Convex Optimization Note

Garrick Lin

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1 Affine Set

1.1 Definition

A set $\mathcal{A}$ is called affine set when it satisfied:

If $\mathbf{x_1} \in \mathcal{A}$ and $\mathbf{x_2} \in \mathcal{A}$, then, $\forall \theta \in \mathcal{R}$, $x = \theta \mathbf{x_1} + (1-\theta)\mathbf{x_2}$ also belong to $\mathcal{A}$.

The expression $\theta \mathbf{x_1} + (1-\theta)\mathbf{x_2}$ represent a line that cross through $\mathbf{x_1}$ and $\mathbf{x_2}$. Hence, affine set can be explained intuitively as:

Affine set is a set that contain the line which cross through any two point within this set.

1.2 Properties

Assume a affine set $\mathcal{A} \subseteq \mathcal{R}^n$. If $\mathcal{A}$ contain original point($\mathbf{0}$), then, $\mathcal{A}$ is a subspace. In order to prove this, $\mathcal{A}$ must satisfied following three rules:

1. $\mathcal{A}$ contain original point.
2. If $\mathbf{v} \in \mathcal{A}$, then, $\forall \theta \in \mathcal{R}$, $\theta \mathbf{v} \in \mathcal{A}$.
3. If $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{A}$, then, $\mathbf{v_1} + \mathbf{v_2} \in \mathbf{A}$.

The first rule is already satisfied. For the second rule, $\forall \theta \in \mathcal{R}$, $\theta \mathbf{v}$ satisfied:

$$\theta \mathbf{v} = \theta \mathbf{v} + (1-\theta)\mathbf{0}$$

Since $\mathcal{A}$ is affine and $\mathbf{v}, \mathbf{0} \in \mathcal{A}$, $\theta \mathbf{v} \in \mathcal{A}$ always true. For the third rule, there is:

$$\mathbf{v_1} + \mathbf{v_2} = 2(\frac{1}{2}\mathbf{v_1} + \frac{1}{2}\mathbf{v_2})$$

$\frac{1}{2}\mathbf{v_1} + \frac{1}{2}\mathbf{v_2}$ is in $\mathcal{A}$ as $\mathcal{A}$ is affine and $\frac{1}{2}\mathbf{v_1}, \frac{1}{2}\mathbf{v_2} \in \mathcal{A}$(According to the second rule, $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{A}$, then, $\frac{1}{2}\mathbf{v_1}, \frac{1}{2}\mathbf{v_2} \in \mathcal{A}$). According to the second rule, $2(\frac{1}{2}\mathbf{v_1} + \frac{1}{2}\mathbf{v_2}) \in \mathcal{A}$

Therefore, any affine set $\mathcal{A}'$ can be thought as a subspace $\mathcal{V}$ with a transportation $\mathbf{v}$, which means, $\mathcal{A}' = \mathcal{V} + \mathbf{v}$. On the other hand, an affine set subtract some constant vector $(\mathcal{A}' - \mathbf{v})$, which make the new set contain original point, can form a subspace. Formally, for a affine set $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}\}$ and $\mathbf{a_i} \in \mathcal{A}$, the set:

$$\mathcal{A} - \mathbf{a_i} = \{\mathbf{a_1} - \mathbf{a_i}, \mathbf{a_2} - \mathbf{a_i}, \cdots, \mathbf{a_{i-1}}, \mathbf{0}, \mathbf{a_{i+1}}, \cdots, \mathbf{a_n} - \mathbf{a_i}\}$$

form a subspace.