NAME

Key

CS/ECE/ISyE 524 Midterm #2

April 15, 2024
4:00PM-5:15PM

1. Please spread out in the room as far as possible. (We may ask you to move to another location)
2. Please put your student ID out on your desk—we will be checking IDs during the exam
3. The exams will be scanned into Gradescope.
   - **Please write your name *very neatly* in the box at the top of this page.**
   - Please write your name (or initials) neatly on the top of each page of the exam. This will help us ensure we associate exams with the right students
   - If you do not write your answer in the box in the space provided, you are unlikely to get full credit for your answer
   - If you need additional scratch space, you should use the blank (back) pages of the exam.
4. Be sure to write valid *mathematical* syntax in your written answers (unless Julia/JuMP syntax is specifically required).
5. The number of points for each problem is displayed at the end of the problem's title. The time required for me to complete each question is also listed. Please use your time wisely. Scan through the exam and do the problems you are sure to know how to do and complete.
6. When time is up, put down your writing utensil and stay in your seat. We will collect the exams. If you write on your exam (even to put your name) after the time is up, we will reduce your final grade by 25%.
7. **Good luck!** Don't panic. We are rooting for you.

Here are the point available for each problem and the time it took me to do it.

Problem	**Points**	**Prof. Linderoth Time** (min.)
1	18	2
2	27	7
3	8	2
4	10	4
5	14	2
Total:	77	17

Cheat Sheet

Minimization LP	Maximization LP
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

- Second-Order (Lorentz) Cone: (of dimension $n+1$)

$$\mathcal{L} := \{(t, x) \in \mathbb{R}^{1+n} : t \geq \|x\|_2\}$$

- Rotated Second-Order Cone: (of dimension $n+2$)

$$\mathcal{R} := \{(u, t, x) \in \mathbb{R}^{2+n} : 2ut \geq x^{\mathsf{T}}x, u \geq 0, t \geq 0\}$$

- Positive Semidefinite Cone: (of order n)

$$\mathcal{S} := \{X \in \mathbb{R}^{n \times n} : X = X^{\mathsf{T}}, w^T X w \geq 0 \;\forall w \in \mathbb{R}^n\}$$

- Exponential Cone: (always in dimension 3)

$$\mathcal{E} := \{(x, y, z) \in \mathbb{R}^3 : y \exp(x/y) \leq z, y > 0\}$$

1 True/False

To answer these questions, circle **True** or **False**

1.1 Problem (1 points)

True or False: For a maximum flow problem, the value of *every* cut in the network provides an upper bound on the maximum flow.

True **False**

1.2 Problem (1 points)

True or False: When solving a shortest path problem as a linear program in a connected network, there is always an optimal solution to the linear program that has *all* variables integer-valued.

True **False**

1.3 Problem (1 points)

True or False: Every real symmetric matrix $Q = Q^\mathsf{T} \in \mathbb{R}^{n\times n}$ can be decomposed into a product:

$$Q = U\Lambda U^\mathsf{T}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix, and $U^\mathsf{T} U = I$.

True **False**

1.4 Problem (1 points)

True or False: Every real matrix $Q \in \mathbb{R}^{n\times n}$ can be decomposed into a product:

$$Q = U\Lambda U^\mathsf{T}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix, and $U^\mathsf{T} U = I$.

True **False**

1.5 Problem (1 points)

True or False: Every real, symmetric, positive-definite matrix $Q = Q^\mathsf{T} \in \mathbb{R}^{n\times n}, Q \succ 0$ can be decomposed into a product:

$$Q = U\Lambda U^\mathsf{T}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix, and $U^\mathsf{T} U = I$.

True **False**

1.6 Problem (1 points)

True or False: Every real, symmetric, positive-definite matrix $Q = Q^\top \in \mathbb{R}^{n\times n}, Q \succ 0$ can be decomposed into a product of two identical matrices

$$Q = RR$$

True **False**

1.7 Problem (1 points)

True or False: $\log(x)$ is a convex function of x for $x > 0$

True **False**

1.8 Problem (1 points)

True or False: $x\log(x)$ is a convex function of x for $x > 0$

True **False**

1.9 Problem (1 points)

True or False: $f(x) := \|x\|_p$ is a convex function of x for $p = 1, 2, \infty$.

True **False**

1.10 Problem (1 points)

True or False: $f(x) := a^\top x + b$ is a convex function of x.

True **False**

1.11 Problem (1 points)

True or False: $f(x) := a^\top x + b$ is a concave function of x.

True **False**

1.12 Problem (1 points)

True or False: If there is an optimal solution to a quadratic programming problem with linear constraints, then there is always an optimal solution that occurs at an extreme point of the feasible region.

True **False**

1.13 Problem (1 points)

True or False: The feasible region of a semidefinite program is a called a polyhedron.

True **False**

1.14 Problem (1 points)

True or False: The set of points

$$R := \{(t, x) \in \mathbb{R}^{1+n} : t \leq \|x\|^2\}$$

is a convex set. (Where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.)

True **False**

1.15 Problem (1 points)

True or False: The set of points

$$R := \{(u, t, x) \in \mathbb{R}^{2+n} : 2ut \geq \|x\|^2\}$$

is a convex set. (Where $u \in \mathbb{R}$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.)

True **False**

1.16 Problem (1 points)

True or False: The expected value of a portfolio return is a quadratic function of the amount invested in each asset in the portfolio.

True **False**

1.17 Problem (1 points)

True or False: The variance of a portfolio return is a quadratic function of the amount invested in each asset in the portfolio.

True **False**

1.18 Problem (1 points)

True or False: All cones are convex sets.

True **False**

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2 Short Answer

2.1 Problem (2 points)

In general, which class of problems is more difficult to solve, linear programs or integer programs? Or would you expect them to take the same amount of time to solve? Circle one below.

Linear Programs   Integer Programs   Equal Difficulty

2.2 Problem (2 points)

Name one software package that solves semidefinite programs in JuMP.

Answer.

SCS

2.3 Problem (2 points)

You wish to implement the constraint $t \geq x^2$ in JuMP. Complete the following code

Answer.

```
@constraint(m, [0.5; t ; x ] in RotatedSecondOrderCone())
```

2.4 Problem (2 points)

Your friend Ralph Wiggum wanted to solve an optimization problem (P)

$$\min_{x \in X} \{x^\mathsf{T} Q x + c^\mathsf{T} x\},$$

that involved an *asymmmetric* matrix Q

$$Q = \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix}.$$

You told him that we *never* use asymmetric matrices in optimization, and you told him a *symmetric* matrix Q' with the property that the optimal solution of

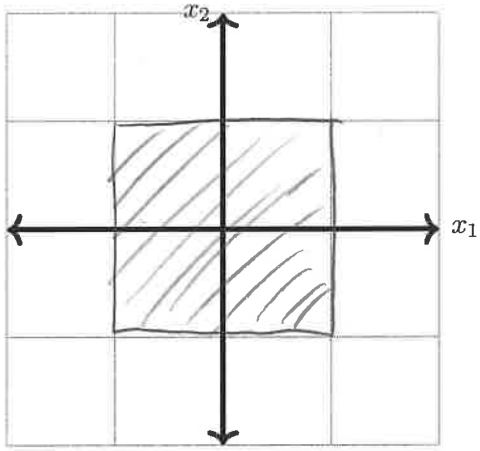
$$\min_{x \in X} \{x^\mathsf{T} Q' x + c^\mathsf{T} x\},$$

was the same as the optimal solution to P. What is your matrix Q'?

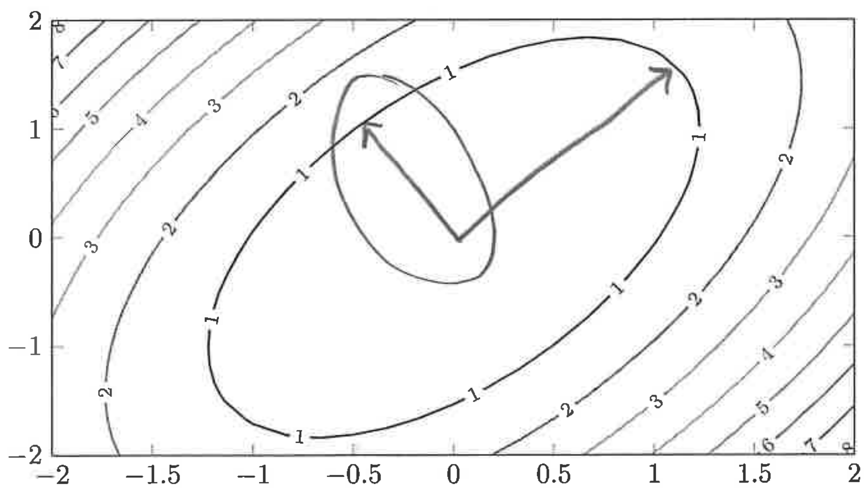
$$Q' = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

2.5 Problem (2 points)
Draw the $\|\cdot\|_\infty$-ball of radius 1 (in dimension $n = 2$) below. The grid lines depict one unit.

Answer.



Consider the following (contour) plot of the function $x^\mathsf{T} Qx$.



2.6 Problem (2 points)
How many *distinct* eigenvalues does the matrix Q have?

Answer.
\# Distinct Eigenvalues: 2

2.7 Problem (2 points)
Draw the eigenvectors of Q on the contour plot above and *circle* the eigenvector associated with largest eigenvalue of Q

Answer.
Draw answer on contour plot

2.8 Problem (2 points)
Given three binary variables $z_1, z_2, z_3 \in \{0, 1\}$, write the the *contrapositive* of the following logical statement:

$$z_3 = 1 \Rightarrow z_1 + z_2 \leq 1$$

Answer.

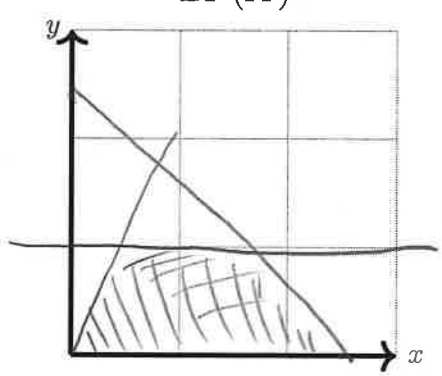
$z_1 + z_2 \geq 2 \Rightarrow z_3 = 0$

Consider the following feasible region of a pure integer programming problem:

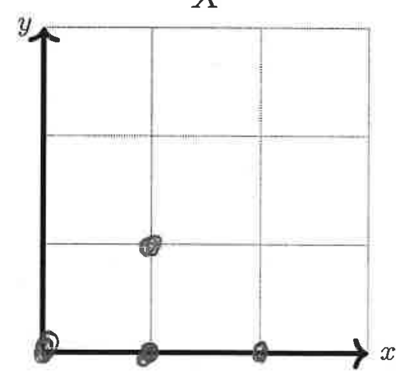
$$X = \left\{(x, y) : -2x + y \leq 0, y \leq 1, x + y \leq \frac{5}{2}, x \geq 0, y \geq 0, x \text{ integer}, y \text{ integer}\right\}$$

2.9 Problem (2 points)
Draw the linear programming relaxation of X ($LP(X)$) and the set X itself on the two graphs below:

$LP(X)$



X



2.10 Problem (3 points)
Write an algebraic description of the convex hull of X (conv(X)) (using linear inequalities)

Answer.

$$\{ (x,y) : \; y \le x, \; x+y \le 2, \; y \ge 0 \}$$

Consider the following regularized least squares problem with an ℓ_1 regularizer:

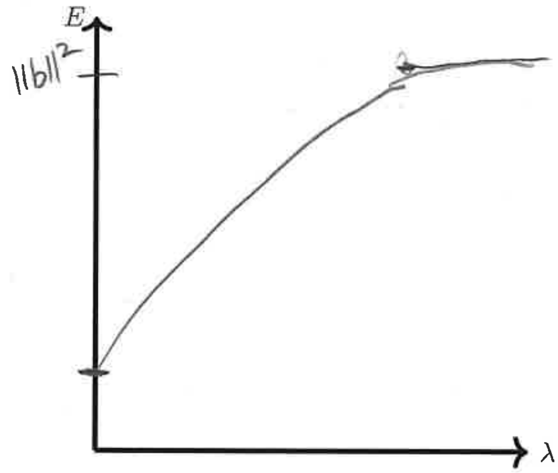
$$\min_x ||Ax - b||_2^2 + \lambda ||x||_1$$

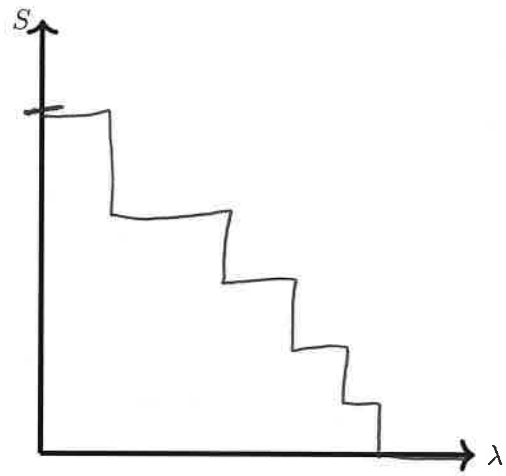
where A is a tall matrix (i.e., the problem is overdetermined with more equations than variables) and λ is a positive parameter. We are interested in how the following quantities will change as λ is varied from 0.

- The optimal solution error:
$$E := \|Ax^* - b\|_2^2$$
- The optimal solution sparsity:
$$S := |\{j : |x_j^*| > 0\}|$$

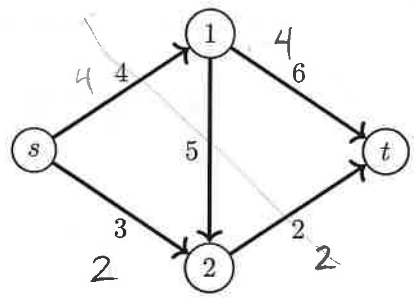
2.11 Problem (2 points)
Draw graphs of how E and S will change as λ goes from $0 \to \infty$.

Answer.





Consider the network below in which the arc capacities are labeled on the arcs.



2.12 Problem (2 points)
The maximum flow from s-t in this network has value 6. Write the full optimal solution x^*:

Answer.

$x^*_{s1} =$ 4

$x^*_{s2} =$ 2

$x^*_{12} =$ 0

$x^*_{1t} =$ 4

$x^*_{2t} =$ 2

2.13 Problem (2 points)

Write the values of the optimal dual multipliers π^* for the flow balance constraints:

Answer.

$\pi_s^* =$ 0

$\pi_1^* =$ 1

$\pi_2^* =$ 0

$\pi_t^* =$ 1

3 Second Order Cones in Springfield

At the Springfield Nuclear Plant, robots move fuel rods from a hub location to K (given) locations whose coordinates are $(a_1, b_1), \ldots, (a_K, b_K)$. Each day r_k fuel rods must be moved from the hub to location k, and the robot can carry only one rod per trip. In this problem, we will write an optimization model to locate the hub to minimize the total distance the robots travel each day. Distance in the plant is measured using Euclidean distance, so the distance between any two locations in the plant with coordinates (p_1, q_1) and (p_2, q_2) is $\sqrt{(p_1 - p_2)^2 + (q_1 - q_2)^2}$. We will write down a **second-order cone program** whose solution gives the location of the hub that minimizes the total distance the robots travel each day. Use the following decision variables in your model.

- x: x-coordinate of the hub
- y: y-coordinate of the hub
- t_k: distance between the hub and location $k \in \{1, 2, \ldots K\}$

3.1 Problem (4 points)

Write the objective function, to minimize the total distance robots travel each day.

Answer.

$$\min \sum_{k=1}^{K} r_k t_k$$

3.2 Problem (4 points)

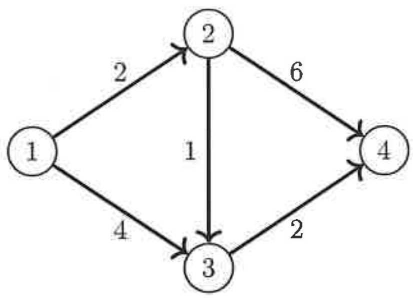
Write the remaining constraints of the problem. (You may write any second-order cone constraints in their natural algebraic form using the provided decision variables).

Answer.

$$t_k \geq \sqrt{(a_k - x)^2 + (b_k - y)^2} \quad \forall k = 1, \ldots, K$$

4 Shortest Path

Consider the following network in which we would like to find a shortest path from node 1 to node 4.



A linear program to find the shortest path from 1 to 4 is the following:

$$
\begin{array}{llllllll}
\min & 2x_{12} & +4x_{13} & +x_{23} & +6x_{24} & +2x_{34} & & \\
\text{s.t} & x_{12} & +x_{13} & & & & =1 \\
& -x_{12} & & +x_{23} & +x_{24} & & =0 \\
& & -x_{13} & -x_{23} & & +x_{34} & =0 \\
& & & & -x_{24} & -x_{34} & =-1 \\
& x_{12}, & x_{13}, & x_{23}, & x_{24}, & x_{34} & \geq 0
\end{array}
$$

4.1 Problem (4 points)

Write the dual linear program for this shortest path problem, using a vector π for the dual variables.

Answer.

$$
\begin{array}{ll}
\max & \pi_1 - \pi_4 \\
& \pi_1 - \pi_2 \leq 2 \\
& \pi_1 - \pi_3 \leq 4 \\
& \pi_2 - \pi_3 \leq 1 \\
& \pi_2 - \pi_4 \leq 6 \\
& \pi_3 - \pi_4 \leq 2 \\
& \pi_i \text{ FREE}
\end{array}
$$

4.2 Problem (4 points)

The optimal (primal) solution x^* to the shortest path problem has $x^*_{12} = x^*_{23} = x^*_{34} = 1$, other $x^*_{ij} = 0$. Use complementary slackness to determine the optimal value of dual variables π^*. You may assume that $\pi^*_1 = 0$. Show your work in the space provided. But write your values for duals where specified.

Answer.

$x^*_{12} = x^*_{23} = x^*_{34} = 1 \Rightarrow$

$\pi_1 - \pi_2 = 2$

$\pi_2 - \pi_3 = 1$

$\pi_3 - \pi_4 = 2$

$\pi_1 = 0 \Rightarrow \pi_2 = -2$

$\pi_3 = \pi_2 - 1 = -3$

$\pi_4 = \pi_3 - 2 = -5$

$\pi^*_1 =$ 0 $\quad \pi^*_2 =$ -2 $\quad \pi^*_3 =$ -3 $\quad \pi^*_4 =$ -5

4.3 Problem (2 points)

Give a (short) interpretation of the of meaning of the dual variables you found in Problem 4.2.

Answer.

These are (negative) of the shortest path distance to each node.

5 Can't Get Enuff of That Ice Cold Duff

Duff Man: "Duff beer is brewed from hops, barley, and sparkling clear mountain what?"

Duff Girl: "Goat!"

Duff Man: "Close enough!"

Duff Beer (DB) needs your help determining which types of beer to make this week. DB must choose three different types of beer among the 5 they have: Duff (beer 1), Duff Lite (beer 2), Duff Dry (beer 3), Duff Adequate (beer 4), and Duff Peanut Butter Lager (beer 5).

The table below shows the revenue per barrel and the pounds of hops required per barrel for each of the five types of beer. It also shows the fixed cost associated with producing any of that particular beer type.

Beer (j)	1	2	3	4	5
Revenue/barrel (r_j)	10	8	12	15	14
Hops/barrel (h_j)	3	2	2	5	4
Fixed Cost (f_j)	20	25	15	30	25

DB has 600 pounds of hops available this week. Each type of beer is made in a batch process, and so for the selected beers the amount produced *must* be between 60 and 120 barrels.

Use the following decision variables:

- x_j: The barrels of beer type $j \in \{1, 2, 3, 4, 5\}$ to make this week
- z_j: A binary variable that indicates if beer type $j \in \{1, 2, 3, 4, 5\}$ is produced this week

5.1 Problem (3 points)

Write the objective function—To maximize the profit from the week's operations.

Answer.

$$\max \sum_{j=1}^{5} (r_j x_j - f_j z_j)$$

5.2 Problem (3 points)

Write a constraint that ensures DB will not use more hops than it has available this week

Answer.

$$\sum_{j=1}^{5} h_j x_j \le 600$$

5.3 Problem (4 points)
Write constraints that ensure that the beers selected to be brewed are brewed in match sizes between 60 and 120 barrels

Answer.

$$60 z_j \le x_j \le 120 z_j \quad \forall j = 1, \ldots, 5$$

5.4 Problem (2 points)
Write a constraint to model a condition that at least two of beers 3,4, and 5 must be produced

Answer.

$$z_3 + z_4 + z_5 \ge 2$$

5.5 Problem (2 points)
Write a constraint to model the restriction that if beer 1 is produced, then beer 2 must also be produced.

Answer.

$$z_1 \le z_2$$

6 Extra Credit

6.1 Problem (2 points)

There will be very little partial credit for this, only work on it if you are confident in your other answers and you have time. (If Prof. Linderoth doesn't understand your proof within 30 seconds of looking at the answer, you won't get any credit)

Prove that $S := \{X : X \succeq 0\}$ is a convex cone.

Answer.

To show S is a cone, take $X \in S$. ($w^T X w \geq 0$ $\forall\, w \in \mathbb{R}^n$)

Then $w^T(\alpha X) w = \alpha w^T X w \geq 0$ for all $\alpha \geq 0$, so $(\alpha X) \in S$ $\forall\, \alpha \geq 0$.

To show S is convex, take $X, Y \in S$.

$w^T(\alpha X + (1-\alpha) Y) w = \alpha w^T X w + (1-\alpha) w^T Y w.$

This quantity is ≥ 0 $\forall\, \alpha \in [0,1]$, since $X, Y \in S$.

Thus $(\alpha X + (1-\alpha) Y) \in S$, so S is convex. $\boxed{\text{JL}}$