NAME:

CS/ECE/ISyE524 Random Attempted Practice Problems

0.1 Older

0.1 Problem (2 points)

Consider the following constraint involving two variables x_1 and x_2:

$$2x_1^2 + 2x_2^2 + 2bx_1x_2 \leq 3,$$

where b is a real number. For what values of b does this constraint define an ellipse?

Answer.

0.2 Problem (2 points)

The following constraint is *not*, by itself, equivalent to a second-order cone constraint:

$$(2x_1 - x_2 + 1)^2 + (-x_1 + x_2 + 1)^2 \leq (4x_1 - x_2 + 2)^2.$$

Write down an additional constraint so that the combination of the given constraint and the additional constraint is equivalent to a second-order cone constraint.

Constraints that appear complicated (or even nonconvex) can sometimes be simplified. For the cases below, your task is to simplify the constraint as much as possible. Describe what kind of constraint you have after simplification (e.g. linear constraint, convex quadratic constraint, second order cone constraint, etc.)

(a) Suppose that $x, y \in \mathbb{R}^n$ are decision variables and $a, b \in \mathbb{R}^n$ are constants. Simplify and categorize the constraint $\|x - a\| + \|y - b\|^2 \leq 0$.

(b) Suppose $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$ are the decision variables and $a \in \mathbb{R}^n$ is a constant. Express the pair of constraints $\|x - a\|^2 - r^2 \leq 0$, $r \geq 0$ as a single constraint, and categorize it.

(c) Suppose $x \in \mathbb{R}^n$ is the decision variable and $a, b \in \mathbb{R}^n$ are constant vectors, with $a \neq b$. Simplify and categorize the constraint $\|x - a\| \leq \|x - b\|$.

Suppose that J_1 and J_2 are two objective functions defined on $\mathbb{R}^n$, and consider the following function obtained by combining these two objectives via a tradeoff parameter $\lambda > 0$:

$$J_1(x) + \lambda J_2(x).$$

Denote the minimizer of this function by x_λ, for each $\lambda > 0$.

(a) Consider the Pareto curve for J_1 and J_2 defined in the lectures. Prove that for any given $\lambda > 0$, it is *not possible* to find a point x such that *both* $J_1(x) < J_1(x_\lambda)$ and $J_2(x) < J_2(x_\lambda)$.

(b) Suppose that both J_1 and J_2 attain their minimizer at the *same point*, which we denote by x^*. Describe the Pareto curve in this situation.

We are interested in the set of points (x, y) that satisfy the inequality:

$$73x^2 - 72xy + 52y^2 \leq 625. \tag{1}$$

To aid in our analysis, we can divide (1) by 625 and rewrite it as:

$$\begin{bmatrix} x \\ y \end{bmatrix}^\mathsf{T} \begin{bmatrix} \frac{73}{625} & -\frac{36}{625} \\ -\frac{36}{625} & \frac{52}{625} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 1.$$

Then, we can find the eigenvalue decomposition of the symmetric matrix, which turns out to be:

$$\begin{bmatrix} \frac{73}{625} & -\frac{36}{625} \\ -\frac{36}{625} & \frac{52}{625} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{25} & 0 \\ 0 & \frac{4}{25} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}^\mathsf{T}.$$

Answer the following questions.

1. **[3 pts]** What is the name given to shape formed by the points satisfying the inequality (1) in $\mathbb{R}^2$?

   [1pt] Is the set of points (x, y) satisfying Equation (1) a convex set in $\mathbb{R}^2$?

2. **[3 pts]** Find a matrix A and scalar b such that (1) can be written in the equivalent form $\left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq b$.

3. **[3 pts]** The set of points satisfying (1) is centered at the origin $(0, 0)^T$. Consider how far we can move (backward and forward) along each of the two directions $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ and

$\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$ while staying inside the set. Which of the following options is true? Circle the correct one (no need for explanation).

(a) The set is longer in the direction $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ than in the direction $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

(b) The set is shorter in the direction $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ than in the direction $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

(c) The set is equally long in both directions.

1 Relaxations (10 total points)

The questions in this section refer to the optimal objective function values of the following problems. The parameters $c \in \mathbb{R}^n, b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ do not change from problem to problem. We also assume the feasible region of each linear program is not empty. In each case write the strongest relationship that can be deduced from the facts.

$$z_A = \max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \qquad i = 1, 2, \ldots, m \\ x_j &\in \{0, 1\} \qquad j = 1, 2, \ldots, n \end{aligned}$$

$$z_B = \max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \qquad i = 1, 2, \ldots, m \\ x_j &\ge 0 \qquad j = 1, 2, \ldots n \\ x_j &\le 1 \qquad j = 1, 2, \ldots n \end{aligned}$$

$$z_C = \max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \qquad i = 1, 2, \ldots, m \\ x_j &\ge 0 \qquad j = 1, 2, \ldots n \end{aligned}$$

$$z_D = \max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \qquad i = 1, 2, \ldots, m \\ x_j &\ge 0 \qquad j = 1, 2, \ldots n \\ x_j &\le 0.5 \qquad j = 1, 2, \ldots n \end{aligned}$$

$$z_E = \min \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \qquad i = 1, 2, \ldots, m \\ x_j &\in \{0, 1\} \qquad j = 1, 2, \ldots n \end{aligned}$$

$$z_F = \min \sum_{i=1}^{m} b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \qquad j = 1, 2, \ldots, n \\ y_i &\geq 0 \qquad i = 1, 2, \ldots m \end{aligned}$$

There are now a series of questions asking whether the values $z_A, z_B, \ldots z_F$ are comparable. Answer each problem with either a relation $(\leq, =, \geq)$ if the values can be compared, or put an **X** if we cannot be sure of the relation between the numbers.

1.1 Problem (2 points)

$z_A \sim z_B$?

$$z_A \quad \underset{\le,\ge,=,\mathbf{x}}{\underline{\qquad\qquad}} \quad z_B$$

1.2 Problem (2 points)

$z_A \sim z_D$?

$$z_A \quad \underset{\le,\ge,=,\mathbf{x}}{\underline{\qquad\qquad}} \quad z_D$$

1.3 Problem (2 points)

$z_A \sim z_E$?

$$z_A \quad \underset{\le,\ge,=,\mathbf{x}}{\underline{\qquad\qquad}} \quad z_E$$

1.4 Problem (2 points)

$z_F \sim z_C$?

$$z_F \quad \underset{\le,\ge,=,\mathbf{x}}{\underline{\qquad\qquad}} \quad z_C$$

1.5 Problem (2 points)

$z_D \sim z_B$?

$$z_D \quad \underset{\le,\ge,=,\mathbf{x}}{\underline{\qquad\qquad}} \quad z_B$$

2 Beer Location

Your dreams have come true. You own a beer distributorship. You are trying to decide on locations of warehouse to meet your demand for beer. Specifically, you have a set of customers $J = \{1, 2, \dots n\}$ that must be served from warehouses $I = \{1, 2, \dots, m\}$. Each customer has a demand $b_j, j \in J$ (measured in gallons) that must be met from open facilities. There is a per unit cost of c_{ij} of shipping one gallon of beer from warehouse $i \in I$ to your customer location $j \in J$. In order to open warehouse $i \in I$, you must pay a fixed charge h_i.

In this problem, we will write an integer programming problem that will determine the set of warehouses to open and shipping amounts from warehouses to customers in order to minimize the total cost of meeting all demands. We will initially use the following decision variables:

- x_{ij}: Gallons of beer shipped from warehouse $i \in I$ to customer $j \in J$
- z_i: Binary variable indicating if warehouse $i \in I$ is opened.

2.1 Problem (4 points)

Write the objective function, to minimize the sum of the total transportation cost plus warehouse opening costs:

Answer.

2.2 Problem (4 points)

Write constraints that ensure that demand b_j is met for every customer

Answer.

2.3 Problem (4 points)

Write constraints that ensure that beer is shipped to customers only from open facilities

Answer.

2.4 Problem (4 points)

Write constraints that enforce that a non-negativity quantity of beer must be shipped, and that enforce binary restrictions on warehouse opening variables

Answer.