

# Convex Optimization Note

Garrick Lin

March 13, 2019

## 1 Affine Set

### 1.1 Definition

A set  $\mathcal{A}$  is called affine set when it satisfied:

If  $\mathbf{x}_1 \in \mathcal{A}$  and  $\mathbf{x}_2 \in \mathcal{A}$ , then,  $\forall \theta \in \mathcal{R}$ ,  $x = \theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2$  also belong to  $\mathcal{A}$ .

The expression  $\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2$  represent a line that cross through  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . Hence, affine set can be explained intuitively as:

Affine set is a set that contain the line which cross through any two point within this set.

### 1.2 Properties

Assume a affine set  $\mathcal{A} \subseteq \mathcal{R}^n$ . If  $\mathcal{A}$  contain original point( $\mathbf{0}$ ), then,  $\mathcal{A}$  is a subspace. In order to prove this,  $\mathcal{A}$  must satisfied following three rules:

1.  $\mathcal{A}$  contain original point.
2. If  $\mathbf{v} \in \mathcal{A}$ , then,  $\forall \theta \in \mathcal{R}$ ,  $\theta\mathbf{v} \in \mathcal{A}$ .
3. If  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}$ , then,  $\mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{A}$ .

The first rule is already satisfied. For the second rule,  $\forall \theta \in \mathcal{R}$ ,  $\theta\mathbf{v}$  satisfied:

$$\theta\mathbf{v} = \theta\mathbf{v} + (1 - \theta)\mathbf{0}$$

Since  $\mathcal{A}$  is affine and  $\mathbf{v}, \mathbf{0} \in \mathcal{A}$ ,  $\theta\mathbf{v} \in \mathcal{A}$  always true. For the third rule, there is:

$$\mathbf{v}_1 + \mathbf{v}_2 = 2(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2)$$

$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$  is in  $\mathcal{A}$  as  $\mathcal{A}$  is affine and  $\frac{1}{2}\mathbf{v}_1, \frac{1}{2}\mathbf{v}_2 \in \mathcal{A}$  (According to the second rule,  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}$ , then,  $\frac{1}{2}\mathbf{v}_1, \frac{1}{2}\mathbf{v}_2 \in \mathcal{A}$ ). According to the second rule,  $2(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2) \in \mathcal{A}$

Therefore, any affine set  $\mathcal{A}'$  can be thought as a subspace  $\mathcal{V}$  with a transportation  $\mathbf{v}$ , which means,  $\mathcal{A}' = \mathcal{V} + \mathbf{v}$ . On the other hand, an affine set subtract some constant vector ( $\mathcal{A}' - \mathbf{v}$ ), which make the new set contain original point, can form a subspace. Formally, for a affine set  $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  and  $\mathbf{a}_i \in \mathcal{A}$ , the set:

$$\mathcal{A} - \mathbf{a}_i = \{\mathbf{a}_1 - \mathbf{a}_i, \mathbf{a}_2 - \mathbf{a}_i, \dots, \mathbf{a}_{i-1} - \mathbf{a}_i, \mathbf{0}, \mathbf{a}_{i+1} - \mathbf{a}_i, \dots, \mathbf{a}_n - \mathbf{a}_i\} \quad (1)$$

form a subspace. Furthermore, any affine set  $\mathcal{A}$  has the form:

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}\mathbf{x} = \mathbf{c}\} \quad (2)$$

**Proof:** According to equation 1,  $\mathcal{A} - \mathbf{a}_i$  is a subspace. Denote this subspace as  $\mathcal{L} = \mathcal{A} - \mathbf{a}_i$ . Assume the space that perpendicular to  $\mathcal{L}$ , denote as  $\mathcal{L}^\perp$ , has basis  $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ . Then,  $\mathcal{L}$  is the set of vectors that perpendicular to the basis of  $\mathcal{L}^\perp$ . As a result, if we write the basis of  $\mathcal{L}^\perp$  in matrix form, which is:

$$\mathcal{B} = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix}$$

We have:

$$\mathcal{L} = \{\mathbf{y} | \mathcal{B}\mathbf{y} = 0\} \quad (3)$$

Equation 3 always hold as all vector in  $\mathcal{L}^\perp$  is the linear combination of the basis, hence, if a vector  $\mathbf{v}$  satisfied  $\mathbf{b}_1^T \mathbf{v} = 0, \mathbf{b}_2^T \mathbf{v} = 0, \dots, \mathbf{b}_n^T \mathbf{v} = 0$ , then,  $(\theta_1 \mathbf{b}_1 + \theta_2 \mathbf{b}_2 + \dots \theta_n \mathbf{b}_n)^T \mathbf{v} = 0$ , which means,  $\mathbf{v}$  is perpendicular to any vector in  $\mathcal{L}^\perp$  if  $\mathbf{v}$  is perpendicular to the basis of  $\mathcal{L}^\perp$ .

Since the vector in  $\mathcal{L}$  come from  $\mathcal{A}$  subtract  $\mathbf{a}_i$ , for all  $\mathbf{y} \in \mathcal{L}$ , there must exists  $\mathbf{x} \in \mathcal{A}$  such that:

$$\mathbf{y} = \mathbf{x} - \mathbf{a}_i \quad (4)$$

By combining equation 4 and equation 3, we have:

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}(\mathbf{x} - \mathbf{a}_i) = 0\}$$

Which is equivalent to :

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}\mathbf{x} = \mathcal{B}\mathbf{a}_i\}$$

and has the same form as equation 2, where  $\mathbf{c} = \mathcal{B}\mathbf{a}_i$ .