{b}^T \mathbf{b}}{1 - p}$$

Which is the same as:

$$r^2 = \frac{p(\mathbf{a}^T \mathbf{a} - 2\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{b})}{(1 - p)^2}$$

Since $\mathbf{a}^T \mathbf{a} - 2\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{b} = \|\mathbf{a} - \mathbf{b}\|_2^2 \geq 0$, r^2 is also greater or equal to zero. Hence, equation 7 describe a ball, which is a convex set.

Question 4

In this question, we firstly define the vector:

$$\mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \quad (9)$$

Which contain all possible value that random variable x may take. After that, the vector \mathbf{p} which describe the probability distribution of x :

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \quad (10)$$

Where the element p_i within \mathbf{p} corresponding to the probability that $x = a_i$. It is obvious that the set $\mathcal{P} = \{\mathbf{p} | \|\mathbf{p}\| = 1 \text{ and for each } p_i \text{ within } \mathbf{p}, p_i \geq 0\}$ is a convex set.

(a)

Follow the definition used in 9 and 10, we can define vector \mathbf{f} as:

$$\mathbf{f} = \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_n) \end{bmatrix}$$

The condition $\alpha \leq \mathcal{E}f(x) \leq \beta$ define a set \mathcal{Q} which is a intersection of two half-space:

$$\mathcal{Q} = \{\mathbf{p} | -\mathbf{f}^T \mathbf{p} \leq -\alpha\} \cap \{\mathbf{p} | \mathbf{f}^T \mathbf{p} \leq \beta\}$$

As a result, the set of \mathbf{p} that satisfied the condition $\alpha \leq \mathcal{E}f(x) \leq \beta$ is the intersection of \mathcal{P} and \mathcal{Q} , which produce a convex set.

(b)

We firstly define a indicator vector \mathbf{i} as:

$$\mathbf{i} = \begin{bmatrix} \mathbb{I}(a_1 > \alpha) \\ \mathbb{I}(a_2 > \alpha) \\ \vdots \\ \mathbb{I}(a_n > \alpha) \end{bmatrix}$$

Where $\mathbb{I}(s)$ is defined as:

$$\mathbb{I}(s) = \begin{cases} 1 & \text{if } s \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the constrain can be expressed as:

$$\mathbf{i}^T \mathbf{p} \leq \beta$$

Which obviously, define a half-space. Therefore, the set defined by this half-space: $\mathcal{Q} = \{\mathbf{p} | \mathbf{i}^T \mathbf{p} \leq \beta\}$ is a convex set. Consequently, the set of \mathbf{p} that satisfied the constrain can be expressed as $\mathcal{P} \cap \mathcal{Q}$, which is convex.

(c)

Let $f(x)$ in (a) equal to x^2 . Hence, there is:

$$\mathbf{f} = \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix}$$

The constrain $\mathcal{E}x^2 \geq \alpha$ can then be expressed as:

$$-\mathbf{f}^T \mathbf{p} \leq -\alpha$$

Which define a half-space. Hence, the set of \mathbf{p} under this constrain is convex.

(d)

Assume $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Set $\alpha = 0.2$. Therefore, for \mathbf{p}_1 , we have:

$$\text{Var}(\mathbf{x}) = 0 < 0.2$$

For \mathbf{p}_2 , we have:

$$\text{Var}(\mathbf{x}) = 0 < 0.2$$

However, for $\mathbf{p}_3 = 0.5\mathbf{p}_1 + 0.5\mathbf{p}_2$, there is:

$$\text{Var}(\mathbf{x}) = 0.25 > 0.2$$

Hence, the set of \mathbf{p} that satisfied this constrain is not convex.

Question 5

(a)

As the function $f(\mathbf{x}, \mathbf{z})$ is convex over \mathbf{x} , it's Hessian Matrix over \mathbf{x} is:

$$\mathcal{H}_{\mathbf{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Which satisfied $\mathcal{H}_{\mathbf{x}} \succeq 0$. The Hessian Matrix of f over \mathbf{z} : $\mathcal{H}_{\mathbf{z}}$ has the similar form as $\mathcal{H}_{\mathbf{x}}$ and satisfied: $\mathcal{H}_{\mathbf{z}} \preceq 0$. As a result, the Hessian Matrix over \mathbf{x} and \mathbf{z} should be:

$$\nabla^2 f(\mathbf{x}, \mathbf{z}) = \begin{bmatrix} \mathcal{H}_{\mathbf{x}} & \mathcal{H}_{\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}} & \mathcal{H}_{\mathbf{x}} \end{bmatrix}$$

(b)

Firstly, we fix $\mathbf{x} = \tilde{\mathbf{x}}$. Since f is concave over \mathbf{z} and $\nabla f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$, according to the first-order condition of concave function, we have, for any \mathbf{z} :

$$f(\tilde{\mathbf{x}}, \mathbf{z}) < f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \quad (11)$$

In order to prove $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) < f(\mathbf{x}, \tilde{\mathbf{z}})$. We fix $\mathbf{z} = \tilde{\mathbf{z}}$. Since f is convex over \mathbf{x} , according to the first-order condition of convex function, there is:

$$f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) + \nabla f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})(\mathbf{x} - \tilde{\mathbf{x}}) < f(\mathbf{x}, \tilde{\mathbf{z}})$$

Since $\nabla f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$, we can prove that:

$$f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) < f(\mathbf{x}, \tilde{\mathbf{z}})$$

Then, we start to prove $\sup_{\mathbf{z}} \inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) = f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$: For any \mathbf{z} , if $\inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) = f(\mathbf{x}^*, \mathbf{z})$, we have:

$$f(\mathbf{x}^*, \mathbf{z}) \leq f(\tilde{\mathbf{x}}, \mathbf{z})$$

Which means, $\inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) \leq f(\tilde{\mathbf{x}}, \mathbf{z})$. The equality hold when $\mathbf{x}^* = \tilde{\mathbf{x}}$. Furthermore, according to equation 11, $\tilde{\mathbf{z}}$ maximize $f(\tilde{\mathbf{x}}, \mathbf{z})$ over \mathbf{z} . Therefore, we can conclude that the infimum of $f(\mathbf{x}, \mathbf{z})$ is smaller than or equal to $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$, which has the maximum value $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$. This conclusion is the same as:

$$\sup_{\mathbf{z}} \inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) = f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$$

The similar method can be used to prove:

$$\inf_{\mathbf{x}} \sup_{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$$

Hence, $\sup_{\mathbf{z}} \inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) = \inf_{\mathbf{x}} \sup_{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$

(c)

We firstly fix $\mathbf{x} = \tilde{\mathbf{x}}$. According to $f(\tilde{\mathbf{x}}, \mathbf{z}) \leq f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$, $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ is the maximum value of function $f(\tilde{\mathbf{x}}, \mathbf{z})$ over \mathbf{z} . Hence, there is:

$$\nabla_{\mathbf{z}} f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$$

Furthermore, we can identify that $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ is the minimum of $f(\mathbf{x}, \tilde{\mathbf{z}})$ over \mathbf{x} when we fix $\mathbf{z} = \tilde{\mathbf{z}}$, there is:

$$\nabla_{\mathbf{x}} f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$$

Combine the above two equation, we have: $\nabla f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$

Question 6

(a)

This function is convex and quasi-convex. Furthermore, this function is also quasi-concave but not concave.

(b)

This function is neither convex nor concave as it's Hessian Matrix is not semi-definite or negative semi-definite. However, this function is quasi-convex but not quasi-concave.

(c)

This function is convex and quasi-convex as it's Hessian Matrix is semi-definite. This function is neither concave nor quasi-concave.

(d)

This function is neither convex nor concave. However, this function is both quasi-convex and quasi-concave.

(e)

This function is convex and quasi-convex but not concave and quasi-concave.

(f)

This function is concave and quasi-concave but not convex and quasi-convex.

Question 7