Tutorial - Week 8

Please read the related material and attempt these questions before attending your allocated tutorial. Solutions are released on Friday 4pm.

Question 1

(a) Show that sampling n observations from a multivariate normal $N_p(\mu, \Sigma)$ distribution and then calculating a sample covariance matrix is equivalent to sampling from a Wishart distribution using the 'rWishart' function. Consider the distribution of the eigenvalues to show this. How are the function parameters 'df' and 'Sigma' related to the multivariate normal distribution?

(b) Write a function that samples from the Wishart distribution $W_p(n, I_p)$ using the Bartlett decomposition. That is, sample a lower-triangular matrix T where the off-diagonals are distributed like $T_{ij} \sim N(0, 1)$ and the diagonal entries T_{ii} satisfy $T_{ii}^2 \sim \chi^2_{n-i+1}$. Then $U = TT' \sim W_p(n, I_p)$.

(c) Write a more general function that handles the case where the population matrix Σ is not the identity. Do this by setting $W = \Sigma^{1/2} U (\Sigma^{1/2})'$ where $\Sigma^{1/2}(\Sigma^{1/2})' = \Sigma$, to obtain $W \sim W_p(n, \Sigma)$. Test in the case where $\Sigma = I_p$ and the case where Σ has a compound symmetric covariance structure (see last week's tutorial).

(d) Show through a simulation that if $U_1 \sim W_p(n_1, \Sigma)$ and $U_2 \sim W_p(n_2, \Sigma)$, then $U_1 + U_2 \sim W_p(n_1 + n_2, \Sigma)$.

(e) Show through a simulation that if $U \sim W_p(n, \Sigma)$ and A is a constant $p \times p$ matrix, then $\mathbb{E}[\mathrm{tr}(AU)] = n \,\mathrm{tr}(A\Sigma)$.

(f) Show through a simulation that if $U \sim W_p(n, \Sigma)$ and A is a constant $p \times p$ matrix, then $\mathbb{E}[\mathrm{tr}(AU^{-1})] = \frac{1}{n-p-1} \mathrm{tr}(A\Sigma^{-1})$ for $n - p - 1 > 0$.

(g) Show through a simulation that if $U \sim W_p(n, \Sigma)$ then

$$\mathbb{E}[|U|^h] = 2^{ph} |\Sigma|^h \prod_{i=1}^{p} \frac{\Gamma(\frac{1}{2}(n-i+1) + h)}{\Gamma(\frac{1}{2}(n-i+1))}.$$

Initially take $p = 2$, comment on what happens when p becomes larger.