

# Convex Optimization Note

Garrrick Lin

April 19, 2019

## 1 Affine Set

### 1.1 Definition

A set  $\mathcal{A}$  is called affine set when it satisfied:

If  $\mathbf{x}_1 \in \mathcal{A}$  and  $\mathbf{x}_2 \in \mathcal{A}$ , then,  $\forall \theta \in \mathcal{R}$ ,  $x = \theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2$  also belong to  $\mathcal{A}$ .

The expression  $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2$  represent a line that cross through  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . Hence, affine set can be explained intuitively as:

Affine set is a set that contain the line which cross through any two point within this set.

### 1.2 Properties

Assume a affine set  $\mathcal{A} \subseteq \mathcal{R}^n$ . If  $\mathcal{A}$  contain original point( $\mathbf{0}$ ), then,  $\mathcal{A}$  is a subspace. In order to prove this,  $\mathcal{A}$  must satisfied following three rules:

1.  $\mathcal{A}$  contain original point.
2. If  $\mathbf{v} \in \mathcal{A}$ , then,  $\forall \theta \in \mathcal{R}$ ,  $\theta \mathbf{v} \in \mathcal{A}$ .
3. If  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}$ , then,  $\mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{A}$ .

The first rule is already satisfied. For the second rule,  $\forall \theta \in \mathcal{R}$ ,  $\theta \mathbf{v}$  satisfied:

$$\theta \mathbf{v} = \theta \mathbf{v} + (1 - \theta) \mathbf{0}$$

Since  $\mathcal{A}$  is affine and  $\mathbf{v}, \mathbf{0} \in \mathcal{A}$ ,  $\theta \mathbf{v} \in \mathcal{A}$  always true. For the third rule, there is:

$$\mathbf{v}_1 + \mathbf{v}_2 = 2(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2)$$

$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$  is in  $\mathcal{A}$  as  $\mathcal{A}$  is affine and  $\frac{1}{2}\mathbf{v}_1, \frac{1}{2}\mathbf{v}_2 \in \mathcal{A}$  (According to the second rule,  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}$ , then,  $\frac{1}{2}\mathbf{v}_1, \frac{1}{2}\mathbf{v}_2 \in \mathcal{A}$ ). According to the second rule,  $2(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2) \in \mathcal{A}$

Therefore, any affine set  $\mathcal{A}'$  can be thought as a subspace  $\mathcal{V}$  with a transportation  $\mathbf{v}$ , which means,  $\mathcal{A}' = \mathcal{V} + \mathbf{v}$ . On the other hand, an affine set subtract some constant vector ( $\mathcal{A}' - \mathbf{v}$ ), which make the new set contain original point, can form a subspace. Formally, for a affine set  $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  and  $\mathbf{a}_i \in \mathcal{A}$ , the set:

$$\mathcal{A} - \mathbf{a}_i = \{\mathbf{a}_1 - \mathbf{a}_i, \mathbf{a}_2 - \mathbf{a}_i, \dots, \mathbf{a}_{i-1} - \mathbf{a}_i, \mathbf{0}, \mathbf{a}_{i+1} - \mathbf{a}_i, \dots, \mathbf{a}_n - \mathbf{a}_i\} \quad (1)$$

form a subspace. Furthermore, any affine set  $\mathcal{A}$  has the form:

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}\mathbf{x} = \mathbf{c}\} \quad (2)$$

**Proof:** According to equation 1,  $\mathcal{A} - \mathbf{a}_i$  is a subspace. Denote this subspace as  $\mathcal{L} = \mathcal{A} - \mathbf{a}_i$ . Assume the space that perpendicular to  $\mathcal{L}$ , denote as  $\mathcal{L}^\perp$ , has basis  $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ . Then,  $\mathcal{L}$  is the set of vectors that perpendicular to the basis of  $\mathcal{L}^\perp$ . As a result, if we write the basis of  $\mathcal{L}^\perp$  in matrix form, which is:

$$\mathcal{B} = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix}$$

We have:

$$\mathcal{L} = \{\mathbf{y} | \mathcal{B}\mathbf{y} = 0\} \quad (3)$$

Equation 3 always hold as all vector in  $\mathcal{L}^\perp$  is the linear combination of the basis, hence, if a vector  $\mathbf{v}$  satisfied  $\mathbf{b}_1^T \mathbf{v} = 0, \mathbf{b}_2^T \mathbf{v} = 0, \dots, \mathbf{b}_n^T \mathbf{v} = 0$ , then,  $(\theta_1 \mathbf{b}_1 + \theta_2 \mathbf{b}_2 + \dots \theta_n \mathbf{b}_n)^T \mathbf{v} = 0$ , which means,  $\mathbf{v}$  is perpendicular to any vector in  $\mathcal{L}^\perp$  if  $\mathbf{v}$  is perpendicular to the basis of  $\mathcal{L}^\perp$ .

Since the vector in  $\mathcal{L}$  come from  $\mathcal{A}$  subtract  $\mathbf{a}_i$ , for all  $\mathbf{y} \in \mathcal{L}$ , there must exists  $\mathbf{x} \in \mathcal{A}$  such that:

$$\mathbf{y} = \mathbf{x} - \mathbf{a}_i \quad (4)$$

By combining equation 4 and equation 3, we have:

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}(\mathbf{x} - \mathbf{a}_i) = 0\}$$

Which is equivalent to :

$$\mathcal{A} = \{\mathbf{x} | \mathcal{B}\mathbf{x} = \mathcal{B}\mathbf{a}_i\}$$

and has the same form as equation 2, where  $\mathbf{c} = \mathcal{B}\mathbf{a}_i$ .

## 2 Hyper-plane

### 2.1 Definition

In a  $\mathcal{R}^n$  space, the hyperplane is defined as a  $n-1$  dimension subspace plus a translate  $\mathbf{x}_0$ , which is perpendicular to a one dimension vector  $\mathbf{w}$ . The formal mathematical expression of hyper-plane  $\mathcal{H}$  is:

$$\mathcal{H} = \{\mathbf{y} + \mathbf{x}_0 | \mathbf{w}^T \mathbf{y} = 0\} \quad (5)$$

We can denote  $\mathbf{x} = \mathbf{y} + \mathbf{x}_0$ , hence, equation 5 become:

$$\mathcal{H} = \{\mathbf{x} | \mathbf{w}^T (\mathbf{x} - \mathbf{x}_0) = 0\} \quad (6)$$

It is clearly that when  $\mathbf{x} = \mathbf{x}_0$ ,  $\mathbf{w}^T (\mathbf{x}_0 - \mathbf{x}_0) = 0$  always hold, hence,  $\mathbf{x}_0$  is a point within hyper-plane. As a result, equation 6 can be explained intuitively as: Hyper-plane is a set of any point  $\mathbf{x}$  which satisfied the vector  $\mathbf{x} - \mathbf{x}_0$  (start from a fixed point  $\mathbf{x}_0$  which belong to this hyper-plane, end at  $\mathbf{x}$ ) is perpendicular to a normal vector  $\mathbf{w}$ .

Equation 6 give us another way to represent hyper-plane:

$$\mathcal{H} = \{\mathbf{x} | \mathbf{w}^T \mathbf{x} = \mathbf{b}\} \quad (7)$$

Where  $\mathbf{b} = \mathbf{w}^T \mathbf{x}_0$ .

### 2.2 Half-space

Intuitively, a hyper-plane split the whole space into two part, each of them is called half-space. The formal mathematical expression for half-space, base on equation 6, is:

$$\{\mathbf{x} | \mathbf{w}^T (\mathbf{x} - \mathbf{x}_0) < 0\}$$

or

$$\{\mathbf{x} | \mathbf{w}^T (\mathbf{x} - \mathbf{x}_0) > 0\}$$

Which corresponding to the two split space. In geometric, two vector  $\mathbf{v}_1$  and  $\mathbf{v}_2$  satisfied:  $\mathbf{v}_1^T \mathbf{v}_2 > 0$  if their angle smaller than 90 degree, otherwise,  $\mathbf{v}_1^T \mathbf{v}_2 < 0$ . As a result, the half-space  $\{\mathbf{x} | \mathbf{w}^T (\mathbf{x} - \mathbf{x}_0) < 0\}$  is the set of point  $\mathbf{x}$  which satisfied the vector  $\mathbf{x} - \mathbf{x}_0$  have a angle greater than 90 degree with a normal vector  $\mathbf{w}$ . The half-space  $\{\mathbf{x} | \mathbf{w}^T (\mathbf{x} - \mathbf{x}_0) > 0\}$  can be explained in the same way except the angle is smaller than 90 degree.

Furthermore, according to 7, the expression of half-space can also be written as:

$$\{\mathbf{x} | \mathbf{w}^T \mathbf{x} < \mathbf{b}\}$$

Where  $\mathbf{b} = \mathbf{w}^T \mathbf{x}_0$ .

### 3 Cone

#### 3.1 Definition

A cone  $\mathcal{K}$  is the set of any point  $\mathbf{x}$  that satisfied: If  $\mathbf{x} \in \mathcal{K}$ , then, for any  $\theta \geq 0$ ,  $\theta\mathbf{x} \in \mathcal{K}$ . A **convex** cone, as its' name suggest, is a cone which is also a convex set.

#### 3.2 Property of Convex Cone

Assume a cone  $\mathcal{K}$  is a convex cone, then, it satisfied:

If  $\mathbf{x}_1 \in \mathcal{K}$  and  $\mathbf{x}_2 \in \mathcal{K}$ , then, for any  $0 \leq \theta \leq 1$ ,  $\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2 \in \mathcal{K}$ .

Furthermore, for any  $p, q > 0$ , the expression  $p\mathbf{x}_1 + q\mathbf{x}_2$  can be written as:

$$p\mathbf{x}_1 + q\mathbf{x}_2 = (p + q)\left(\frac{p}{p + q}\mathbf{x}_1 + \frac{q}{p + q}\mathbf{x}_2\right) \quad (8)$$

Following the property of convex set, the point  $(\frac{p}{p+q}\mathbf{x}_1 + \frac{q}{p+q}\mathbf{x}_2) \in \mathcal{K}$ . Additionally, since  $(p + q) \in \mathcal{R}$  and  $p + q > 0$ , the definition of cone indicate  $(p + q)(\frac{p}{p+q}\mathbf{x}_1 + \frac{q}{p+q}\mathbf{x}_2) \in \mathcal{K}$ . As a result, it can be seen within a convex cone  $\mathcal{K}$ :

If  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{K}$ , then, for any  $\theta_1, \theta_2 > 0$ ,  $\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 \in \mathcal{K}$

#### 3.3 Dual Cone

In geometric, a point  $\mathbf{x}$  can be defined as the end point of a vector that start with the original point( $\mathbf{0}$ ) and terminate at point  $\mathbf{x}$ , which means, any point  $\mathbf{x}$  can also be thought as a vector  $\mathbf{x} - \mathbf{0}$ . Base on this, the dual cone of a cone  $\mathcal{K}$ , denote as  $\mathcal{K}^*$ , is defined as the set of any point  $\lambda$  which satisfied  $\lambda^T \mathbf{x} > 0$  for all  $\mathbf{x} \in \mathcal{K}$ , which means, the **vector**  $\lambda$  have the angle smaller than 90 degree for any **vector**  $\mathbf{x} - \mathbf{0}$  defined by the point  $\mathbf{x}$  in  $\mathcal{K}$ .

Furthermore, the expression  $\lambda^T \mathbf{x} > 0$  define a half-space according to equation 6, where the fixed point  $\mathbf{x}_0$  is the original point. As a result, the dual cone of  $\mathcal{K}$  can also be illustrated as:

A point  $\lambda$  is in  $\mathcal{K}^*$  if the half-space with the **vector**  $\lambda$  as normal vector, which expressed as  $\lambda\mathbf{x} > 0$ , contain the original cone  $\mathcal{K}$ .

#### 3.4 Generalized Inequality

We can define the generalized inequality based on the concept of cone:

$\mathbf{x}$  is smaller than  $\mathbf{y}$  respect to a specific cone  $\mathcal{K}$  if  $\mathbf{y} - \mathbf{x} \in \mathcal{K}$ . Denote as:  $\mathbf{x} \prec_{\mathcal{K}} \mathbf{y}$

Start from the definition of generalized inequality, we can define **minimum** and **minimal** value of a set  $\mathcal{S}$  with respect to a specific cone  $\mathcal{K}$ :

The point  $\mathbf{x}_0 \in \mathcal{S}$  is said to be the **minimum** of  $\mathcal{S}$  if for any other  $\mathbf{y} \in \mathcal{S}$ , the expression  $\mathbf{x} \preceq_{\mathcal{K}} \mathbf{y}$  always holds.

Before we move to the definition of **minimal** point, we have to realize an important fact with respect to generalized inequality: When  $\mathbf{y} \not\preceq_{\mathcal{K}} \mathbf{x}$  hold, it **does not** mean  $\mathbf{x} \preceq_{\mathcal{K}} \mathbf{y}$ , which means, even though  $\mathbf{y}$  is not smaller than  $\mathbf{x}$  (with respect to a specific cone  $\mathcal{K}$ ), it **does not** implies that  $\mathbf{x}$  is smaller or equal to  $\mathbf{y}$ . This fact implies that a point smaller than any other element of a set is not equal to there is no other element within this set that smaller than the point. We can now introduce the definition of **minimal** point with respect to cone  $\mathcal{K}$ :

A point  $\mathbf{x}_0$  is minimal point of  $\mathcal{S}$  if for any  $\mathbf{y} \in \mathcal{S}$ ,  $\mathbf{y} \not\preceq_{\mathcal{K}} \mathbf{x}_0$  always hold unless  $\mathbf{y} = \mathbf{x}_0$

Which means, if  $\mathbf{x}_0$  is a minimal point of  $\mathcal{S}$ , there exist no element  $\mathbf{y}$  within  $\mathcal{S}$  which satisfied  $\mathbf{y} \preceq_{\mathcal{K}} \mathbf{x}_0$  (There is no element in  $\mathcal{S}$  that smaller than  $\mathbf{x}_0$ ). Furthermore, according to the fact that mentioned above, for some  $\mathbf{y} \in \mathcal{S}$ , even though  $\mathbf{y} \not\preceq_{\mathcal{K}} \mathbf{x}_0$ ,  $\mathbf{x}_0$  does **not** need to satisfy  $\mathbf{x}_0 \preceq_{\mathcal{K}} \mathbf{y}$ . As a result,  $\mathbf{x}_0$  does not need to be the minimum point.

#### 3.5 Minimum and Minimal with Dual Cone Perspective

If a point  $\mathbf{x}_0$  is a minimum point of  $\mathcal{S}$  with respect to cone  $\mathcal{K}$ , then, for any  $\mathbf{x} \in \mathcal{S}$ , there is:

$$\mathbf{x} - \mathbf{x}_0 \in \mathcal{K}$$

According to the definition of dual cone, for the element  $\mathbf{v}$  within  $\mathcal{K}$ , it must satisfy:

$$\lambda^T \mathbf{b} > 0$$

Where  $\lambda \in \mathcal{K}^*$ . Since  $\mathbf{x} - \mathbf{x}_0 \in \mathcal{K}$ , we have:

$$\lambda^T (\mathbf{x} - \mathbf{x}_0) > 0$$

Which define a hyper-plane and a half-space with  $\mathbf{x}_0$  as the fixed point according to equation 6. As a result, the minimum point of a set  $\mathcal{S}$  with respect to a cone  $\mathcal{K}$  and its' dual cone  $\mathcal{K}^*$  can be illustrated as:

If  $\mathbf{x}_0$  is minimum point, then, for any  $\lambda \in \mathcal{K}^*$ , the hyper-plane and half-space defined by  $\lambda$  and  $\mathbf{x}_0$ :  $\lambda^T (\mathbf{x} - \mathbf{x}_0)$  should contain entire  $\mathcal{S}$ .

## 4 Convex Function

### 4.1 Definition

A function  $f(\mathbf{x})$  is called convex function if it satisfied:

$$f(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) \leq \theta f(\mathbf{x}_1) + (1 - \theta) f(\mathbf{x}_2) \quad (9)$$

for any  $\mathbf{x}_1, \mathbf{x}_2 \in \text{dom} f$  and  $0 \leq \theta \leq 1$ .

### 4.2 Conditions

A function  $f(\mathbf{x})$  is a convex function if it satisfied one of the following conditions:

#### 4.2.1 First-Order Condition

If  $f(\mathbf{x})$  is differentiable, then, for any  $\mathbf{x} \in \text{dom} f$  and an arbitrary  $\mathbf{x}_0$ , the first-order derivative of  $f$  should satisfy:

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \quad (10)$$

The right hand side of this inequality is the first-order Taylor expression, which can be written as:

$$g(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Therefore, the first-order condition can be explained as: For any point  $\mathbf{x}$ , the value of function  $f$  at this point is greater or equal to the value of  $g$  at the same point  $\mathbf{x}$ . In  $\mathcal{R}^2$ , the function  $g(\mathbf{x})$  define a line that is tangent to  $f(\mathbf{x})$  at  $(\mathbf{x}_0, f(\mathbf{x}_0))$ . Moreover, in  $\mathcal{R}^n$ ,  $g(\mathbf{x})$  indeed define a *support* hyper-plane as the function  $g(\mathbf{x})$  can be written as:

$$\nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) - (g(\mathbf{x}) - f(\mathbf{x}_0)) = 0$$

Which is the same as:

$$\begin{bmatrix} \nabla f(\mathbf{x}_0) \\ -1 \end{bmatrix}^T \left( \begin{bmatrix} \mathbf{x} \\ g(\mathbf{x}) \end{bmatrix} - \begin{bmatrix} \mathbf{x}_0 \\ f(\mathbf{x}_0) \end{bmatrix} \right) = 0 \quad (11)$$

It is obvious that the above equation define a hyper-plane. Hence, the first-order condition can also be expressed as: For any  $\mathbf{x} \in \text{dom} f$ , the point  $(\mathbf{x}, f(\mathbf{x}))$  is above the hyper-plane defined by equation 11 through some fixed point  $(\mathbf{x}_0, f(\mathbf{x}_0))$ , which means, it is in the half-space given by this hyper-plane. This can be proved through:

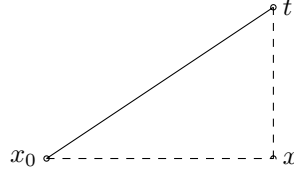
$$\begin{bmatrix} \nabla f(\mathbf{x}_0) \\ -1 \end{bmatrix}^T \left( \begin{bmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{bmatrix} - \begin{bmatrix} \mathbf{x}_0 \\ f(\mathbf{x}_0) \end{bmatrix} \right) \leq 0$$

Which define a half-space. Furthermore, when  $\nabla f(\mathbf{x}_0) = 0$  in some point  $(\mathbf{x}_0, f(\mathbf{x}_0))$ , the equation 10 become:

$$f(\mathbf{x}) \geq f(\mathbf{x}_0)$$

Hence,  $f(\mathbf{x}_0)$  is the minimum value of function  $f$ , which is why for convex function, the point that first-order derivative equal to zero is the minimizer of this function.

Indeed, we can have a geometry interpretation of term  $\nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$  over  $\mathcal{R}^2$ :



The straight line that pass through has the slope  $\nabla f(\mathbf{x}_0)$ , the term  $\nabla f(x_0)(x - x_0)$  is actually the length of the line segment between points  $t$  and  $x$ . Hence, if  $\nabla f(x_0)(x - x_0) \geq 0$ , the length of line segment between  $x$  and  $t$  is positive and hence the point  $t$  is above the point  $x_0$ . By contrast, if  $\nabla f(x_0)(x - x_0) \leq 0$ , the point  $t$  is below the point  $x_0$ . Furthermore, we can also see that the term  $f(x_0) + \nabla f(x_0)(x - x_0)$  is actually the value of  $f(x_0)$  plus the length of line segment between  $t$  and  $x$ .

#### 4.2.2 Second-Order Condition

If function  $f(\mathbf{x})$  is twice differentiable, then,  $f(\mathbf{x})$  is convex if and only if it's second-order Hessian Matrix is positive semi-definite:

$$\nabla^2 f(\mathbf{x}) \succeq 0 \quad (12)$$

This condition is more often be used to verify whether a function is convex than the first-order condition as it is more convenient to compute Hessian Matrix and determine whether it is positive semi-definite. By contrast, the first-order condition works more like a property that indicate for convex function, it's minimum value exist at point  $\mathbf{x}_0$  which satisfied  $\nabla f(\mathbf{x}_0) = 0$ .

Even though the second-order condition provide a good way to verify the convexity of a function  $f$ , a more easier way to verify the convexity is to prove that  $f(\mathbf{x})$  is constructed by some operations that preserve convexity. These operation will be describe in the next part.

### 4.3 Operation that Preserve Convexity

A function  $f(\mathbf{x})$  is convex if it can be constructed through one of the following operation:

#### 4.3.1 Non-negative Weighted Sum

Function  $f(\mathbf{x})$  is convex if it can be constructed through:

$$f(\mathbf{x}) = \sum_{i=1}^n a_i f_i(\mathbf{x})$$

Where  $a_i \geq 0$  and  $f_i(\mathbf{x})$  is convex.

#### 4.3.2 Point-wise maximum

Function  $f(\mathbf{x})$  is convex if it is the point-wise maximum of some convex function:

$$f(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})\}$$

Where  $f_i(\mathbf{x})$  is convex. Based on this property, it can also be proved that function  $f(\mathbf{x})$  is convex if it is the supreme of function  $g(\mathbf{x}, \mathbf{y})$  over  $\mathbf{y}$ :

$$f(\mathbf{x}) = \sup_{\mathbf{y}} g(\mathbf{x}, \mathbf{y})$$

As  $f(\mathbf{x})$  can be constructed through:

$$f(\mathbf{x}) = \max\{g(\mathbf{x}, \mathbf{y}_1), g(\mathbf{x}, \mathbf{y}_2), \dots, g(\mathbf{x}, \mathbf{y}_n)\}$$

Where  $\text{dom}_{\mathbf{y}} g(\mathbf{x}, \mathbf{y}) = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$

#### 4.3.3 Composition

Function  $f(\mathbf{x})$  is convex if it is a composition of other function:

$$f(\mathbf{x}) = h(g(\mathbf{x}))$$

with function  $h(\mathbf{y}), g(\mathbf{y})$  under some constrains, which can be derived directly from:

$$\nabla^2 f(\mathbf{x}) = \nabla^2 h(g(\mathbf{x}))(\nabla g(\mathbf{x}))^2 + \nabla h(g(\mathbf{x}))\nabla^2 g(\mathbf{x}) \geq 0 \quad (13)$$

e.g. If  $g(\mathbf{y}), h(\mathbf{y})$  is convex and  $h(\mathbf{y})$  is non-decreasing, then,  $\nabla_2 h(g(\mathbf{x})) \geq 0$ ,  $\nabla h(g(\mathbf{x})) \geq 0$  and  $\nabla_2 g(\mathbf{x}) \geq 0$ . Hence, equation 13 always hold, which means  $f(\mathbf{x})$  is convex. The other constrains can be derived based on the similarly.

## 4.4 Epigraph

The epigraph is another thing that we can used to prove convexity. Geometrically, the epigraph of a function  $f(\mathbf{x})$  is the area that above the graph of  $f(\mathbf{x})$ . Mathematically, the epigraph of a function  $f(\mathbf{x})$  can be defined as a set of points that satisfied:

$$\{(\mathbf{x}, t) | f(\mathbf{x}) \leq t\}$$

The function  $f(\mathbf{x})$  is convex if the point within its' epigraph form a convex set.

## 5 Conjugate Function

### 5.1 Definition

The conjugate function of  $f(\mathbf{x})$ , denote as  $f^*$ , can be defined as:

$$f^*(\mathbf{y}) = \sup_{\mathbf{x}} (\mathbf{y}^T \mathbf{x} - f(\mathbf{x})) \quad (14)$$

In order to complete the definition of conjugate function  $f^*(\mathbf{y})$ , we need to define the domain of  $f^*$ , i.e. the range of  $\mathbf{y}$ : The domain of  $f^*$  is the  $\mathbf{y}$  that make  $f^*(\mathbf{y})$  finite:

$$\text{dom} f^*(\mathbf{y}) = \{\mathbf{y} | -\infty < f^*(\mathbf{y}) < \infty\}$$

### 5.2 Find Conjugate Function

A simple procedure that may be used to find conjugate function of  $f(\mathbf{x})$  is:

1. Establish the domain  $\text{dom} f^*(\mathbf{y})$ , this may be done by observing or experiment, e.g. start from lower dimension (2-dimension), find the range of  $\mathbf{y}$  that cause  $f^*(\mathbf{y})$  goes to infinity and exclude these range.
2. Find the supreme of  $\mathbf{y}^T \mathbf{x} - f(\mathbf{x})$  over  $\mathbf{x}$ . Some general method to do this are:
  - Take the gradient of  $\mathbf{x}$  and set it to zero.
  - By observing.

## 6 Quasiconvex Function

Before we introduce the definition of quasiconvex function, we need first familiar with the concept of  $\alpha$ -sublevel set. The  $\alpha$ -sublevel set  $\mathcal{S}_\alpha$  of a function  $f(\mathbf{x})$  is the set of  $\mathbf{x}$  that satisfied  $f(\mathbf{x}) \leq \alpha$ :

$$\mathcal{S}_\alpha = \{\mathbf{x} | f(\mathbf{x}) \leq \alpha\} \quad (15)$$

Based on the concept of  $\alpha$ -sublevel set, we can give the definition of quasiconvex function:

The function  $f(\mathbf{x})$  is quasiconvex if and only if for any  $\alpha \in \mathcal{R}$ , the  $\alpha$ -sublevel set of  $f(\mathbf{x})$  is convex.

Indeed, for any  $\mathbf{y} \in \text{dom} f$ , we can make  $\alpha = f(\mathbf{y})$ . Hence, the  $\alpha$ -sublevel set  $\mathcal{S}_\alpha$  is convex. By noticing that  $f(\mathbf{y}) \leq f(\mathbf{y})$  always hold, we can conclude that  $\mathbf{y} \in \mathcal{S}_\alpha$ . Therefore, for any  $\mathbf{x} \in \mathcal{S}_\alpha$  and  $\mathbf{y}$ ,  $\theta \mathbf{x} + (1 - \theta) \mathbf{y} \in \mathcal{S}_\alpha$  and hence,  $f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq f(\mathbf{y})$ . As a result, we can draw a simple conclude about quasiconvex function  $f$ : For any  $\mathbf{x}, \mathbf{y} \in \text{dom} f$  and  $0 \leq \theta \leq 1$ , there is

$$f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\} \quad (16)$$

This is the first order property of quasiconvex function.

## 7 Convex Optimization Problem

After we have described some basic concept, we can now start to formalize the concept of convex optimization problem. We will first give the definition of general optimization problem.

## 7.1 General Optimization Problem

The general optimization problem has the stand form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & h_j(\mathbf{x}) = 0 \quad j = 1, 2, \dots, m \end{aligned} \quad (17)$$

The *domain*  $\mathcal{D}$  of this optimization problem is:

$$\mathcal{D} = \text{dom}f_0 \cap \dots \cap \text{dom}f_n \cap \text{dom}h_1 \cap \dots \cap \text{dom}h_m$$

Indeed, we can think of that the constrains  $f_i(\mathbf{x})$  and  $h_j(\mathbf{x})$  define a set  $\mathcal{C}$  of  $\mathbf{x}$ :

$$\mathcal{C} = \{\mathbf{x} \mid f_i(\mathbf{x}) \leq 0 (i = 1, \dots, n), h_j(\mathbf{x}) = 0 (j = 1, \dots, m)\}$$

And the goal of optimization problem is to find a  $\mathbf{x}^* \in \mathcal{C}$  that minimize  $f_0(\mathbf{x})$ . The set  $\mathcal{C}$  here is called feasible set and a  $\mathbf{x}$  is called feasible if  $\mathbf{x} \in \mathcal{C}$ . Generally, we are more interesting in a special type of optimization problem, which is convex optimization problem.

## 7.2 Convex Optimization Problem

Convex optimization problem is a specific type of optimization problem, in which, the function  $f_i(\mathbf{x}) (i = 0, 1, \dots, n)$  is convex and  $h_j(\mathbf{x}) (j = 1, 2, \dots, m)$  is affine(linear). Mathematically, convex optimization problem can be expressed as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & \mathcal{A}\mathbf{x} = \mathbf{b} \end{aligned} \quad (18)$$

Where  $f_i(\mathbf{x}) (i = 0, 1, \dots, n)$  is convex.

## 7.3 Special Cases

Some convex optimization problem with special constrain have typical solutions. Particularly, the convex optimization problem that only have equality constrains:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & \mathcal{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

In this case, we first notice that if we can find a  $\mathbf{x}_0$  that satisfied  $\mathcal{A}\mathbf{x}_0 = \mathbf{b}$ , then, the following equation:

$$\mathcal{A}(\mathbf{z} + \mathbf{x}_0) = \mathbf{b} \quad (19)$$

always hold for any  $\mathbf{z}$  that satisfied  $\mathcal{A}\mathbf{z} = 0$ . Hence, we can conclude that for every  $\mathbf{x}$  that satisfied  $\mathcal{A}\mathbf{x} = \mathbf{b}$ , it can be decomposed as a variable  $\mathbf{z}$  plus a fixed vector  $\mathbf{x}_0$ :

$$\mathbf{x} = \mathbf{z} + \mathbf{x}_0 \quad (20)$$

Where  $\mathbf{z}$  is in the null space of  $\mathcal{A}$  ( $\mathbf{z} \in \mathcal{N}(\mathcal{A})$ ). Furthermore, we can find the biases of null space of  $\mathcal{A}$ , denote as  $\mathcal{M}$  and express  $\mathbf{z}$  as:

$$\mathbf{z} = \mathcal{M}\mathbf{y} \quad (21)$$

Where  $\mathbf{y} \in \mathcal{R}^n$ . The Equation 21 means that  $\mathbf{z}$  is arbitrary linear combination of the biases of  $\mathcal{N}(\mathcal{A})$ . By the combination of Equation 20 and 21, the original problem can be refined as:

$$\min_{\mathbf{y}} \quad f_0(\mathcal{M}\mathbf{y} + \mathbf{x}_0)$$

With the feasible set  $\mathcal{C}_{\mathbf{y}} = \{\mathbf{y} \mid \mathbf{y} \in \mathcal{R}^n\}$ . By transforming the feasible to  $\mathcal{R}^n$ , we reduce the difficulty to find the optimal value of  $\mathbf{f}_0$ . Indeed, this is the idea that where *Lagrangian* come from, which will be explained in detail later.

## 7.4 Equivalent Problem

By inspecting the transforming we have done in section 7.3, it can be seen that even though we introduce a new variable  $\mathbf{y}$  and define a new feasible set  $\mathcal{C}_{\mathbf{y}}$ , we did not change the original feasible set  $\mathcal{C}_{\mathbf{x}}$  of  $\mathbf{x}$ , i.e.  $\mathcal{C}_{\mathbf{x}} = \{\mathbf{x} | \mathcal{A}\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} | \mathbf{x} = \mathcal{M}\mathbf{y} + \mathbf{x}_0, \mathbf{y} \in \mathcal{R}^n\}$ . Hence, we did not change the optimal point of  $f_0$ .

This observation tell us that we can construct a equivalent optimization problem with the original feasible set and optimal point unchanged. One important technology that used to construct equivalent problem is by introducing new variable. Particularly, when the objective function  $f_0$  has the form  $f_0(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$  and satisfied  $p(t\mathbf{x}) = tp(\mathbf{x}), q(t\mathbf{x}) = tq(\mathbf{x})$  for any  $t$ . Hence, the convex optimization problem is:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})} \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & h_j(\mathbf{x}) = 0 \quad j = 1, 2, \dots, m \end{aligned}$$

The property  $p(t\mathbf{x}) = tp(\mathbf{x}), q(t\mathbf{x}) = tq(\mathbf{x})$  implied  $\frac{p(t\mathbf{x})}{q(t\mathbf{x})} = \frac{p(\mathbf{x})}{q(\mathbf{x})}$  and means that we can multiple  $\mathbf{x}$  with any  $t$  without change the value of  $f_0(\mathbf{x})$  (the optimal value will not be changed under this multiplication operation). As a result, we can always find a  $t$  that satisfied  $q(t\mathbf{x}) = 1$  and  $\frac{p(t\mathbf{x})}{q(t\mathbf{x})} = \frac{p(\mathbf{x})}{q(\mathbf{x})} = p(\mathbf{x})$ , which means, we can construct a equivalent problem as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & p(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & h_j(\mathbf{x}) = 0 \quad j = 1, 2, \dots, m \\ & q(t\mathbf{x}) - 1 = 0 \end{aligned} \tag{22}$$

Since the last constrain  $q(t\mathbf{x}) - 1 = 0$  define the feasible set of  $t$ , which is unrelated to the objective function  $p(\mathbf{x})$ , this constrain can be ignored. Hence, the problem can be further refined as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & p(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & h_j(\mathbf{x}) = 0 \quad j = 1, 2, \dots, m \end{aligned} \tag{23}$$

We can verify the problem is equivalent by checking the optimal value and feasible set is unchanged. More general, we can multiple both  $p(\mathbf{x})$  and  $q(\mathbf{x})$  with a variable  $t$  and force  $tq(\mathbf{x}) = 1$ . As a result, we successfully remove the numerator  $q(\mathbf{x})$  and construct an equivalent problem as follow:

$$\begin{aligned} \min_{\mathbf{x}} \quad & tp(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, n \\ & h_j(\mathbf{x}) = 0 \quad j = 1, 2, \dots, m \\ & tq(\mathbf{x}) - 1 = 0 \end{aligned}$$