# Assignment 3

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## Q1

### (a)

A path is start at node 1 and end at node $n$ if the number of edges that outflow node 1 is equal to the number of edges that flow into node n. Hence, according to this property, the equation can be written as:

$$\sum_{i \neq 1} x_{1,i} = \sum_{j \neq n} x_{j,n} \tag{1}$$

### (b)

The path pass through any node $i$ except node 1 and node $n$ only once indicate the number of edge that flow into node $i$ small or equal to one. Hence, the constrain can be written as:

$$\sum_{j \neq i} x_{j,i} \leq 1 \quad for\ i \neq 1\ and\ i \neq n \tag{2}$$

### (c)

The number of times a path goes into any node $i$ is the same as the number of times a path goes out of the same node indicate that the number of edges that flow into node $i$ is equal to the number of edges that flow out this node. Hence, the constrain can be expressed as:

$$\sum_{j \neq i} x_{i,j} = \sum_{k \neq i} x_{k,i} \quad for\ i \neq 1\ and\ i \neq n \tag{3}$$

### (d)

It is easy to find out that the cost of a path which specified by $x_{i,j}$ is:

$$\sum_{i,j} c_{i,j} x_{i,j} \tag{4}$$

### (e)

Following equation 1 to 4, the *Shortest Path Problem* can be formalized into optimization problem as:

$$
\begin{aligned}
\min_{x_{i,j}} \quad & \sum_{i,j} c_{i,j} x_{i,j} \\
s.t. \quad & \sum_{i \neq 1} x_{1,i} = \sum_{j \neq n} x_{j,n} \\
& \sum_{j \neq i} x_{j,i} \leq 1 \quad for\ i \neq 1\ and\ i \neq n \\
& \sum_{j \neq i} x_{i,j} = \sum_{k \neq i} x_{k,i} \quad for\ i \neq 1\ and\ i \neq n \\
& x_{i,j} \in \{0, 1\}
\end{aligned}
\tag{5}
$$

## (f)

Need modification

## Q2

In order to prove that $\mathbf{x}^*$ is the optimal points of function $f_0(\mathbf{x}) \frac{1}{2}\mathbf{x}^T \mathcal{P} \mathbf{x} + \mathbf{q}^T \mathbf{x} + r$ at point $\mathbf{x}^*$, we need to verify that for any $-1 \preceq \mathbf{x} \preceq 1$, the following equation:

$$\nabla f_0(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0 \tag{6}$$

always hold. Indeed, the gradient of function $f_0$ at $\mathbf{x}^*$ is:

$$\nabla f_0(\mathbf{x}^*) = \mathcal{P}\mathbf{x}^* + \mathbf{q}^T = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \tag{7}$$

By substituting equation 7 to 6, we can rewrite the condition that need to be verified as:

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} (\mathbf{x} - \begin{bmatrix} 1 \\ 0.5 \\ -1 \end{bmatrix}) = -1(x_1 - 1) + 2(x_3 + 1) \tag{8}$$
$$\geq 0$$

As a result, we need to verify the minimum value of $-1(x_1 - 1) + 2(x_3 + 1)$ is greater or equal to zero, which can be formalized as a *Linear Programming Problem* as:

$$\begin{align} \min_{x_1, x_3} \quad & -x_1 + 2x_2 + 3 \\ s.t. \quad & -1 \leq x_1 \leq 1 \\ & -1 \leq x_3 \leq 1 \end{align} \tag{9}$$

The optimal point of this problem can be obtained by simple inspection, which is $(1, -1)$. Hence, the minimum value of $-x_1 + 2x_2 + 3$ is 0, which means $\nabla f_0(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0$ always hold and $\mathbf{x}^*$ is the optimal point.

## Q3

### (a)

The variable $\mathbf{x}$ can be decomposed as:

$$\mathbf{x} = \mathbf{y} + \mathbf{x_0}$$

Where $\mathbf{y} \in \mathcal{N}(\mathcal{A})$ can be any vector within the null space of $\mathcal{A}$ and $\mathbf{x_0}$ is the optimal point of original problem. The reason this equation always hold is:

$$\begin{align} \mathcal{A}\mathbf{x} &= \mathcal{A}(\mathbf{y} + \mathbf{x_0}) \\ &= \mathcal{A}\mathbf{y} + \mathcal{A}\mathbf{x_0} \\ &= 0 + \mathbf{b} \end{align} \tag{10}$$

Based on this decomposition, there is:

$$\nabla f_0(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) = \mathbf{c}^T \mathbf{y} \tag{11}$$

If the optimal point $\mathbf{x_0}$ exist, then, $\mathbf{c}^T \mathbf{y} \geq 0$ for any $\mathbf{y} \in \mathcal{N}(\mathcal{A})$ must be true. However, since both $\mathbf{y}$ and $-\mathbf{y}$ are in $\mathcal{N}(\mathcal{A})$, there is:

$$\begin{cases} \mathbf{c}^T \mathbf{y} \geq 0 \\ -\mathbf{c}^T \mathbf{y} \geq 0 \end{cases} \tag{12}$$

Hence, $\mathbf{c}^T \mathbf{y}$ must equal to 0 for any $\mathbf{y} \in \mathcal{N}(\mathcal{A})$, which means: $\mathbf{c}$ is perpendicular to $\mathcal{N}(\mathcal{A})$. Furthermore, since $\mathcal{R}(\mathcal{A}) \perp \mathcal{N}(\mathcal{A})$, where $\mathcal{R}(\mathcal{A})$ is the row space of $\mathcal{A}$, we can conduct that $\mathbf{c} \in \mathcal{R}(\mathcal{A})$. Therefore, $\mathbf{c}$ is some linear combination of $\mathcal{A}$'s rows:

$$\mathbf{c} = \mathcal{A}^T \lambda \quad (\lambda \in \mathcal{R}^n) \tag{13}$$

Additionally, since the optimal point $\mathbf{x_0}$ must be feasible, $\mathbf{x_0}$ should satisfied $\mathcal{A}\mathbf{x} = \mathbf{b}$. As a result, the optimal solution of original problem under this circumstance is:

$$\begin{align} \mathbf{c}^T \mathbf{x_0} &= \lambda^T \mathcal{A}\mathbf{x_0} \\ &= \lambda^T \mathbf{b} \end{align} \tag{14}$$

Now, we tend to discuss the situation that $\mathbf{c} \notin \mathcal{R}(\mathcal{A})$. We can still decompose variable $\mathbf{x}$ with some $\mathbf{x}'$ which satisfied $\mathcal{A}\mathbf{x}' = \mathbf{b}$ as:

$$\mathbf{x} = \mathbf{y} + \mathbf{x}'$$

Where $\mathbf{y} \in \mathcal{N}(\mathcal{A})$. As a result, the objective function can be written as:

$$f_0 = \mathbf{c}^T(\mathbf{y} + \mathbf{x}')$$

Since $\mathbf{c}^T\mathbf{x}'$ is a constant, we are actually minimize $\mathbf{c}^T\mathbf{y}$. Furthermore, we can always find some $\mathbf{y}^*$ which satisfied $\mathbf{c}^T\mathbf{y}^* < 0$. As a result, by multiple $\mathbf{y}$ with any $t > 0$, $\mathbf{c}^T t\mathbf{y}$ is unbound below, which means, the optimal value is $-\infty$.

In conclusion, the optimal value of this problem is:

$$p^* = \begin{cases} -\infty & \mathbf{c} \notin \mathcal{R}(\mathcal{A}) \\ \lambda^T\mathbf{b} & \mathbf{c} \in \mathcal{R}(\mathcal{A}) \\ \infty & \mathbf{x} \text{ is infeasible} \end{cases} \tag{15}$$

## (b)

The objective function can be written as:

$$f_0(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i \tag{16}$$

It is obvious that for each $c_i$, $f_0$ will achieve the minimum at $\mathbf{x}^*$ if:

$$x_i^* = \begin{cases} u & c_i \leq 0 \\ l & otherwise \end{cases} \tag{17}$$

This can be proved through verifying $\nabla f_0(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0$. Indeed, we have:

$$\nabla f_0(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = \sum_i c_i(x_i - x_i^*) \tag{18}$$

For each $c_i(x_i - x_i^*)$, if $c_i \leq 0$, then, $x_i^* = u$ and $x_i - x_i^* \leq 0$ as $x_i \leq u = x_i^*$. As a result, $c_i(x_i - x_i^*) \geq 0$ when $c_i \leq 0$. Additionally, by using similar method, we can also prove that $c_i(x_i - x_i^*) \geq 0$ when $c_i > 0$. As a result $c_i(x_i - x_i^*) \geq 0$ always hold, hence, $\nabla f_0(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0$.

## (c)

Intuitively, the minimum value of $\sum_{i=1}^{n} c_i x_i$ under constrain $\sum_{i=1}^{n} x_i = 1$ and $0 \leq x_i \leq 1$ is:

$$c_1^* + c_2^* + \cdots + c_\alpha^* \tag{19}$$

Where $c_i^*$ is the $i$th largest element within $\mathbf{c}$. This formulation indicate the minimum value of $\mathbf{c}^T\mathbf{x}$ is the sum of first $\alpha$ smallest element within $\mathbf{c}$. Indeed, in *Question 7(b)* of last assignment, we already proved that the maximum value of $\mathbf{y}^T\mathbf{x}$ for any $\mathbf{y}$ under constrain $\mathbf{1}^T\mathbf{x} = \alpha$ and $0 \preceq \mathbf{x} \preceq 1$ is $\sum_{i=1}^{\alpha} y_i^*$, which is the first $\alpha$ largest element within $\mathbf{y}$. By multiplying each $\mathbf{y}$ with -1, $-y_i^*$ becomes the $i$th smallest element within $-\mathbf{y}$ and $-\sum_{i=1}^{\alpha} y_i^*$ become the minimum value of $-\mathbf{y}^T\mathbf{x}$. Since $\mathbf{y}$ is arbitrary, we can conduct that the minimum value of $\mathbf{y}^T\mathbf{x}$ under constrain $\mathbf{1}^T\mathbf{x} = \alpha$ and $0 \preceq \mathbf{x} \preceq 1$ is the first $\alpha$ smallest number within $\mathbf{y}$. As a result, the optimal value of $\mathbf{c}^T\mathbf{x}$ is $\sum_{i=1}^{\alpha} c_i^*$

Furthermore, if $\alpha$ is not integer, we can decompose $\alpha$ with some fraction $0 \leq q \leq 1$ as:

$$\alpha = \lfloor \alpha \rfloor + q$$

Extend from equation 19, the optimal value under this situation is:

$$\sum_{i=1}^{\lfloor \alpha \rfloor} c_i^* + qc_{\lfloor \alpha \rfloor + 1}^* \tag{20}$$