Assignment 3

STAT3017/STAT6017 - Big Data Statistics - Sem 2 2023

2023-09-11

```
knitr::opts_chunk$set(echo = TRUE)
library(ggplot2)
library(mvnfast)
library(future.apply)
```

```
## Loading required package: future
```

```
plan(multisession, workers = 8)
```

Question 1

Suppose we had two independent p-dimensional vector samples $X := x_1, ..., x_{n_1}$ and $Y := y_1, ..., y_{n_2}$ where $p \leq n_2$. We assume that each sample comes from a (possibly different) population distribution with *i.i.d.* components and finite second moment.

(a) How is the Fisher LSD related to the two vector samples X and Y? What is the relationship between the two parameters (s, t) of $F_{s,t}$ and the three values (p, n_1, n_2) describing the dimensionality and sizes of X and Y?

Denote the S_1 as the sample covariance matrix for the X and the S_2 as the sample covariance matrix for the Y. The random Fisher matrices take the form of $V_n = S_1 S_2^{-1}, n = (n_1, n_2)$. When the $p/n_1 \to y_1$ and $p/n_2 \to y_2$, the empirical spectral distribution (ESD) of $F_n^{V_n}$ of V_n converges to the a limiting spectral distribution (LSD) F_{y_1,y_2}, that is, $s = y_1, t = y_2$.

(b) Now that you explained the relationship between X, Y , the values (p, n_1, n_2) and the parameters (s, t) of $F_{s,t}$ in part (a), what would you expect the empirical density of eigenvalues be for the following three choices of triplets $(p, n1, n2)$: $(50, 100, 100), (75, 100, 200), (25, 100, 200)$. Plot the three densities on the same figure with an appropriate legend.

Let's first calculate the s, t *and* h for each triple. The first one has $s = \frac{1}{2}, t = \frac{1}{2}, h = \sqrt{3}/2$, while the second has $s = \frac{3}{4}, t = \frac{3}{8}, h = 3\sqrt{3/2}/4$. The third has $s = \frac{1}{4}, t = \frac{1}{8}, h = \sqrt{11/2}/4$. We can compute each support for each triple. It is expected that the first triple has the longest tail, as $t \to 1, a \to \frac{1}{4}(1-s)^2, b \to \infty$. Also, it is clear that the second triple is more concentrated around 0 as $s \uparrow, t \to 1, a \to \frac{1}{4}(1-s)^2$, the left support is getting closer to 0. The third triple illustrates the case where the LSD of Fisher matrix is getting closer to the MP distribution with fixed \$s\$ and $t \to 0$. Firstly, we can write a function about LSD of the Fisher matrix.

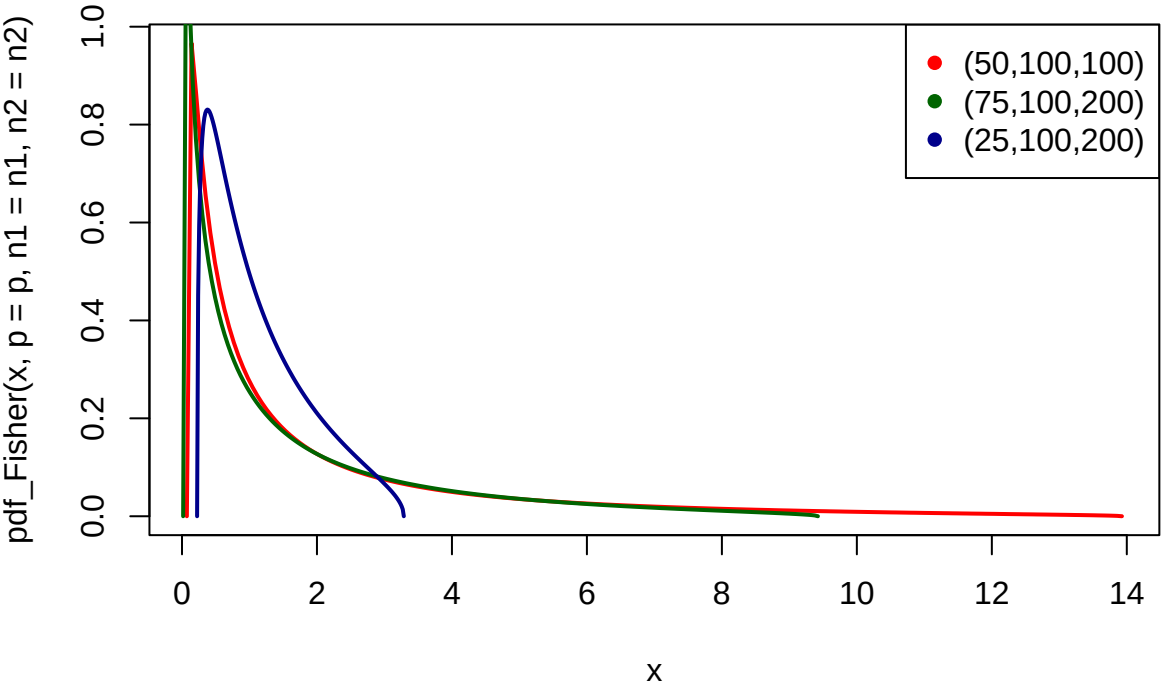
```
pdf_Fisher <- function(x, p = p, n1 = n1, n2 = n2){
  s = p/n1
  t = p/n2
  h = (s + t - s*t)^{1/2}
  a = (1 - h)^2 / (1 - t)^2
  b = (1 + h)^2 / (1 - t)^2
  ifelse(x <= a | x >=b, 0,
         suppressWarnings(
           (1-t)/ (2 * pi * x * (s + t*x)) *sqrt((b - x) * (x -a)), "x")
         )
}
```

Then, we can write a function to create a equally spaced points inside the support of of the Fisher LSD.

```
fisher_support <- function(n_points = 200, p = p, n1 = n1, n2 = n2){
  s = p/n1
  t = p/n2
  h = (s + t - s*t)^{1/2}
  a = (1 - h)^2 / (1 - t)^2
  b = (1 + h)^2 / (1 - t)^2
  seq(a, b, length.out = n_points)
}
```

Let's plot the three cases together.

```
p = 50
n1 = 100
n2 = 100
x = fisher_support(p = p, n1 = n1, n2 = n2)
plot(x, pdf_Fisher(x, p = p, n1 = n1, n2 = n2), type='l', lwd=2, col="red")
p = 75
n1 = 100
n2 = 200
x = fisher_support(p = p, n1 = n1, n2 = n2)
lines(x, pdf_Fisher(x, p = p, n1 = n1, n2 = n2), type='l', lwd=2, col="darkgreen", ylab="")
p = 25
n1 = 100
n2 = 200
x = fisher_support(p = p, n1 = n1, n2 = n2)
lines(x, pdf_Fisher(x, p = p, n1 = n1, n2 = n2), type='l', lwd=2, col="darkblue", ylab="")
legend("topright", c("(50,100,100)", "(75,100,200)","(25,100,200)"),
       col = c("red", "darkgreen","darkblue"),
       pch = c(16,16,16))
```



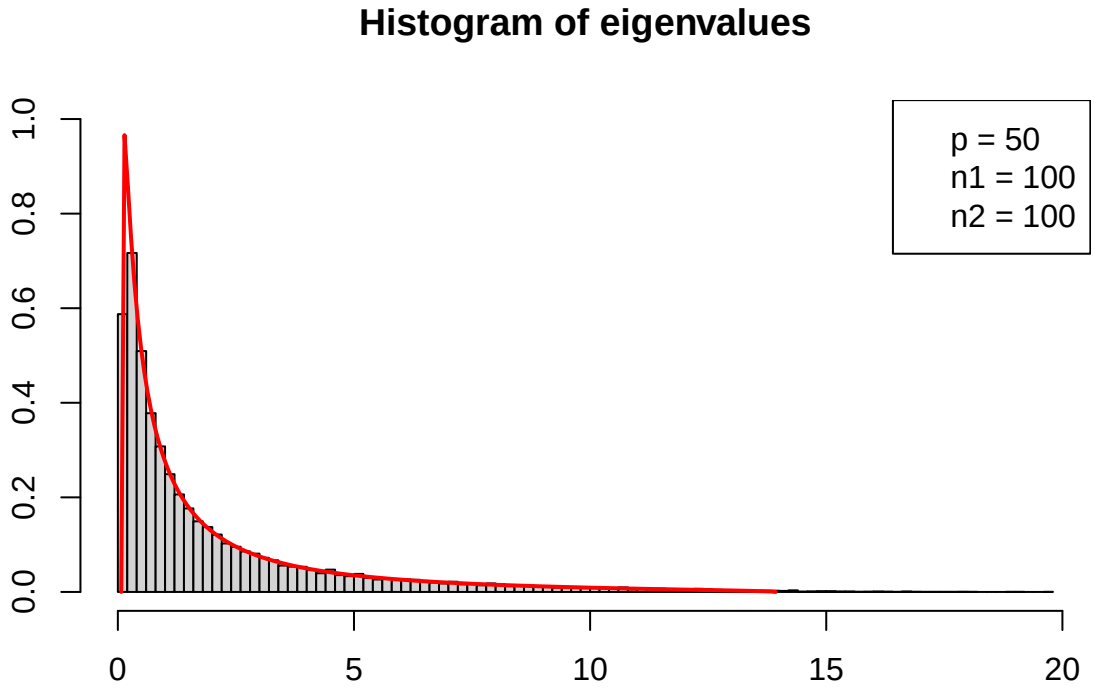
(c) Perform a simulation study for the same choices of the triplets $(p, n1, n2)$ that are given in part (b). Setup your experiment correctly to demonstrate a histogram of eigenvalues and compare them to the appropriately parametrised densities from part (b). For each triplet $(p, n1, n2)$, plot the histogram of eigenvalues, overlay the appropriate density, and ensure the plot is appropriately titled. Show the code for your simulation study.

We first construct a function to simulate the eigenvalues for a number of Fisher matrices. Let's adhere to the traditional assumption that the entries in the F matrix have mean 0 and variance 1.

```
simulate_Fisher <- function(p, n1, n2){
  mu1 = rep(0,p)
  mu2 = rep(0,p)
  Sigma1 = diag(p)
  Sigma2 = diag(p)
  X = rmvn(n1, mu1, Sigma1)
  Y = rmvn(n2, mu2, Sigma1)
  S1 = cov(X)
  S2 = cov(Y)
  Fisher_matrix = S1 %*% solve(S2)
  eigen(Fisher_matrix, only.values = TRUE)$values
}
```

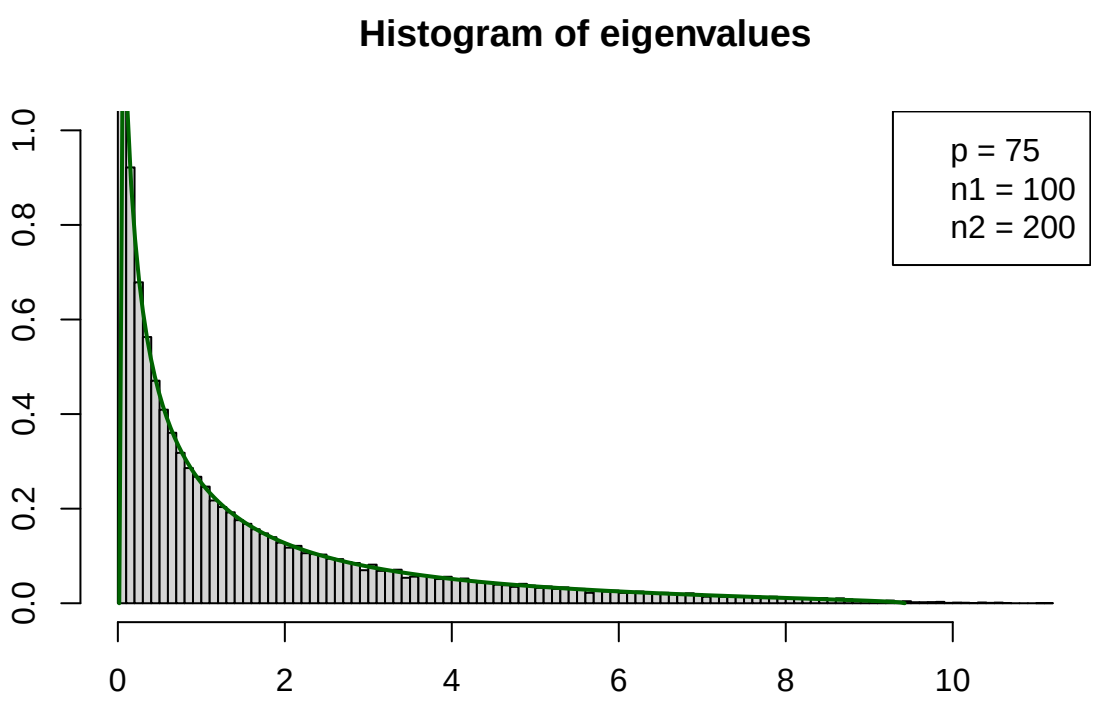
Now, we are ready to plot the each triplet. Let's first plot the case of $(50, 100, 100)$. To make the result smooth, let's make the a number of Fisher matrix large, say 500.

```r
eigenvalues = c(replicate(500, simulate_Fisher(50, 100, 100)))
hist(eigenvalues, breaks = "FD",
     freq = FALSE, ylim = c(0,1),
     main = "Histogram of eigenvalues",
     xlab = "",
     ylab = "")
x = fisher_support(p = 50, n1 = 100, n2 = 100)
lines(x, pdf_Fisher(x, p = 50, n1 = 100, n2 = 100),
      type='l', lwd=2, col="red", ylab="")
legend("topright", c("p = 50", "n1 = 100", "n2 = 100"))
```



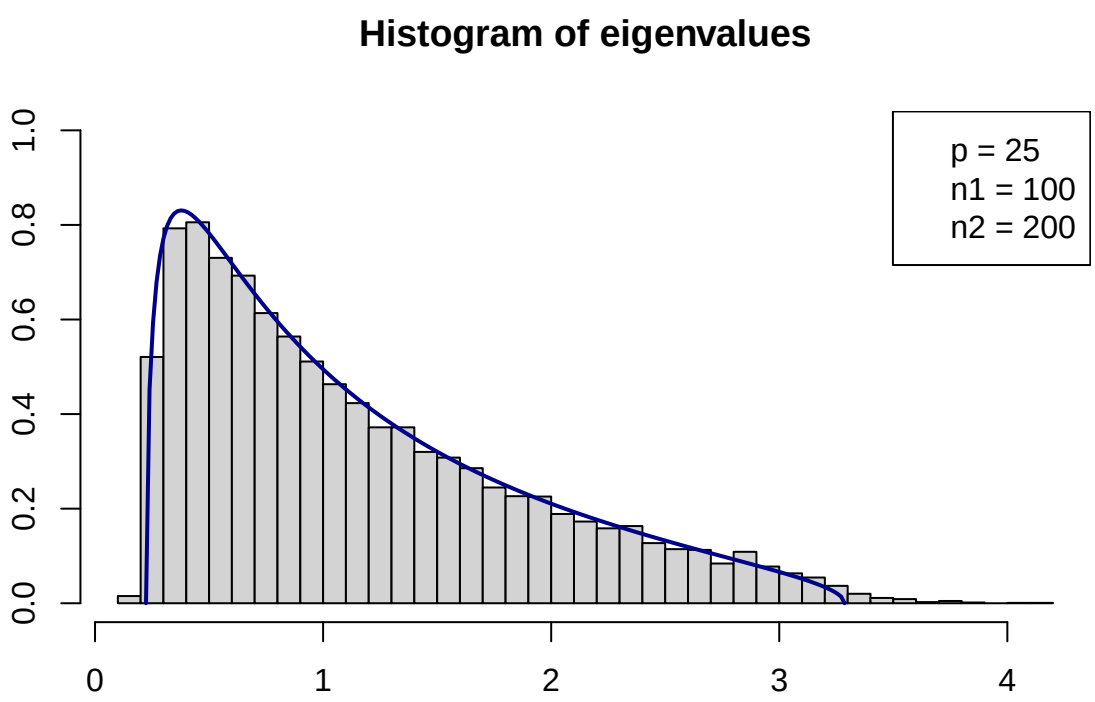
Let's then plot the case of $(75, 100, 200)$ with the number of Fish matrix to be large.

```r
eigenvalues = c(replicate(500, simulate_Fisher(75, 100, 200)))
hist(eigenvalues, breaks = "FD",
     probability = TRUE,
     ylim = c(0,1),
     main = "Histogram of eigenvalues",
     xlab = "",
     ylab = "")
x = fisher_support(p = 75, n1 = 100, n2 = 200)
lines(x, pdf_Fisher(x, p = 75, n1 = 100, n2 = 200),
      type='l', lwd=2, col="darkgreen", ylab="")
legend("topright", c("p = 75", "n1 = 100", "n2 = 200"))
```



Let's lastly plot the case of $(25, 100, 200)$ with the same number of Fish matrix.

```
eigenvalues = c(replicate(500, simulate_Fisher(25, 100, 200)))
hist(eigenvalues, breaks = "FD",
     probability = TRUE,
     ylim = c(0,1),
     main = "Histogram of eigenvalues",
     xlab = "",
     ylab = "")
x = fisher_support(p = 25, n1 = 100, n2 = 200)
lines(x, pdf_Fisher(x, p = 25, n1 = 100, n2 = 200),
      type='l', lwd=2, col="darkblue", ylab="")
legend("topright", c("p = 25", "n1 = 100", "n2 = 200"))
```



(d) Derive a formula for the first moment of the Fisher LSD in terms of its parameters s and t. You can assume that $h > t$ as this always holds by the definition of h.

See hand-written note.

(e) Perform a numerical experiment to confirm the formula you derived in part (d). That is, choose 5 values of (s,t) then use your simulation study code (from part c) to generate empirical eigenvalues for those values and calculate their sample mean. Compare the sample means to your formula.

Let's choose five set of s and t value.

```
s = c(0.5, 1.5, 3, 2, 0.75)
t = c(0.1, 0.3, 0.5, 0.7, 0.9)
```

With regards to the implementation of the code base, let's find the corresponding p, n_1, n_2.

```
p = c(100, 150, 300, 70, 162)
n1 = c(200, 100, 100, 35, 216)
n2 = c(1000, 500, 600, 100, 180)
```

Let's use this parameter to generate eigenvalues.

```r
comparison = data.frame(c(seq(1, 5, by = 1)))
colnames(comparison) = c("Number_of_exp")
Empirical_mean = c()
True_mean = c()
for(i in 1:5){
  eigenvalues = c(replicate(500, simulate_Fisher(p[i], n1[i], n2[i])))
  t = p[i]/n2[i]
  True_mean = c(True_mean, 1/(1-t))
  Empirical_mean = c(Empirical_mean, mean(c(eigenvalues)))
}
comparison$Empirical_mean = Empirical_mean
comparison$True_mean = True_mean
comparison
```

```
##   Number_of_exp Empirical_mean True_mean
## 1             1    1.112043+0i  1.111111
## 2             2    1.433298+0i  1.428571
## 3             3    2.009912+0i  2.000000
## 4             4    3.554814+0i  3.333333
## 5             5   11.245469+0i 10.000000
```

We can see that the empirical mean aligns closely with theoretical expectations.

(f) Describe what happens to the first moment when $t \to 0$ and $t \to 1$.

It is clear from the result in part (d) that the first moment of the LSD of Fisher matrix will approach to 1 as $t \to 0$ and will approach to infinity as $t \to 1$. This happens when p and n_2 are comparable in size.

Question 2

Show that the second moment

$$\int_{-\infty}^{\infty} x^2 p_{s,t}(x) dx = \frac{h^2 + 1 - t}{(1-t)^3}$$

See hand-written note.

Question 3

Show that the variance equals $h2/(1-t)3$.

See hand-written note.