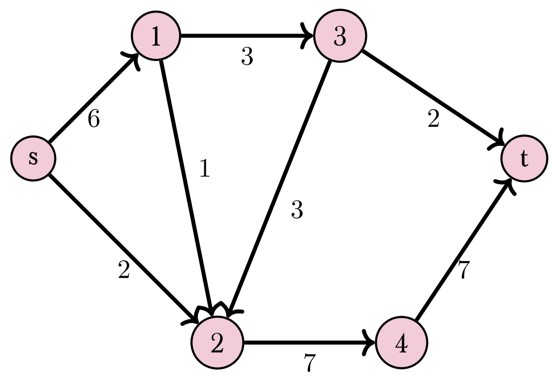
ISyE524: Introduction to Optimization
Problem Set #3
Due: March 17 at 11:59PM

Notes and Deliverables:

- Submit solutions to all problems in the form of a single PDF file of the IJulia notebook to the course dropbox. (If you do not submit as a single PDF you will lose points.)
- Be sure that your answers (including any pictures you include in the notebook) are in the correct order. (If you do not submit the problems in order, you will lose points.)

1 MaxFlow

Consider the network below, in which the label on each arc represents the capacity of the arc.



1-1 Problem Write a linear program formulation to maximize the flow from source s to sink t. Please use a "circulation arc" from t to s in your formulation.

1-2 Problem Write the dual of the linear programming formulation from Problem 1-1.

1-3 Problem Implement your model from Problem 1-1 in Julia and JuMP to find the maximum flow from s to t in the network.

1-4 Problem Use the complementary slackness conditions to find an optimal *dual* solution to your formulation in Problem 1-2. You should note how the optimal dual solution identifies a ***minimum cut*** in the network.

1-5 Problem Find a cut in the network whose capacity equals the value of the maximum flow. Recall a cut can be specified as a set of nodes S that contains the source node. The capacity of the cut is the sum of the capacity of edges that start in S and end outside of S.

2 Lasso

In this problem, we will investigate different approaches for performing polynomial regression on the data in the file `lasso_data.csv`. You can read and plot the data with the following Julia code.

Julia Code

```
using PyPlot, CSV, DataFrames

data = CSV.read("lasso-data.csv", DataFrame)
x = data[:,1]
y = data[:,2]

cla()
figure(figsize=(8,4))
plot(x,y,"r.", markersize=10)
grid("True")
# Only need this line if using vscode?
display(gcf())
```

2-1 Problem Create a Julia/JuMP model to solve a convex optimization problem that finds the best polynomial fit for a polynomial of degree 6. Print the value of the error (total squared residuals) as well as the values of the coefficients in the polynomial.

2-2 Problem Create a Julia/JuMP model to solve a convex optimization problem that finds the best polynomial fit for a polynomial of degree 18. Print the value of the error (total squared residuals) as well as the values of the coefficients in the polynomial.

2-3 Problem Make a plot that shows the data as well as the best fit to the data for polynomials of degree $d = 6$ and $d = 18$ that you created in Problems 2-1 and 2-2

2-4 Problem Our model is too complicated because it has too many parameters. One way to simplify our model is to look for a sparse model (where many of the parameters are zero). Solve the $d = 18$ problem once more, but this time use the Lasso (L_1 regularization). Start with a small λ and progressively make λ larger until you obtain a model with at most $K = 6$ coefficients positive. Print the resulting (squared) error. And plot the resulting fit.

Warning: Prof. Linderoth had some trouble (likely a bug) in using HiGHS for this one. You could use the Gurobi solver (but need to follow instructions for getting it installed in your environment), or the solver "Ipopt", which is a general nonlinear optimization solver.

3 Beam Me Up

In treating a cancerous tumor with radiotherapy, physicians want to bombard the tumor with radiation while avoiding the surrounding normal tissue as much as possible. They must therefore select which of a number of possible radiation beams to use, and figure the “weight” (actually the exposure time) to allot for each beam.

Abstractly, in a mathematical model of this decision problem, we are given a set of "beams" B, and a set of regions R, where $R = N \cup T$, where N is the set of normal regions, and T is the set of cancerous (tumor) regions.

If the weight/exposure of beam $b \in B$ is x_b (a continuous decision variable), then beam $b \in B$ delivers an amount of radiation of $a_{br}x_b$ to region $r \in R$. The total amount of radiation delivered to a particular region $r \in R$ is obtained by adding the contributions of each beam:

$$\sum_{b \in B} a_{br} x_b.$$

The maximum weight that can be applied for any beam $b \in B$ is W_b. Ideally, each tumor region $r \in T$ should receive as large a dose as possible, while each non-tumor region $r \in N$ should receive a dose of at most p_r.

The ***tumor score*** of any tumor region is the amount of radiation received:

$$\tau_r := \sum_{b \in B} a_{br} x_b \quad \forall r \in T.$$

The ***tissue damage*** of any normal region is the total amount delivered to that region over the prescribed amount

$$\Delta_r := \max\left(\sum_{b \in B} a_{br} x_b - p_r, 0\right) \quad \forall r \in N$$

Doctors would like to simultaneously maximize $\sum_{r \in T} \tau_r$ while minimizing $\sum_{r \in N} \Delta_r$.

3-1 Problem Write a weighted multi-objective optimization model (a linear program) that shows the tradeoff between the amount of good-dose ($\sum_{r \in T} \tau_r$) delivered and the amount of damaging dose delivered ($\sum_{r \in N} \Delta_r$)

Hint: You will need to write the problem using linear inequalities. Do not use the $\max$ operator in your constraints. Recall how to model (some) piecewise linear functions using linear inequalities.

3-2 Problem Suppose there are a collection of 6 beams, each with a maximum weight/intensity of $W_b = 3 \; \forall b \in B$. There are also 6 regions, regions 1-3 are normal and regions 4-6 are cancerous. The amount of radiation delivered by beam b to region r is given in the matrix below:

	normal1	normal2	normal3	tumor1	tumor2	tumor3
beam1	15	7	8	12	12	6
beam2	13	4	12	19	15	14
beam3	9	8	13	13	10	17
beam4	4	12	12	6	18	16
beam5	9	4	11	13	6	14
beam6	8	7	7	10	10	10

All normal regions have a desired upper bound of $p_r := 65 \; \forall r \in N$. Implement your model from Problem 3-1 in JuMP and print out total dose to the tumor region and the total dose to the normal region when your tradeoff parameter is $\lambda = 1$.

Note: Since you are trying to maximize the dose to the tumor and minimize the penalty dose to the normal region, your objective should be something like

$$\max \text{Total Tumor Dose} - \lambda \text{Total Overage in Normal Regions}$$

3-3 Problem Plot the Pareto frontier/tradeoff curve:

If you make a vector called `tumor_dose` and a vector called `normal_penalty` that contain the tumor dose and normal tissue penalty amounts for different values of tradeoff parameter, then this code will make a pareto plot for you.

Julia Code

```
using PyPlot

function paretoPlot(x,y)
    figure(figsize=(10,10))
    plot( x, y, "b.-", markersize=4 )
    xlabel("Tumor Dose")
    ylabel("Normal Penalty")
    # Only need this in vscode?
    display(gcf())
end
;

paretoPlot(tumor_dose, normal_penalty)
```

4 Nonconvex QP

The following matrix is indefinite:

$$Q = \begin{pmatrix} 0 & 0 & -2 & -4 & 0 & 1 \\ 0 & 1 & -1 & -1 & 3 & -4 \\ -2 & -1 & -1 & -5 & 7 & -4 \\ -4 & -1 & -5 & -3 & 7 & -2 \\ 0 & 3 & 7 & 7 & -1 & -2 \\ 1 & -4 & -4 & -2 & -2 & 0 \end{pmatrix}$$

Consider the following box-constrained quadratic program.

$$\begin{aligned} &\min x^\top Q x + c^\top x \\ &\text{s.t.} 0 \le x_j \le 1 \quad \forall j = 1, \dots, 6 \end{aligned} \tag{1}$$

and $c^\top = [-1, 0, 2, -2, 4, 0]$

4-1 Problem Use Julia to compute the eigenvalues of Q and print them out. You will need to use the LinearAlgebra package, and then `eigen(Q)` returns the eigenvalues and eigenvectors of Q

4-2 Problem Create a model of (1) in Julia and attempt to solve it using HiGHS. Include the output from the solver.

4-3 Problem Show that if a real symmetric matrix $Q \prec 0$ with minimum eigenvalue $\lambda_1 < 0$, then the matrix obtained by subtracting λ_1 from the diagonal is PSD, i.e. $Q - \lambda_1 I \succeq 0$.

4-4 Problem Subtract the minimum eigenvalue you found in Problem 4-1 from the diagonal of Q, and solve the resulting model in HiGHS. Report the objective value and solution.

5 Pod Racing Rendezvous

Anakin and Palpatine are cruising the Dune Sea in their pods. Each pod has the following dynamics:

$$\text{Dynamics of each hovercraft:} \qquad \begin{aligned} x_{t+1} &= x_t + \tfrac{1}{3600} v_t \\ v_{t+1} &= v_t + u_t \end{aligned}$$

At time t (in seconds), $x_t \in \mathbb{R}^2$ is the position (in miles), $v_t \in \mathbb{R}^2$ is the velocity (in miles per hour), and $u_t \in \mathbb{R}^2$ is the thrust in normalized units. At $t = 1$, Anakin has a speed of 20 mph going North, and Palpatine is located half a mile East of Anakin, moving due East at 30 mph. Anakin and Palpatine would like to rendezvous at exactly $t = 60$ seconds. The location where they meet is up to you.

5-1 Problem Find the sequence of thruster inputs for Anakin (u^{A}) and Palpatine (u^{P}) that achieves a rendezvous at $t = 60$ while minimizing the total energy used by both hovercraft:

$$\text{total energy} = \sum_{t=1}^{60} \left\| u_t^{\mathrm{A}} \right\|^2 + \sum_{t=1}^{60} \left\| u_t^{\mathrm{P}} \right\|^2$$

Implement and solve your model in Julia, print their final (rendezvous) location, assuming Anakin starts at the origin and the minimum total energy required.

5-2 Problem Plot the trajectories of each hovercraft to verify that they do indeed rendezvous.

5-3 Problem In addition to arriving at the same place at the same time, Anakin and Palpatine should also make sure that their velocities are both zero when they rendezvous — otherwise, they might crash! Solve the rendezvous problem again with these additional constraints on the final velocities. Again, print the final (rendezvous) location and the minimum total energy required.

5-4 Problem Plot the trajectories of each hovercraft in this instance