ISyE524: Introduction to Optimization
Problem Set #1
Due: February 25 at 11:59PM

Notes and Deliverables:

- Submit solutions to all problems in the form of a single PDF file of the IJulia notebook to the course dropbox.

1 Snitches'R'Us

I am sure that you all know that Quidditch is the most popular game in the wizarding world. In order to play quidditch, you need a snitch. We are going to write a linear program to help Angelina Johnson, former chaser for Gryffindor, and now production manager for Snitches'R'Us. From past data, Angelina knows that if the production rate of snitches changes, additional costs are incurred. She estimates that the cost of increasing production is $1.50 per snitch increased from one month to the next. Similarly, the cost of decreasing production is $1.00 per unit decreased from one month to the next. It costs only $0.10/snitch in production costs. Sales forecasts (in thousands) for the next six months are given in Table 1.

Table 1: Snitch Sales Forecasts

Month	**Demand**
July	4
August	8
September	20
October	12
November	6
December	2

For the purpose of this problem, assume the current month is June, the production level in June was 4000 units, and the June 30 inventory level is projected to be 2000 units. Storage of the first 8000 units costs $0.20/unit/month. Additional storage (for units in excess of 8000) can be rented for $0.50/unit/month. Demand in each month *must* be met.

1-1 Problem Formulate a linear program to determine the optimal production schedule.
: Hint. I had decision variables for the following quantities for each period:

- Number of snitches to produce during period t
- The increase in production rates between periods $t-1$ and t
- The decrease in production rates between periods $t-1$ and t
- Total inventory level at end of period t
- Number of snitches (up to 8000) at the end of period t
- Number of snitches (over 8000) at the end of period t

1-2 Problem Create your model from Problem 1-1 in the Julia and JuMP modeling language and solve it to determine the monthly production levels. Be sure to print the optimal solution value and the monthly production levels.

1-3 Problem Modify your answer from Problem 1-1 so that demand may be backlogged at a cost of \$.25 per unit per month. Be sure to enforce that demand for all snitches must eventually be met.

1-4 Problem Implement your model from Problem 1-3 in JuMP to determine an optimal production schedule. Print the objective value, the number of snitches in inventory, and the backlogged amounts in each period.

2 Least "Squares."

The set of six equations in four variables (1)—(6) does not have a unique solution.[1]

$$\begin{aligned}
8x_1 - 2x_2 + 4x_3 - 9x_4 &= 17 & (1)\\
x_1 + 6x_2 - x_3 - 5x_4 &= 16 & (2)\\
x_1 - x_2 + x_3 &= 7 & (3)\\
x_1 + 2x_2 - 7x_3 + 4x_4 &= 15 & (4)\\
x_3 - x_4 &= 6 & (5)\\
x_1 + x_3 - x_4 &= 0 & (6)
\end{aligned}$$

For each equation i, and values of variables $x = (x_1, x_2, x_3, x_4)$, let r_i be the ***absolute*** difference (error), or residual, between the left hand side and the right hand side. For example, for $i = 2$ and $x = (-5, 3, 1, 4)$, the error is

$$r_2 = |(1)(-5) + 6(3) - (1)(1) - 5(4) - 16| = |-24| = 24.$$

2-1 Problem Write a linear programming instance that will minimize the total absolute error (sum of residuals)

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6$$

2-2 Problem Create the instance you wrote down for Problem 2-1 in JuMP and solve it, printing the minimum total residual and the values of x that achieve it.

2-3 Problem I asked GPTv3.5 to answer this problem for me, and here is what is returned:

[1] Most six equations with four variables don't.

$$
\begin{aligned}
\text{Minimize } & r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \\
\text{subject to } & r_1 \geq 8x_1 - 2x_2 + 4x_3 - 9x_4 - 17, \\
& r_1 \geq -(8x_1 - 2x_2 + 4x_3 - 9x_4 - 17), \\
& r_2 \geq x_1 + 6x_2 - x_3 - 5x_4 - 16, \\
& r_2 \geq -(x_1 + 6x_2 - x_3 - 5x_4 - 16), \\
& r_3 \geq x_1 - x_2 + x_3 - 7, \\
& r_3 \geq -(x_1 - x_2 + x_3 - 7), \\
& r_4 \geq x_1 + 2x_2 - 7x_3 + 4x_4 - 15, \\
& r_4 \geq -(x_1 + 2x_2 - 7x_3 + 4x_4 - 15), \\
& r_5 \geq x_3 - x_4 - 6, \\
& r_5 \geq -(x_3 - x_4 - 6), \\
& r_6 \geq x_1 + x_3 - x_4, \\
& r_6 \geq -(x_1 + x_3 - x_4), \\
& r_1, r_2, r_3, r_4, r_5, r_6 \geq 0.
\end{aligned}
$$

Is this a correct model for the problem? If so, state how (if at all) it is different from your answer. If not, please state how you would correct the model.

2-4 Problem For the same instance, write a *linear* program that will minimize the *maximum* error in any one equation. Namely find values of x that will

$$\min \max\{r_1, r_2, r_3, r_4, r_5, r_6\}$$

for the definition of error (residual) r_i I gave above.

2-5 Problem I asked GPTv3.5 to solve the problem, and here is what it returned

$$
\begin{aligned}
&\text{Minimize max\_residual} \\
&\text{subject to } r_1 = 8x_1 - 2x_2 + 4x_3 - 9x_4 - 17 \\
&\qquad r_2 = x_1 + 6x_2 - x_3 - 5x_4 - 16 \\
&\qquad r_3 = x_1 - x_2 + x_3 - 7 \\
&\qquad r_4 = x_1 + 2x_2 - 7x_3 + 4x_4 - 15 \\
&\qquad r_5 = x_3 - x_4 - 6 \\
&\qquad r_6 = x_1 + x_3 - x_4 \\
&\qquad \text{max\_residual} \geq r_1 \\
&\qquad \text{max\_residual} \geq r_2 \\
&\qquad \text{max\_residual} \geq r_3 \\
&\qquad \text{max\_residual} \geq r_4 \\
&\qquad \text{max\_residual} \geq r_5 \\
&\qquad \text{max\_residual} \geq r_6 \\
&\qquad x_1, x_2, x_3, x_4, r_1, r_2, r_3, r_4, r_5, r_6, \text{max\_residual} \geq 0.
\end{aligned}
$$

Is this a correct model for the problem? If so, state how (if at all) it is different from your answer. If not, please state how you would correct the model.

2-6 Problem Implement your model from Problem 2-4 in JuMP and solve it. What is the minimum max-residual that can be achieved?

3 Stadium Building

A town council wishes to construct a small stadium in order to improve the services provided to the people living in the district. After the invitation to tender, a local construction company is awarded the contract and wishes to complete the task within the shortest possible time. All the major tasks are listed in the following table. Some tasks can only start after the completion of certain other tasks, as indicated by the “Predecessors” column.

Task	Description	Duration (in weeks)	Predecessors	Maximum reduction (in weeks)	Cost of reduction ($1k/wk)
1	Installing the construction site	2	none	0	–
2	Terracing	16	1	3	30
3	Constructing the foundations	9	2	1	26
4	Access roads and other networks	8	2	2	12
5	Erecting the basement	10	3	2	17
6	Main floor	6	4,5	1	15
7	Dividing up the changing rooms	2	4	1	8
8	Electrifying the terraces	2	6	0	–
9	Constructing the roof	9	4,6	2	42
10	Lighting of the stadium	5	4	1	21
11	Installing the terraces	3	6	1	18
12	Sealing the roof	2	9	0	–
13	Finishing the changing rooms	1	7	0	–
14	Constructing the ticket office	7	2	2	22
15	Secondary access roads	4	4,14	2	12
16	Means of signalling	3	8,11,14	1	6
17	Lawn and sport accessories	9	12	3	16

3-1 Problem Draw the directed-acyclic graph corresponding to the precedences between tasks

3-2 Problem Make the following definitions:

- $T := \{1, 2, \dots, 17\}$ the set of tasks
- d_t: The duration of task $t \in T$
- $P := \{(j, i) : j \in T$ directly follows $i \in T\}$

Write a general LP model to determine the minimum project completion time. (To make the rest of the problem a bit easier, I suggest you write the constraints all in $\geq$ form.)

3-3 Problem What is the earliest possible date of completion for the construction? Note that the last two columns of the table are not relevant for this part of the problem. Build a linear programming model in Julia and JuMP model to solve the instance. Print the minimum number of weeks required to complete construction. Also print the value dual variables associated with each constraint (if the dual variable is not equal to zero).

Recall that you can print the optimal dual value of a constraint named ILove524SoMuch after solving the instance as `dual(ILove524SoMuch)`

3-4 Problem What are the activities that are on the *critical path* for project completion? (You cannot reduce the project completion time unless you reduce the time of an activity on the critical path)

3-5 Problem Write the dual of the linear program that you wrote in Problem 3-2.

Hint: It may be useful to make a small instance/example, write the dual of this, and then generalize the pattern you see.

3-6 Problem Check/show that the dual solution you obtained from your solution to Problem 3-3 satisfies the constraints in your dual from Problem 3-5.

4 Broom Rental

Oliver Wood retired from his career as keeper at Puddlemere United to start the Wood Broom Rental Company wbc, [2] and he needs your help. There is a fleet of brooms that are distributed among a set of locations I. For each location $i \in I$, there is a current number of brooms c_i and a required number of brooms r_i. The distance between any pair of locations $i \in I$, $j \in I$ is d_{ij} (in km—Wizards use the metric system). The cost of transporting a broom from one location to another is κ galleon/km.

4-1 Problem Write a min-cost network flow problem that will determine the movement of all brooms to establish the required number of brooms at all agencies in a minimum cost manner.

4-2 Problem For the 10 locations in the table below, the (x, y) coordinates of each broom rental agency (in a grid based on kilometers) is given in below, as are the number of brooms currently in each location c_i and the number of brooms required for tomorrow's rentals r_i. Naturally brooms travel "as the crow flies," so distances between agencies are Euclidean distances. Use $\kappa = 0.5$\$/km for this instance. Implement your model from Problem 4-1 in Julia and JuMP to find the number of brooms to transport as well as the minimum cost.

Place	x **Coor.**	y **Coor.**	**Required Brooms**	**Current Brooms**
Hogwarts	0	0	10	8
Godric's Hollow	20	20	6	13
Little Whinging	18	10	8	4
Shell Cottage	30	12	11	8
The Leaky Cauldron	35	0	9	12
Ollivander's	33	25	7	2
Zonko's Joke Shop	5	27	15	14
Dervish and Banges	5	10	7	11
Little Hangleton	11	0	9	15
Weasley's Wizard Wheezes	2	15	12	7

5 Chess Sets and Duality

ViditChess is a customer maker of boxwood chess sets, and two different sizes of chess sets are available for order. The small set requires 3 hours of machining on a lathe, and the large set requires 2 hours. There are four lathes with skilled operators who each work a 40 hour week, so ViditChess production has 160 lathe-hours per week. The small chess set requires 1 kg of boxwood, and the large set requires 4 kg. Unfortunately, boxwood is scarce and only 200 kg per week can be obtained. When sold, each of the large chess sets yields a profit of \$8, and one of the small chess set has a profit of \$5. The problem is to decide how many sets of each kind should be made each week so as to maximize profit.

[2] In the Wizarding World, brooms may be rented and used, much like cars, in our normal Muggle world.

5-1 Problem Write out both the primal and dual linear programs.

5-2 Problem Chess guru MagzyBogues claims that the optimal solution to this problem is $(0, 50)$. Is he correct? Prove your answer using complementary slackness and linear programming duality.

5-3 Problem Plot the feasible set and solve the primal LP graphically. Label the optimal point and find the optimal objective.

5-4 Problem Plot the feasible set and solve the dual LP graphically. Label the optimal point and find the optimal objective.

5-5 Problem ViditChess could increase its profit if there were more boxwood available for production. What is the maximum amount that ViditChess's owner AnishGiri should be willing to pay for an extra kg of boxwood?

5-6 Problem There are currently 200 units of boxwood available. Find the *range* of this value for which the current solution remains optimal.

Hint: changing this value changes the objective function of the dual. Determine for which objective *directions* your current dual solution remains optimal.