NAME: Key

CS/ECE/ISyE 524 Midterm #1

February 28, 2024
4:00PM-5:15PM

READ THIS!

1. Please spread out in the room as far as possible. (We may ask you to move to another location)
2. Please put your student ID out on your desk—we will be checking IDs during the exam
3. The exams will be scanned into Gradescope.
4. Please write your name *very neatly* at the top page of the exam.
5. Please write your name on the top of each page of the exam. This will help us ensure we associate exams with the right students
6. You have access to a single, handwritten letter-sized sheet of paper with notes.
7. Be sure to write valid *mathematical* syntax in your written answers (unless Julia/JuMP syntax is specifically required).
8. Write your answers in the space provided. All exams will be scanned into Gradescope. If you need additional scratch space, you should use the blank (back) pages of the exam.
9. The number of points for each problem is displayed at the end of the problem's title. The time required for me to complete each question is also listed. Please use your time wisely. Scan through the exam and do the problems you are sure to know how to do and complete.
10. **Good luck!** Don't panic. We are rooting for you.

Here are the point available for each problem and the time it took me to do it.

Problem	**Points**	**Prof. Linderoth Time** (min.)
1	14	2
2	29	8
3	20	5
4	19	5
EC	4	8
4 **Total:**	82	20

1 True/False

To answer these questions, circle **True** or **False**

1.1 Problem (1 points)

If the dual of a linear program is infeasible, then the primal must be unbounded.

True **False**

1.2 Problem (1 points)

Every linear program can be put into the following standard form:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

True **False**

1.3 Problem (1 points)

Consider a minimum cost network flow problem in which the upper bound of flow on each arc is 10, the lower bound of flow on each arc is 0, and the node supplies are all zero except for three nodes, having net supply 4.5, -3, and -1.5. If this problem has a feasible solution, then there exists an optimal solution to this problem with all decision variables having integer values.

True **False**

1.4 Problem (1 points)

The node-arc incidence matrix of every network $G = (V, E)$ is a totally-unimodular matrix.

True **False**

1.5 Problem (1 points)

Consider the primal constraint $2x_1 - 3x_2 \leq 12$ and its associated dual variable λ. If the optimal dual solution satisfies $\lambda^* = 0$, then the optimal primal solution (x_1^*, x_2^*) must satisfy $2x_1^* - 3x_2^* < 12$.

True **False**

1.6 Problem (1 points)

Consider the primal constraint $2x_1 - 3x_2 \leq 12$ and its associated dual variable λ. If the optimal primal solution satisfies $2x_1^* - 3x_2^* < 12$, then the optimal dual solution must satisfy $\lambda^* = 0$.

True **False**

1.7 Problem (1 points)

The goal of solving a linear programming problem is to determine the values of the parameters that yield the best objective function value while satisfying all the constraints.

True **False**

1.8 Problem (1 points)

If a linear program has two different optimal solutions, then it is always possible to find a third different optimal solution.

True **False**

1.9 Problem (1 points)

If a linear program has a feasible solution, then it must have an extreme point optimal solution.

True **False**

1.10 Problem (1 points)

We call a linear program *unbounded* if it has an unbounded feasible region

True **False**

1.11 Problem (1 points)

True or False: The feasible region of every linear programming instance is a convex set.

True **False**

1.12 Problem (1 points)

The set of points in $\mathbb{R}^2$ that satisfy the following inequalities

$$x_1 + 3x_2 \leq 10$$
$$\frac{2x_1 + 4x_2}{x_1 + x_2} \geq 1$$
$$x_1 \geq 1, x_2 \geq 0$$

forms a convex set.

True **False**

1.13 Problem (1 points)

The set of points in $\mathbb{R}^2$ that satisfy the following inequalities

$$\begin{aligned} x_1 + 3x_2 &\leq 10 \\ \frac{2x_1 + 4x_2}{x_1 + x_2} &\geq 1 \\ x_1 \leq 1, x_2 &\geq 0 \end{aligned}$$

forms a convex set.

True **False**

1.14 Problem (1 points)

The *weak duality theorem* of linear programming states that if x^* is a feasible solution of value p^* to the primal problem, and λ^* is a feasible solution of value d^* to the dual problem, then $p^* = d^*$.

True **False**

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2 Short Answer

2.1 Problem (2 points)

What are the three components of every optimization model?

Answer.

Decision Variable, Objective Function, Constraints

2.2 Problem (2 points)

After modeling your LP that you know should have an optimal solution, the solver unexpectedly returns "INFEASIBLE". Which of the following modeling errors could have caused this? (circle all that apply)

1. incorrect objective function
2. forgot to include a constraint
3. added an extra constraint by mistake.

2.3 Problem (2 points)

For $x, y \in \mathbb{R}^n$, we call the point $w = \alpha x + (1-\alpha)y$ for some $0 \leq \alpha \leq 1$ a __________ of x and y. Please write your answer in the space below.

Answer.

Convex combination

2.4 Problem (2 points)

Name two software packages that solve linear programs.

Answer.

HiGHS, Clp, Gurobi, Cplex, ECos, SCS, CVX

NOT: Jump, GAMS

2.5 Problem (2 points)

We have a linear program whose optimal primal solution has value $p^* = 36$. At the optimal solution, there is a constraint $a^\top x \leq b$ whose associated optimal dual variable is $\lambda^* = 3$. If the constraints was changed to $a^\top x \leq (b+2)$, estimate the new optimal solution value p^*_{NEW}.

Answer. $36 + 2(3)$

$p^*_{\text{NEW}} = $ 42

2.6 Problem (2 points)

Consider the following snippet of output from the solver HiGHS for a linear program. Does the instance have an optimal solution? If so, what is its optimal value?

Julia Code

```
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
Solving the original LP with primal simplex to determine infeasible or unbounded
Using EKK primal simplex solver
  Iteration        Objective     Infeasibilities num(sum)
          0    -4.1207498688e-08 Ph1: 7(1920); Du: 9(100.207) 0s
          3     5.0000000000e+01 Pr: 0(0); Du: 1(0.5) 0s
Using EKK primal simplex solver
  Iteration        Objective     Infeasibilities num(sum)
          3     5.0000006313e+01 Pr: 0(0); Du: 1(0.5) 0s
          3     5.0000000000e+01 Pr: 0(0); Du: 1(0.5) 0s
Model   status      : Unbounded
Simplex   iterations: 3
Objective value     :  5.0000000000e+01
HiGHS run time      :          0.00
```

Answer.

Instance has Optimal Solution?: (Y/N) N

Optimal Objective Value (if applicable): —

2.7 Problem (2 points)

Consider the primal linear program below, where the right-hand side values b_1 and b_2 are numbers:

$$\begin{aligned} \max\ z = & 4x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 - 5x_2 \le b_1 \\ & x_1 - x_2 \le b_2 \\ & x_1 \ge 1, x_2 \ge 0 \end{aligned}$$

There is feasible dual solution that has dual objective value $w = 20$. What can you say about the point $(x_1, x_2) = (5, 1)$? Circle one option below.

(a) It is a feasible solution to the primal problem.
(b) It is not a feasible solution to the primal problem.
(c) We do not know if it is feasible to the primal problem, but if it is feasible, it is an optimal solution to the primal problem.
(d) It is not possible to say any of the above for sure.

2.8 Problem (2 points)

After modeling your LP that you know should have a solution, the solver unexpectedly returns "UNBOUNDED". Which of the following modeling errors could have caused this? (circle all that apply)

1. incorrect objective function
2. forgot to include a constraint
3. added an extra constraint by mistake.

2.9 Problem (3 points)

Indicate which of the following problems can be reformulated as a linear program. (Circle all that apply.)

1. $\min_{x_1,x_2} 2x_1 + 5 + |x_2|$
2. $\min_{x_1,x_2} \max\{2x_1 + 3x_2 - 5, 4x_2 - 6, -x_1 + 5x_2 - 3\}$
3. $\min_{x_1,x_2} \min\{2x_1 + 3x_2 - 5, 4x_2 - 6, -x_1 + 5x_2 - 3\}$

2.10 Problem (3 points)

Convert the following mathematical description of constraints into the correct Julia/JuMP syntax.

$$\sum_{j \in J} a_{ij} x_j \geq b_i \quad \forall i \in I$$

You may assume that all JuMP variables and parameters exist. For example, you may assume that the following code has been written:

Julia Code

```julia
using JuMP, HiGHS
m = Model(HiGHS.Optimizer)

nrows = 20
ncols = 50
A = rand(nrows,ncols)
b = rand(nrows)

I = 1:nrows
J = 1:ncols

@variable(m, x[J] >= 0)
```

Answer.

@constraint(m, [i in I], sum(A[i,j]*x[j] for j in J)
>= b[i])

2.11 Problem (3 points)

Write the dual of this linear program:

$$\begin{aligned} \max\ z = \quad & x_2 + x_3 \\ & x_2 - 2x_3 \le 5 \\ -\ 2x_1 + & 3x_2 - 2x_3 = 0 \\ 2x_1 \quad & \quad -4x_3 \le 1 \\ x_1,\ & x_2,\ x_3 \ge 0 \end{aligned}$$

Answer.

$$\begin{aligned} \min\ & 5\lambda_1 + \lambda_3 \\ \text{s.t.}\ & -2\lambda_2 + 2\lambda_3 \ge 0 \\ & \lambda_1 + 3\lambda_2 \ge 1 \\ & -2\lambda_1 - 2\lambda_2 - 4\lambda_3 \ge 1 \\ & \lambda_1 \ge 0,\ \lambda_2 \text{ FREE},\ \lambda_3 \ge 0 \end{aligned}$$

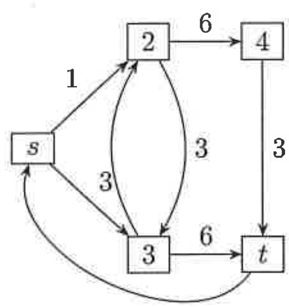
OR

(after convert to standard form)

$$\begin{aligned} & 5\lambda_1 + \lambda_4 \\ & -2\lambda_2 + 2\lambda_3 + 2\lambda_4 \ge 0 \\ & \lambda_1 + 3\lambda_2 - 3\lambda_3 \ge 1 \\ & -2\lambda_1 - 2\lambda_2 + 2\lambda_3 - 4\lambda_4 \ge 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \ge 0 \end{aligned}$$

2.12 Problem (2 points)

Consider the following network $G = (N, A)$ on which we would like to solve a maximum flow problem from the node s to the node t. The node labels are inside the boxes for $i \in N$, and the numbers above the arcs are the arc capacities.



Write the flow balance constraint (from the min-cost-network flow model) corresponding to node 3. You should use flow varaibles x_{ij} for $(i, j) \in A$.

Answer.

$$x_{32} + x_{3t} - x_{23} - x_{s3} = 0$$

2.13 Problem (2 points)
What is the right hand side (b) vector associated with solving the max flow problem on the network above?

$$b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(No need to write upper bound on x)

3 524 Mining

The 524 Mining Company (524MC) can mine up to 200 tons of ore per day at a cost of \$800 per ton. Mining a ton of ore requires 2.5 hours of labor and yields 300 pounds of iron and 50 pounds of carbon. Iron can be sold for \$3 per pound and carbon can be sold for \$40 per pound. Iron and carbon can also be used to make steel beams which sell for \$1000 per beam. Making a steel beam requires 2 hours of labor, and uses 200 pounds of iron and 5 pounds of carbon. 524MC has 750 labor hours available per day. Formulate a linear program to help 524MC determine how to maximize its daily profit. Use (only) the following decision variables in your solution:

- x = tons of ore to mine per day.
- y_I = pounds of iron sold per day.
- y_C = pounds of carbon sold per day.
- y_B = steel beams made and sold per day.

3.1 Problem (3 points)

Write the objective function to maximize daily profit:

Answer.

$$\max \quad 3y_I + 40y_C + 1000y_B - 800x$$

3.2 Problem (3 points)

Write the constraint that ensures that 524MC does not exceed its mining capacity per day

Answer.

$$x \le 200$$

3.3 Problem (3 points)

Write the constraint that ensures 524MC does not exceed its total number of available labor hours per day

Answer.

$$2.5x + 2y_B \le 750$$

3.4 Problem (3 points)
Write the constraint that ensures the amount of carbon mined balances the carbon used to make steel beams and the carbon sold.

Answer.

$$50x = y_C + 5y_B$$

3.5 Problem (3 points)
Write the constraint that ensures the amount of iron mined balances the iron to make steel beams and the iron sold.

Answer.

$$300x = y_I + 200y_B$$

3.6 Problem (1 points)
Did we forget any constraints? If we did, write them here, otherwise write *None.*

Answer.

Non-negativity. $x, y_I, y_C, y_B \geq 0$.

3.7 Problem (4 points)
Now write a linear inequality constraint that says that the ratio of the tons of carbon sold to the tons of iron sold can be at most 10%

Answer.

$$\frac{y_C}{y_I} \leq 0.1 \Rightarrow \boxed{y_C \leq 0.1\, y_I}$$

4 Freakin' Tuition

Man, is college tuition expensive. I am trying to figure out how to pay my son's college tuition for the next $T = \{1, 2, \ldots, 8\}$ semesters. I have some money saved, and there is some amount of my UW salary that I can afford to pay, but I am going to have to supplement my income working as a consultant. Here is some data that I have to help me make the decisions

- d_t: Tuition payment due during semester $t \in T$ (\$)
- s_t: The UW salary that I make during semester $t \in T$ that I can afford to pay for tuition (\$)
- α_t: My consulting rate during semester t—I can sometimes charge more if my clients are super desparate that semester. (\$/hour)
- C_0: The amount of cash I have saved (\$). I will use all of this cash to pay his tuition if necessary.
- H: The maximum number of hours that UW will allow we to work as a consultant during the semester (hours)

Obviously I would like to minimize the total number of extra hours I have to work in order to pay my son's tuition[1]. Please help me write a linear program to figure out the minimum amount I have to work over the next 8 semesters.

Please use the following decision variables in your linear program.

- x_t: The number of hours Prof. L works as a consultant during semester t
- C_t: The total amount of cash Prof. L has on hand at the end of period t

4.1 Problem (4 points)

In proper mathematical notation, write the objective function: To minimize the total number of hours Prof. L works as a consultant over the next 8 semesters.

Answer.

$$\min \sum_{t \in T} x_t$$

4.2 Problem (4 points)

In proper mathematical notation, write the constraints that ensure that Prof. Linderoth never works more than H total hours as a consultant in any semester

Answer. $$x_t \leq H \quad \forall t \in T$$

[1] So I have more time to spending preparing the 524 class!

4.3 Problem (4 points)
Write the constraints that balance the amount of cash Prof. Linderoth has on hand to pay tuition between semesters.

Answer.

$$C_0 + S_1 + \alpha_1 x_1 = d_1 + C_1$$
$$C_{t-1} + S_t + \alpha_t x_t = d_t + C_t \quad \forall t \in T$$

(OK, if combine)

4.4 Problem (1 points)
Write any other constraints that may be necessary for the problem. If none are needed, write *None*

Answer.

Non-negativity

$$x_t \geq 0 \quad \forall t \in T$$
$$C_t \geq 0 \quad \forall t \in T$$

4.5 Problem (6 points)
Now suppose I wish to minimize the maximum number of hours I work in any semester. Using additional variables and constraints as necessary, write the problem as a linear program. You do *not* need to rewrite the constraints from Problems 4.2—4.4.

Answer. New Variable z.

$$\min \quad z$$
$$z \geq x_t \quad \forall t \in T.$$

5 Extra Credit

5.1 Problem (4 points)

There will be very little partial credit for this, only work on it if you are confident in your other answers and you have time.

Consider the linear program

$$\begin{aligned} \max\ & 4x_1 + c_2x_2 + c_3x_3 \\ \text{s.t.}\ & x_1 + 2x_2 \le 5 \\ & 2x_1 - x_2 + x_3 \le 5 \\ & x_1,\ x_2,\ x_3 \ge 0. \end{aligned}$$

For what values of c_2 and c_3 will the solution $x^* = (3, 1, 0)$ be optimal? Show your work, but write your answer where specified at the bottom of the page.

Answer.

Dual: $\min\ 5\lambda_1 + 5\lambda_2$

s.t. $\lambda_1 + 2\lambda_2 \ge 4$

$2\lambda_1 - \lambda_2 \ge c_2$

$\lambda_2 \ge c_3$

$\lambda_1, \lambda_2 \ge 0$

$x^* = (3,1,0) \Rightarrow \lambda_1 + 2\lambda_2 = 4$, $2\lambda_1 - \lambda_2 = c_2$

$\Downarrow$

$\lambda_1 = \frac{4+2c_2}{5}$, $\lambda_2 = \frac{8-c_2}{5}$

Need $\lambda_1 = \frac{4+2c_2}{5} \ge 0$, $\lambda_2 = \frac{8-c_2}{5} \ge 0$, $\lambda_2 = \frac{8-c_2}{5} \ge c_3$

$\Downarrow$ $c_2 \ge -2$

$\Downarrow$ $c_2 \le 8$

$-2 \le c_2 \le 8$

$\cancel{\quad} \le c_3 \le$ ______ $\rightarrow \frac{8-c_2}{5}$