Assignment 2

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Question 1

According to the definition of norm, the equation $\|\mathbf{x}-\mathbf{x_0}\|_2 \leq \|\mathbf{x}-\mathbf{x_1}\|_2$ is the same as:

$$(\mathbf{x}-\mathbf{x_0})^T(\mathbf{x}-\mathbf{x_0}) \leq (\mathbf{x}-\mathbf{x_1})^T(\mathbf{x}-\mathbf{x_1})$$

Which can be further expressed as:

$$\mathbf{x}^T(\mathbf{x_1}-\mathbf{x_0}) + (\mathbf{x_1}-\mathbf{x_0})^T\mathbf{x} \leq \mathbf{x_1}^T\mathbf{x_1} - \mathbf{x_0}^T\mathbf{x_0} \tag{1}$$

Since $\mathbf{x}^T(\mathbf{x_1}-\mathbf{x_0}) = \left((\mathbf{x_1}-\mathbf{x_0})^T\mathbf{x}\right)^T$, which produce a constant, we have:

$$\mathbf{x}^T(\mathbf{x_1}-\mathbf{x_0}) = (\mathbf{x_1}-\mathbf{x_0})^T\mathbf{x} \tag{2}$$

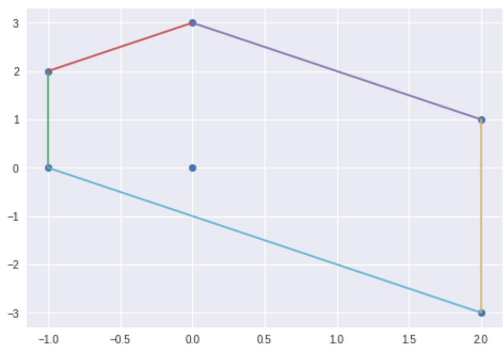
Replace equation 2 to equation 1, there is:

$$2(\mathbf{x_1}-\mathbf{x_0})^T\mathbf{x} \leq \mathbf{x_1}^T\mathbf{x_1} - \mathbf{x_0}^T\mathbf{x_0}$$

Which follow the definition of half-space: $\lambda^T\mathbf{x} \leq \mathbf{b}$, where $\lambda = 2(\mathbf{x_1}-\mathbf{x_0})$ and $\mathbf{b} = \mathbf{x_1}^T\mathbf{x_1} - \mathbf{x_0}^T\mathbf{x_0}$.

Question 2

The polyhedron constructed by the convex hull is the area with the 5 color line as boundary, shown in following figure:



Denote $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the point within $\mathcal{R}^2$, the hyper-plane defined by the 5 color line shown in above figure are:

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$$

As a result, the polyhedron constructed by these hyper-plane can be expressed as:

$$\mathcal{A}\mathbf{x} \preceq \mathbf{b}$$

Where $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$

Question 3

(a)

Denote set $\{\mathbf{x}|\,\alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$ as $\mathcal{C}$, then, $\mathcal{C}$ is a convex set. The prove is as follow:

Assume $\mathbf{x_1}, \mathbf{x_2} \in \mathcal{C}$, their convex combination follow:

$$\mathbf{a}^T(\theta\mathbf{x_1} + (1-\theta)\mathbf{x_2}) = \theta\mathbf{a}^T\mathbf{x} + (1-\theta)\mathbf{a}^T\mathbf{x_2} \tag{3}$$

Since $\mathbf{x_1}, \mathbf{x_2} \in \mathcal{C}$, there is:

$$\begin{cases} \mathbf{a}^T\mathbf{x_1} \geq \alpha \\ \mathbf{a}^T\mathbf{x_2} \geq \alpha \end{cases} \tag{4}$$

Combine equation 3 and 4, we have:

$$\begin{aligned} \theta\mathbf{a}^T\mathbf{x} + (1-\theta)\mathbf{a}^T\mathbf{x_2} &\geq \theta\alpha + (1-\theta)\alpha \\ &= \alpha \end{aligned}$$

The equation $\theta\mathbf{a}^T\mathbf{x} + (1-\theta)\mathbf{a}^T\mathbf{x_2} \leq \beta$ can be proved by using the same method. As a result, when $\mathbf{x_1}, \mathbf{x_2} \in \mathcal{C}$, $\theta\mathbf{x_1} + (1-\theta)\mathbf{x_2} \in \mathcal{C}$ hold for any $0 \leq \theta \leq 1$. Thus, $\mathcal{C}$ is a convex set.

(b)

The *rectangle* set $\mathcal{G}$ can be thought as the intersection of different sets that follow:

$$\mathcal{G} = \bigcap_{i=1}^{n} \mathcal{G}_i$$

Where $\mathcal{G}_i$ is the set: $\{\mathbf{x}|\,\alpha \leq x_i \leq \beta\}$. It is obvious that $\mathcal{G}_i$ is the *slab* defined in (a) with

$$\mathbf{a} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

In which, the ith element a_i that corresponding to x_i within $\mathbf{x}$ is set to be 1 and the other elements is 0. Therefore, $\mathcal{G}_i$ is convex, which means, $\mathcal{G}$ is the intersection of convex set, which is also convex.

(c)

The set $\{\mathbf{x}|\,\mathbf{a_1}^T\mathbf{x} \leq b_1, \mathbf{a_2}^T\mathbf{x} \leq b_2\}$ is the same as:

$$\{\mathbf{x}|\,\mathbf{a_1}^T\mathbf{x} \leq b_1\} \cap \{\mathbf{x}|\,\mathbf{a_2}^T\mathbf{x} \leq b_2\}$$

For any $i \in \{0, 1\}$, the set $\{\mathbf{x}|\,\mathbf{a_i}^T\mathbf{x} \leq b_i\}$ define a half-space, which is a convex set. As a result, the set $\{\mathbf{x}|\,\mathbf{a_1}^T\mathbf{x} \leq b_1, \mathbf{a_2}^T\mathbf{x} \leq b_2\}$ is the intersection of two convex set, which is also convex.

(d)

Assume $\mathcal{S} = \{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_n}\}$. The original set $\{\mathbf{x}|\,\|\mathbf{x} - \mathbf{x_0}\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2$ for any $\mathbf{s} \in \mathcal{S}\}$ can be expressed as:

$$\bigcap_{i=1}^{n} \{\mathbf{x}|\,\|\mathbf{x} - \mathbf{x_0}\|_2 \leq \|\mathbf{x} - \mathbf{s_i}\|_2\}$$

The same concept can be expanded to set $\mathcal{S}$ which has infinite element($n \to \infty$). According to Question 1, the single set $\mathbf{x}|\,\|\mathbf{x} - \mathbf{x_0}\|_2 \leq \|\mathbf{x} - \mathbf{s_i}\|_2$ define a half-space and is convex. As a result, the intersection of these convex is also convex, which means, $\{\mathbf{x}|\,\|\mathbf{x} - \mathbf{x_0}\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2$ for any $\mathbf{s} \in \mathcal{S}\}$ is convex.

(e)

We first consider the set:

$$\mathcal{G}_i = \{\mathbf{x} | \, \|\mathbf{x} - \mathbf{s_i}\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2 \text{ } \textit{for all } \mathbf{s} \in \mathcal{S}\} \tag{5}$$

Where $\mathbf{s_i} \in \mathcal{S}$ is a fixed point within $\mathcal{S}$. This set is consist of points $\mathbf{x}$ that have the minimum distance to $\mathcal{S}$ through $\mathbf{s_i}$. It is obvious that

$$\{\mathbf{x} | \, \|\mathbf{x} - \mathbf{s_i}\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2\} \cap \{\mathbf{x} | \, \|\mathbf{x} - \mathbf{s_j}\|_2 \leq \|\mathbf{x} - \mathbf{s}\|_2\} = \emptyset \tag{6}$$

when $\mathbf{s_i}, \mathbf{s_j} \in \mathcal{S}$ and $\mathbf{s_i} \neq \mathbf{s_j}$ as a point $\mathbf{x}$ can only achieve the minimum distance to $\mathcal{S}$ by hold $\|\mathbf{x} - \mathbf{s_i}\| < \|\mathbf{x} - \mathbf{s_j}\|$ or the inverse. Now assume $\mathcal{S} = \{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_n}\}$ and $\mathcal{T} = \{\mathbf{t_1}, \mathbf{t_2}, \cdots, \mathbf{t_k}\}$. We define the set:

$$\mathcal{Q}_i = \{\mathbf{x} | \, \|\mathbf{x} - \mathbf{s_i}\|_2 \leq \|\mathbf{x} - \mathbf{t}\|_2 \text{ } \textit{for all } \mathbf{t} \in \mathcal{T}\}$$

Where $\mathbf{s_i} \in \mathcal{S}$. This set describe the point $\mathbf{x}$ that the distance between $\mathbf{x}$ and a point $\mathbf{s_i} \in \mathcal{S}$ is less than the distance between $\mathbf{x}$ and set $\mathcal{T}$. As a result, the set $\{\mathbf{x} | \, \mathbf{dist}(\mathbf{x}, \mathcal{S}) < \mathbf{dist}(\mathbf{x}, \mathcal{T})\}$ can be then expressed as:

$$\bigcup_{i=1}^{n} (\mathcal{G}_i \cap \mathcal{Q}_i)$$

According to equation 6, for any $1 \leq i, j \leq n$, $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$. Hence, $(\mathcal{G}_i \cap \mathcal{Q}_i) \cup (\mathcal{G}_j \cap \mathcal{Q}_j) = \emptyset$, which means, the set $\{\mathbf{x} | \, \mathbf{dist}(\mathbf{x}, \mathcal{S}) < \mathbf{dist}(\mathbf{x}, \mathcal{T})\}$ is separated and therefore can not be convex.

(f)

Assume $\mathcal{S}_1 = \{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_n}\}$. Then, the set $\mathcal{C} = \{\mathbf{x} | \, \mathbf{x} + \mathcal{S}_1 \subset \mathcal{S}_2\}$ can be replaced as the intersection of different set as:

$$\mathcal{C} = \bigcap_{i=1}^{n} \mathcal{C}_i$$

Where $\mathcal{C}_i = \{\mathbf{x} | \, \mathbf{x} + \mathbf{s_i} \in \mathcal{S}_2\}$. It can be proved that $\mathcal{C}_i$ is convex: Assume $\mathbf{x_1}, \mathbf{x_2} \in \mathcal{C}_i$. The equation $\theta\mathbf{x_1} + (1-\theta)\mathbf{x_2} + \mathbf{s_i}$ can be expressed as:

$$\theta\mathbf{x_1} + (1-\theta)\mathbf{x_2} + \mathbf{s_i} = \theta(\mathbf{x_1} + \mathbf{s_i}) + (1-\theta)(\mathbf{x_2} + \mathbf{s_i})$$

Since $\mathbf{x_1} + \mathbf{s_i}, \mathbf{x_2} + \mathbf{s_i} \in \mathcal{S}_2$ and $\mathcal{S}_2$ is convex, we have $\theta(\mathbf{x_1} + \mathbf{s_i}) + (1-\theta)(\mathbf{x_2} + \mathbf{s_i}) \in \mathcal{S}_2$. Hence, $\mathcal{C}_i$ is convex. As a result, $\mathcal{C}$ is convex as it is the intersection of convex set.

(g)

The equation $\|\mathbf{x} - \mathbf{a}\|_2 \leq \theta\|\mathbf{x} - \mathbf{b}\|_2$ is the same as $\|\mathbf{x} - \mathbf{a}\|_2^2 \leq \theta^2\|\mathbf{x} - \mathbf{b}\|_2^2$. Now, let $\theta^2 = p$, the equation mentioned before can be expanded as:

$$\mathbf{x}^T\mathbf{x} + \frac{2(p\mathbf{b} - \mathbf{a})^T}{1-p}\mathbf{x} + \frac{\mathbf{a}^T\mathbf{a} - p\mathbf{b}^T\mathbf{b}}{1-p} \leq 0 \tag{7}$$

Which has the same form as the equation that describe a ball:

$$\|\mathbf{x} - \mathbf{x_0}\|_2^2 = \mathbf{x}^T\mathbf{x} - 2\mathbf{x_0}^T\mathbf{x} + \mathbf{x_0}^T\mathbf{x_0} \leq r^2 \tag{8}$$

Compare equation 7 with 8, it is easy to figure out that:

$$\mathbf{x_0} = \frac{\mathbf{a} - p\mathbf{b}}{1-p} \text{ } \textit{and } r^2 = \mathbf{x_0}^T\mathbf{x_0} - \frac{\mathbf{a}^T\mathbf{a} - p\mathbf{b}^T\mathbf{b}}{1-p}$$

If $r^2 \geq 0$ can be verified, then, equation 7 always describe a ball. Indeed, r^2 can be expanded as:

$$r^2 = \frac{(\mathbf{a} - p\mathbf{b})^T(\mathbf{a} - p\mathbf{b})}{(1-p)^2} - \frac{\mathbf{a}^T\mathbf{a} - p\mathbf{b}^T\mathbf{b}}{1-p}$$

Which is the same as:

$$r^2 = \frac{p(\mathbf{a}^T\mathbf{a} - 2\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{b})}{(1-p)^2}$$

Since $\mathbf{a}^T\mathbf{a} - 2\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{b} = \|\mathbf{a} - \mathbf{b}\|_2^2 \geq 0$, r^2 is also greater or equal to zero. Hence, equation 7 describe a ball, which is a convex set.

Question 4

It is obvious that the probability distributions define a convex set $\mathcal{P}$, which satisfied, for any $\mathbf{p} \in \mathcal{P}$, $\|\mathbf{p}\| = 1$

(a)

Denote $f(x)$ in vector form as:

$$\mathbf{f} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

The condition $\alpha \leq \mathcal{E}f(x) \leq \beta$ define a set $\mathcal{Q}$ which is a intersection of two half-space:

$$\mathcal{Q} = \{\mathbf{p} | -\mathbf{f}^T\mathbf{p} \leq -\alpha\} \cap \{\mathbf{p} | \mathbf{f}^T\mathbf{p} \leq \beta\}$$

As a result, the set of $\mathbf{p}$ that satisfied the condition $\alpha \leq \mathcal{E}f(x) \leq \beta$ is the intersection of $\mathcal{P}$ and $\mathcal{Q}$, which produce a convex set.

(b)

Denote the vector $\mathbf{x}$ as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Which satisfied that for any $0 \leq i, j \leq n$, if $i < j$, then, $x_i < x_j$. Furthermore, we define vector $\tilde{\mathbf{x}}$ as:

$$\tilde{\mathbf{x}} = \begin{bmatrix} 0 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_n \end{bmatrix}$$

In which, x_i is the first element that greater than α. Therefore, the set of $\mathbf{p}$ that satisfied the condition in this question can be expressed as:

$$\mathcal{P} \cap \{\mathbf{p} | \tilde{\mathbf{x}}^T\mathbf{p} \leq \beta\}$$

Which is a convex set.