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On the Realized Risk of High-Dimensional Markowitz Portfolios*

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Abstract. We study the realized risk of Markowitz portfolios computed using parameters estimated from data and generalizations to similar questions involving the out-of-sample risk in quadratic programs with linear equality constraints. We do so under the assumption that the data is generated according to an elliptical model, which allows us to study models with heavy tails, tail dependence, and leptokurtic marginals for the data. We place ourselves in the setting of high-dimensional inference where the number of assets in the portfolio, p, is large and comparable to the number of samples, n, we use to estimate the parameters. Our approach is based on random matrix theory. We consider the impact of both the estimation of the mean and of the covariance. Our work shows that risk is underestimated in this setting and, further, that in the class of elliptical distributions, the Gaussian case yields the least amount of risk underestimation. The problem is more pronounced for genuinely elliptical distributions. Therefore, results and intuition based on Gaussian computations give an overoptimistic view of the risk-underestimation situation. We also propose a robust estimator of realized risk and investigate its performance in simulations.

Key words. covariance matrices, quadratic programs, multivariate statistical analysis, high-dimensional inference, random matrix theory, concentration of measure, Markowitz problem, elliptical distributions

AMS subject classifications. Primary, 62H10; Secondary, 90C20

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1. Introduction.

The Markowitz problem (Markowitz (1952)) is a classic portfolio optimization problem in finance, where investors choose to invest according to the following framework: one picks assets in such a way that the portfolio guarantees a certain level of expected returns but minimizes the "risk" associated with them. In the standard framework, this risk is measured by the variance of the portfolio. Markowitz's paper was highly influential, and much work has followed. It is now of course part of the standard textbook literature on these issues (Ruppert (2006), Campbell, Lo, and MacKinlay (1996)).

Naturally, many variants exist now, involving various notions of risk. The most common ones seem to involve Value-at-Risk (VaR) and conditional Value-at-Risk (cVaR) as alternatives to variance. We discuss here only the classical problem.

In the ideal (or, in statistical parlance, population) solution, the covariance and the mean of the returns are known. The mathematical formulation is then the following simple quadratic program: we wish to find the weights w by solving the problem

$$\begin{cases} \min \frac{1}{2} w'\Sigma w \ , \\ w'\mu = \mu_P \ , \\ w'\mathbf{e} = 1 \ . \end{cases}$$

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Here, $\mathbf{e}$ is a p-dimensional vector with 1 in every entry, μ is the vector of mean returns, Σ is the covariance between the returns of the assets, and μ_P is the level of expected returns the investor wishes to achieve. If Σ is invertible, the solution is known explicitly (see section 2). If we call w_{theo} the solution of this problem, the curve $w'_{\text{theo}}\Sigma w_{\text{theo}}$, seen as a function of μ_P, is called the *efficient frontier*. (The problems we investigate in this paper are of course generalizations of this classic Markowitz formulation.)

We note that the problem is sometimes formulated slightly differently, i.e., with the constraint $w'\mu \geq \mu_P$ instead of $w'\mu = \mu_P$. However, this has minimal consequences in our setting since knowing the solution of the problem we consider (for all μ_P) will yield the solution of the problem involving the inequality constraint. If needed, another motivation for studying the above mentioned problem for all μ_P's is that in the case where we want to incorporate a riskless asset in the portfolio, the shape of the efficient frontier changes and becomes a straight line. If the variance of the portfolio is on the x-axis and the level of expected returns is on the y-axis, this straight line goes through $(0, r)$ and is tangent to the efficient frontier computed above, which is a parabola. The line touches the parabola at a point called the *tangency portfolio*, and we naturally need the whole efficient frontier to compute it (see, e.g., Ruppert (2006)).

Going back to the original problem, in practice, we of course do not know μ and Σ and we need to estimate them. An interesting question is therefore to understand what happens in the Markowitz problem when we replace population quantities by corresponding estimators. In this paper we will be especially concerned with the following risk management question: if we choose our strategy by solving the Markowitz problem with μ (resp., Σ) replaced by the sample mean $\widehat{\mu}$ (resp., the sample covariance matrix $\widehat{\Sigma}$), what is the risk of our portfolio? Related interesting questions are naturally the respective contributions of $\widehat{\mu}$ and $\widehat{\Sigma}$ in our measures of risk and the problems they may create.

Naturally, we can ask a similar question for general quadratic programs with linear equality constraints (see below or Boyd and Vandenberghe (2004) for a definition), the Markowitz problem in the form presented here being a particular instance of such a problem. This more general question is relevant for various very central and important statistical problems (such as classification (using linear discriminant analysis) or weighted regression) where we are interested in out-of-sample measures of risk.

It has been observed by many that there are problems in practice when replacing population quantities by standard estimators (see Lai and Xing (2008), section 3.5), and alternatives have been proposed. A famous one is the Black–Litterman model (see Black and Litterman (1990) and, e.g., Meucci (2008)). Adjustments to the standard estimators have also been proposed: Ledoit and Wolf (2004), partly motivated by portfolio optimization problems, proposed "shrinking" the sample covariance matrix towards another positive definite matrix (often the identity matrix properly scaled), while Michaud (1998) proposed using the bootstrap and to average bootstrap weights to find better-behaved weights for the portfolio. We note that this latter reference considers as "canonical" (and hence works with) the case where portfolio optimization is performed under a no short-selling constraint (see Michaud (1998), pp. 9–10), and this completely changes the behavior of the estimator from what we will describe here. Important and very closely related papers to those referenced here in the statistics literature are Efron and Morris (1976), Friedman (1989), and Breiman (1996).

An aspect of the problem that is of particular interest to us is the study of large-dimensional portfolios (or quadratic programs with linear equality constraints). To make matters clear, we focus on a portfolio with $p = 100$ assets. If we use a year of daily data to estimate Σ, the covariance between the daily returns of the assets, we have $n \simeq 250$ observations at our disposal. In modern statistical parlance, we are therefore in a "large n, large p" setting, and we know from random matrix theory that $\widehat{\Sigma}$, the sample covariance matrix, is a poor estimator of Σ, especially when it comes to spectral properties of Σ. There is now a developing statistical literature on properties of sample covariance matrices when n and p are both large—and it is now understood that, though $\widehat{\Sigma}$ is unbiased for Σ, the eigenvalues and eigenvectors of $\widehat{\Sigma}$ behave very differently from those of Σ. We refer the interested reader to Marčenko and Pastur (1967), Wachter (1976), and Wachter (1978) and, more recently, to Johnstone (2001), El Karoui (2007), El Karoui (2008a), Bickel and Levina (2008), Paul (2007), Rothman, Bickel, Levina, and Zhu (2008), and El Karoui (2009) for a partial introduction to these problems. Some of the classic statistics paper mentioned in the previous paragraph are also naturally relevant. The literature on high-dimensional statistics is currently exploding, so our list of references is naturally partial.

Another interesting aspect of this problem is that the high-dimensional setting does not allow, by contrast to the classical "small p, large n" setting, a perturbative approach to go through. In the "small p, large n" setting, the classic paper by Jobson and Korkie (1980) is concerned, in the Gaussian case, with issues similar to those we will be investigating, namely statistical properties of the empirical solution of the Markowitz problem. However, it does not seem that so far there has been much interest in this high-dimensionality question in the finance literature. For instance, a book-length treatment of asset allocation questions (Meucci, 2005) gives only a rather cursory one page discussion of these issues.

Before we proceed, we would like to discuss briefly the issue of no short-selling constraints. The problem we consider in this paper supposes that short-selling is allowed. Hence it is generally relevant to hedge-fund managers (who tend to not be subject to short-selling constraints) and not so to long-only funds (for instance, mutual funds and pension funds). The mathematics of no short-selling portfolios is essentially trivial (see Fan, Zhang, and Yu (2008)—the key being the inequality—for w a vector and A a symmetric matrix - $|w'Aw| \leq \max_{i,j} |A_{i,j}| \|w\|_1^2$ and the remark that for no short-selling portfolios whose weights must sum to 1, $\|w\|_1 = 1$), even in high dimensions and guarantees (at least at a theoretical level) that naive estimates of risk are very close to realized risk. Hence, no short-selling mitigates the problems we will consider and investigate here considerably, though it has also been remarked that no short-selling portfolios have their own share of problems, for instance, tending to yield portfolios that are not well diversified. Furthermore, there is potentially a trade-off between this "stability" property of no short-selling portfolios (in terms of risk) and their optimality in the classic Markowitz sense detailed above. Hence, we believe it is important to understand the realized risk properties of "classic" Markowitz portfolios.

The "large n, large p" setting is the one with which random matrix theory is concerned—and the practical aspects of the high-dimensional Markowitz problem have therefore been of interest to random matrix theorists for some time now. We note in particular the paper by Laloux, Cizeau, Bouchaud, and Potters (2000), where a random matrix-inspired (shrinkage) approach to improved estimation of the sample covariance matrix is proposed in the context of the Markowitz problem (at a high level, the idea in that paper was to posit that the population correlation matrix of the asset returns was a finite rank perturbation of the identity, decide which part of the empirical spectrum corresponds to the Marchenko–Pastur law (see below), replace the corresponding empirical eigenvalues by the same constant (estimated from these empirical eigenvalues), and keep the eigenvalues which were unlikely to come from the identity part of the population correlation matrix). We also note that other random matrix–based approaches to covariance estimation were later proposed (El Karoui (2008b)), with asymptotic theoretical guarantees on the estimation of the spectral distribution of the covariance matrix.

Let us now remind the reader of some basic facts of random matrix theory that suggest that serious problems may arise if one solves the high-dimensional Markowitz problem or other quadratic programs with linear equality constraints naively. A key result in random matrix theory is the Marčenko–Pastur equation (Marčenko and Pastur (1967)), which characterizes the limiting distribution of the eigenvalues of the sample covariance matrix and relates it to the spectral distribution of the population covariance matrix. We give in this introduction only its simplest form and refer the reader to Marčenko and Pastur (1967), El Karoui (2008b), and El Karoui (2009) for a more thorough introduction and very recent developments, as well as potential geometric and statistical limitations of the models usually considered in random matrix theory. (As we will see, these geometric implications have a strong impact on the results we will present.)

In the simplest setting, we consider data $\{X_i\}_{i=1}^n$, which are p-dimensional. In a financial context, these vectors are vectors of (log)-returns of assets, the portfolio consisting of p assets. To simplify the exposition, let us assume that the X_i's are independent and identically distributed (i.i.d.) with distribution $\mathcal{N}(0, \mathrm{Id}_p)$—the normality assumption for the data being close to assuming a Black–Scholes model for the underlying diffusion of stock prices. We call X the $n \times p$ matrix whose ith row is the vector X_i. Let us consider the sample covariance matrix

$$\widehat{\Sigma} = \frac{1}{n-1}(X - \bar{X})'(X - \bar{X}) ,$$

where $\bar{X}$ is a matrix whose rows are all equal to the column mean of X. Now let us call F_p the spectral distribution of $\widehat{\Sigma}$, i.e., the probability distribution that puts mass $1/p$ at each of the p eigenvalues of $\widehat{\Sigma}$. A graphical representation of this probability distribution is naturally the histogram of eigenvalues of $\widehat{\Sigma}$. A consequence of the main result of the very profound paper by Marčenko and Pastur (1967) is that F_p, though a random measure, is asymptotically nonrandom, and its limit, in the sense of weak convergence of distributions, F has a density (when $p < n$) that can be computed. F depends on $\rho = \lim_{n\to\infty} p/n$ in the following manner: if $p < n$, the density of F is

$$f_\rho(x) = \frac{1}{2\pi\rho}\frac{\sqrt{(y_+ - x)(x - y_-)}}{x} ,$$

where $y_+ = (1+\sqrt{\rho})^2$ and $y_- = (1-\sqrt{\rho})^2$. Figure 1 presents a graphical illustration of this result.

What is striking about this result is that it implies that the largest eigenvalue of Σ, λ_1, will be overestimated by l_1, the largest eigenvalue of $\widehat{\Sigma}$. Also, the smallest eigenvalue of Σ, λ_p,

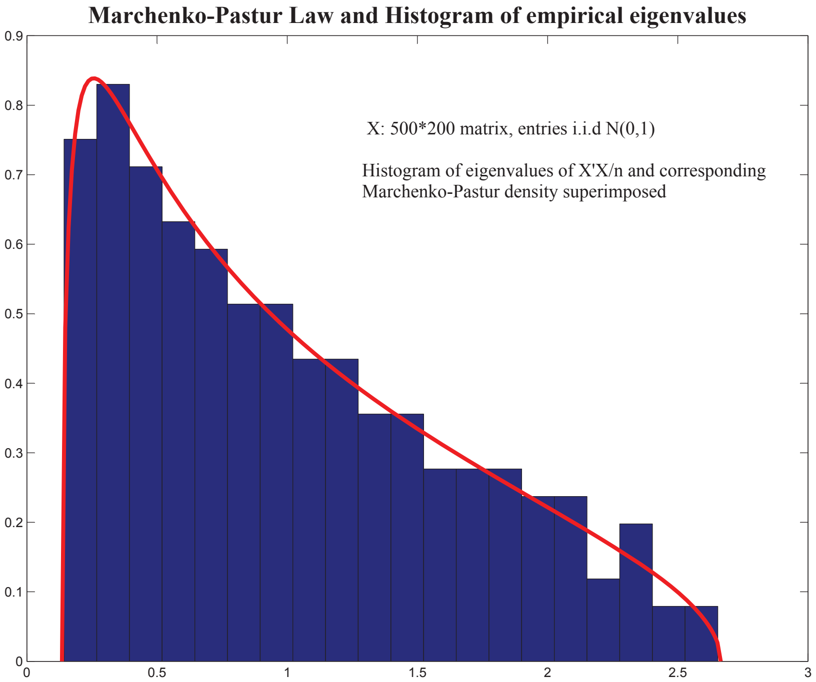


Figure 1. *Illustration of Marčenko–Pastur law, $n = 500$, $p = 200$. The red curve is the density of the Marčenko–Pastur law for $\rho = 2/5$. The simulation was done with i.i.d. Gaussian data. The histogram is the histogram of eigenvalues of $X'X/n$.*

will be underestimated by the smallest eigenvalue of $\widehat{\Sigma}$, l_p. As a matter of fact, in the model described above, Σ has all its eigenvalues equal to 1, so $\lambda_1(\Sigma) = \lambda_p(\Sigma) = 1$, while l_1 will asymptotically be larger than or equal to $(1 + \sqrt{\rho})^2$ and l_p smaller than or equal to $(1 - \sqrt{\rho})^2$ (in the Gaussian case and several others, l_1 and l_p converge to those limits). We note that the result of Marčenko and Pastur (1967) is not limited to the case where Σ is identity, as presented here, but holds for general covariance Σ (F_p then of course has a different limit).

Perhaps more concretely, let us consider a projection of the data along a vector v, with $\|v\|_2 = 1$, where $\|v\|_2$ is the Euclidean norm of v. Here it is clear that if $X_{n+1} \sim \mathcal{N}(0, \mathrm{Id}_p)$, then $\mathrm{var}\,(v'X_{n+1}) = 1$ for all v since $v'X_{n+1} \sim \mathcal{N}(0, 1)$. However, if we do not know Σ and estimate it by $\widehat{\Sigma}$, a naive (and wrong) reasoning suggests that we can find directions with variances lower than 1, namely those corresponding to eigenvectors of $\widehat{\Sigma}$ associated with eigenvalues that are less than 1. In particular, if v_p is the eigenvector associated with l_p, the smallest eigenvalue of $\widehat{\Sigma}$, by naively estimating, for X_{n+1} independent of $\{X_i\}_{i=1}^{n}$, the variance in the direction of v_p, $\mathrm{var}\left(v_p'X_{n+1}\right)$, by the empirical version $v_p'\widehat{\Sigma}v_p$, one would commit a severe mistake: the variance in any direction is 1, but it would be estimated by something roughly equal to $(1 - \sqrt{p/n})^2$ in the direction of v_p.

In a portfolio optimization context, this suggests that by using standard estimators, such as the sample covariance matrix, when solving the high-dimensional Markowitz problem one might underestimate the variance of certain portfolios (or "optimal" vectors of weights). As a matter of fact, in the previous toy example, thinking (wrongly) that there is low variance in the direction v_p, one might (numerically) "load" this direction more than warranted, given that the true variance is the same in all directions. Naturally, this will also lead us to choose a portfolio that is suboptimal and should therefore have higher realized risk than the optimal portfolio.

This simple argument suggests that severe problems might arise in the high-dimensional Markowitz problem and other quadratic programs with linear equality constraints. In particular, risk might be underestimated. While this heuristic argument is probably clear to specialists of random matrix theory, as far as we know, the problem had not been investigated at a mathematical level of rigor in that literature until the very recent papers by Bai, Liu, and Wong (2009) and El Karoui (2010). (Bai, Liu, and Wong (2009) appeared after this paper was submitted and is concerned with Gaussian-like cases for the data distribution, so it has no overlap (except in the Gaussian case where no random matrix theory is needed) with the current paper.) It has received some attention at a physical level of rigor (see, e.g., Pafka and Kondor (2003), where the authors treat only the Gaussian case and neglect the effect of the mean, which as we show below creates problems of its own). We note that there are claims of universality in the physics literature; in other words, the model for the data would not matter (only the population parameters (and n and p) would) and we show below that there is no universality in these problems. Finally, we personally found the physics literature very hard to read because of the lack of detailed and rigorous proofs and its reliance on a vocabulary that is different from that in mathematical random matrix theory.

In this paper, which is a companion to El Karoui (2010), where we analyzed (among many other things) the issue of naive risk estimation in the Markowitz problem (i.e., how the naive estimate of risk of the empirical portfolio relates to the actual risk of the (population) optimal portfolio), we propose a theoretical analysis of the problem in an elliptical framework (which incorporates the Gaussian case as a subcase) for general quadratic programs with linear equality constraints, one of them involving the parameter μ. We treat the problem at this level of generality because, beyond the finance settings, the results should be interesting in statistics and several areas of applied mathematics where one needs to solve optimization problems based on estimated parameters. Our results are severalfold. We relate the realized risk of the portfolios to the theoretical efficient frontier that is key to the Markowitz theory. We quantify this realized risk as a function of the population parameters, n and p, and a quantity characterizing the ellipticity of the data. Finally, we propose an estimator of this realized risk that is easy to compute. We show its performance in some simulations.

The elliptical framework for modeling returns of financial stocks has been advocated in the literature for some time now (see Frahm and Jaekel (2005)) and is still, as a random matrix problem, quite interesting to theoreticians (see El Karoui (2009)). One of its many benefits is that it allows us to incorporate heavy-tailed modeling in our analysis, and it yields marginal distributions for the returns of individual assets that are all leptokurtic. Finally, elliptical distributions have a nonzero coefficient of tail dependence (see McNeil, Frey, and Embrechts (2005)), whereas the Gaussian distribution has zero dependence on the tail.

Interestingly, there seems to be a consensus in the finance literature that estimation of covariance is "easy" and that the more difficult aspect of the Markowitz problem (and other portfolio optimization problems) comes from estimating the mean. By contrast, statisticians working in high-dimensional inference have recently devoted a lot of effort to improving covariance estimation, which is thought to be a hard task. We show here (and showed in El Karoui (2010)) that estimating the mean and the covariance matrix both matter and create quantifiable (and first-order) problems and biases.

The paper is divided into three main parts and a conclusion. In section 2, we give some preliminaries on the problem we are tackling here and the key results of El Karoui (2010) that will prove useful in the paper. In section 3, we state the main technical results of the paper and discuss them briefly. Because the proofs are rather technical and self-contained, they are given in an appendix. In section 4, we compare the results in the Gaussian and genuinely elliptical settings. The main conclusion there is that the Gaussian case gives an overoptimistic view of risk underestimation and that risk underestimation is more pronounced in the elliptical setting. This shows in particular that no "universality" should be expected in these questions. We also present some simulations that illustrate both the accuracy and potential limitations of our work. We discuss our results and possible extensions in the conclusion.

2. Preliminaries.

In this section we remind the reader of some classical and well-known results concerning quadratic programs with linear equality constraints. We also briefly remind the reader of the key results in El Karoui (2010) we will need later on.

Setup of the problem. We get to observe data $X_1, \dots, X_n$, with distribution

$$X_i = \mu + \lambda_i \Sigma^{1/2} Y_i \ , \tag{2.1}$$

where Σ is a $p \times p$ covariance matrix, $\lambda_i \geq 0$, and the Y_i are i.i.d. $\mathcal{N}(0, \mathrm{Id}_p)$. Here we assume that the λ_i's are independent of the Y_i's, but they might be correlated with one another. We note that $\mathbf{E}\left(X_i\right) = \mu$ and $\operatorname{cov}\left(X_i\right) = \mathbf{E}\left(\lambda_i^2\right)\Sigma$. If $\mathbf{E}\left(\lambda_i^2\right) = 1$, all these models lead to the same covariance for the data, Σ. We note that X_i has as many moments as λ_i and in particular can have much heavier tails than Gaussian data. It is also easy to see that the marginals of X_i are leptokurtic. By a slight abuse of language, we call data distributed according to the model stipulated in (2.1) elliptical, though the model is slightly different from the standard model for elliptical data in statistics. (With our normalization, the standard statistical formulation of elliptical models replaces Y_i by $Y_i/\|Y_i\|\sqrt{p}$ and requires the independence of λ_i and $Y_i/\|Y_i\|$. Our abuse of notation is slight because it is well known (see Anderson (2003)) that if $Y_i \sim \mathcal{N}(0, \mathrm{Id}_p)$, $Y_i/\|Y_i\|$ is independent of $\|Y_i\|$. Therefore, our model is genuinely elliptical since it can be formulated as $X_i = \mu + \lambda_i \frac{\|Y_i\|}{\sqrt{p}} \Sigma^{1/2}(\sqrt{p}Y_i/\|Y_i\|)$ but does not necessarily encompass all elliptical distributions (it is a scale mixture of normals model). However, in high dimensions, it is possible (using, for instance, properties of χ^2 random variables) to show that $\sup_{1 \leq i \leq n} |\|Y_i\|/\sqrt{p} - 1|$ is very small when p/n has a finite limit as $p \to \infty$. Using that fact, it is not difficult (after writing everything in matrix form) to show that one can easily extend the results of the model we consider to the full class of elliptical distributions, provided the moment requirements we have below are satisfied.

Known results. We are interested in the solution of the following problem, which is a generalization of the Markowitz problem:

$$\text{(QP-eqc-Pop)} \qquad \begin{cases} \min_{w \in \mathbb{R}^p} \frac{1}{2} w' \Sigma w \ , \\ w' v_i = u_i \, , \ 1 \leq i \leq k-1 \ , \\ w' \mu = u_k \ . \end{cases}$$

By contrast to the standard Markowitz formulation, we have added extra constraints. They are here to include potential diversification requirements. For instance, one might want to have 30% of their investment in the energy sector. In that case, the corresponding vector v_i would be a vector of 0's and 1's, having 1's for assets related to the energy sector. The corresponding u_i would be equal to .3.

We remind the reader of the following fact.

Fact 2.1. *The solution of Problem* (QP-eqc-Pop) *is given by*

$$w_{\text{theo}} = \Sigma^{-1} V M^{-1} U \ ,$$

where V is the $p \times k$ matrix containing the v_i's, U is the $k \times 1$ vector containing the u_i's, and $M = (V'\Sigma^{-1}V)^{-1}$, provided all these quantities exist.

Throughout, we will assume that the number of constraints k stays fixed in the asymptotics we consider. Our assumptions will also guarantee that Σ^{-1} and M^{-1} exist. Unless otherwise noted, these assumptions will be made implicitly throughout the paper.

In the situation we care most about, $v_k = \widehat{\mu}$ and $\Sigma = \widehat{\Sigma}$ are estimated from data. In other word, we will seek the solution of the problem

$$\text{(QP-eqc-Emp)} \qquad \begin{cases} \min_{w \in \mathbb{R}^p} \frac{1}{2} w' \widehat{\Sigma} w \ ' \\ w' v_i = u_i \ , \ 1 \leq i \leq k-1 \ , \\ w' \widehat{\mu} = u_k \ . \end{cases}$$

We will call $\widehat{V}$ the $p \times k$ matrix containing the v_i's. The notation $\widehat{V}$ includes the "hat" because $\widehat{V}$ is now a random $p \times k$ matrix since its kth column is $\widehat{\mu}$.

We call w_{emp} the vector of weights obtained by solving the problem (QP-eqc-Emp). A very important question is to understand how w_{emp} and functions of this vector relate to w_{theo} and functions of w_{theo}. We will do so in the setting where p and n are large and do asymptotic computations when $p \to \infty$ and $n \to \infty$, while $p/n \to \rho \in (0,1)$. A reason for doing double asymptotic computations of this sort is that they might yield better insights than standard (i.e., fixed p, large n) asymptotics when p and n are moderately large, i.e., in the few 100's. This is verified in our simulations.

In El Karoui (2010), we focused on the issue of relating the naive estimator of risk $w'_{\text{emp}} \widehat{\Sigma} w_{\text{emp}}$ to the population risk $w'_{\text{theo}} \Sigma w_{\text{theo}}$. By contrast, in this paper for reasons that are explained below we will be focusing on $w'_{\text{emp}} \Sigma w_{\text{emp}}$.

On the realized risk of portfolios. From a risk management standpoint, a natural quantity to estimate is the realized risk (or out-of-sample risk) of a vector obtained by solving problem (QP-eqc-Emp). We first place ourselves in the setting where the X_i's are independent. By realized risk, we mean

$$\text{RRisk} = \text{var}\left(w'_{\text{emp}} X_{n+1} | X_1, \ldots, X_n\right) = w'_{\text{emp}} \Sigma w_{\text{emp}} \ , \tag{2.2}$$

namely the risk that we will subjected to in the future if we choose w_{emp} as our allocation today, and the returns are independent. The previous result naturally holds under the milder assumption that X_{n+1} is independent of $\{X_i\}_{i=1}^n$. If the λ_i's are dependent, while the Y_i's are independent, we have

$$\text{var}\left(w'_{\text{emp}} X_{n+1} | X_1, \ldots, X_n\right) = \mathbf{E}\left(\lambda_{n+1}^2 | \{\lambda_i\}_{i=1}^n\right) w'_{\text{emp}} \Sigma w_{\text{emp}} \ .$$

So most of our work will focus on understanding the random variable $w'_{\text{emp}}\Sigma w_{\text{emp}}$, which we will call the realized risk, while keeping in mind that our analysis would also give us (through a simple modification) results concerning the case where the λ_i's are dependent.

In particular, we might want to compare this risk to the naive estimator of risk, $w'_{\text{emp}}\widehat{\Sigma}w_{\text{emp}}$, and to the actual optimal risk, $w'_{\text{theo}}\Sigma w_{\text{theo}}$. These latter two estimators were considered in El Karoui (2010); therefore we need only compare RRisk and $w'_{\text{theo}}\Sigma w_{\text{theo}}$. Clearly, to do so, we will need only focus on understanding $w'_{\text{emp}}\Sigma w_{\text{emp}}$. Note that this is not such an easy quantity to estimate since Σ is unknown, and as we said earlier, the sample covariance matrix is a poor estimator of Σ in high dimensions, the setting we will consider here.

From now on we will assume unless otherwise noted that $\widehat{\mu}$ is the sample mean and $\widehat{\Sigma}$ is the sample covariance matrix. In other words, X is the data matrix whose ith row is X_i, $\widehat{\mu}' = \mathbf{e}'X/n$, and $\widehat{\Sigma} = (X - \mathbf{e}\widehat{\mu}')'(X - \mathbf{e}\widehat{\mu}')/n - 1$. We will sometime refer to the $n \times p$ matrix $\mathbf{e}\widehat{\mu}'$ as $\bar{X}$. The sample covariance matrix could also be rescaled by $1/n$ instead of $1/(n-1)$, in which case it is not unbiased, but since this might lead to rather less cumbersome expressions, we will sometime choose this normalization. Note that the normalization will have no effect on our asymptotic results.

A simple but key observation in what follows is the following fact.

Fact 2.2. *Suppose the observed data can be written in matrix form as*

$$X = \mathbf{e}\mu' + \Sigma_1 Y \Sigma = \mathbf{e}\mu' + Y_1\Sigma^{1/2}\ , \textit{where } Y_1 = \Sigma_1 Y\ . \tag{2.3}$$

Then, if $H = \mathrm{Id}_n - \mathbf{e}\mathbf{e}'/n$, and if $\widehat{M} = \widehat{V}'\widehat{\Sigma}^{-1}\widehat{V}$,

$$\text{RRisk} = w'_{\text{emp}}\Sigma w_{\text{emp}} = U'\widehat{M}^{-1}\widehat{V}'\Sigma^{-1/2}\left(\frac{Y_1'HY_1}{n-1}\right)^{-2}\Sigma^{-1/2}\widehat{V}\widehat{M}^{-1}U\ . \tag{2.4}$$

The simple fact above is an immediate consequence of the fact that $\widehat{\Sigma} = X'HX/(n-1)$, and we therefore have the stochastic representation $\widehat{\Sigma} = \Sigma^{1/2}(\frac{Y_1'HY_1}{n-1})\Sigma^{1/2}$.

This observation is relevant to our problem because the model described in (2.1) can be written in matrix form as in (2.3), with Σ_1 a diagonal matrix whose entries are $\sqrt{\lambda_i}$ and Y an $n \times p$ matrix whose entries are i.i.d. $\mathcal{N}(0,1)$. The equality

$$\widehat{\Sigma} = \Sigma^{1/2}\left(\frac{Y_1'HY_1}{n-1}\right)\Sigma^{1/2}$$

can be understood either in law or algebraically. Naturally, we observe neither Σ, Y nor Σ_1 in practice, just X. But the stochastic representation we have here for $\widehat{\Sigma}$ is very important for the rest of the paper.

The expression we give in (2.4) might look somewhat involved. However, it simplifies the problem considerably. As a matter of fact, U and $\widehat{M}$ are finite-dimensional objects, which are now well understood (see El Karoui (2010)). For instance, we showed in El Karoui (2010) that under technical assumptions similar to those we will be making in this paper, if $\kappa = \rho/(1-\rho)$,

$$\widehat{M} \simeq \mathfrak{s}V'\Sigma^{-1}V + \kappa e_k e_k'\ ,$$

where $\mathfrak{s}$ is defined below in (3.1). Our assumptions guarantee that the approximate equality above holds in probability and that we can also take the inverses of the two matrices on both sides of the approximate equality and have approximate equality for the inverses.

So the only real difficulty we will have to deal with is to understand the $k \times k$ matrix

$$\widehat{V}'\Sigma^{-1/2}\left(\frac{Y_1'HY_1}{n-1}\right)^{-2}\Sigma^{-1/2}\widehat{V} .$$

On closer inspection, and with insights coming from El Karoui (2010), it turns out that we need only understand mainly two things: first, for certain well-chosen deterministic vectors α, we need to grapple with

$$\alpha'\left(\frac{Y_1'HY_1}{n-1}\right)^{-2}\alpha .$$

We will see that the ideas we developed in El Karoui (2010) will be extremely helpful in that context. Second, we will have to consider

$$\widehat{\mu}'\Sigma^{-1/2}\left(\frac{Y_1'HY_1}{n-1}\right)^{-2}\Sigma^{-1/2}\widehat{\mu} .$$

This quantity will require substantially more work.

The key insight from random matrix theory we will need in this context is the fact that these random quantities converge to (deterministic) constants in the asymptotic setting we consider. Hence, it turns out that RRisk will be the product of five essentially deterministic matrices, and we will be able to relate this product to the "population" quantity (or theoretical efficient frontier) $w_{\text{theo}}'\Sigma w_{\text{theo}} = U'M^{-1}U$.

In section 3, we present our main results and apply them to compare the Gaussian and elliptical cases in section 4.

A remark on the time-dependent case. Before we proceed, we should also note that another benefit of our matrix formulation is that it allows a certain amount of (time) dependence in the observed data, $\{X_i\}_{i=1}^n$. As a matter of fact, when Y is an $n \times p$ matrix whose entries are i.i.d. $\mathcal{N}(0,1)$, if O is any $n \times n$ orthogonal matrix, $Y \stackrel{\mathcal{L}}{=} OY$. Therefore,

$$Y_1'HY_1 = Y'\Sigma_1'H\Sigma_1'Y \stackrel{\mathcal{L}}{=} Y'O'\Sigma_1'H\Sigma_1'OY .$$

Picking O to be an orthogonal matrix that diagonalizes the positive semidefinite matrix $\Sigma_1'H\Sigma_1$, we see that if we call $D_{\Sigma_1'H\Sigma_1}$ the diagonal matrix containing the eigenvalues of $\Sigma_1'H\Sigma_1$, we have

$$Y_1'HY_1 \stackrel{\mathcal{L}}{=} Y'D_{\Sigma_1'H\Sigma_1}Y .$$

Hence, to understand quantities like

$$\alpha'\left(\frac{Y_1'HY_1}{n-1}\right)^{-2}\alpha ,$$

even in the case when Σ_1 is not diagonal, and hence when we have time dependence in the X_i's, it is in fact enough to understand the case where Σ_1 is diagonal. So the work we do here can actually be used to derive corresponding results for the time-dependent case, as long as the time dependence follows the description outlined above. For quadratic forms involving $\widehat{\mu}$, the time-dependent situation is a bit more complicated: one needs to perform the singular value decomposition of Σ_1, i.e., $\Sigma_1 = ADB'$, and then one realizes that the key vector turns out to be equal in law to

$$\left(\frac{Y'(D^2 - \frac{\beta\beta'}{n})Y}{n-1}\right)^{-2} Y'\beta ,$$

where $\beta = DA'\mathbf{e}/n$. In the case of diagonal Σ_1, β is just $D\mathbf{e}/n$, so the results change a bit. The method of proof we present here nonetheless stays applicable, but some care needs to be applied, and dealing with all these new details is beyond the scope of this paper. (The situation is the same in El Karoui (2010), so we refer the interested reader there for more details.)

Notation. Before we start presenting our results, let us describe some notation we will be using. $\|v\|$ is by default the Euclidean norm of the vector v. We sometime also write $\|v\|_2$. $\succeq$ represents the positive semidefinite ordering for matrices: so if $A \succeq B$, $A - B$ is positive semidefinite. $|||A|||_2$ is the operator norm or largest singular value of the matrix A. $\mathrm{o}_P(1)$ means that the corresponding random variable goes to zero in probability. $a \vee b$ stands for $\max(a, b)$. The vector $\mathbf{e}$ is a vector whose entries are all equal to 1; it is generally of dimension n. Unless otherwise noted, e_i represents the ith canonical basis vector (the dimension of this vector is context dependent). The notation $\stackrel{\mathcal{L}}{=}$ represents equality in law.

3. Main results.

In this section, we state our main results so as to extract them from the technical details of the proofs. We wish to note that we will work with the assumption that Y_i's (see (2.1)) are Gaussian. We chose to do so to limit the technical details and to bypass standard methods of random matrix theory, in the hope that our proofs would show more clearly the phenomena at play. As explained in El Karoui (2009), most of the results of random matrix theory we will rely on depend strongly on the geometry that the purported model implies on the data. By working with our model we are able to capture all the richness of elliptical models from a random matrix theoretic point of view—in particular, the different models induce differences in the geometry of the data—while keeping proofs relatively clear. It is very likely possible to use other well-known techniques (based on Stieltjes transforms, so more specialized and less broadly accessible) to weaken the assumptions on the Y_i's and replace the current ones by assumptions concerning concentration of convex Lipschitz functions of the Y_i's (see Ledoux (2001) and El Karoui (2009) for concrete examples in the present context). However, this would change essentially nothing about the geometry of the data—the key driver of the results in our opinion—and hence we expect to get the same results under these weaker assumptions. Work in this direction is currently underway. In the present paper we really want to focus on the key phenomena and not on what is now for serious practitioners of random matrix theory essentially manageable technical details.

As said earlier, we will need two main results to draw conclusions about the realized risk of high-dimensional Markowitz portfolios. We now present them.

3.1. On $V'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}V$.

The first question is to get a good understanding of the quantity

$$V'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}V ,$$

where V is a $p \times k$ deterministic and given matrix. (Note that for some forms of our problem we need to estimate $\widehat{\mu}$ from the data, in which case the matrix containing the vectors of constraints becomes random. When our constraints involve a random constraint, we use the notation $\widehat{V}$. When it does not, and all the constraints are deterministic, we use the notation V. In this subsection, we do not deal with the more involved situation where one constraint is random, and hence we use the V notation.)

We have the following results, proven in Appendix A.

Theorem 3.1. *Suppose we observe n i.i.d. observations X_i, where X_i has the form $X_i = \mu + \lambda_i Y_i$, with Y_ii.i.d. $\mathcal{N}(0, \mathrm{Id}_p)$, and $\{\lambda_i\}_{i=1}^n$ is independent of $\{Y_i\}_{i=1}^n$. We assume that $\mathbf{E}\left(\lambda_i^2\right) = 1$. We call X the $n \times p$ data matrix containing the X_i's.*

We call $\rho_n = p/n$ and assume that $\rho_n \to \rho \in (0,1)$.

We use the notation $\tau_i = \lambda_i^2$ and assume that the empirical distribution, G_n, of τ_i converges weakly in probability to a deterministic limit G. We also assume that $\tau_i \neq 0$ for all i.

If $\tau_{(i)}$ is the ith largest τ_k, we assume that we can find a random variable $N \in \mathbb{N}$ and positive real numbers ϵ_0 and C_0 such that

$$\text{(Assumption-BB)} \qquad \begin{cases} P(p/N < 1 - \epsilon_0) \to 1 \text{ as } n \to \infty\ , \\ P(\tau_{(N)} > C_0) \to 1\ , \\ \exists \eta_0 > 0 \text{ such that } P(N/n > \eta_0) \to 1 \text{ as } n \to \infty\ . \end{cases}$$

Under these assumptions, if α is a (sequence of) deterministic vectors with norm 1, then

$$\alpha' \left(\frac{X'HX}{n-1}\right)^{-2} \alpha \to \xi \text{ in probability}\ ,$$

whereas if $\mathfrak{s}$ satisfies

$$\int \frac{dG(\tau)}{1 + \rho\tau\mathfrak{s}} = 1 - \rho\ , \text{then} \tag{3.1}$$

ξ is defined as

$$\xi = \frac{1}{\frac{1}{\mathfrak{s}^2} - \rho \int \frac{\tau^2 dG(\tau)}{(1+\tau\rho\mathfrak{s})^2}}\ . \tag{3.2}$$

We note that if v is a given deterministic vector and $\alpha = \Sigma^{-1/2}v/\sqrt{v'\Sigma^{-1}v}$, the previous theorem means that, under the appropriate technical conditions,

$$\frac{v'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v}{v'\Sigma^{-1}v} \to \xi \text{ in probability}\ .$$

It is in this latter form that the theorem is going to be used.

To flesh out the result a little bit, let us point out that in the Gaussian case, $G = \delta_1$, so $\mathfrak{s} = 1/(1-\rho)$ and $\xi = (1-\rho)^{-3}$.

Perhaps remarkably, even though it is stated with the condition $\mathbf{E}\left(\lambda_i^2\right) = 1$, this theorem does not actually require any assumptions on the moments of λ_i, nor do we need to assume that the λ_i's are independent of each other. All that really matters is the existence of a deterministic limiting empirical distribution for $\tau_i = \lambda_i^2$. (Assumption-BB) essentially means that the distribution of the λ_i's does not put "too much" mass near 0, which will guarantee that the smallest singular values of certain random matrices appearing in our computations are bounded away from 0. This is very important in the proof.

To get a sense of differences between the elliptical and Gaussian cases, let us mention the following fact.

Fact 3.1. *We have*

$$\xi \geq \frac{\mathfrak{s}^2}{1-\rho} .$$

This latter value corresponds to the Gaussian case.

Let us now turn to the more general situation. Let us call P1 the following condition on the population parameters:

$$\forall 1 \leq i \neq j \leq k, \frac{v_i'\Sigma^{-1}v_j}{(v_i+v_j)'\Sigma^{-1}(v_i+v_j)} \text{ and}$$

$$\text{(Condition-P1)} \qquad \frac{v_i'\Sigma^{-1}v_j}{(v_i-v_j)'\Sigma^{-1}(v_i-v_j)} \text{ stay bounded away from } 0 .$$

Let us call P2 the following alternative condition:

$$\text{(Condition-P2)} \qquad \forall 1 \leq i \neq j \leq k, \text{ if } v_i'\Sigma^{-1}v_j \to 0 , \quad v_i'\Sigma^{-1}v_i \text{ and } v_j'\Sigma^{-1}v_j \text{ stay bounded} .$$

As an immediate corollary of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Suppose now that $X_i = \mu + \lambda_i \Sigma^{1/2} Y_i$ and the conditions of Theorem 3.1 hold. With the definitions above, and if the $p \times k$ deterministic matrix V and Σ are such that* (Condition-P1) *and/or* (Condition-P2) *are satisfied, we have asymptotically*

$$V'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}V = \xi\, V'\Sigma^{-1}V + \mathrm{o}_P(V'\Sigma^{-1}V \vee 1) = \xi\, M + \mathrm{o}_P(M \vee 1) .$$

We note that since M is a $k \times k$ matrix and k is held fixed in our asymptotics, all norms on M are equivalent. So $\mathrm{o}_P(M)$ just means that the maximal entry of $V'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}V - \xi M$ is negligible compared to the largest entry of M, or equivalently here that the largest singular value of the difference of the two matrices is negligible compared to that of M.

3.2. Quadratic forms in $\widehat{\mu}$ and $\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}$. Because we need to estimate the mean of the data, we also have to deal with forms of types $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$ and $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v$ for fixed v. Our main result in this direction is the following. The result is proven in Appendix B. (Given the results in the previous subsection, the set of results obtained there and here will allow us to conclude about the behavior of the quantity $\widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V}$ that is of central interest in this paper.)

Theorem 3.3. *Suppose that $X_i = \mu + \lambda_i \Sigma^{1/2} Y_i$, where Y_i are i.i.d. $\mathcal{N}(0, \mathrm{Id}_p)$ and $\{\lambda_i\}_{i=1}^n$ are random variables, independent of $\{Y_i\}_{i=1}^n$ and with $\mathbf{E}\left(\lambda_i^2\right) = 1$. Let v be a deterministic vector. Suppose that $\rho_n = p/n$ has a finite nonzero limit, ρ, and that $\rho \in (0,1)$.*

We denote $\tau_i = \lambda_i^2$. We assume that $\tau_i \neq 0$ for all i as well as that

(Assumption-BL)
$$\frac{1}{n^2}\sum_{i=1}^n \lambda_i^4 \to 0 \textit{ in probability , and } \frac{1}{n}\sum_{i=1}^n \lambda_i^2 \textit{ remains bounded in probability .}$$

If $\tau_{(i)}$ is the ith largest τ_k, we assume that we can find a random variable $N \in \mathbb{N}$ and positive real numbers ϵ_0 and C_0 such that

$$\begin{cases} P(p/N < 1-\epsilon_0) \to 1 \textit{ as } n \to \infty \ , \\ P(\tau_{(N)} > C_0) \to 1 \ , \\ \exists \eta_0 > 0 \textit{ such that } P(N/n > \eta_0) \to 1 \textit{ as } n \to \infty \ . \end{cases}$$

We also assume that the empirical distribution of τ_i's converges weakly in probability to a deterministic limit G.

We call Λ the $n \times n$ diagonal matrix with $\Lambda(i,i) = \lambda_i$, Y is the $n \times p$ matrix whose ith row is Y_i, $W = \Lambda Y$, and $\mathcal{S} = W'W/n$. Finally, we use the notation $\widehat{m} = W'\mathbf{e}/n$, $\widetilde{\mu} = \Sigma^{-1/2}\mu$.

Then we have, for ξ defined as in (3.2) and $\mathfrak{s}$ defined as in (3.1),

$$\frac{\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v}{\sqrt{v'\Sigma^{-1}v}} = \frac{\mu'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v}{\sqrt{v'\Sigma^{-1}v}} + \mathrm{o}_P(1) = \xi\frac{\mu'\Sigma^{-1}v}{\sqrt{v'\Sigma^{-1}v}} + \mathrm{o}_P\left(1 \vee \frac{\mu'\Sigma^{-1}v}{\sqrt{v'\Sigma^{-1}v}}\right) \ . \tag{3.3}$$

Also,

$$\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu} = \mu'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\mu + \frac{\rho_n}{(1-\rho_n)^2}\mathfrak{s} + 2\widetilde{\mu}'(\mathcal{S} - \widehat{m}\widehat{m}')^{-2}\widehat{m} + \mathrm{o}_P(1) \ , \tag{3.4}$$

and we recall that $\widetilde{\mu}'(\mathcal{S} - \widehat{m}\widehat{m}')^{-2}\widehat{m}/\|\widetilde{\mu}\| = \mathrm{o}_P(1)$.

The following remarks should help with the use of (3.4) in practice. We can consider three cases having to do with the size of $\mu'\Sigma^{-1}\mu = \|\widetilde{\mu}\|_2^2$.

1. If $\mu'\Sigma^{-1}\mu \to 0$, then $\widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu} = \frac{\rho_n}{(1-\rho_n)^2}\mathfrak{s} + \mathrm{o}_P(1)$.
2. If $\mu'\Sigma^{-1}\mu \to \infty$, then $\widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu} \sim \xi\mu'\Sigma^{-1}\mu$.
3. Finally, if $\mu'\Sigma^{-1}\mu$ stays bounded away from 0 and infinity, then

$$\widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu} = \xi\mu'\Sigma^{-1}\mu + \frac{\rho}{(1-\rho)^2}\mathfrak{s} + \mathrm{o}_P(1) \ .$$

We note that when the λ_i's are independent, (Assumption-BL) will be satisfied, as soon as λ_i^2 has two moments, by the Marcienkiewicz–Zygmund strong law of large numbers (see Chow and Teicher (1997), p. 125). Since two moments are required for X_i to have a covariance, the existence of a second moment is also necessary for the population quantities to exist. (Assumption-BL) is here to guarantee that certain quadratic forms involving random projections are asymptotically deterministic.

4. Applications to computing the realized risk of portfolios. We now combine our results to reach conclusions about the realized risk of portfolios selected by solving the problem (QP-eqc-Emp).

As a matter of notation, all of our approximation statements hold with high probability asymptotically, unless otherwise noted. We will carry out our work assuming that the data is generated from the models described in section 2 and our theorems and under the following assumptions:

1. Assumption A0: $p/n \to \rho \in (0,1)$, and the empirical distribution of the λ_i's converges weakly in probability to a deterministic limit G.
2. Assumption A1: For all $i \in \{1,\dots,k\}$, $v_i'\Sigma^{-1}v_i$ stays bounded away from 0. v_k is assumed to be equal to μ.
3. Assumption A2: The smallest eigenvalue of $M = V'\Sigma^{-1}V$ stays bounded away from 0, and the condition number of M remains bounded.
4. Assumption A3: For all i, $v_i'\Sigma^{-1}v_i$ stays bounded.
5. Assumption A4: (Assumption-BB) and (Assumption-BL) hold. (See Theorems 3.1 and 3.3 for definitions.)
6. Assumption A5: The operator norm of Σ, $|||\Sigma|||_2$, remains bounded.

We note that under these assumptions, the conclusions of our main theorems above are immediately applicable. In particular, A2 and A3 imply that (Condition-P1) and (Condition-P2) are satisfied. A5 could be relaxed and will simply be needed for estimation purposes later. Also, the part of A2 concerning the smallest off-diagonal entry of M could also likely be relaxed.

Let us now take a moment to recall some key relevant results from El Karoui (2010). Under assumptions similar to those we are now operating under, we showed the following:

- $\mu' w_{\text{emp}}$, the realized returns of our portfolio, were not a consistent estimator of μ_P, the target returns for our portfolio. We proposed in El Karoui (2010) an estimator of $\mu' w_{\text{emp}}$ which seems to perform well in (perhaps limited) simulations. The corrections we proposed there (or others) should be used if one wants to plot efficient frontier graphs that reflect the correct level of returns of portfolios.
- We showed in El Karoui (2010) that $\mathfrak{s}^{(E)} \geq \mathfrak{s}^{(G)} = (1-\rho)^{-1}$ (see (3.1) for a definition of $\mathfrak{s}$). In other words, the $\mathfrak{s}$ corresponding to genuinely elliptical models is greater than the $\mathfrak{s}$ corresponding to Gaussian models.

4.1. Theoretical predictions. We recall the notations $\rho = \lim p/n$ and $\kappa = \rho/(1-\rho)$. Applying the results of our theorems above, we have the following fact.

Fact 4.1. *Let us denote $M = V'\Sigma^{-1}V$. When Assumptions* A0–A5 *are satisfied, we have asymptotically*

$$\widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V} \simeq \xi V'\Sigma^{-1}V + \frac{\kappa}{1-\rho}\mathfrak{s} e_k e_k' \,,$$

$$\widehat{M} = \widehat{V}'\widehat{\Sigma}^{-1}\widehat{V} \simeq \mathfrak{s} V'\Sigma^{-1}V + \kappa e_k e_k' \,.$$

Also, under our assumptions, as shown in El Karoui (2010),

$$\widehat{M}^{-1} \simeq \frac{1}{\mathfrak{s}}\left(V'\Sigma^{-1}V + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1} \,.$$

Before we proceed, let us also recall that

$$\frac{\xi}{\mathfrak{s}^2} \geq \frac{1}{1-\rho}$$

and that, in the Gaussian case, $\xi = (1-\rho)^{-3}$ and $\mathfrak{s} = (1-\rho)^{-1}$. So the right-hand side of the previous inequality is achieved in the Gaussian setting.

Fact 4.1 allows us to give the following characterization of the realized risk of "Markowitz" portfolios.

Theorem 4.1. *Recall that in matrix form, the optimal risk for* (QP-eqc-Pop), $w'_{\text{theo}}\Sigma w_{\text{theo}}$, *is equal to* $U'M^{-1}U$.

Suppose that a portfolio is chosen by allocating weights w_{emp} *to each asset according to the solution of* (QP-eqc-Emp)*. Under assumptions* A0–A5*, we have asymptotically*

$$w'_{\text{emp}}\Sigma w_{\text{emp}} \simeq \frac{1}{\mathfrak{s}^2}\left(\xi U'\left(M + \frac{\kappa}{\mathfrak{s}}e_k e'_k\right)^{-1} U + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho} - \xi\right)\left[U'\left(M + \frac{\kappa}{\mathfrak{s}}e_k e'_k\right)^{-1} e_k\right]^2\right) . \tag{4.1}$$

Furthermore, in the situation where μ *is assumed to be known or, equivalently, if all the elements of* V *are deterministic and given, asymptotically,*

$$w'_{\text{emp}}\Sigma w_{\text{emp}} \simeq \frac{\xi}{\mathfrak{s}^2}U'M^{-1}U \simeq \frac{\xi}{\mathfrak{s}^2}w'_{\text{theo}}\Sigma w_{\text{theo}} .$$

As noted above, $\xi \geq \mathfrak{s}^2/(1-\rho)$, with equality in the Gaussian case. So the second term in the approximation to $w'_{\text{emp}}\Sigma w_{\text{emp}}$ in (4.1) is negative, except in the Gaussian case, where it is zero.

Furthermore, in cases where μ does not need to be estimated, $w'_{\text{emp}}\Sigma w_{\text{emp}}$ is at least $1/(1-\rho)$ times as large as the population optimum $w'_{\text{theo}}\Sigma w_{\text{theo}}$, the coefficient $1/(1-\rho)$ corresponding to the Gaussian case.

We also recall one of the main results of El Karoui (2010): under the same assumptions as those of Theorem 4.1, we have, for the (extremely) naive estimate of risk,

$$w'_{\text{emp}}\widehat{\Sigma}w_{\text{emp}} \simeq \frac{1}{\mathfrak{s}}U'\left(M + \frac{\kappa}{\mathfrak{s}}e_k e'_k\right)^{-1} U .$$

Furthermore, if μ does not need to be estimated, we then have $w'_{\text{emp}}\widehat{\Sigma}w_{\text{emp}} \simeq \frac{1}{\mathfrak{s}}w'_{\text{theo}}\Sigma w_{\text{theo}}$.

These remarks combined with Theorem 4.1 allow us to quantify the difference between the naive estimate of risk of our portfolio and its realized risk. We will later discuss an estimator of this realized risk.

Proof. Let us denote $\widehat{N} = \widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V}$. The analyses we performed show that

$$\begin{aligned}\widehat{N} &\simeq \xi M + \frac{\kappa}{1-\rho}\mathfrak{s}e_k e'_k \\ &\simeq \xi\left(M + \frac{\kappa}{\mathfrak{s}}\right) + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho} - \xi\right)e_k e'_k .\end{aligned}$$

Therefore, since $\widehat{M}^{-1} \simeq 1/\mathfrak{s}\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1}$, we have

$$\widehat{M}^{-1}\widehat{N} \simeq \frac{1}{\mathfrak{s}}\left(\xi \mathrm{Id}_p + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho}-\xi\right)\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} e_k e_k'\right) .$$

So we also have

$$\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1} \simeq \frac{1}{\mathfrak{s}^2}\left(\xi\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho}-\xi\right)\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} e_k e_k'\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1}\right) .$$

Since $w_{\text{emp}}'\Sigma w_{\text{emp}} = U'\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1}U$, we finally conclude that

$$w_{\text{emp}}'\Sigma w_{\text{emp}} \simeq \frac{1}{\mathfrak{s}^2}\left(\xi U'\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} U + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho}-\xi\right)\left[U'\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} e_k\right]^2\right) .$$

This proves (4.1).

Let us now turn to the second part of the theorem. When $\widehat{V} = V$, $\widehat{M} = V'\widehat{\Sigma}^{-1}V \simeq \mathfrak{s}M$. Also, $\widehat{N} = V'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}V \simeq \xi M$. Therefore, in this situation, where μ does not need to be estimated,

$$w_{\text{emp}}'\Sigma w_{\text{emp}} \simeq \frac{\xi}{\mathfrak{s}^2}U'M^{-1}U .$$

Now $U'M^{-1}U$ is equal to $w_{\text{theo}}'\Sigma w_{\text{theo}}$, the population optimum and efficient frontier. ∎

Theorem 4.2 (comparison between Gaussian and elliptical cases). *The realized risk of a portfolio computed by solving* (QP-eqc-Emp)*, where the data is elliptical, is greater than that of its Gaussian counterpart (under our assumptions and asymptotically in probability).*

If $w_{\text{emp}}^{(E)}$ *is the solution of* (QP-eqc-Emp) *when* X_i *are elliptical and* $w_{\text{emp}}^{(G)}$ *is the solution when* X_i *are Gaussian, we have, under our assumptions and asymptotically in probability,*

$$(w_{\text{emp}}^{(E)})'\,\Sigma\, w_{\text{emp}}^{(E)} \geq (w_{\text{emp}}^{(G)})'\,\Sigma\, w_{\text{emp}}^{(G)} .$$

The inequality is strict if X_i *is genuinely elliptical and not Gaussian (i.e.,* $\lambda_i \neq 1$ *with strictly positive probability).*

It is interesting to compare this theorem to its counterpart in El Karoui (2010), Theorem 5.1 there. That theorem shows that the (very) naive estimator of risk, $(w_{\text{emp}}^{(E)})'\,\widehat{\Sigma}\, w_{\text{emp}}^{(E)}$, underestimates the true risk, and that this underestimation is more pronounced in the elliptical case than in the Gaussian case. So we conclude that

$$\frac{(w_{\text{emp}}^{(E)})'\,\Sigma\, w_{\text{emp}}^{(E)}}{(w_{\text{emp}}^{(E)})'\,\widehat{\Sigma}\, w_{\text{emp}}^{(E)}} \geq \frac{(w_{\text{emp}}^{(G)})'\,\Sigma\, w_{\text{emp}}^{(G)}}{(w_{\text{emp}}^{(G)})'\,\widehat{\Sigma}\, w_{\text{emp}}^{(G)}} \simeq \frac{1}{(1-\rho)^2} .$$

We note that the analysis presented in the proof below actually shows that

$$(w_{\text{emp}}^{(E)})'\,\Sigma\, w_{\text{emp}}^{(E)} \geq \frac{1}{1-\rho}U'\left(M+\frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1} U ,$$

and therefore, asymptotically and with high probability,

$$\frac{(w_{\text{emp}}^{(E)})' \, \Sigma \, w_{\text{emp}}^{(E)}}{(w_{\text{emp}}^{(E)})' \, \widehat{\Sigma} \, w_{\text{emp}}^{(E)}} \geq \frac{\mathfrak{s}}{1-\rho} \, .$$

Since $\mathfrak{s} \geq 1/(1-\rho)$, with equality in the Gaussian case, this inequality gives a sharper notion of the impact of ellipticity on risk underestimation.

Finally, let us say that

$$\frac{(w_{\text{emp}}^{(E)})' \, \Sigma \, w_{\text{emp}}^{(E)}}{(w_{\text{emp}}^{(E)})' \, \widehat{\Sigma} \, w_{\text{emp}}^{(E)}}$$

is a measure of how accurate the (very) naive estimator of risk $(w_{\text{emp}}^{(E)})' \, \widehat{\Sigma} \, w_{\text{emp}}^{(E)}$ is at predicting the actual risk of our strategy (in the setting of i.i.d. data), $(w_{\text{emp}}^{(E)})' \, \Sigma \, w_{\text{emp}}^{(E)}$. What our computations show is that it is never terribly accurate and it is least inaccurate in the Gaussian case. This also suggests that doing corrections or predictions based on Gaussian computations will yield poor results (and still risk underestimation!) in the class of elliptical distributions considered here.

We now turn to the proof.

Proof of Theorem 4.2. Recall that $\mathfrak{s}^{(E)} \geq \mathfrak{s}^{(G)}$. Recall also that $\xi^{(E)}/(\mathfrak{s}^{(E)})^2 \geq 1/(1-\rho)$. Finally, our theorems show that

$$\widehat{N} = \widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V} \simeq \xi M + \frac{\kappa}{1-\rho}\mathfrak{s} e_k e_k' \, .$$

Therefore,

$$\frac{1}{\mathfrak{s}^2}\widehat{N} \simeq \frac{\xi}{\mathfrak{s}^2} M + \frac{\kappa}{1-\rho}\frac{1}{\mathfrak{s}} e_k e_k' \succeq \frac{1}{1-\rho}\left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right) \, .$$

Now $\widehat{M} = \widehat{V}'\widehat{\Sigma}^{-1}\widehat{V} \simeq \mathfrak{s}(M + \frac{\kappa}{\mathfrak{s}} e_k e_k')$. So

$$\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1} \simeq \left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1}\left[\frac{1}{\mathfrak{s}^2}\widehat{N}\right]\left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1} \, .$$

Therefore, asymptotically with high probability,

$$\begin{aligned}
\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1} &\succeq \left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1}\left[\frac{1}{1-\rho}\left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)\right]\left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1} \\
&= \frac{1}{1-\rho}\left(M + \frac{\kappa}{\mathfrak{s}} e_k e_k'\right)^{-1} \, .
\end{aligned}$$

Now $M + \frac{\kappa}{\mathfrak{s}^{(E)}} e_k e_k' \preceq M + \frac{\kappa}{\mathfrak{s}^{(G)}} e_k e_k'$, so

$$\left(M + \frac{\kappa}{\mathfrak{s}^{(E)}} e_k e_k'\right)^{-1} \succeq \left(M + \frac{\kappa}{\mathfrak{s}^{(G)}} e_k e_k'\right)^{-1} \, .$$

We conclude that asymptotically with high probability,

$$\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1} \succeq \frac{1}{1-\rho}\left(M + \frac{\kappa}{\mathfrak{s}^{(G)}} e_k e_k'\right)^{-1} \, .$$

Now recall that in the Gaussian case, $\xi^{(G)}/(\mathfrak{s}^{(G)})^2 = 1/1-\rho$, so that

$$(\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1})^{(G)} \simeq \frac{1}{1-\rho}\left(M + \frac{\kappa}{\mathfrak{s}^{(G)}}e_k e_k'\right)^{-1} .$$

Since $w_{\text{emp}}'\Sigma w_{\text{emp}} = U'\widehat{M}^{-1}\widehat{N}\widehat{M}^{-1}U$, we conclude that, asymptotically with high probability,

$$(w_{\text{emp}}^{(E)})'\Sigma w_{\text{emp}}^{(E)} \geq (w_{\text{emp}}^{(G)})'\Sigma w_{\text{emp}}^{(G)} .$$

The proof shows that in the case where X_i are genuinely elliptical, since $\mathfrak{s}^{(E)} > \mathfrak{s}^{(G)}$, we have a strict inequality in the conclusion. ∎

4.2. Improved estimation of the realized risk. We now seek a robust estimator—in the class of elliptical distributions considered here—for the realized risk of our Markowitz portfolios. From a practical standpoint, it could be useful to help assess the actual risk of a portfolio constructed using all the data available, i.e., the n observations $\{X_i\}_{i=1}^n$. Naturally, statistically, one might want to use techniques like cross-validation to make this assessment empirically, but this would reduce the effective number of samples of the procedure and hence yield even fewer optimal allocations than those we could get by using all the data. (Note also that empirical procedures that effectively change the ratio p/n—5-fold and 10-fold cross-validation are typical examples—are problematic in our setting since all the risk results depend on this ratio.)

For the purposes of the discussion that follows, we now assume that the λ_i's are independent.

Recall that (4.1) showed that

$$\text{RRisk} \simeq \frac{1}{\mathfrak{s}^2}\left(\xi U'\left(M + \frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1}U + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho} - \xi\right)\left[U'\left(M + \frac{\kappa}{\mathfrak{s}}e_k e_k'\right)^{-1}e_k\right]^2\right) .$$

Furthermore, $\widehat{M}^{-1} \simeq \frac{1}{\mathfrak{s}}(M + \frac{\kappa}{\mathfrak{s}}e_k e_k')^{-1}$, so it turns out that

$$\text{RRisk} \simeq \frac{\xi}{\mathfrak{s}}U'\widehat{M}^{-1}U + \frac{\kappa}{\mathfrak{s}}\left(\frac{\mathfrak{s}^2}{1-\rho} - \xi\right)\left[U'\widehat{M}^{-1}e_k\right] .$$

Now we recall that in El Karoui (2010), we proposed an estimator of $\mathfrak{s}$ and the λ_i^2's: our proposal, motivated by concentration of measure results for Gaussian random vectors (see Ledoux (2001)), was to do the following:

1. Estimate λ_i^2 by
$$\widehat{\tau}_i = \widehat{\lambda_i^2} = \frac{\|X_i - \widehat{\mu}\|_2^2}{\sum_{i=1}^n \|X_i - \widehat{\mu}\|_2^2/n} .$$
2. If we denote $\rho_n = p/n$, we then proposed to estimate $\mathfrak{s}$ by $\widehat{\mathfrak{s}}$, the positive solution of
$$g(x) = 1 - \rho_n ,$$
$$\text{where } g(x) = \frac{1}{n}\sum_{i=1}^n \frac{1}{1 + x\widehat{\lambda_i^2}\rho_n} .$$

In light of (3.2), we propose estimating ξ by $\widehat{\xi}$ with

$$\frac{1}{\widehat{\xi}} = \frac{1}{\widehat{\mathfrak{s}}^2} - \rho_n \frac{1}{n} \sum_{i=1}^{n} \frac{\widehat{\tau}_i^{\,2}}{(1+\widehat{\tau}_i \rho_n \widehat{\mathfrak{s}})^2} \,. \tag{4.2}$$

Note that $U'\widehat{M}^{-1}U$ is the plug-in (and very naive) estimator of risk. Let us denote it by f_{emp}. We propose using

$$\widehat{\text{RRisk}} = \frac{\widehat{\xi}}{\widehat{\mathfrak{s}}} f_{\text{emp}} + \frac{\kappa}{\widehat{\mathfrak{s}}} \left(\frac{\widehat{\mathfrak{s}}^2}{1-\rho_n} - \widehat{\xi} \right) \left[U'\widehat{M}^{-1} e_k \right] \,. \tag{4.3}$$

The simulation work that follows illustrates the performance of this estimator.

4.3. Simulation results. To investigate the quality of our proposed estimator in practice, we now present some simulations. We compare two situations, one where the data generated are normally distributed and one where they have a distribution close to a multivariate t_6 distribution. In this latter case, the λ_i's are i.i.d. and have a univariate t_6 distribution, scaled to have a second moment equal to 1. We note that the t_6 distribution has only 5 moments, so the corresponding X_i's also have only 5 moments and are much more heavy-tailed than in the situation where they have a Gaussian distribution.

Before we present pictures of our simulations, let us briefly summarize our findings. The estimator presented in (4.3) seems to work quite or reasonably well in expectation when n and p are in the low 100's. However, in this situation, the variance is still quite large. We also note that the "true" realized risk of our portfolios, which is itself a conditional variance and hence varies from one simulation to another, is also quite variable. Furthermore, in this situation, the difference in realized risks between the Gaussian and the t_6 cases is considerable, as our theoretical results suggest. We make this remark to emphasize how assuming Gaussianity would lead to vastly overoptimistic conclusions.

When we increase dimensionality to the low 1000's for both p and n (something that is probably unrealistic at this point in financial applications but might be relevant in other areas of applications), the variance issue becomes less important or significant (as we would expect since we are closer to the limiting setting and therefore our predictions should be more accurate), and the quality of our predictions of realized risk is even better. Of course, the advantage of having an estimator that is robust in the class of elliptical distributions is that we do not need to specify which distribution likely generated the data—as long as it is elliptical, we will be fine. We note that again here the difference between the realized risks in the Gaussian and t_6 cases is very large.

We present simulations in the situation where Σ is a Toeplitz matrix with $\sigma(i,j) = .4^{|i-j|}$. Two dimensionality settings are investigated. In the first case, illustrated in Figure 2, $n = 250$ and $p = 100$. In the second case, illustrated in Figure 3, $n = 2500$, $p = 1000$. The first constraint vector, v_1, was chosen to be the eigenvector associated with the $.9 * p$ largest eigenvalue of Σ. We also relied on the vector v_2, chosen to be the eigenvector associated with the $.15 * p$ largest eigenvalue of Σ. The second constraint vector, which played the role of μ, was chosen as $\sqrt{.3}v_1 + \sqrt{.7}v_2$.

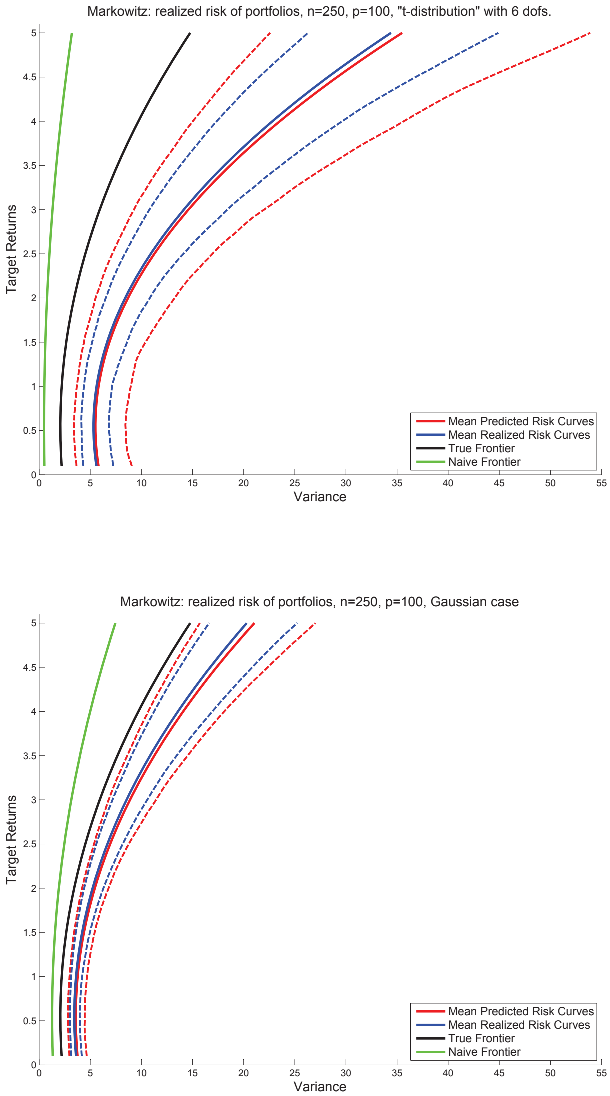
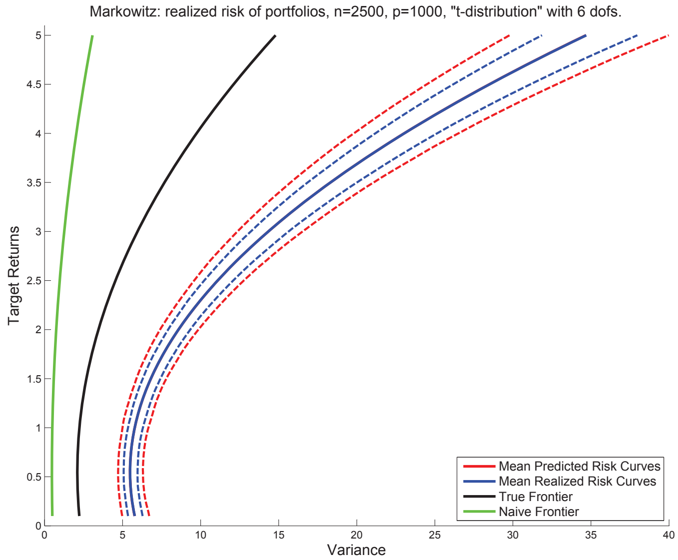


Figure 2. *Performance of naive and corrected frontiers for scaled "t_6" (upper picture) and Gaussian returns. Here, $n = 250$ and $p = 100$. The number of simulations is 1000 in all pictures. The dashed lines represent (empirical) 95% confidence bands. (The confidence bands are computed for a fixed level of expected returns y.) The x-axis represents our estimate of the realized variance of the optimal portfolios. The y-axis represents the target returns for the portfolios. The plots show both the average realized risk of the naive Markowitz portfolios (blue curves) and the fact that our estimator (in red) is nearly unbiased (red solid curves near the blue curves). They also illustrate the robustness of our corrections. Another striking feature is the lack of robustness of Gaussian computations since the average realized risk computed with "t_6" returns is very different from that computed with the Gaussian ones. The fact that, as our theoretical work predicts, Gaussian computations leads to underestimation of the realized risk in the class of elliptical distributions considered in the paper is illustrated by the fact that the "t_6" curves are to the right of the Gaussian curves. The population (or true) efficient frontier is in black. The green curve is the mean estimated risk if estimated through the naive estimator $w'_{\text{emp}}\widehat{\Sigma}w_{\text{emp}}$.*



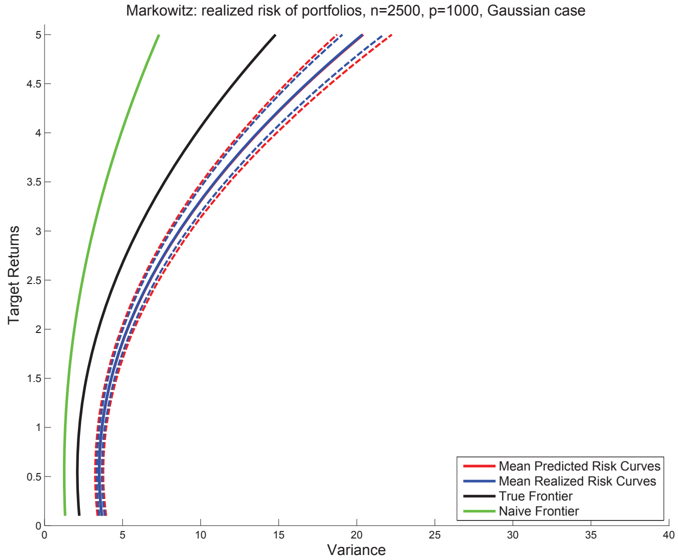


Figure 3. *Performance of naive and corrected frontiers for scaled "t_6" (upper picture) and Gaussian returns. Here, $n = 2500$ and $p = 1000$. The number of simulations is 1000 in all pictures. The dashed lines represent (empirical) 95% confidence bands. (The confidence bands are computed for a fixed level of expected returns y.) The x-axis represents our estimate of the realized variance of the optimal portfolios. The y-axis represents the target returns for the portfolios. The plots show both the average realized risk of the naive Markowitz portfolios (blue curves) and the fact that our estimator (in red) is nearly unbiased (red solid curves below the blue curves and not visible on the plots). They also illustrate the robustness of our corrections.*

We also present simulations that are closer to real data. In this second set of simulations, we took the daily returns of 48 Fama–French industry portfolios for the year 2005. We computed the corresponding sample mean and sample covariance matrix and took them as our new population parameters. From them, as in a parametric bootstrap, we generated 1000 datasets, with $n = 252$ and $p = 48$, under the Gaussian and t_6 models. In this situation, the matrix M had a relatively large condition number, equal to 40, and the setting is more difficult for our estimators. As we show in Figure 4, the variability of our estimator is quite large and the average performance is not as good (the performance of the medians are quite similar) as the one we observed on the other synthetic problems. The smaller p and relatively large condition number of M might explain some of these problems.

4.4. Independent mean and covariance estimation. Our previous analyses have explained the somewhat involved risk phenomenon that happens (under our assumptions) when mean and covariance are estimated from a common dataset. This is a natural way to go about solving our problem and is a recurrent problem in various areas of statistics, so it is important to understand this interaction.

In a more specific finance context, practitioners often argue that the mean can be estimated independently of the covariance matrix. Probabilistically and statistically, this is a much simpler situation than the one we have investigated, as it essentially amounts to simply understanding the case where the constraints are deterministic, which is the technically simpler part of our work.

However, a referee suggested that this be investigated for the benefits of practitioners, so we now spell out the results.

We will consider the case where $\widehat{\mu} \sim \mathcal{N}(\mu, \frac{\Sigma_{\widehat{\mu}}}{N_{\widehat{\mu}}})$ and $\widehat{\mu}$ is independent of $\widehat{\Sigma}$. More complicated situations can be investigated along the lines we outline below, but we leave them to the interested reader.

Theorem 4.3. *Suppose* $\widehat{\mu} \sim \mathcal{N}(\mu, \frac{\Sigma_{\widehat{\mu}}}{N_{\widehat{\mu}}})$*, with* $N_{\widehat{\mu}} \to \infty$*, and, further,* $\operatorname{trace}\left((\Sigma_{\widehat{\mu}}\Sigma^{-1})^2\right)/N_{\widehat{\mu}}^2 \to 0$, $\operatorname{trace}\left(\Sigma_{\widehat{\mu}}\Sigma^{-1}\right)/N_{\widehat{\mu}} \to C$*, with* $0 \leq C < \infty$ *and* $\|\Sigma^{-1/2}\Sigma_{\widehat{\mu}}\Sigma^{-1/2}\|$ *remaining bounded.*

Then, under assumptions A0–A5*, we have (asymptotically in probability)*

$$w_{\text{emp}}'\Sigma w_{\text{emp}} \simeq \frac{\xi}{\mathfrak{s}^2}U'(M + Ce_k e_k')^{-1}U = \frac{\xi}{\mathfrak{s}^2}\left[U'M^{-1}U - C\frac{(U'M^{-1}e_k)^2}{1 + Ce_k'M^{-1}e_k}\right] .$$

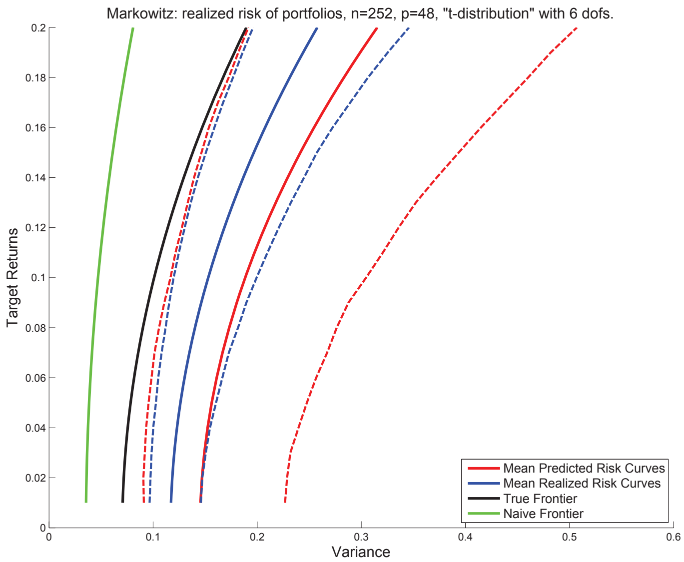
An informal argument is as follows: because $\widehat{\Sigma}$ and $\widehat{\mu}$ are independent by assumption, we can condition on $\widehat{\mu}$ when studying quantities of the form $\widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V}$ without affecting the statistical properties of $\widehat{\Sigma}$, and we can therefore do as if $\widehat{V}$ were deterministic.

Therefore, when $\widehat{\mu}$ is independent of $\widehat{\Sigma}$, if we can proceed with the results we have developed above,

$$\widehat{V}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{V}|\widehat{\mu} \simeq \xi\widehat{V}'\Sigma^{-1}\widehat{V}|\widehat{\mu} ,$$
$$\widehat{V}'\widehat{\Sigma}^{-1}\widehat{V}|\widehat{\mu} \simeq \mathfrak{s}\widehat{V}'\Sigma^{-1}\widehat{V}|\widehat{\mu} .$$

And finally,

$$w_{\text{emp}}'\Sigma w_{\text{emp}}|\widehat{\mu} \simeq \frac{\xi}{\mathfrak{s}^2}U'(\widehat{V}'\Sigma^{-1}\widehat{V})^{-1}U|\widehat{\mu} .$$



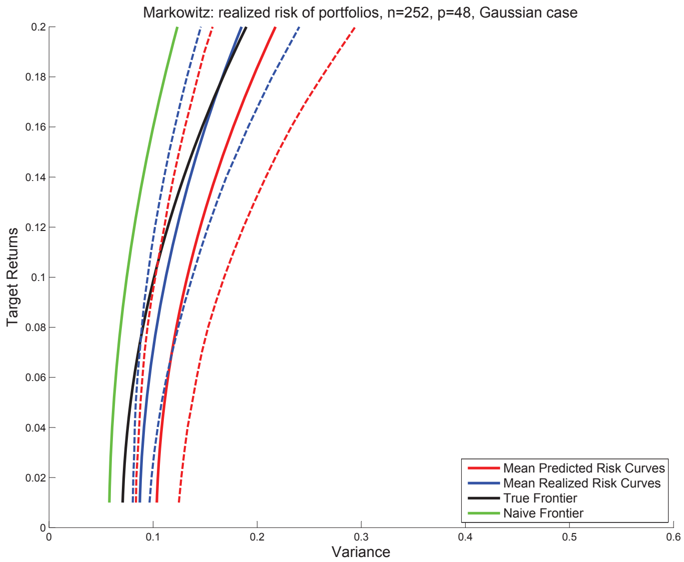


Figure 4. *Performance of naive and corrected frontiers for scaled "t_6" (upper picture) and Gaussian returns. Here, $n = 252$ and $p = 48$ and the population parameters were computed from real data. The number of simulations is 1000 in all pictures. The dashed lines represent (empirical) 95% confidence bands. (The confidence bands are computed for a fixed level of expected returns y.) The x-axis represents our estimate of the realized variance of the optimal portfolios. The y-axis represents the target returns for the portfolios. The plots show both the average realized risk of the naive Markowitz portfolios (blue curves) and the average of our estimators (in red). Here our estimator is still biased.*

Now we have to decondition on $\widehat{\mu}$, but under our assumptions it is clear that (in probability)

$$(\widehat{V}'\Sigma^{-1}\widehat{V})^{-1} \simeq (V'\Sigma^{-1}V + Ce_k e_k')^{-1} \, .$$

That is how one arrives at the result. We check in Appendix C (subsection C.1) the few details that remain to be verified.

4.4.1. On time dependence of observations. As we have mentioned at the end of section 2, our framework is rich enough to deal with observations that are dependent in time; in other words, the vectors of returns from one day to another are dependent. (More details about the kind of time dependence our approach can handle can be found in our discussion there.) In particular, the computations and proofs done in this paper immediately apply to the situation where we have time dependence and deterministic constraints. Furthermore, when the constraints are independent of the sample covariance matrix, we can apply the same strategy as we just did in subsection 4.4 to measure the realized risk: condition on the constraints, find equivalents of the risk that involve only Σ, and then decondition on the constraints.

Hence, the results of subsection 4.4 also apply to observations that are correlated in time, after we make the adjustments mentioned at the end of section 2 (i.e., the role of the λ_i^2 is now taken by the eigenvalues of a certain positive semidefinite matrix mentioned in section 2).

As explained there, the time-dependence structure we investigate here is simple (it covers some basic multivariate time series), but interestingly, the paper shows that neglecting this time dependence can be rather harmful: for instance, when the returns are assumed to be (marginally) Gaussian, the presence of time correlation essentially turns them—from a risk point of view—into elliptical data. And as we have seen, risk underestimation is more pronounced for elliptical data than it is for Gaussian data. So taking the "independent Gaussian view" on Gaussian data that are correlated in time is potentially harmful, as it might lead to an overoptimistic view of risk.

4.4.2. On the role of the empirical estimate of the mean. It is very reasonable statistically to think that a working estimate of the mean returns could be obtained by combining the sample mean from the data used to compute the sample covariance matrix and an estimate that is independent of the sample covariance matrix.

In that case, our analysis of the quadratic forms involving $\widehat{\mu}$ and $\widehat{\Sigma}$ again becomes relevant and the same strategy as the one we have used in subsection 4.4 is again useful for coming up with theoretical predictions of the realized risk of a portfolio constructed using these three estimators.

4.5. Mitigating the ellipticity problem.

From a practical standpoint, we may ask whether one could mitigate the ellipticity problem and build estimators that behave as if the data were Gaussian instead of genuinely elliptical. The answer to this question is—under the assumptions on the data made in this paper—positive, as far as the covariance matrix is concerned. We refer the interested reader to El Karoui (2010), subsubsection 5.4.4 for such a proposal, specifically targeted to the high-dimensional setting. (The gist of the idea is to use concentration of Gaussian random vectors (see Ledoux (2001)) in practice.) When the constraints are independent of the sample covariance matrix, this estimator can easily be shown to (asymptotically) transform an elliptical problem into a Gaussian one, yielding lower risk for the allocation. However, when the estimate of the mean is not independent of our estimate of the covariance the interaction between the estimate of the mean and the estimate of the covariance remains to be studied.

5. Conclusion. We have analyzed in this paper the impact of high dimensionality and ellipticity of the data vectors on the risk of Markowitz portfolios obtained by seeking the solution of the generalized Markowitz problem (QP-eqc-Emp), a quadratic program with linear equality constraints. The situation we investigate allows short-selling; the no short-selling situation is completely different from a risk standpoint and far less subject to the problems of risk estimation we describe here, though no short-selling optimization comes with its shares of practical problems and limitations. One of our main results is that we have provided an estimator of this realized risk that is robust in the class of elliptical distributions and appears to work reasonably well, in practice, in the limited simulations we have investigated. An interesting by-product of our results is that they show that there is no "universality" in the random matrix sense of the word for this problem. The details of the model do matter, and we cannot limit ourselves to specifying the population parameters if we want to get a general and robust answer. Our results therefore suggest that claims of universality found in the physics literature are unfounded.

Elliptical models are a relatively rich class of models and allow us to incorporate in the modeling features of financial data that are often found in practice. For instance, our models have leptokurtic marginals and can have heavy tails (only two moments are needed for our results to be valid in the i.i.d. case) and tail dependence. However, it seems to us that what drives the key results are global geometric features of the data and not details about the marginals. These global geometric features play an important role in random matrix theory (see El Karoui (2009)), and it is not surprising that they play a key role here. We expect that the results obtained in our paper can be shown to be valid under weaker and less restrictive distributional assumptions, something we are currently working on, as long as the geometry of the data implied by the model is conserved.

Interestingly, we show that both the estimation of the mean and the covariance of the data vectors are important and create their share of problems. Both yield first-order effects and biases which cannot be ignored. Naturally, if extra information is available and used, these problems could be partially alleviated by, for instance, shrinking the sample mean and covariance matrix towards appropriate targets. But an analysis that takes into account both sources of problems is indeed necessary.

We note that our work could also be adapted to study the performance of portfolios obtained by using weighted estimators of covariance or by bootstrapping. As shown in El Karoui (2010) for the bootstrap, these problems are essentially covered by the elliptical framework. We note that the more complicated aspects of the issue come from having to understand $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$, which in general will behave differently from how it did in the situations we have considered in this paper (see the bootstrap analysis in El Karoui (2010) for an example in a different but related problem). The possible choice of different weights for the estimation of the mean and the covariance also complicates the situation, though the tools used in this paper seem suitable for analyzing this problem. (We postpone its analysis to another paper, as this one is already quite long.) On the other hand, we note that forms of the type $v'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v$ should still behave as we have described in this paper, after we properly account for the "ellipticity" generated by weighting different observations differently.

It is natural to ask whether the analysis can be extended to shrinkage estimates of covariance, for instance. This question was investigated by the author while this paper was being reviewed, and it was solved in El Karoui and Koesters (2011) in an extremely general setting, allowing a wide range of distributions (such as log-normal, for instance) and shrinkage towards essentially any covariance matrix. We refer the interested reader to this paper for more information.

Random matrix theory appears to be a convenient and valuable tool for the study of the optimization problem we were concerned with. It has helped shed some light on an otherwise quite difficult problem and might be helpful in the analysis of other optimization problems that are sensitive to spectral properties of the input data.

Recall that two main quantities of central interest in this paper are $v'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v$, where v is a deterministic vector, and $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$.

If $Y_1 = \Sigma_1 Y$, where Y is an $n \times p$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries, understanding $v'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v$ is equivalent to understanding $\alpha'(\frac{Y_1'HY_1}{n-1})^{-2}\alpha$ for an appropriately chosen α. This question is the focus of the following section.

Appendix A. On $\alpha'(\frac{Y_1'HY_1}{n-1})^{-2}\alpha$. The analysis of this quantity is closely connected the work that was done in El Karoui (2010), Theorem 4.1. In this section we prove Theorem 3.1, Fact 3.1, and Corollary 3.2. We refer the reader to subsection 3.1 for their statements.

Proof of Corollary 3.2. The proof of the corollary follows exactly the same steps as the proof of Lemma 4.1 in El Karoui (2010). ■

Proof of Fact 3.1. Let us consider the integral

$$I = \int \frac{\rho^2\tau^2\mathfrak{s}^2}{(1+\tau\rho\mathfrak{s})^2}dG(\tau)\ .$$

By writing $\rho\tau\mathfrak{s}/(1+\rho\tau\mathfrak{s}) = (1-1/(1+\rho\tau\mathfrak{s}))$, we see that

$$I = 1 - 2\int \frac{dG(\tau)}{1+\rho\tau\mathfrak{s}} + \int \frac{dG(\tau)}{(1+\rho\tau\mathfrak{s})^2}\ .$$

Now, by the definition of $\mathfrak{s}$, $\int \frac{dG(\tau)}{1+\rho\tau\mathfrak{s}} = (1-\rho)$. On the other hand, by the convexity of the square function, we have

$$\int \frac{dG(\tau)}{(1+\rho\tau\mathfrak{s})^2} \geq \left(\int \frac{dG(\tau)}{(1+\rho\tau\mathfrak{s})}\right)^2 = (1-\rho)^2\ .$$

Therefore,

$$I \geq 1 - 2(1-\rho) + (1-\rho)^2 = (1-(1-\rho))^2 = \rho^2\ ,$$

or

$$\int \frac{\tau^2\mathfrak{s}^2}{(1+\rho\tau\mathfrak{s})^2}dG(\tau) \geq 1\ .$$

Therefore,

$$\frac{1}{\mathfrak{s}^2}\left(1-\rho\int\frac{\tau^2\mathfrak{s}^2 dG(\tau)}{(1+\rho\tau\mathfrak{s})^2}\right)\leq\frac{1-\rho}{\mathfrak{s}^2}\,.$$

Now we know that $\xi \geq 0$ by construction (see below). So we conclude that

$$\xi \geq \frac{\mathfrak{s}^2}{(1-\rho)}\,. \qquad \blacksquare$$

Proof of Theorem 3.1. Our proof follows closely the proof of Theorem 4.1 in El Karoui (2010). The matrix X can be written as

$$X = \mathbf{e}\mu' + \Lambda Y\,,$$

where Y is an $n \times p$ data matrix having the Y_i's as its rows and hence i.i.d. $\mathcal{N}(0,1)$ entries under our assumptions, and Λ is the diagonal matrix containing the λ_i's. Therefore,

$$\mathcal{S} = \frac{1}{n-1}X'HX = \frac{1}{n-1}Y'\Lambda'H\Lambda Y \triangleq \frac{1}{n-1}Y'LY\,,$$

where $L = \Lambda'H\Lambda$. It has been argued in El Karoui (2010) that $Y'LY$ is invertible with probability 1 under our assumptions. Note that the quantity we care about is

$$\alpha'\mathcal{S}^{-2}\alpha\,.$$

We will first get results conditional on Λ and then will argue that we can decondition and get results unconditionally. We call γ_i the ordered eigenvalues of $\mathcal{S}$, γ_p being the smallest.

Results conditionally on Λ. Let us call $\mathcal{L}_{\epsilon,\delta}$ the set of matrices Λ such that $p/N < 1-\epsilon$ and $C(N-1)/(n-1) > \delta$. Under (Assumption-BB), for a δ bounded away from 0 (e.g., $\delta = 1/2\liminf C_0 N/n$), $P(\Lambda \in \mathcal{L}_{\epsilon,\delta}) \to 1$. We assume that the Λ we condition on below belongs to $\mathcal{L}_{\epsilon,\delta}$. In such a situation, one can show (see Lemma B-1 in El Karoui (2010)) that if P_Λ denotes probability conditional on Λ, and if $\Lambda \in \mathcal{L}_{\epsilon,\delta}$, then

$$P_\Lambda\left(\sqrt{\gamma_p}\leq\sqrt{\delta}\left[(1-\sqrt{1-\epsilon})-t\right]\right)\leq\exp\left(-(n-1)\delta t^2/C\right)\,.$$

In other words, the smallest eigenvalue of $\mathcal{S}$ is uniformly bounded away from 0 with very high (conditional) probability. We will see that these uniform bounds on the smallest eigenvalue of $\mathcal{S}$ will eventually allow us to go from the conditional results on Λ to unconditional ones.

Let us write the spectral decomposition of $\mathcal{S}$:

$$\mathcal{S} = \sum_{i=1}^p \gamma_i \upsilon_i \upsilon_i'\,.$$

As was explained in the proof of Theorem 4.1 in El Karoui (2010), the eigenvalues and eigenvectors of $\mathcal{S}$ are independent, and the matrix of eigenvectors is uniformly distributed on the orthogonal group (after taking proper care of sign indeterminacy—see Chikuse (2003), p. 40). Therefore, the random variable we care about can be written as

$$\alpha' \mathcal{S}^{-2}\alpha = \sum_{i=1}^{p} \frac{1}{\gamma_i^2} \left(\alpha' v_i\right)^2 .$$

Going through the proof of Theorem 4.1 in El Karoui (2010), we see that if we consider, for a given function h, the random variable

$$Z_h = \sum_{i=1}^{p} h(\gamma_i) \left(\alpha' v_i\right)^2 ,$$

it is shown there that

$$\operatorname{var}\left(Z_n | \gamma_i, \Lambda\right) \leq C \frac{1}{p^2} \sum_{i=1}^{p} \left(h(\gamma_i)\right)^2 .$$

In our case here, $h(x) = x^{-2}$. Under our assumptions, according to Lemma B-1 in El Karoui (2010), we have $\gamma_i^2 \geq \mathfrak{C}_n (1 - \sqrt{p/(N-1)})^2/2$, where $\mathfrak{C}_n = C_0 (N-1)/(n-1)$, with high $(\{Y_i\}_{i=1}^n)$-probability, so we conclude that

$$\operatorname{var}\left(\alpha' \mathcal{S}^{-2}\alpha \,|\{\gamma_i\}, \Lambda\right) \to 0 .$$

Since (see, e.g., El Karoui (2010))

$$\mathbf{E}\left(\alpha' \mathcal{S}^{-2}\alpha | \{\gamma_i\}, \Lambda\right) = \frac{1}{p} \sum_{i=1}^{p} \frac{1}{\gamma_i^2} ,$$

we conclude that

$$\left. \alpha' \mathcal{S}^{-2}\alpha - \frac{1}{p} \sum_{i=1}^{p} \frac{1}{\gamma_i^2} \right| \{\gamma_i\}, \Lambda \to 0 \text{ in probability} .$$

The same arguments as those used in the proof in Theorem 4.1 of El Karoui (2010) then show that the same result can be obtained conditionally on Λ only.

Identifying the limit. It is now clear that at least conditionally on Λ, our problem reduces to understanding the limit of

$$\frac{1}{p} \sum_{i=1}^{p} \frac{1}{\gamma_i^2} .$$

The Stieltjes transform (see, e.g., Bai (1999)) of the spectral distribution of $\mathcal{S}$ is

$$s_p(z) = \frac{1}{p} \sum_{i=1}^{p} \frac{1}{\gamma_i - z} .$$

As was stated in El Karoui (2010), Theorem 4.1, according to results of Marčenko and Pastur (1967), Wachter (1978), and Silverstein (1995), for any fixed $z \in \mathbb{C}^+$, $s_p(z) \to s(z)$ in probability, where $s(z)$ satisfies if G is the limiting spectral distribution of $L = \Lambda' H \Lambda$,

$$-\frac{1}{s(z)} = z - \int \frac{\tau dG(\tau)}{1 + \tau \rho s(z)} . \tag{A.1}$$

Note that under our assumptions, all the L's have the same limiting spectral distribution, G. So the result does not depend asymptotically on the sequence of Λ's we are conditioning on. Because we know that, given $\Lambda \in \mathcal{L}_{\epsilon,\delta}$, the smallest eigenvalue of $\mathcal{S}$ is asymptotically bounded away from 0, and hence the limiting spectral distribution of $\mathcal{S}$, $\mathcal{K}$, has support bounded away from 0, we know that s is analytic in a neighborhood of zero. Also, because the pointwise convergence of Stieltjes transforms implies weak convergence of spectral distributions, we see that if $\mathcal{K}$ is the limiting spectral distribution of $\mathcal{S}$, by taking $m_\eta(x) = \inf(1/\eta^2, 1/x^2)$ as a test function, for any given η smaller than the left endpoint of the support of $\mathcal{K}$,

$$\frac{1}{p}\sum_{i=1}^p m_\eta(\gamma_i) \to \int m_\eta(x) d\mathcal{K}(x) = \int \frac{1}{x^2} d\mathcal{K}(x) \text{ in probability} .$$

Because the smallest γ_i is bounded away from 0 with high probability, we also see that if η is small enough, and p and n are large enough,

$$\frac{1}{p}\sum_{i=1}^p m_\eta(\gamma_i) = \frac{1}{p}\sum_{i=1}^p \frac{1}{\gamma_i^2} \text{ with high probability.}$$

Now, because s is analytic in a neighborhood of 0, we have

$$s'(0) = \int \frac{1}{x^2} d\mathcal{K}(x) ,$$

and we have finally established that

$$\frac{1}{p}\sum_{i=1}^p \frac{1}{\gamma_i^2} \to s'(0) \text{ in probability} ,$$

conditionally on any Λ belonging to $\mathcal{L}_{\epsilon,\delta}$.

Now, according to (A.1) and using the fact that $s(z)$ is analytic in a neighborhood of 0, we have, for z in a neighborhood of zero,

$$\frac{s'(z)}{s^2(z)} = 1 + \rho s'(z) \int \frac{\tau^2 dG(\tau)}{(1+\rho\tau s(z))^2} .$$

From this equation and the fact that we know that $s(0) \neq 0$ and is finite, we conclude that $s'(0) \neq 0$, for otherwise we would have $0 = 1$. We finally obtain

$$\frac{1}{s'(0)} = \frac{1}{s^2(0)} - \rho \int \frac{\tau^2 dG(\tau)}{(1+\rho\tau s(0))^2} .$$

Note that as seen in Theorem 4.1 of El Karoui (2010), $s(0) = \mathfrak{s}$. We see that $s'(0)$ is the value of ξ announced above. Also, since $s'(0) = \lim \sum \gamma_i^{-2}/p$, $s'(0) \geq 0$, and therefore $\xi \geq 0$.

Getting results unconditionally on Λ. So far we have worked with matrices Λ belonging to $\mathcal{L}_{\epsilon,\delta}$. Following exactly the deconditioning arguments given at the end of the proof of Theorem 4.1 in El Karoui (2010), we see that the results hold also unconditionally on Λ under the assumptions of the theorem. The theorem is shown. ∎

Appendix B. Quadratic forms in $\widehat{\mu}$ and $\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}$. In this section, we will use the definition $\widehat{\Sigma} = X'HX/n$ if X is the data matrix. This mild change of scaling has no asymptotic consequences but makes the notation less cumbersome in our proofs and theorems.

B.1. Understanding $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$, when $\mu = 0$. The aim of this and the following subsections is to show the following theorem.

Theorem B.1. *Suppose Y is an $n \times p$ matrix whose rows are the vectors Y_i, which are i.i.d. $\mathcal{N}(0, \mathrm{Id}_p)$.*

Suppose Λ is a diagonal matrix whose ith entry is λ_i, which is possibly random and is independent of Y. We use the notation $\tau_i = \lambda_i^2$ and assume that the empirical distribution, G_n, of τ_i converges weakly in probability to a deterministic limit G. We also assume that $\tau_i \neq 0$ for all i and

$$\text{(Assumption-BLa)} \qquad \frac{1}{n^2}\sum_{i=1}^n \lambda_i^4 = \frac{1}{n^2}\sum_{i=1}^n \tau_i^2 \to 0 \textit{ in probability}\,.$$

If $\tau_{(i)}$ is the ith largest τ_k, we assume that we can find a random variable $N \in \mathbb{N}$ and positive real numbers ϵ_0 and C_0 such that

$$\text{(Assumption-BB)} \qquad \begin{cases} P(p/N < 1-\epsilon_0) \to 1 \textit{ as } n \to \infty\,, \\ P(\tau_{(N)} > C_0) \to 1\,, \\ \exists \eta_0 > 0 \textit{ such that } P(N/n > \eta_0) \to 1 \textit{ as } n \to \infty\,. \end{cases}$$

Let us denote $\rho_n = p/n$ and $\rho = \lim_{n\to\infty} \rho_n$. We assume that $\rho \in (0,1)$. We denote

$$\zeta_{n,p} = \frac{1}{n^2}\mathbf{e}'\Lambda Y(Y'\Lambda^2 Y/n)^{-2}Y'\Lambda\mathbf{e}\,.$$

Then we have

$$\zeta_{n,p} \to \rho\mathfrak{s} \textit{ in probability}\,.$$

If the $n \times p$ data matrix $\widetilde{X}$ is written as $\widetilde{X} = \Lambda Y \Sigma^{1/2}$, if $\widehat{m} = \Sigma^{1/2}Y'\Lambda\mathbf{e}/n$ is the vector of column means of $\widetilde{X}$, and if $\widehat{\Sigma}$ is the sample covariance matrix computed from $\widetilde{X}$, we have

$$\widehat{m}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{m} \to \frac{\kappa}{1-\rho}\mathfrak{s} \textit{ in probability}\,.$$

We note that under our assumptions $\zeta_{n,p}$ will exist with probability 1 since the λ_i's are all different from 0 and the Y_i's have a continuous distribution.

B.1.1. Linear algebraic preliminaries. Before we deal with the central issues of this problem, we need the following preliminary lemma.

Lemma B.2. *If $X = Y_1\Sigma^{1/2}$ and $\widehat{\mu} = X'\mathbf{e}/n$, then*

$$Z_{n,p} = \widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu} = \frac{\mathbf{e}'Y_1}{n}\left(\frac{Y_1'HY_1}{n}\right)^{-2}\frac{Y_1'\mathbf{e}}{n}$$

and is therefore independent of Σ. Further, if M is an invertible matrix and u is a vector such that $u'M^{-1}u \neq 1$, we have

$$u'(M-uu')^{-2}u = \frac{u'M^{-2}u}{(1-u'M^{-1}u)^2}\,.$$

In particular, if $\widehat{\mu}_{Y_1} = Y_1'\mathbf{e}/n$, $\mathcal{S} = Y_1'Y_1/n$, *and* $\widehat{\Sigma}_{Y_1} = \mathcal{S} - \widehat{\mu}_{Y_1}\widehat{\mu}_{Y_1}' = Y_1'HY_1/n$, *we have*

$$Z_{n,p} = \widehat{\mu}_{Y_1}'\widehat{\Sigma}_{Y_1}^{-2}\widehat{\mu}_{Y_1} = \frac{\widehat{\mu}_{Y_1}'\mathcal{S}^{-2}\widehat{\mu}_{Y_1}}{(1-\widehat{\mu}_{Y_1}'\mathcal{S}^{-1}\widehat{\mu}_{Y_1})^2} .$$

This lemma shows that the problems we are considering do not involve Σ when $\mu = 0$ and that therefore we can assume that $\Sigma = \mathrm{Id}_p$ without loss of generality in our analysis. The central object of our study will be the random variable

$$\zeta_{n,p} = \widehat{\mu}_W'\mathcal{S}^{-2}\widehat{\mu}_W = \frac{\mathbf{e}'W}{n}\left(\frac{W'W}{n}\right)^{-2}\frac{W'\mathbf{e}}{n} , \tag{B.1}$$

where $W = \Lambda Y$, and Y is a random matrix whose entries are i.i.d. $\mathcal{N}(0,1)$ and Λ is a diagonal random matrix whose entries are independent of Y.

Under the assumptions of Theorem B.1, we showed in El Karoui (2010) that

$$\widehat{\mu}_W'\mathcal{S}^{-1}\widehat{\mu}_W \to \rho$$

in probability. So all that is left to do is understand $\zeta_{n,p}$.

We now prove Lemma B.2.

Proof of Lemma B.2. The first part of the lemma is almost immediate after we realize that $\widehat{\mu}' = \mathbf{e}'X/n = \mathbf{e}'Y\Sigma^{1/2}/n$. All the matrices Σ cancel, and we are left with an expression that does not involve Σ.

For the second part, we use a differentiation trick we will use repeatedly in this paper. Let us denote $M_t = M + t\mathrm{Id}$, where $t \geq 0$ and M is assumed to be positive definite (this is all we need for this paper, but the results below hold in more generality). In this case, M_t is also invertible for any $t \geq 0$ (and actually for t in a neighborhood of 0). Simple calculus shows that, for M positive definite and any $t \geq 0$, M_t^{-1} is differentiable on $[0,\infty)$ and

$$u'\left(M_t'\right)^{-2}u = -\frac{\partial}{\partial t}u'\left(M_t'\right)^{-1}u ,$$

M being invertible guaranteeing differentiability at 0.

Now, using the classic expansion of the inverse of a rank-1 perturbation of M_t (see Horn and Johnson (1990), p. 19), we have, if $u'M_t^{-1}u \neq 1$,

$$(M_t - uu')^{-1} = M_t^{-1} + \frac{M_t^{-1}uu'M_t^{-1}}{1-u'M_t^{-1}u} ,$$
$$u'(M_t - uu')^{-1}u = \frac{1}{1-u'M_t^{-1}u} - 1 .$$

Differentiating the last equality and multiplying by (-1), we get, for any $t \geq 0$,

$$u'(M_t - uu')^{-2}u = \frac{u'M_t^{-2}u}{(1-u'M_t^{-1}u)^2} .$$

Applying the previous equality at $t = 0$ gives the result.

We note that another method of proof would be to use a rank one update and then take squares. This is a bit more tedious than the simple trick we presented here, but also shows that the result applies for any invertible M. ∎

B.1.2. Structure of the proof. The proof of convergence of $\zeta_{n,p}$ is based on regularization ideas. In particular, we will focus on

$$\zeta_{n,p}(t) = \widehat{\mu}'_W(\mathcal{S} + t\mathrm{Id}_p)^{-2}\widehat{\mu}_W \ , \tag{B.2}$$

where t is a positive number. The overall strategy is the following:

1. Show that, for any $\epsilon > 0$, we can find t_ϵ such that $|\zeta_{n,p} - \zeta_{n,p}(t_\epsilon)| < \epsilon$ with high probability.
2. Show that, for any given t_ϵ, $\zeta_{n,p}(t_\epsilon)$ converges in probability to $\rho\mathfrak{s}(t_\epsilon)$, where this quantity is deterministic.
3. Show that $\mathfrak{s}(t_\epsilon)$ can be made arbitrarily close to $\mathfrak{s}$ (by picking ϵ small enough), so that in the end one concludes that $\zeta_{n,p} - \rho\mathfrak{s}$ tends to 0 in probability.

Technically, $\zeta_{n,p}(t)$ is a much nicer object to work with than $\zeta_{n,p}$ is since it is bounded and amenable to variance computations (at least conditionally on Λ). This is especially useful because to show convergence in probability of $\zeta_{n,p}(t)$ we rely, among other things, on conditional variance computations, based on the Efron–Stein inequality (see Lugosi (2006), Theorem 9).

B.1.3. Approximating $\zeta_{n,p}$ by $\zeta_{n,p}(t_n)$. To show that we can approximate $\zeta_{n,p}$ by $\zeta_{n,p}(t)$, we rely on the following lemma.

Lemma B.3. *Suppose* (Assumption-BB) *is satisfied. Then we have, as $n \to \infty$,*

$$\forall \epsilon > 0,\ \exists t_\epsilon : P(|\zeta_{n,p} - \zeta_{n,p}(t_\epsilon)| > \epsilon) \to 0 \ .$$

In plain English, the lemma means that for any $\epsilon > 0$ we can find $\zeta_{n,p}(t_\epsilon)$, which approximates $\zeta_{n,p}$ to within ϵ with high probability.

In this and other proofs, it will be convenient to work with the matrix $P_2(t)$ defined as

$$P_2(t) = \frac{W}{\sqrt{n}}\left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2}\frac{W'}{\sqrt{n}} \ . \tag{B.3}$$

Proof. Recall that $\zeta_{n,p} = \zeta_{n,p}(0)$ and that, if $W = \Lambda Y$ and $\nu' = \mathbf{e}'W/n$, then

$$\begin{aligned}
\zeta_{n,p}(t) &= \nu'\left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2}\nu \\
&= \frac{\mathbf{e}'W}{n}\left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2}\frac{W'\mathbf{e}}{n} \\
&= \frac{\mathbf{e}'}{\sqrt{n}}\left(\frac{W}{\sqrt{n}}\left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2}\frac{W'}{\sqrt{n}}\right)\frac{\mathbf{e}}{\sqrt{n}} \\
&\triangleq \frac{\mathbf{e}'}{\sqrt{n}}P_2(t)\frac{\mathbf{e}}{\sqrt{n}} \ .
\end{aligned}$$

Let us write the singular value decomposition of $W/\sqrt{n}$ as

$$\frac{W}{\sqrt{n}} = UDV' \ .$$

Then, using the fact that V is orthogonal, we have

$$\left(\frac{W'W}{n} + t\mathrm{Id}_p\right) = V(D^2 + t\mathrm{Id}_p)V' \ .$$

Therefore,

$$\left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2} = V(D^2 + t\mathrm{Id}_p)^{-2}V' \ ,$$

$$P_2(t) = UD(D^2 + t\mathrm{Id}_p)^{-2}DU' = \sum_{i=1}^{p} \frac{d_i^2}{(d_i^2+t)^2} u_i u_i' \ .$$

Recall that our aim is to compare $P_2(0) - P_2(t)$, or, more precisely, quadratic forms involving this difference of matrices. Towards this end, we make the following simple remark: suppose that the d_i's are decreasingly ordered and $t > 0$:

$$0 \leq \frac{1}{d_i^2} - \frac{d_i^2}{(d_i^2+t)^2} = \frac{2td_i^2 + t^2}{d_i^2(d_i^2+t)^2} = \frac{2t}{(d_i^2+t)^2} + \frac{t^2}{d_i^2(d_i^2+t)^2} \leq \frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2} \ .$$

We can hence conclude that

$$0 \leq \zeta_{n,p} - \zeta_{n,p}(t) \leq \frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2} \tag{B.4}$$

since

$$\begin{aligned} 0 \leq \zeta_{n,p} - \zeta_{n,p}(t) &= \frac{\mathbf{e}'}{\sqrt{n}}(P_2(0) - P_2(t))\frac{\mathbf{e}}{\sqrt{n}} \\ &\leq \left(\frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2}\right)\sum_{i=1}^{p}\left(u_i'\frac{\mathbf{e}}{\sqrt{n}}\right)^2 \leq \frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2} \end{aligned}$$

because $\|\mathbf{e}/\sqrt{n}\|_2 = 1$ and U is an orthogonal matrix.

Let us now explain why d_p is bounded away from 0 under our assumptions. Let us call $\mathcal{L}_{\epsilon_0,\delta}$ the set of matrices Λ such that $p/N < 1 - \epsilon_0$ and $C_0(N-1)/(n-1) > \delta$. Under our assumptions, for a δ_0 bounded away from 0 (e.g., $\delta_0 = 1/2 \liminf C_0(N-1)/(n-1)$), $P(\Lambda \in \mathcal{L}_{\epsilon_0,\delta_0}) \to 1$. Let us pick such a δ_0. If $\Lambda \in \mathcal{L}_{\epsilon_0,\delta_0}$, according to Lemma B-1 and the proof of Theorem 4.1 in El Karoui (2010), if P_Λ denotes probability conditional on Λ,

$$P_\Lambda\left(d_p \leq \sqrt{\delta_0}\left[(1-\sqrt{1-\epsilon_0}) - r\right]\right) \leq \exp\left(-(n-1)\delta_0 r^2/C_0\right) \ .$$

Hence, when $\Lambda \in \mathcal{L}_{\epsilon_0,\delta_0}$, d_p, the smallest singular value of $W/\sqrt{n}$, is bounded away from 0 with high probability.

We conclude that

$$\forall \epsilon > 0, \ \exists t_\epsilon \text{ such that } P(|\zeta_{n,p} - \zeta_{n,p}(t_\epsilon)| > \epsilon) \to 0 \ . \qquad \blacksquare$$

B.1.4. About $\mathbf{E}\left(\zeta_{n,p}(t)\right)$ and $\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right)$. Recall that

$$\zeta_{n,p}(t) = \frac{1}{n}\mathbf{e}'P_2(t)\mathbf{e}$$

and is therefore the scaled sum of the entries of $P_2(t)$. Let us call $p_{2,t}(i,j)$ the (i,j)th entry of $P_2(t)$. We have the following lemma.

Lemma B.4. *When Y_i are i.i.d. and have a symmetric distribution (i.e., $Y_i \stackrel{\mathcal{L}}{=} -Y_i$), we have, if $i \neq j$,*

$$\mathbf{E}\left(p_{2,t}(i,j)|\Lambda\right) = \mathbf{E}\left(p_{2,t}(i,j)\right) = 0\ .$$

Further, for any given $t > 0$,

$$\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) - \frac{1}{n}\mathrm{trace}(P_2(t)) \to 0 \textit{ in probability}\ ,$$

and $\frac{1}{n}\mathrm{trace}(P_2(t))$ has a deterministic limit, which depends only on G, the limiting spectral distribution of Λ.

Proof. In all of the proof, we assume that $t > 0$ is given. We use an invariance idea similar to that used in a corresponding situation in El Karoui (2010). Let us first note that the expectations we are referring to are well defined. As a matter of fact, because $W'W/n+t\mathrm{Id}_p \succeq t\mathrm{Id}_p$, we have

$$|||P_2(t)|||_2 \leq \frac{1}{t}|||W(W'W)^{-1}W'|||_2 = \frac{1}{t}$$

since the matrix appearing in the right-hand side is an orthogonal projection matrix. Therefore, all the entries of $P_2(t)$ are bounded and less than $1/t$.

We now work conditionally on Λ and focus on the case $i = 1$, $j \neq 1$. Let us denote

$$p_{2,t}(1,j) = \Theta_\Lambda(Y_1, \ldots, Y_n) = \lambda_1\lambda_j Y_1'\left(\frac{1}{n}\sum_{i=1}^n \lambda_i^2 Y_i Y_i' + t\mathrm{Id}_p\right)^{-2} Y_j\ .$$

We clearly have $\Theta_\Lambda(Y_1, Y_2, \ldots, Y_n) = -\Theta_\Lambda(-Y_1, Y_2, \ldots, Y_n)$. In other respects, because $Y_i \stackrel{\mathcal{L}}{=} -Y_i$, and the Y_i's are independent, we have $\Theta_\Lambda(Y_1, Y_2, \ldots, Y_n) \stackrel{\mathcal{L}}{=} \Theta_\Lambda(-Y_1, Y_2, \ldots, Y_n)$. Because $\Theta_\Lambda(Y_1, Y_2, \ldots, Y_n)$ is bounded, we can take expectations in the previous equality and we have, if $i \neq j$,

$$\mathbf{E}\left(p_{2,t}(1,j)|\Lambda\right) = 0\ . \tag{B.5}$$

We conclude that the same result holds unconditionally and $\mathbf{E}\left(p_{2,t}(i,j)\right) = 0$ since under our assumptions P_2 is defined with probability 1.

Hence, for any $t > 0$,

$$\mathbf{E}\left(\zeta_{n,p}(t)\right) = \frac{1}{n}\sum_{i=1}^n \mathbf{E}\left(p_{2,t}(i,i)\right) = \frac{1}{n}\mathbf{E}\left(\mathrm{trace}\left(P_2(t)\right)\right)\ .$$

Let us now argue that, for any fixed $t > 0$,

$$\zeta_{n,p}(t) - \frac{1}{n}\mathrm{trace}\left(P_2(t)\right) \to 0 \text{ in probability}\ .$$

If $\mathcal{S} = W'W/n$, and if l_i are the eigenvalues of $\mathcal{S}$,

$$\frac{1}{n}\mathrm{trace}\,(P_2(t)) = \frac{p}{n}\frac{1}{p}\sum_{i=1}^{p}\frac{l_i}{(l_i+t)^2} = \rho_n\left(\frac{1}{p}\sum_{i=1}^{p}\frac{1}{l_i+t} - t\frac{1}{(l_i+t)^2}\right) .$$

Under our assumptions on the convergence of the spectral distribution of Λ, we have the convergence of the spectral distribution of $\mathcal{S}$ in probability (see Marčenko and Pastur (1967), Wachter (1978), Silverstein (1995), El Karoui (2009), and El Karoui (2010)) to a deterministic probability measure $\mathcal{K}$, and therefore

$$\frac{1}{n}\mathrm{trace}\,(P_2(t)) \to \rho\mathfrak{s}(t) \text{ in probability .}$$

Though we will not need it later for any explicit computations, let us mention that if $\mathcal{K}$ is the limiting spectral distribution of $\mathcal{S}$, we have, since $a(x) = 1/(x+t)$ is continuous and bounded on $[0,\infty)$,

$$\mathfrak{s}(t) = \int \frac{d\mathcal{K}(l)}{l+t} . \tag{B.6}$$

Because $\frac{1}{n}\mathrm{trace}\,(P_2(t)) \leq 1/t$, we also have $\mathrm{var}\left(\frac{1}{n}\mathrm{trace}\,(P_2(t))\right) \to 0$ and $\mathbf{E}\left(\frac{1}{n}\mathrm{trace}\,(P_2(t))\right) \to \rho\mathfrak{s}(t)$. Note also that since

$$\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) = \frac{1}{n}\mathbf{E}\left(\mathrm{trace}\,(P_2(t))\,|\Lambda\right) ,$$

we also have, under our assumptions, that

$$\mathrm{var}\left(\frac{1}{n}\mathbf{E}\left(\mathrm{trace}\,(P_2(t))\,|\Lambda\right)\right) \to 0$$

since $\frac{1}{n}\mathbf{E}\left(\mathrm{trace}\,(P_2(t))\,|\Lambda\right)$ is a bounded random variable, depending on Λ, which converges in probability to a limit that is independent of Λ.

So for any given t,

$$\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) - \rho\mathfrak{s}(t) \to 0 \;\text{ in probability,}$$
$$\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) - \frac{1}{n}\mathrm{trace}\,(P_2(t)) \to 0 \;\text{ in probability .}$$

The first statement is to be understood in Λ-probability. Note that $\mathfrak{s}(t)$ depends only on the limiting spectral distribution of Λ, which is the same for all the Λ's we consider. In other words, $\mathfrak{s}(t)$ is deterministic, and the random variable $\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right)$ is asymptotically nonrandom. The second statement is to be understood with respect to the probability induced by the joint distribution of the λ_i's and the Y_i's (thus the joint $(\Lambda \times Y)$-probability). ■

B.1.5. Conditional variance and convergence of $\zeta_{n,p}(t)$. We now place ourselves in the setting where $\mu = 0$. To understand $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$, we simply need to understand, if $W = \Lambda Y$,

$$\zeta_{n,p} = \frac{\mathbf{e}'W}{n}\left(\frac{W'W}{n}\right)^{-2}\frac{W'\mathbf{e}}{n} .$$

Lemma B.5. *Suppose that $Y_i i.i.d. \mathcal{N}(0, \mathrm{Id}_p)$ and $\zeta_{n,p}(t)$ is defined as above in (B.2). Then we can find a constant C and two (explicit) functions g_1 and g_2 such that, for any $t > 0$,*

$$\mathrm{var}\left(\zeta_{n,p}(t)|\Lambda\right) \leq C\frac{1}{n^2}\sum_{i=1}^{n}\left(\lambda_i^4 g_1(t) + g_2(t)\right) .$$

Furthermore, for any $t > 0$, when (Assumption-BLa) and (Assumption-BB) are satisfied,

$$\zeta_{n,p}(t) - \frac{1}{n}\mathrm{trace}(P_2(t)) \to 0 \textit{ in probability} .$$

The key to the proof of this lemma is the Efron–Stein inequality (Efron and Stein, 1981), as stated, for instance, in Theorem 9 of Lugosi (2006): it says that if X is a random variable such that $X = f(\xi_1, \dots, \xi_n)$, where the ξ_i's are independent random variables, then, if $X_i = f_i(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n)$,

$$\mathrm{var}\left(X\right) \leq \sum_{i=1}^{n}\mathrm{var}\left(X - X_i\right) .$$

We will therefore try to approximate $\zeta_{n,p}(t)$ by random variables $\zeta_i(t)$ involving all but one of the Y_i's to get our conditional variance bound—the key to the lemma.

Proof. We denote

$$\mathcal{S} = W'W/n = \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 Y_i Y_i' .$$

We denote $\mathcal{S}_i = \mathcal{S} - \lambda_i^2 Y_i Y_i'$. We also denote $\mathcal{S}(t) = \mathcal{S} + t\mathrm{Id}_p$ and $\mathcal{S}_i(t) = \mathcal{S}(t) - \lambda_i^2 Y_i Y_i'/n$.

Let us denote $\widehat{m} = W'\mathbf{e}/n$ and $\widehat{m}_i = W_i'\mathbf{e}/n$, where $W_i = W - \lambda_i e_i Y_i'$ (e_i is the ith canonical basis vector in $\mathbb{R}^n$). Note that W_i is simply W with the ith row replaced by 0.

We denote

$$\begin{aligned} Z_{n,p}(t) &\triangleq \widehat{m}'\left(\mathcal{S}(t)\right)^{-1}\widehat{m} , \\ Z_i(t) &\triangleq \widehat{m}_i'\left(\mathcal{S}_i(t)\right)^{-1}\widehat{m}_i . \end{aligned}$$

Note that

$$\begin{aligned} \frac{\partial Z_{n,p}(t)}{\partial t} &= -\widehat{m}'\left(\mathcal{S}(t)\right)^{-2}\widehat{m} = -\zeta_{n,p}(t) , \\ \frac{\partial Z_i(t)}{\partial t} &= -\widehat{m}_i'\left(\mathcal{S}_i(t)\right)^{-2}\widehat{m}_i = -\zeta_i(t) . \end{aligned}$$

Now recall the results of El Karoui (2010), Theorem 4.2 and its proof:

$$
\begin{aligned}
Z_{n,p}(t) &= Z_i(t) + \frac{1}{n}\left(1 - \frac{(1-\lambda_i w_i(t))^2}{1+\lambda_i^2 q_i(t)}\right) \;, \text{ where} \\
w_i(t) &= \widehat{m}_i' \left(\mathcal{S}_i(t)\right)^{-1} Y_i \;, \\
q_i(t) &= \frac{Y_i' \left(\mathcal{S}_i(t)\right)^{-1} Y_i}{n} \;.
\end{aligned}
$$

Note that $q_i(t) \geq 0$. Taking derivatives with respect to t in the previous expression, we get

$$
\frac{\partial Z_{n,p}(t)}{\partial t} = \frac{\partial Z_i(t)}{\partial t} - \frac{1}{n}\left[\frac{2(\lambda_i w_i(t) - 1)\lambda_i w_i'(t)}{1+\lambda_i^2 q_i(t)} - \frac{(1-\lambda_i w_i(t))^2}{(1+\lambda_i^2 q_i(t))^2}\lambda_i^2 q_i'(t)\right] \;.
$$

Hence,

$$
|\zeta_{n,p}(t) - \zeta_i(t)| = \frac{1}{n}\left|\frac{2(\lambda_i w_i(t) - 1)\lambda_i w_i'(t)}{1+\lambda_i^2 q_i(t)} - \frac{(1-\lambda_i w_i(t))^2}{(1+\lambda_i^2 q_i(t))^2}\lambda_i^2 q_i'(t)\right| \;.
$$

We notice that, trivially, $\mathcal{S}(t) \succeq t\mathrm{Id}_p$ and $\mathcal{S}_i(t) \succeq t\mathrm{Id}_p$. Also,

$$
\begin{aligned}
w_i'(t) &= -\widehat{m}_i'(\mathcal{S}_i(t))^{-2} Y_i \;, \\
q_i'(t) &= -\frac{Y_i'(\mathcal{S}_i(t))^{-2} Y_i}{n} \;.
\end{aligned}
$$

Therefore,

$$
|q_i'(t)| \leq \frac{|||(\mathcal{S}_i(t))^{-1}|||_2 \|(\mathcal{S}_i(t))^{-1/2} Y_i\|_2^2}{n} = q_i(t)\frac{1}{t} \;.
$$

Consequently,

$$
\left|\frac{|1-\lambda_i w_i(t)|}{(1+\lambda_i^2 q_i(t))^2}\lambda_i^2 q_i'(t)\right| \leq \frac{1}{t}\frac{|1-\lambda_i w_i(t)|}{1+\lambda_i^2 q_i(t)} \;.
$$

Let us write

$$
\Delta_i(t) = n(\zeta_{n,p}(t) - \zeta_i(t)) = \frac{\lambda_i w_i(t) - 1}{1+\lambda_i^2 q_i(t)}\left(2\lambda_i w_i'(t) - \frac{\lambda_i w_i(t) - 1}{1+\lambda_i^2 q_i(t)}\lambda_i^2 q_i'(t)\right) \;. \tag{B.7}
$$

The remarks we made above show that

$$
\begin{aligned}
|\Delta_i(t)| &\leq \left[\frac{(1-\lambda_i w_i(t))^2}{t} + 2\lambda_i^2 |w_i(t) w_i'(t)| + 2|\lambda_i||w_i'(t)|\right]\frac{1}{1+\lambda_i^2 q_i(t)} \\
&\leq \frac{2}{t} + \lambda_i^2\left(2\frac{w_i^2(t)}{t} + |w_i(t) w_i'(t)|\right) + 2|\lambda_i||w_i'(t)| \;.
\end{aligned}
$$

Using the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, as well as the equally well known $(a+b)^2 \leq 2(a^2+b^2)$ and $|ab| \leq (a^2+b^2)/2$, we have

$$
\Delta_i(t)^2 \leq 3\left[\frac{4}{t^2} + \lambda_i^4\left(\frac{8w_i^4(t)}{t^2} + w_i^4(t) + (w_i'(t))^4\right) + 4\lambda_i^2 (w_i'(t))^2\right] \;.
$$

Note from El Karoui (2010) that, conditional on $Y_{(i)} = (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ and Λ, $w_i'(t) \sim \mathcal{N}(0, \widehat{m}_i'(\mathcal{S}_i(t))^{-4}\widehat{m}_i)$. Recall also from that under the same conditioning, $w_i(t) \sim \mathcal{N}(0, \widehat{m}_i'(\mathcal{S}_i(t))^{-2}\widehat{m}_i)$. So in particular, since

$$\widehat{m}_i'(\mathcal{S}_i(t))^{-4}\widehat{m}_i \leq \frac{1}{t^3} ,$$

because, as explained in El Karoui (2010), $\widehat{m}_i'(\mathcal{S}_i(t))^{-1}\widehat{m}_i \leq 1$, we have, using the same arguments as in El Karoui (2010),

$$\mathbf{E}\left((w_i'(t))^2|\Lambda\right) \leq \frac{1}{t^3} \quad \text{and} \quad \mathbf{E}\left((w_i'(t))^4|\Lambda\right) \leq \frac{3}{t^6} .$$

Similarly,

$$\mathbf{E}\left(w_i^4(t)|\Lambda\right) \leq \frac{3}{t^2} .$$

So we conclude that, for C a constant (independent of p and n),

$$\mathbf{E}\left((\Delta_i(t))^2|\Lambda\right) \leq C(\lambda_i^4 f_1(t) + \lambda_i^2 f_2(t) + f_3(t)) .$$

For completeness, let us say that a possible choice of such functions is $f_1(t) = 8t^{-4} + t^{-2} + t^{-6}$, $f_2(t) = t^{-3}$, and $f_3(t) = t^{-2}$.

After using the very coarse bound $\lambda_i^2 \leq 1 + \lambda_i^4$, this inequality can be rewritten as (for appropriate functions g_1 and g_2)

$$\mathbf{E}\left((\Delta_i(t))^2|\Lambda\right) \leq C(\lambda_i^4 g_1(t) + g_2(t)) .$$

Applying the Efron–Stein inequality (see Lugosi (2006), Theorem 9), we see that

$$\operatorname{var}\left(\zeta_{n,p}(t)|\Lambda\right) \leq \sum_{i=1}^n \mathbf{E}\left((\zeta_{n,p}(t) - \zeta_i(t))^2|\Lambda\right) \leq C\frac{1}{n^2}\sum_{i=1}^n \left(\lambda_i^4 g_1(t) + g_2(t)\right) .$$

Let us now consider $\mathcal{L}_\eta = \left\{\Lambda : \frac{1}{n^2}\sum_{i=1}^n \lambda_i^4 \leq \eta\right\}$. If $\Lambda \in \mathcal{L}_\eta$ and P_Λ denotes probability conditional on Λ, we have, for all $x > 0$,

$$P_\Lambda\left(|\zeta_{n,p}(t) - \mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right)| > x\right) \leq \frac{1}{x^2}C\left(\eta g_1(t) + \frac{g_2(t)}{n}\right) .$$

In particular, when n is large enough and $\epsilon > 0$ is given, we see that we can pick an $\eta(t, x, \epsilon)$ such that for any given x and t,

$$P_\Lambda\left(|\zeta_{n,p}(t) - \mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right)| > x\right) \leq 4C\epsilon .$$

Now, given x, t, and ϵ, $P(\Lambda \in \mathcal{L}_{\eta(t,x,\epsilon)}) \to 1$ under our assumptions. Also, we have shown above that $\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) \to \rho\mathfrak{s}(t)$ in Λ-probability, where $\mathfrak{s}(t)$ is deterministic, and the same for a set of Λ's of probability 1. In other words, if we denote $\mathcal{L}_{\delta,t} = \{\Lambda : |\mathbf{E}\left(\zeta_{n,p}(t)|\Lambda\right) - \rho\mathfrak{s}(t)| \leq \delta\}$, for any $\delta > 0$ and $t > 0$, we have $P(\mathcal{L}_{\delta,t}) \to 1$. Of course, if $\mathcal{L}_{\eta(t,x,\epsilon),\delta} = \mathcal{L}_{\eta(t,x,\epsilon)} \bigcap \mathcal{L}_{\delta,t}$, we also have $P(\mathcal{L}_{\eta(t,x,\epsilon),\delta}) \to 1$. So we conclude, by elementary conditioning arguments (similar to those found at the end of the proof of Theorem 4.2 in El Karoui (2010)) that under our assumptions

$$\text{for any given } t > 0, \qquad \zeta_{n,p}(t) - \rho\mathfrak{s}(t) \to 0 \text{ in probability },$$

where the last statement is of course to be understood unconditionally on Λ.

Note that because $\frac{1}{n}\text{trace}\,(P_2(t)) \to \rho\mathfrak{s}(t)$ in (unconditional) probability, this also implies that, for any given $t > 0$,

$$\zeta_{n,p}(t) - \frac{1}{n}\text{trace}\,(P_2(t)) \to 0 \text{ in probability }. \qquad \blacksquare$$

B.1.6. Convergence of $\zeta_{n,p}$. We now put together all the previous arguments to show the first result announced in Theorem B.1, namely that, under our assumptions,

$$\zeta_{n,p} \to \rho\mathfrak{s} \text{ in probability }.$$

As a matter of fact, we have, for any $t > 0$,

$$\begin{aligned}\left|\zeta_{n,p} - \frac{1}{n}\text{trace}\,(P_2(0))\right| &\leq |\zeta_{n,p} - \zeta_{n,p}(t)| + \left|\zeta_{n,p}(t) - \frac{1}{n}\text{trace}\,(P_2(t))\right| \\ &\quad + \left|\frac{1}{n}\text{trace}\,(P_2(t)) - \frac{1}{n}\text{trace}\,(P_2(0))\right| .\end{aligned}$$

We have already established that if d_p is the smallest singular value of $\mathcal{S} = W'W/n$, then

$$0 \leq \zeta_{n,p} - \zeta_{n,p}(t) \leq \frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2} ,$$

$$\zeta_{n,p}(t) - \frac{1}{n}\text{trace}\,(P_2(t)) \to 0 \text{ in probability }.$$

Now we also have by the same argument as the one we used to bound $\zeta_{n,p} - \zeta_{n,p}(t)$

$$\left|\frac{1}{n}\text{trace}\,(P_2(t)) - \frac{1}{n}\text{trace}\,(P_2(0))\right| \leq \frac{2t}{(d_p^2+t)^2} + \frac{t^2}{d_p^2(d_p^2+t)^2} .$$

Since under our assumptions d_p is bounded away from 0 with high probability, for any $\epsilon > 0$, we can find t_ϵ such that, in probability,

$$\begin{aligned}|\zeta_{n,p} - \zeta_{n,p}(t_\epsilon)| &< \epsilon , \\ \left|\zeta_{n,p}(t_\epsilon) - \frac{1}{n}\text{trace}\,(P_2(t_\epsilon))\right| &< \epsilon , \\ \left|\frac{1}{n}\text{trace}\,(P_2(t_\epsilon)) - \frac{1}{n}\text{trace}\,(P_2(0))\right| &< \epsilon .\end{aligned}$$

Hence,

$$\left|\zeta_{n,p} - \frac{1}{n}\text{trace}\,(P_2(0))\right| \to 0 \text{ in probability }.$$

Now

$$\frac{1}{n}\text{trace}\,(P_2(0)) = \rho_n \frac{1}{p}\sum_{i=1}^{p} d_i^{-1} \to \rho\mathfrak{s} \text{ in probability}$$

by the analysis done in El Karoui (2010) and the result is shown.

B.1.7. On $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$. Let us now focus on the second result announced in Theorem B.1. Recall that the statistic we were interested in was

$$Z_{n,p} = \widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu} \, .$$

Calling $T_{n,p} = \widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu}$, we have shown in Lemma B.2 that

$$Z_{n,p} = \frac{\zeta_{n,p}}{(1-T_{n,p})^2} \, .$$

In El Karoui (2010), we showed that $T_{n,p} \to \rho$ under our assumptions. And we just showed that

$$\zeta_{n,p} \to \rho\mathfrak{s} \, .$$

Recalling the notation $\kappa = \rho/(1-\rho)$, we finally have

$$Z_{n,p} \to \frac{\kappa}{1-\rho}\mathfrak{s} \, ,$$

as announced in Theorem B.1.

B.2. On $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v$. We use here the normalization $\widehat{\Sigma} = \frac{1}{n}X'HX$. Note that $\widehat{\Sigma}$ is shift-invariant (it does not depend on μ), so we denote $\widehat{m} = W'\mathbf{e}/n$, where $W = \Lambda Y$. $\widehat{m}$ is the sample mean in the case where $\mu = 0$. Let us denote $\mathcal{S} = W'W/n$. We have

$$\widehat{\Sigma} = \Sigma^{1/2}\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)\Sigma^{1/2} \, .$$

Therefore, we have

$$\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu} = \mu'\Sigma^{-1/2}\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\Sigma^{-1/2}\mu + 2\mu'\Sigma^{-1/2}\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\widehat{m} + \widehat{m}'\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\widehat{m} \, .$$

On the other hand, if v is a deterministic vector,

$$\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}v = \mu'\Sigma^{-1/2}\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\Sigma^{-1/2}v + \widehat{m}'\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\Sigma^{-1/2}v \, .$$

The work that was done earlier in this paper gives results concerning quadratic forms of types $v'\Sigma^{-1/2}\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\Sigma^{-1/2}v$, for a fixed sequence of vectors v, and $\widehat{m}'\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}\widehat{m}$. To complete the study, we now need to understand quantities of the type

$$\widehat{m}'\left(\mathcal{S} - \widehat{m}\widehat{m}'\right)^{-2}v \, .$$

We now turn to studying these objects and begin with the following lemma.

Lemma B.6. *Let u be a vector and M be a positive definite matrix such that $M - uu'$ is invertible. Then*

$$u'(M-uu')^{-2}v = \frac{u'M^{-2}v}{1-u'M^{-1}u} + \frac{u'M^{-2}u}{(1-u'M^{-1}u)^2}u'M^{-1}v \, . \tag{B.8}$$

In our applications of this lemma we will have that $M = \mathcal{S}$, $u = \widehat{m}$, and v is deterministic. Because we have studied in El Karoui (2010) quantities of the type $v'\mathcal{S}^{-1}\widehat{m}$ and $\widehat{m}'\mathcal{S}^{-1}\widehat{m}$, and because in light of the results above we now understand $\widehat{m}'\mathcal{S}^{-2}\widehat{m}$, we will have to focus just on quantities of the type $v'\mathcal{S}^{-2}\widehat{m}$ to get a general understanding of statistics of the form $\widehat{\mu}'\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}\widehat{\mu}$.

Proof. We use the same trick as above. Because M is invertible, $M_t = M + t\mathrm{Id}$ is such that M_t^{-1} is well defined and differentiable in a neighborhood of 0. Recall that the rank one update formula gives

$$u'\left(M_t - uu'\right)^{-1} = \frac{u'M_t^{-1}}{1 - u'M_t^{-1}u} .$$

Let us denote $g(t) = -u'\left(M_t - uu'\right)^{-1} v$. We have

$$g'(0) = u'\left(M - uu'\right)^{-2} v .$$

However, $g(t) = (u'M_t^{-1}v)/(1 - u'M_t^{-1}u)$. Therefore,

$$g'(0) = \frac{u'M^{-2}v}{1 - u'M^{-1}u} + \frac{u'M^{-1}v}{(1 - u'M^{-1}u)^2} u'M^{-2}u . \qquad \blacksquare$$

B.2.1. On $\widehat{m}'\mathcal{S}^{-2}v$. Our aim is to show the following theorem.

Theorem B.7. *Suppose that v is a deterministic vector with $\|v\|_2 = 1$. Suppose that the assumptions stated in Theorem B.1 hold and also that*

$$\text{(Assumption-BLb)} \qquad \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 \textit{ remains bounded with probability going to } 1.$$

Consider, if $W = \Lambda Y$, $\mathcal{S} = W'W/n$,

$$\psi = \frac{1}{n}\mathbf{e}'\Lambda Y \mathcal{S}^{-2} v = \widehat{m}'\mathcal{S}^{-2} v .$$

Then

$$\psi \to 0 \textit{ in probability} . \tag{B.9}$$

Furthermore, we have

$$\widehat{m}'(\mathcal{S} - \widehat{m}\widehat{m}')^{-2} v \to 0 \textit{ in probability} . \tag{B.10}$$

Proof. As before, our proof will rely on an approximation argument and the Efron–Stein inequality. We denote, for $t > 0$,

$$\psi(t) = \frac{1}{n}\mathbf{e}'\Lambda Y (\mathcal{S} + t\mathrm{Id}_p)^{-2} v .$$

As before, we denote $S(t) = \mathcal{S} + t\mathrm{Id}_p$ and we will be using the same notation as in the proof of Theorem B.1. Note that

$$\psi(t) = \frac{\mathbf{e}}{\sqrt{n}} \frac{W}{\sqrt{n}} \left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-2} v .$$

We first remark that if W is changed to $-W$, then $\psi(t)$ is changed to $-\psi(t)$. We also note that since $|||S(t)|||_2 \leq 1/t$, $\psi(t)$ has an expectation, conditional on Λ. Since $Y \stackrel{\mathcal{L}}{=} -Y$, we immediately have

$$\mathbf{E}\left(\psi(t)|\Lambda\right) = 0$$

since, conditional on Λ, $\psi(t) \stackrel{\mathcal{L}}{=} -\psi(t)$. Let us call $\psi_i(t)$ the random variable obtained by replacing λ_i by zero in the definition of $\psi(t)$. Note that $\psi_i(t)$ does not involve Y_i.

Now the proof of Theorem 4.3 in El Karoui (2010) shows that if

$$\Psi(t) = \frac{\mathbf{e}}{\sqrt{n}} \frac{W}{\sqrt{n}} \left(\frac{W'W}{n} + t\mathrm{Id}_p\right)^{-1} v \ ,$$

and if $\Psi_i(t)$ is the random variable obtained from Ψ by replacing λ_i by 0, then

$$\Psi(t) - \Psi_i(t) = \frac{1}{n}\left(\frac{\lambda_i \theta_i(t)(1-\lambda_i w_i(t))}{1+\lambda_i^2 q_i(t)}\right) \ , \text{ where}$$
$$\theta_i(t) = Y_i'(\mathcal{S}_i(t))^{-1} v \ .$$

Naturally, we have $\psi(t) = -\frac{\partial \Psi(t)}{\partial t}$ and similarly for $\Psi_i(t)$ and $\psi_i(t)$. Therefore, we have

$$n(\psi_i(t) - \psi(t)) = \lambda_i \left[\frac{\theta_i(t)'(1-\lambda_i w_i(t)) - \lambda_i w_i'(t)\theta_i(t)}{1+\lambda_i^2 q_i(t)} - \frac{\theta_i(t)(1-\lambda_i w_i(t))}{(1+\lambda_i^2 q_i(t))^2}\lambda_i^2 q_i'(t)\right] \ .$$

So we conclude that

$$n|\psi_i(t) - \psi(t)| \leq |\lambda_i| \left(|\theta_i'(t)||1-\lambda_i w_i(t)| + |\lambda_i w_i'(t)\theta_i(t)| + \frac{|\theta_i(t)||1-\lambda_i w_i(t)|}{t}\right) \ .$$

A key aspect of this equation for our purposes is that λ_i appears in it with powers at most 2. We also note that $\theta_i'(t)|Y_{(i)}, \Lambda \sim \mathcal{N}(0, v'(\mathcal{S}_i(t))^{-4} v)$ and recall that $\theta_i(t)|Y_{(i)}, \Lambda \sim \mathcal{N}(0, v'(\mathcal{S}_i(t))^{-2} v)$. Finally, as seen many times before, $\mathcal{S}_i(t)^{-1} \leq t\mathrm{Id}_p$.

Therefore, $\mathbf{E}\left((\theta_i'(t))^{-2k}\right) \leq C_k t^{-4k}$ and $\mathbf{E}\left((\theta_i(t))^{-2k}\right) \leq C_k t^{-2k}$. We conclude that

$$\mathbf{E}\left(n^2(\psi_i(t) - \psi(t))^2 | \Lambda\right) \leq C(\lambda_i^4 h_1(t) + h_2(t)) \ .$$

Hence, the Efron–Stein inequality guarantees that

$$\mathrm{var}\left(\psi(t)|\Lambda\right) \leq C\frac{1}{n^2}\sum_{i=1}^n \left(\lambda_i^4 h_1(t) + h_2(t)\right) \ .$$

So if Λ is such that $\sum_{i=1}^n \lambda_i^4/n^2 \to 0$, we have

$$\psi(t) \to 0 \text{ conditionally on } \Lambda \ .$$

Note that under (Assumption-BLa), the set of matrices Λ such that $\sum_{i=1}^n \lambda_i^4/n^2 \to 0$ has (asymptotically) measure 1.

We now recall that under (Assumption-BB), we can find sets of matrices $\mathcal{L}_{\epsilon_0,\delta_0}$ whose measures go to 1 asymptotically, for which the smallest singular value of $\mathcal{S}$ is bounded away from 0 with high probability. Recall also from El Karoui (2010) that conditionally on Λ, $\widehat{m} = W'\mathbf{e}/n$ is $\mathcal{N}(0, \frac{\sum_{i=1}^n \lambda_i^2}{n^2}\mathrm{Id}_p)$, and therefore $\|\widehat{m}\|_2^2 \sim \chi_p^2/n(\sum_{i=1}^n \lambda_i^2/n)$. Now

$$|\psi - \psi(t)| \leq \|\widehat{m}\| \, |||\mathcal{S}^{-2} - (\mathcal{S}(t))^{-2}|||_2 \, \|v\| \ .$$

So if $\Lambda \in \mathcal{L}_{\epsilon_0,\delta_0}$ and is such that $\sum_{i=1}^n \lambda_i^2/n$ stays bounded, we see that for any $\epsilon > 0$, we can find t_ϵ such that $|\psi - \psi(t_\epsilon| < \epsilon$ with high probability. For such a Λ, satisfying also the conditions of (Assumption-BLa), we have

$$\psi \to 0 \text{ in probability, conditionally on } \Lambda\ .$$

By using deconditioning arguments similar to those we presented above, we can conclude that under (Assumption-BB), (Assumption-BLa), and (Assumption-BLb), we have

$$\psi \to 0 \text{ in probability}\ ,$$

where this last statement is unconditional on Λ. Equation (B.9) is shown.

To show that the result announced in (B.10) holds, we just notice that according to Lemma B.6,

$$\widehat{m}'(\mathcal{S}-\widehat{m}\widehat{m}')^{-2}v = \frac{\widehat{m}'\mathcal{S}^{-2}v}{1-\widehat{m}'\mathcal{S}^{-1}\widehat{m}} + \frac{\widehat{m}'\mathcal{S}^{-2}\widehat{m}}{(1-\widehat{m}'\mathcal{S}^{-1}\widehat{m})^2}\widehat{m}'\mathcal{S}^{-1}v\ .$$

Under our assumptions, the results in El Karoui (2010) show that $\widehat{m}'\mathcal{S}^{-1}\widehat{m} \to \rho$ in probability. We have also just established that $\widehat{m}'\mathcal{S}^{-2}\widehat{m}$ has a finite limit in probability. And finally, we know from El Karoui (2010), Theorem 4.3 that $\widehat{m}'\mathcal{S}^{-1}v \to 0$ in probability. Because $\psi = \widehat{m}'\mathcal{S}^{-2}v$, we conclude that

$$\widehat{m}'(\mathcal{S}-\widehat{m}\widehat{m}')^{-2}v \to 0 \text{ in probability}\ . \qquad \blacksquare$$

B.2.2. Combining all the arguments together.

Proof of Theorem 3.3. Theorem 3.3 summarizes our findings and follows immediately from Theorems B.1 and B.7. ■

Appendix C. On the situation where the constraints are independent of the sample covariance matrix.

C.1. Checking conditions required for Theorem 4.3.

We really need to check only two things to make the arguments we outlined in subsection 4.4 go through. We need to make sure that conditions A0–A5 hold in probability (because of conditioning they now involve $\widehat{\mu}$, which is a random variable, and they are therefore not deterministic conditions); we need to show that

$$|||\widehat{V}'\Sigma^{-1}\widehat{V} - (V'\Sigma^{-1}V + Ce_ke_k')|||_2 \to 0 \text{ in probability}$$

and, furthermore, that the same holds true when we replace these matrices by their inverses. Because we made the somewhat simplifying assumption that $\widehat{\mu} \sim \mathcal{N}(\mu, \Sigma_{\widehat{\mu}}/N_{\widehat{\mu}})$, this basically amounts only to elementary variance computations. As a matter of fact, if we denote $\epsilon_{\widehat{\mu}} = \widehat{\mu} - \mu$, then

$$\widehat{\mu}'\Sigma^{-1}\widehat{\mu} - \mu'\Sigma^{-1}\mu - \epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}} = 2\epsilon_{\widehat{\mu}}'\Sigma^{-1}\mu\ .$$

The right-hand side is a Gaussian random variable with mean 0 and variance $4\mu'\Sigma^{-1}\Sigma_{\widehat{\mu}}\Sigma^{-1}\mu/N_{\widehat{\mu}}$. Using elementary matrix algebra (for instance, the fact that the largest eigenvalue of a symmetric matrix is larger than any of its diagonal entries and the fact that for a positive semidefinite matrix, its largest eigenvalue is also its spectral norm), it is clear that

$$\mu'\Sigma^{-1}\Sigma_{\widehat{\mu}}\Sigma^{-1}\mu \leq |||V'\Sigma^{-1}V|||_2|||\Sigma^{-1/2}\Sigma_{\widehat{\mu}}\Sigma^{-1/2}|||_2\ .$$

Our assumption is that these two quantities are bounded, and since $N_{\widehat{\mu}} \to \infty$, too, we see that

$$\widehat{\mu}'\Sigma^{-1}\widehat{\mu} - \mu'\Sigma^{-1}\mu - \epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}} \to 0 \text{ in probability }.$$

A similar analysis shows that

$$\widehat{V}'\Sigma^{-1}V - V'\Sigma^{-1}V \to 0 \text{ in probability }.$$

So to get the matrix approximation we seek, all we need to show is that

$$\epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}} - C \to 0 \,.$$

Naturally, we have the stochastic representation

$$\epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}} \stackrel{\mathcal{L}}{=} \frac{1}{N_{\widehat{\mu}}}\sum_{i=1}^{p} l_i(\Sigma_{\widehat{\mu}}^{1/2}\Sigma^{-1}\Sigma_{\widehat{\mu}}^{1/2})Z_i^2 \,,$$

where $\{l_i(N)\}_{i=1}^{p}$ are the eigenvalues of the matrix N and Z_i are i.i.d. $\mathcal{N}(0,1)$. In other words, $\epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}}$ is a weighted χ^2 random variable. Hence, we have

$$\operatorname{var}\left(\epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}}\right) = \frac{2}{N_{\widehat{\mu}}^2}\operatorname{trace}\left((\Sigma_{\widehat{\mu}}\Sigma^{-1})^2\right) \to 0$$

by our assumptions. On the other hand,

$$\epsilon_{\widehat{\mu}}'\Sigma^{-1}\epsilon_{\widehat{\mu}} = \frac{1}{N_{\widehat{\mu}}}\operatorname{trace}\left(\Sigma_{\widehat{\mu}}\Sigma^{-1}\right) = C \,.$$

Hence,

$$\widehat{V}'\Sigma^{-1}\widehat{V} - (V'\Sigma^{-1}V + Ce_ke_k') \to 0 \text{ in probability }.$$

We have just established entrywise convergence in probability for these $k \times k$ matrices, and this becomes operator (or spectral) norm convergence since we are assuming that k, the number of constraints, and hence the number of columns of V, remains fixed. So we have

$$|||\widehat{V}'\Sigma^{-1}\widehat{V} - (V'\Sigma^{-1}V + Ce_ke_k')|||_2 \to 0 \text{ in probability }.$$

To get the approximations for the inverse, we use the first resolvent identity; i.e., if A and B are invertible matrices, $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$. Recall that Weyl's inequality (Bhatia (1997)) guarantees that if l_i denotes the ith eigenvalue of a symmetric matrix,

$$|l_i(A) - l_i(B)| \leq |||B - A|||_2 \,.$$

Our assumption is that the smallest eigenvalue of the positive semidefinite matrix $M = V'\Sigma^{-1}V$ is bounded away from 0. Therefore, the same is true of the smallest eigenvalue of $(V'\Sigma^{-1}V + Ce_ke_k')$ since $(V'\Sigma^{-1}V + Ce_ke_k') - M \succeq 0$ in the semidefinite ordering ($C \geq 0$, of course). And therefore, the smallest eigenvalue of $\widehat{V}'\Sigma^{-1}\widehat{V}$ is also bounded away from 0 in probability since, by Weyl's inequality,

$$l_{\min}(\widehat{V}'\Sigma^{-1}\widehat{V}) \geq l_{\min}(V'\Sigma^{-1}V + Ce_ke_k) - |||V'\Sigma^{-1}V + Ce_ke_k - \widehat{V}'\Sigma^{-1}\widehat{V}|||_2 \,.$$

Putting all of this together with the first resolvent identity, we conclude (using the fact that the spectral norm is a matrix norm and hence is submultiplicative) that

$$|||(\widehat{V}'\Sigma^{-1}\widehat{V})^{-1} - (V'\Sigma^{-1}V + Ce_ke_k)^{-1}|||_2 \to 0 \text{ in probability }.$$

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