NAME

Prof. L

CS/ECE/ISyE 524 Final Exam

May 10, 2024
7:45AM-9:45AM

1. Please spread out in the room as far as possible. (We may ask you to move to another location)
2. Please put your student ID out on your desk—we will be checking IDs during the exam
3. The exams will be scanned into Gradescope.
   - **Please write your name *very neatly* inside the box at the top of this page**.
   - Please write your name (or initials) neatly on the top of each page of the exam. This will help us ensure we associate exams with the right students
   - If you do not write your answer in the box in the space provided, you are unlikely to get full credit for your answer
   - If you need additional scratch space, you should use the blank (back) pages of the exam.
4. Be sure to write valid *mathematical* syntax in your written answers (unless Julia/JuMP syntax is specifically required).
5. The number of points for each problem is displayed at the end of the problem's title. The time required for me to complete each question is also listed. Please use your time wisely. Scan through the exam and do the problems you are sure to know how to do and complete.
6. When time is up, put down your writing utensil and stay in your seat. We will collect the exams. If you write on your exam (even to put your name) after the time is up, we will reduce your final grade by 25%.
7. **Good luck!** Don't panic. We are rooting for you.

Here are the point available for each problem and the time it took me to do it.

Problem	**Points**	**Prof. Linderoth Time** (min.)
1	28	3
2	33	6
3	10	3
4	10	3
5	28	7
6	16	5
Total:	125	27

Cheat Sheet

Minimization LP	Maximization LP
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

- Second-Order (Lorentz) Cone: (of dimension $n+1$)

$$\mathcal{L} := \{(t, x) \in \mathbb{R}^{1+n} : t \geq \|x\|_2\}$$

- Rotated Second-Order Cone: (of dimension $n+2$)

$$\mathcal{R} := \{(u, t, x) \in \mathbb{R}^{2+n} : 2ut \geq x^\mathsf{T} x, u \geq 0, t \geq 0\}$$

- Positive Semidefinite Cone: (of order n)

$$\mathcal{S} := \{X \in \mathbb{R}^{n \times n} : X = X^\mathsf{T}, w^T X w \geq 0\ \forall w \in \mathbb{R}^n\}$$

- Exponential Cone: (always in dimension 3)

$$\mathcal{E} := \{(x, y, z) \in \mathbb{R}^3 : y \exp(x/y) \leq z, y > 0\}$$

Logic statement	**Constraint**
if $z = 0$ then $a^\mathsf{T} x \leq b$	$a^\mathsf{T} x - b \leq Mz$
if $z = 0$ then $a^\mathsf{T} x \geq b$	$a^\mathsf{T} x - b \geq mz$
if $z = 1$ then $a^\mathsf{T} x \leq b$	$a^\mathsf{T} x - b \leq M(1 - z)$
if $z = 1$ then $a^\mathsf{T} x \geq b$	$a^\mathsf{T} x - b \geq m(1 - z)$
if $a^\mathsf{T} x \leq b$ then $z = 1$	$a^\mathsf{T} x - b \geq mz + \epsilon(1 - z)$
if $a^\mathsf{T} x \geq b$ then $z = 1$	$a^\mathsf{T} x - b \leq Mz - \epsilon(1 - z)$
if $a^\mathsf{T} x \leq b$ then $z = 0$	$a^\mathsf{T} x - b \geq m(1 - z) + \epsilon z$
if $a^\mathsf{T} x \geq b$ then $z = 0$	$a^\mathsf{T} x - b \leq M(1 - z) - \epsilon z$

1 True/False

To answer these questions, circle **True** or **False**

1.1 Problem (1 points)

True or False: If the feasible region of a linear program is not unbounded and there is at least one point in the feasible region, then the linear program must have an optimal solution.

True **False**

1.2 Problem (1 points)

True or False: The following linear program has an optimal solution for any values of the parameters $a_{11}, a_{12}, a_{21}, a_{22}$.

$$
\begin{aligned}
\min z = \quad & 5x_1 + 6x_2 \\
\text{subject to:} \quad & a_{11}x_1 + a_{22}x_2 \leq 3 \\
& a_{21}x_1 + a_{22}x_2 \leq 10 \\
& x_1, \quad x_2 \geq 0
\end{aligned}
$$

Note $z^* \geq 0$ and $x = (0, 0)$ is feasible.

True **False**

1.3 Problem (1 points)

Let y be the dual variable for a constraint $x_1 + 3x_2 - x_3 \leq 9$ in a maximization problem.
True or False: If $x_1 + 3x_2 - x_3 = 9$ at an optimal primal solution x_1, x_2, x_3, then its dual variable $y > 0$.

True **False**

Could be $= 0$.

1.4 Problem (1 points)

If a linear program has a feasible point, then it must have an optimal solution.

True **False**

unbounded.

1.5 Problem (1 points)

In a min cost flow problem with the standard flow balance constraints

$$
\sum_{j \in N \ | \ (i,j) \in A} x_{ij} - \sum_{j \in N \ | \ (j,i) \in A} x_{ji} = b_i \quad \forall i \in N
$$

$$
0 \leq x_{ij} \leq U_{ij} \quad \forall (i,j) \in A,
$$

if $\sum_{i \in N} b_i \neq 0$, then the instance *must* be infeasible.

True **False**

1.6 Problem (1 points)

In a min cost flow problem with the standard flow balance constraints

$$\sum_{j \in N \,\mid\, (i,j) \in A} x_{ij} - \sum_{j \in N \,\mid\, (j,i) \in A} x_{ji} = b_i \quad \forall i \in N$$

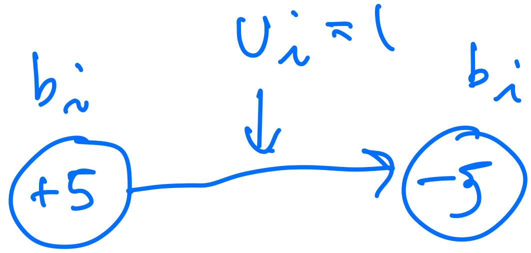
$$0 \le x_{ij} \le U_{ij} \quad \forall (i,j) \in A,$$

if $\sum_{i \in N} b_i = 0$, then the instance *must* be feasible.

True **False**

---

e.g.



2 Short Answer

2.1 Problem (4 points)

In the following linear program with unknown parameter c_3, give a value c_3 such that the optimal primal solution is $x_1^* = 3, x_2^* = 0$, and $x_3^* = 1$ and the optimal dual solution is $y_1^* = 0, y_2^* = 1$. If this is not possible, say "Not Possible," and justify your answer.

$$\begin{aligned} \max 2x_1 + 2x_2 + c_3x_3 \\ x_1 + 3x_2 - x_3 \leq 9 \\ 3x_1 - 2x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Dual: $\min\ 9y_1 + 7y_2$

$y_1 + 3y_2 \geq 2$

$\boxed{3y_1 \geq 2}$

$-y_1 - 2y_2 \geq c_3$

$y_1, y_2 \geq 0$

Answer.

NOT POSSIBLE

$c_3 =$ __________

Impossible: Justification

$y_1^* = 0$ can never be a dual feasible sol'n

2.2 Problem (3 points)

Write the following linear program in the standard form

$$\max\{c^T x \mid Ax \leq b, x \geq 0\},$$

(A maximization problem with all variables ≥ 0 and all inequalities of the form $\leq$).

$$\begin{aligned} \min \quad & 4x_1 - 3x_2 - 7x_3 \\ \text{s.t.} \quad & x_1 - x_3 = 8 \\ & 2x_1 + 2x_2 \leq 7 \\ & x_1, x_2 \geq 0 \\ & x_3 \quad \text{free} \end{aligned}$$

Answer.

$$\begin{aligned} \min\ & 4x_1 - 3x_2 - 7(p_3 - n_3) \\ & x_1 - (p_3 - n_3) \leq 8 \\ & -x_1 + (p_3 - n_3) \leq -8 \\ & 2x_1 + 2x_2 \leq 7 \\ & x_1, x_2, p_3, n_3 \geq 0. \end{aligned}$$

2.3 Problem (2 points)

Suppose we have a minimum-cost network flow problem of the form min $c^T x$ subject to $Ax = b$ and $0 \le x \le q$, where A is the incidence matrix for the network, x are the flows on the edges, q are the capacity constraints, and b is the vector indicating how much material is produced or consumed at each node. Which of the following combination of properties are sufficient to guarantee that any solution x of the problem with have *integer* flows? (circle all that apply)

1. b contains all integer values
2. b and q contain all integer values
3. c and b contain all integer values
4. c and q contain all integer values

2.4 Problem (2 points)

Given two points x and y, we define w as

$$w = (1 - \alpha)x + \alpha y$$

for $\alpha \in \mathbb{R}$. Circle the correct answer(s):

- w is called an affine combination.
- w defines a line.
- w is called a convex combination.

2.5 Problem (2 points)

For a linear program in $n = 2$ variables, we know that the solutions $(4, 1)$ and $(2, 9)$ are both optimal solutions. Find a third optimal solution.

Answer. $\frac{1}{2}\left(\begin{pmatrix}4\\1\end{pmatrix}+\begin{pmatrix}2\\9\end{pmatrix}\right) = \begin{pmatrix}3\\5\end{pmatrix}$ is one

Any $\lambda\begin{pmatrix}4\\1\end{pmatrix} + (1-\lambda)\begin{pmatrix}2\\9\end{pmatrix}$ with $0 \le \lambda \le 1$

2.6 Problem (2 points)

Complete the following definition of a convex combination:? The set X is convex if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0, 1]$,

Answer. $\lambda x_1 + (1-\lambda) x_2 \in X$

2.7 Problem (2 points)

It is possible for a linear program to have exactly this many different optimal solutions: (circle **all** answers that are possible)

(a) None (0)
(b) 1
(c) 2
(d) 100,000
(e) Infinitely many (∞)

2.8 Problem (2 points)

If there are no solutions that satisfy all the constraints of a linear program, what is such a linear program called?

(a) Multiple optimal solutions
(b) Unbounded
(c) Extreme point
(d) Infeasible
(e) Convex
(f) Degenerate

2.9 Problem (6 points)

In the following linear program with unknown parameters c_2 and c_3, give values for parameters c_2 and c_3 such that the optimal primal solution is $x_1^* = 3, x_2^* = 2,$ and $x_3^* = 1$ and the optimal dual solution is $y_1^* = 2, y_2^* = 0$. If this is not possible, say "Not Possible," and justify your answer.

$$\begin{aligned} \max\ & 2x_1 + c_2x_2 + c_3x_3 \\ & x_1 + 3x_2 - x_3 \le 9 \\ & 3x_1 - 2x_3 \le 7 \end{aligned}$$

Dual

$$\begin{aligned} \min\ & 9y_1 + 7y_2 \\ & y_1 + 3y_2 \ge 2 \\ & 3y_1 \ge c_2 \\ & -y_1 - 2y_2 \ge c_3 \\ & y_1 \ge 0,\ y_2 \ge 0. \end{aligned}$$

Answer.

Note: $x_1 + 3x_2 - x_3 = 3 + 6 - 1 = 8 < 9$

so $y_1^* = 0$ at optimal sol'n.

So by Comp. Slackness, this is

$\boxed{\text{NOT POSSIBLE}}$

2.10 Problem (4 points)

Consider the linear program below, where the coefficients c_1 and c_2 are (unknown to you) numbers:

$$\begin{aligned} \max\ z = & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \le 4 \quad (y_1) \\ & x_1 + 5x_2 \le 6 \quad (y_2) \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

There is a primal feasible solution (values of x_1 and x_2) that has objective value $z = 20$. What can you say about the candidate dual solution $(y_1, y_2) = (2, 1)$? Circle one option below and briefly explain.

(a) It is a feasible solution to the dual problem.
(b) It is not a feasible solution to the dual problem.
(c) It is an optimal solution to the dual problem.
(d) It is not possible to say any of the above for sure.

Dual objective of sol'n (2,1) is 4(2)+1(6) = 14

since 14 < 20, (2,1) CANNOT be feasible by weak duality

3 Buuuuuuuuuuuuuuuuuuuuuuuuuurp

Barney Gumble has decided to turn his passion into profit. As such, he started the company Barney's Fine Bourbon (BFB)—a bourbon distilling company specializing in cheap, low-quality bourbon for distribution in Southern Wisconsin. Barney is trying to determine his long range production and inventory policy for the next T years. Each year $t \in \{1, 2, \dots, T\}$, Barney must decide how many tons of corn mash to purchase at a cost of α_t dollars/ton. To produce 50 barrels of bourbon, it takes 1 ton of corn mash. The mash is then distilled and fermented for one year, producing bourbon. In each year $t \in \{1, 2, \dots, T\}$, the maximum amount of corn mash that Barney can purchase is given by the parameter u_t. Corn mash rots if stored, so any mash purchased must be immediately turned into bourbon. However, in any period, Barney can choose to carry bourbon over in inventory at a cost of γ dollars per barrel. Each period $t \in \{1, 2, \dots, T\}$, Barney must meet a demand of d_t from a combination of production and inventory. Initially Barney has Q barrels of bourbon in inventory. You will write a linear program using the following decision variables:

- x_t : Tons of mash to buy in year $t \in T$
- I_t : Inventory of bourbon (in barrels) at the end of yeat $t \in T$

These are the only decision variables that you need.

3.1 Problem (5 points)

Write Barney's objective in a linear program: to minimize the total cost of purchasing corn mash plus his total inventory holding cost

Answer.

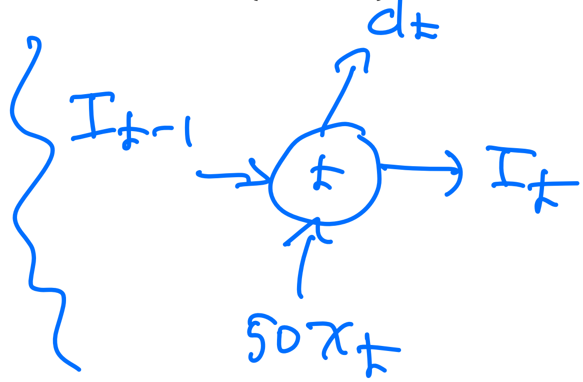
$$\min \sum_{t \in T} (\alpha_t x_t + \gamma I_t)$$

3.2 Problem (5 points)

Write a constraint that says that Barney must meet his demand in each period $t \in \{2, 3, \dots T\}$

Answer.

$$50 x_t + I_{t-1} = d_t + I_t \quad \forall t = 2, \dots, T$$



3.3 Problem (5 points)

Write a constraint that says that Barney must meet his demand in the period $t = 1$

Answer.

$$Q + 50x_1 = d_1 + I_1$$

3.4 Problem (5 points)

Write constraints that limit the number of tons of corn mash Barney can purchase in each period.

Answer.

$$x_t \leq U_t \quad \forall\, t \in \{1, 2, \ldots, T\}$$

3.5 Problem (2 points)

Write constraints that ensure that the purchase amounts and inventory amounts are non-negative quantities

Answer.

$$x_t, I_t \geq 0 \quad \forall\, t \in \{1, 2, \ldots, T\}$$

4 Transporting for the Office

At Dunder-Mifflin, Dwight Schrute is in charge of operations. There is a set of suppliers I that must produce paper to meet demand for a set of customers J. Each supplier $i \in I$ has a capacity b_i, and each customer $j \in J$ has a demand d_j. There is a cost of c_{ij} for transporting one unit of demand from supplier $i \in I$ to customer $j \in J$. We will help Dwight keep customers loyal by formulating a linear programming model to determine how to meet each customer's demand at a minimum transportation cost, while not exceeding the supplier capacities. In formulating your answers, please use the following decision variables:

- Let x_{ij} be a decision variable representing the amount of product $i \in I$ to be shipped to customer $j \in J$.

4.1 Problem (3 points)

Write the objective function, to minimize total transportation cost

Answer.

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

4.2 Problem (3 points)

Write the constraints that ensure that each customer has their demand met

Answer.

$$\sum_{i \in I} x_{ij} \geq d_j \quad \forall j \in J$$

4.3 Problem (3 points)

Write the constraints that ensure that each supplier does not supply more than their capacity.

Answer.

$$\sum_{j \in J} x_{ij} \leq b_i \quad \forall i \in I$$

4.4 Problem (4 points)

Now suppose instead of minimizing the total transportation cost, Dwight is interested in minimizing the *maximum* transportation cost that Dunder-Mifflin would pay to any one customer. Adapt your answer to write this new problem as a linear program. State any new variables and new constraints you may need, as well as how you would change the objective function.

Answer.

Note transportation cost to customer j

is $\sum_{i \in I} c_{ij} x_{ij}$

Add New var z

Obj:

$$\min \quad z$$

New Cons:

$$z \geq \sum_{i \in I} c_{ij} x_{ij} \quad \forall j \in J$$

5 HOYVIN-GLAVIN

Prof. John Frink needs your help to schedule a set of Montgomery Burns' power plants. Burns has a set of 10 power plants $P = \{1, 2, \ldots 10\}$. The first four are coal plants $C = \{1, 2, 3, 4\}$. The next four are "nuculur" plants $N = \{5, 6, 7, 8\}$, and the final 2 are wind generation facilities $W = \{9, 10\}$. If a power plant $i \in P$ is "turned on," it is able to produce any amount of power in the numerical range given by values $[\ell_i, u_i]$ $\forall i \in P$

We need to meet Springfield's power demand, and Springfield is broken into a set of districts D. Each district $j \in D$ requires a total amount of power b_j (in MWh). There is a *fixed cost* c_i of operating plant $i \in P$, and we will seek to meet demand that minimizes the sum of these fixed costs, subject to a variety of operating constraints

For this problem, we will need the following decision variables:

- x_{ij}: The MWh of power transmit from plant $i \in P$ to district $j \in D$
- y_i: The total power produced (in MWh) at plant $i \in P$
- z_i: A binary variable that is one if and only if plant $i \in P$ is operated.

5.1 Problem (5 points)

Write Frink's objective function: to minimize the total cost of operating all plants.

Answer.

$$\min \sum_{i \in P} c_i z_i$$

5.2 Problem (5 points)

Write constraints that relate the total total power produced at plant $i \in P$ to the total power shipped from plant $i \in P$

Answer.

$$y_i = \sum_{j \in D} x_{ij} \quad \forall i \in P$$

5.3 Problem (5 points)

Write constraints that ensure that each district $j \in D$ receives at least its required total amount of power.

Answer.

$$\sum_{i \in P} x_{ij} \geq b_j \quad \forall j \in D$$

5.4 Problem (5 points)

Write constraints that say we cannot produce power at plant $i \in P$ unless it is turned on.

Answer.

$$y_i \leq U_i z_i \quad \forall i \in P$$

5.5 Problem (5 points)

Write constraints that say that if plant $i \in P$ is turned on, then it must produce at least at a level ℓ_i

Answer.

$$\ell_i z_i \leq y_i \quad \forall i \in P$$

Now we will add a few additional operating requirements:

5.6 Problem (5 points)

Write constraints that each district must receive at least 5% of its power from the wind plants $W = \{9, 10\}$

Answer.

$$\sum_{i \in W} x_{ij} \geq .05 b_j \quad \forall j \in D$$

$|C| = 4, \quad |W| = 2$

5.7 Problem (10 points)

Write constraints that say that if all of the coal plants are opened, then we must operate both of the wind plants. (You may need to introduce an auxiliary binary variable).

Answer. Model: $\sum_{i \in C} z_i \geq |C| \Rightarrow \sum_{i \in W} z_i \geq |W|$

$\Downarrow \delta = 1 \Rightarrow$

A $\sum_{i \in C} z_i \leq |C| - (1 - \delta)$

C $\delta \in \{0, 1\}$

B $\sum_{i \in W} z_i \geq |W| \delta$

5.8 Problem (4 points)

Bonus Problem: (Only do if you have time). Mayor Quimby wants to ensure that all districts are treated in an equitable manner. Write a constraint that says that if the absolute difference in total nuclear power delivered to any pair of districts is more than γ (in MHw), then a penalty of Δ is imposed in the objective. You will need to introduce two new variables. State the constraints you need to add and the new objective.

Answer. Note: $\left| \sum_{i \in P} x_{ij} - \sum_{i \in P} x_{ik} \right|$ is absolute difference for districts j, k.

Model:

A $\sum_{i \in P} x_{ij} - \sum_{i \in P} x_{ik} \geq \gamma \Rightarrow u_{jk} = 1$

B $\sum_{i \in P} x_{ik} - \sum_{i \in P} x_{ij} \geq \gamma \Rightarrow v_{jk} = 1$

Then add to objective term

$\sum_{j \in D} \sum_{k \in D} (u_{jk} + v_{jk}) \cdot \Delta$

$j \in D \quad k \in D$

(I interpreted as must invoke penalty for each violation)

$M = \left(\sum_{i \in P} U_i - \gamma\right)$

$\varepsilon = 0.0001$

A is

$$\sum_{i \in P} x_{ij} - \sum_{i \in P} x_{ik} \leq \gamma + M\,U_{jk} - \varepsilon\left(1 - U_{jk}\right)$$

is

(same M & ε)

B

$$\sum_{i \in P} x_{ik} - \sum_{i \in P} x_{ij} \leq \gamma + M\,U_{jk} - \varepsilon\left(1 - U_{jk}\right)$$

Note: can use same variable!

Rewriting:

$$\sum_{i \in P} x_{ij} - \sum_{i \in P} x_{ik} \leq \gamma + \left[\sum_{i \in P} U_i - \gamma\right] U_{jk} - 0.0001\left(1 - U_{jk}\right) \quad \forall j \in D,\ \forall k \in D,\ j \neq k$$

$$U_{jk} \in \{0, 1\} \quad \forall j \in D \quad \forall k \in D,\ j \neq k$$

Add $\sum_{j \in D} \sum_{k \in D} \Delta \cdot U_{jk}$ to objective

6 Frink-A-Linear

Prof. Frink would like to build a piecewise-linear approximation of the function x^3 using three pieces on the interval $0 \le x \le 3$. Specifically he would like to model the relationship:

$$t = \begin{cases} x & 0 \le x \le 1 \\ 7x - 6 & 1 \le x \le 2 \\ 19x - 30 & 2 \le x \le 3 \end{cases}$$

6.1 Problem (10 points)

Write this same relationship for t using integer and continuous variables. Introduce binary variables $y_i, i = 1, 2, 3$ to indicate the interval in which x lies, and continuous variables $w_i, i = 1, 2, 3$ to complete the model.

Answer.

$y_1 + y_2 + y_3 = 1$

$w_1 + w_2 + w_3 = x$

$0 \le w_1 \le y_1$

$y_2 \le w_2 \le 2y_2$

$2y_3 \le w_3 \le 3y_3$

$t = w_1 + 7w_2 + 19w_3 - 6y_2 - 30y_3$

7 A Two-Variable LP (18 total points)

Lockheed-Martin has debris from a UFO crash that they have used to reverse engineer anti-gravity technology. They now produce two types of AI-controlled drones that use the antigrav tech: drone A and drone B. Each A-type drone requires 3 tons of metal, and each B-type drone requires 2 tons of metal. The sale of an A-type drone makes \$3 profit, and the sale of a B-type drone makes \$1.5 profit. The number of A-type drones produced should be at least twice the number of B-type number produced. There are 12 tons of metal available to produce drones. Assume that all drones produced are sold, and facility has a production schedule that allows production of fractional numbers of drones.

7.1 Problem (5 points)

Let x_1 be the number of A-type drones produced, and let x_2 be the number of B-type drones produced. Formulate a linear program to maximize Lockheed Martin's profit under its production constraints.

Answer.

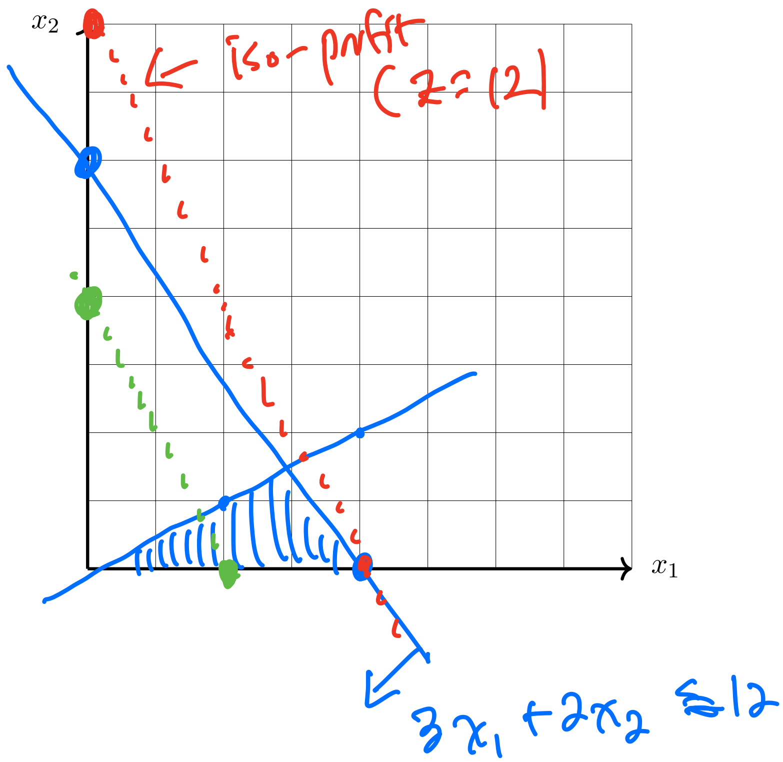
$$
\begin{aligned}
\text{Max } \quad & 3x_1 + \tfrac{3}{2}x_2 \\
\text{s.t. } \quad & 3x_1 + 2x_2 \le 12 \\
& x_1 - 2x_2 \ge 0 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

7.2 Problem (5 points)

Graph the feasible region for this problem in the space provided. Shade the feasible region. Draw and label one iso-profit line.

iso-line $\quad 6x_1 + 3x_2 = 12$

$= 24$


x_2
x_1
Iso-profit
(z = 12)
$3x_1 + 2x_2 \leq 12$

7.3 Problem (4 points)

Solve this LP using the graphical method. What is the optimal solution (x_1, x_2)?

Answer.

$(x_1, x_2) =$ $(4, 0)$

7.4 Problem (4 points)

Given the feasible region in 7.1, Identify one objective function that would result in the instance having multiple optimal solutions. If this is impossible, write *impossible*.

Answer.

Max $3x_1 + 2x_2$

(parallel to metal constraint)

7.5 Problem (4 points)

Take the dual of your linear program in 7.1.

Answer.

$$\text{Min } 12y_1$$
$$3y_1 + y_2 \geq 3$$
$$2y_1 - 2y_2 \geq 3/2$$
$$y_1, y_2 \geq 0$$

$y = (1, 0)$ is optimal

8 Winter is Here!

Sansa Stark needs your help planning to defend the North against a possible attack from the White Walkers. She needs to decide which of five potential forts ($i \in F$) to station soldiers at, and how many soldiers to place at each fort. Before soldiers can be placed at a fort, the fort must restored which takes some number of labor hours. She has 60 labor hours available. The table below shows, for each fort $i \in F$, the maximum number of soldiers (M_i, in thousands) that could be placed at the fort, and the number of labor hours (L_i) required to open the fort. The North is divided into 4 districts ($j \in D$), and in order to provide adequate defense, Sansa has determined that at least 10 thousand soldiers must be placed at forts within 100 miles of each district. The table below also indicates which districts are within 100 miles of each fort. Note that the soldier coverage can be split across multiple forts (e.g., district 1 is considered covered if forts F1 and F5 each have 5 thousand soldiers placed there).

Fort ($i \in F$)	Max Soldiers (M_i)	Labor Hrs (L_i)	Districts within 100 miles
F1	10	35	D1,D2,D3
F2	8	20	D2,D4
F3	5	15	D1,D4
F4	6	18	D2,D3
F5	4	14	D1,D2

In this problem you will develop an optimization model to help Sansa determine how to meet these requirements using the fewest number of soldiers. Use the following decision variables as part of your solution:

- x_i: Binary variable with $x_i = 1$ indicates fort $i \in F$ is restored, and $x_i = 0$ otherwise

8.1 Problem (3 points)

Define the additional decision variables needed for your model here.

y_i = # soldiers placed at fort $i \in F$.

8.2 Problem (3 points)

Write the objective of your model here.

$$\min \sum_{i \in F} y_i$$

8.3 Problem (3 points)

Write the constraint that models the limited labor availability here.

$$\sum_{i \in F} L_i x_i \le 60$$

8.4 Problem (10 points)

Write the remaining constraints required for the model here.

(Also max # soldiers)

Cannot place soldiers unless restored:

$$y_i \le M_i x_i \quad \forall i \in F$$

Make sure 10K soldiers within 100 miles of each district:

$$y_1 + y_3 + y_5 \ge 10$$

$$y_1 + y_2 + y_4 + y_5 \ge 10$$

$$y_1 + y_4 \ge 10$$

$$y_2 + y_3 \ge 10$$

$$\left.\begin{aligned} x_i &\in \{0,1\} \\ y_i &\ge 0 \end{aligned}\right\} \forall i \in F$$

8.5 Problem (2 points)

Extend the model to include the additional restriction that if fort F1 is restored, then fort F2 cannot also be restored.

$$x_1 = 1 \Rightarrow x_2 = 0$$

$$x_2 \le 1 - x_1 \qquad \text{OR} \qquad x_1 + x_2 \le 1$$

8.6 Problem (3 points)

Extend the model to include the additional restriction that if fort F2 is not restored, then at least 14 thousand soldiers must be placed at forts that are within 100 miles of district D2.

$$x_2 = 0 \Rightarrow y_1 + y_2 + y_4 + y_5 \ge 14$$

(Can remove y_2 if you want)

$$y_1 + y_4 + y_5 \ge 14(1 - x_2)$$

change to F4

8.7 Problem (6 points)

Extend the model to include the additional restriction that if 8 thousand or more soldiers are assigned to fort F1, then at most 4 thousand soldiers can be assigned to fort ~~F5~~..

(already at most 4K → F5!)

$$y_1 \geq 8 \Rightarrow y_4 \leq 4$$

A $\Downarrow$ $\delta = 1$ $\Nearrow$ B

A

Logic from slide says $a^T x - b \leq Mz - \varepsilon(1-z)$

"M" $= M_1 - 8 = 2$

"ε" $= 0.001$ $\left(\frac{1}{1000}\right)$

$$\boxed{y_1 \leq 8 + 2\delta - 0.001(1-\delta)}$$

B

Logic from slide says $a^T x - b \leq M(1-z)$

"M" $= M_4 - 4 = 2$

$$\boxed{y_4 \leq 4 + 2(1-\delta)}$$

$$\boxed{\delta \in \{0, 1\}}$$