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The mean–variance ratio test—A complement to the coefficient of variation test and the Sharpe ratio test

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ABSTRACT

To circumvent the limitations of the tests for coefficients of variation and Sharpe ratios, we develop the mean–variance ratio statistic for testing the equality of mean–variance ratios, and prove that our proposed statistic is the uniformly most powerful unbiased statistic. In addition, we illustrate the applicability of our proposed test for comparing the performances of stock indices.

1. Introduction and motivation

The coefficient of variation (CV), the ratio of the standard deviation and the mean of a random variable, provides a measure of relative variability. On the other hand, the Sharpe ratio (SR), the ratio of the excess expected return to its standard deviation, provides a measure of relative excess return. Populations can have the same relative variability or the same relative excess return even if the means and variances of the variables of interest are different.

Miller and Karson (1977) presented the likelihood ratio test for the equality of two CVs whereas Jobson and Korkie (1981) developed a Sharpe ratio statistic to test for the equality of two SRs. The CV test statistic is very important in medical sciences, physical sciences, and social sciences whereas the SR test statistic is important in finance as it provides a formal statistical comparison for the performances among portfolios. However, as both the CV and SR statistics possess only asymptotic distributions, one can only obtain their properties for large samples, and not for small samples. Nevertheless, it is important to compare the performances of variables by using small samples as sometimes large samples are not available. Also it is, sometimes, not so meaningful to measure CVs and SRs for too large samples as the means and standard deviations of the underlying variables could change when the samples become large (Hamilton and Susmel, 1994). The main obstacle to developing the CV and SR tests for small samples is that it is impossible to obtain a uniformly most powerful unbiased (UMPU) statistic for testing the equality of CVs or SRs in the case of small samples. To circumvent this problem, in this paper we propose to use the mean–variance ratio (MVR) test for the comparison.

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The MVR has its own importance in many areas. We illustrate its importance to Markowitz optimal portfolio theory. Let us consider a random p-vector $\mathbf{r}$ with mean vector $\boldsymbol{\mu}$ and covariance matrix V, a positive definite matrix. The random vector $\mathbf{r}$ can be regarded as the returns of p branches of stocks. According to Markowitz optimal portfolio theory (Markowitz, 1952; Bai et al., 2009, forthcoming), the objective of the investors is to find the optimal portfolio $\mathbf{a} = (a, \ldots, a_p)^T$ such that the overall expected return $\mathbf{a}^T\boldsymbol{\mu}$ reaches its maximum subject to a given risk value $\sigma_0^2 = \mathbf{a}^T V\mathbf{a}$ or, equivalently, the risk $\mathbf{a}^T V\mathbf{a}$ reaches its minimum subject to a proleptic return $R_0 = \mathbf{a}^T\boldsymbol{\mu}$. For both problems, the solutions are $\hat{\mathbf{a}} = bV^{-1}\boldsymbol{\mu}$ where $b = \sigma_0/\sqrt{\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu}}$. When the components of $\mathbf{r}$ are independent and have means μ_i and variances σ_i^2, the optimal solution for each component is equal to $\hat{a}_i = b\mu_i/\sigma_i^2$. This shows that the optimal portfolio component is proportional to its mean–variance ratio, and thus, the MVR is useful in making investment decisions.

In the remainder of the paper, we shall discuss the evaluation of MVRs for small samples by providing a theoretical framework and then invoking both one-sided and two-sided UMPU MVR tests.

2. The UMPU test for mean–variance ratio ordering

Let X_i and Y_i $(i = 1, 2, \ldots, n)$ be independent variables or excess returns drawn from the corresponding normal distributions $N(\mu, \sigma^2)$ and $N(\eta, \tau^2)$ with joint density $p(x, y)$ such that

$$p(x, y) = k \times \exp\left(\frac{\mu}{\sigma^2}\sum x_i - \frac{1}{2\sigma^2}\sum x_i^2 + \frac{\eta}{\tau^2}\sum y_i - \frac{1}{2\tau^2}\sum y_i^2\right) \tag{1}$$

where $k = (2\pi\sigma^2)^{-n/2}(2\pi\tau^2)^{-n/2}\exp(-\frac{n\mu^2}{2\sigma^2})\exp(-\frac{n\eta^2}{2\tau^2})$. To evaluate the performances of X and Y, practitioners and academics are interested in testing the hypotheses

$$H_0^{\#} : \frac{\sigma}{\mu} \le \frac{\tau}{\eta} \quad \text{versus} \quad H_1^{\#} : \frac{\sigma}{\mu} > \frac{\tau}{\eta} \tag{2}$$

in order to compare the performances of their corresponding CVs, σ/μ and τ/η; or in testing the hypotheses

$$H_0^{*} : \frac{\mu}{\sigma} \le \frac{\eta}{\tau} \quad \text{versus} \quad H_1^{*} : \frac{\mu}{\sigma} > \frac{\eta}{\tau} \tag{3}$$

in order to compare the performances of their corresponding SRs, μ/σ and η/τ.

Miller and Karson (1977) developed a test statistic for testing the hypotheses in (2) for large samples whereas Jobson and Korkie (1981) developed a test statistic for testing the hypotheses in (3) for large samples, but their tests are not appropriate for testing small samples as the distributions of their test statistics are only valid asymptotically. However, it is important to test the hypotheses in (3) for small samples but it is impossible to obtain any UMPU test statistic for testing the inequality of the CVs in (2) or the SRs in (3) for small samples. To circumvent this problem, in this paper we propose to alter the hypothesis in order to test the inequality of the mean–variance ratios as shown in the following:

$$H_0 : \frac{\mu}{\sigma^2} \le \frac{\eta}{\tau^2} \quad \text{versus} \quad H_{11} : \frac{\mu}{\sigma^2} > \frac{\eta}{\tau^2}. \tag{4}$$

Rejecting H_0 suggests X to possess a higher mean–variance ratio (or better investment prospect in finance) when X possesses either a smaller variance or bigger mean (or higher excess mean return in finance) or both. As, sometimes, academics and practitioners do conduct the two-sided test to compare the MVRs, to complete the theory we also consider the following hypotheses in this context:

$$H_0 : \frac{\mu}{\sigma^2} = \frac{\eta}{\tau^2} \quad \text{versus} \quad H_{12} : \frac{\mu}{\sigma^2} \neq \frac{\eta}{\tau^2}. \tag{5}$$

The beauty of using the CV (SR) is that it provides a scale-free measure of the relative variability (relative excess return). Here, we illustrate the beauty of using the MVR in finance. One could easily illustrate the beauty of applying the MVR in other areas like the physical sciences. We note that in some financial processes, the mean change in a short period of time is proportional to its variance change. For example, many financial processes Y_t could follow $\mathrm{d}Y_t = \mu^P(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t^P$, where μ^P is an N-dimensional function, σ is an $N \times N$ matrix, and W_t^P is an N-dimensional standard Brownian motion under the objective probability measure P. When $N = 1$, the SR will be close to 0 while the MVR will be independent of $\mathrm{d}t$. Thus, when the time period $\mathrm{d}t$ is small, it is better to consider the MVR rather than the SR. In addition, one could easily modify the work from Bai et al. (2009) in the Markowitz mean–variance (MV) optimization (Markowitz, 1952) to show that the investment allocation is proportional to the MVR when the covariance matrix of stock returns is a diagonal matrix. Hence, when the asset is concluded to be superior in performance by utilizing the MVR test, its corresponding weight could then be computed on the basis of the corresponding MVR test value, but not on the basis of the SR value. In this sense, our proposed test is better.

We first develop the one-sided UMPU test to test the hypothesis in (4).

Theorem 1. *Let X_i and Y_i ($i = 1, 2, \dots, n$) be independent random variables with the joint distribution function defined in (1). For the hypotheses set up in (4), there exists a UMPU level-α test with the critical function $\phi(u, t)$ such that*

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \geq C_0(t) \\ 0, & \text{when } u < C_0(t) \end{cases} \tag{6}$$

where C_0 is determined by

$$\int_{C_0}^{\infty} f^*_{n,t}(u)\,\mathrm{d}u = K_1; \tag{7}$$

with $f^*_{n,t}(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$ and $K_1 = \alpha \int_{\Omega} f^*_{n,t}(u)\,\mathrm{d}u$, in which $U = \sum_{i=1}^n X_i$, $T_1 = \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i$, $T_2 = \sum_{i=1}^n X_i^2$, $T_3 = \sum_{i=1}^n Y_i^2$, and $T = (T_1, T_2, T_3)$, with $\Omega = \{u \mid \max(-\sqrt{nt_2}, t_1 - \sqrt{nt_3}) \leq u \leq \min(\sqrt{nt_2}, t_1 + \sqrt{nt_3})\}$ as the support of the joint density function of (U, T).

Next, we introduce the two-sided UMPU test statistic as stated in the following theorem in order to test the equality of the MVRs listed in (5).

Theorem 2. *Let X_i and Y_i ($i = 1, 2, \dots, n$) be independent random variables with the joint distribution function defined in (1). Then, for the hypotheses set up in (5), there exists a UMPU level-α test with critical function*

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \leq C_1(t) \text{ or } \geq C_2(t) \\ 0, & \text{when } C_1(t) < u < C_2(t) \end{cases} \tag{8}$$

in which C_1 and C_2 satisfy

$$\begin{cases} \displaystyle\int_{C_1}^{C_2} f^*_{n,t}(u)\,\mathrm{d}u = K_2 \\ \displaystyle\int_{C_1}^{C_2} u f^*_{n,t}(u)\,\mathrm{d}u = K_3 \end{cases} \tag{9}$$

where $K_2 = (1-\alpha)\int_{\Omega} f^*_{n,t}(u)\,\mathrm{d}u$ and $K_3 = (1-\alpha)\int_{\Omega} u f^*_{n,t}(u)\,\mathrm{d}u$. The terms $f^*_{n,t}(u)$, T_i ($i = 1, 2, 3$) and T are defined in Theorem 1.

We call the statistic U in Theorem 1 (Theorem 2) the one-sided (two-sided) MVR test statistic for the hypotheses set up in (4) ((5)). One could easily apply numerical methods to look for the numerical solutions to equations in (9) as stated in the following problem.

Problem 3. Compute the values of the constants C_1 and C_2 in $\Omega = [I_d, I_u]$ such that

$$\int_{C_1}^{C_2} f^*_{n,t}(u)\,\mathrm{d}u = K_2 \tag{10}$$

and

$$\int_{C_1}^{C_2} u f^*_{n,t}(u)\,\mathrm{d}u = K_3 \tag{11}$$

where $f^*_{n,t}(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$, $K_2 = (1-\alpha)\int_{\Omega} f^*_{n,t}(u)\,\mathrm{d}u$, and $K_3 = (1-\alpha)\int_{\Omega} u f^*_{n,t}(u)\,\mathrm{d}u$.

3. An illustration

In this section, we demonstrate the superiority of the mean–variance ratio test developed in this paper over the traditional Sharpe ratio test by illustrating the applicability of our test to the decision making process as regards investing in Shanghai SE Composite (China) and Australia All Ordinary Index (Australia).[1] For simplicity, we only demonstrate the two-sided UMPU test. The data analyzed in this section are the monthly returns of China and Australia for the sample period from January 2003 to December 2007 in which the data from January 2005 to December 2005 are used to compute the mean–variance ratio in January 2006, while the data from February 2005 to January 2006 are used to compute the mean–variance ratio in February 2006, and so on.[2] However, using too short periods to compute the Sharpe ratio would not

1 Similarly, one could conduct the CV test for comparison. We use the SR test rather than the CV test here because it is more common for investors to use the SR test in finance.

2 For this period, the p-values of the Jarque–Bera test are 0.7102 and 0.1063 for China and Australia, respectively, and the p-values of the Ljung–Box test for lags 6 and 12 are 0.5786 and 0.8843 for China, and 0.7347 and 0.6701 for Australia, respectively. These findings lead us not to reject the hypothesis that the data are drawn from an iid normal distribution.

Table 1
The results of the mean–variance ratio test and Sharpe ratio test for China versus Australia from 2006 to 2007.

China versus Australia							
Time	Mean–variance ratio test					Sharpe ratio test	
		$\alpha = 5\%$		$\alpha = 10\%$			
	U	C_1	C_2	C_1^*	C_2^*	Z	p-value
Jan 06	−0.0869	−0.1311	0.2382	−0.1058	0.211	−0.9890	0.3227
Feb 06	0.0541	−0.0043	0.3722	0.0241	0.3457	−1.0527	0.2925
Mar 06	−0.0054**	−0.0633	0.3081	−0.0370	0.2808	−1.0852	0.2778
Apr 06	0.0945	0.0438	0.3751	0.0687	0.3519	−1.0868	0.2771
May 06	0.2171*	0.2168	0.5094	0.2421	0.4918	−1.0349	0.3007
Jun 06	0.4365	0.3337	0.6071	0.3569	0.5910	−0.8973	0.3696
Jul 06	0.4363	0.3275	0.6015	0.3445	0.5793	−0.8400	0.4009
Aug 06	0.3982	0.2660	0.5730	0.2872	0.5503	−0.8209	0.4117
Sep 06	0.3552	0.2289	0.5401	0.2487	0.5141	−0.7679	0.4425
Oct 06	0.4164	0.2723	0.5673	0.2926	0.5441	−0.7112	0.4770
Nov 06	0.5199	0.4307	0.6427	0.4395	0.6216	−0.6764	0.4988
Dec 06	0.6470	0.5553	0.7685	0.5672	0.7501	−0.7066	0.4798
Jan 07	0.8348	0.7537	1.0284	0.7750	1.0097	−0.6796	0.4968
Feb 07	0.7952	0.7160	0.9949	0.7352	0.9729	−0.7301	0.4653
Mar 07	0.7965	0.7245	0.9998	0.7434	0.9782	−0.7223	0.4701
Apr 07	0.8971	0.8197	1.0698	0.8379	1.0516	−0.7111	0.4770
May 07	0.9810	0.9063	1.1695	0.9256	1.1501	−0.6167	0.5374
Jun 07	0.9178**	0.9307	1.1513	0.9433	1.1312	−0.5792	0.5625
Jul 07	0.8263**	0.8367	1.1056	0.8560	1.0849	−0.5102	0.6099
Aug 07	1.0197*	1.0161	1.2565	1.0290	1.2330	−0.3741	0.7083
Sep 07	1.1463	1.1348	1.3626	1.1448	1.3383	−0.2923	0.7701
Oct 07	1.1532*	1.1384	1.3722	1.1572	1.3578	−0.2911	0.7710
Nov 07	1.1755*	1.1646	1.3880	1.1806	1.3721	−0.2038	0.8385
Dec 07	0.8419	0.8159	1.1562	0.8404	1.1300	−0.3015	0.7630

The MVR test U and the corresponding critical values C_1 and C_2 (C_1^* and C_2^*) are defined in Theorem 2 while the SR test Z is defined in (12).
** $p < 5\%$.
* $p < 10\%$.

be meaningful, as discussed in the previous sections. Thus, we utilize a longer period from January 2003 to December 2005 to compute the Sharpe ratio in January 2006, from February 2003 to January 2006 to compute that in February 2006, and so on.[3]

Let X_t and Y_t be the returns of China and Australia at time t with means μ and η and variances σ^2 and τ^2, respectively. We test the hypothesis in (5). To test the hypothesis, we first compute the values of the test function U for the mean–variance ratio statistic as shown in Theorem 2 and display the values in Table 1. We then compute the corresponding critical values C_1 and C_2 under the test levels of 5% and 10% to test the hypothesis. In addition, in order to illustrate the performances of the indices and the corresponding test results visually, we exhibit the returns of the two indices and their difference in Fig. 1(A), and display their corresponding values of U with C_1 and C_2 in Fig. 1(B).

For comparison, we also compute the corresponding Sharpe ratio statistic (see, e.g., Leung and Wong (2008)), such that

$$z_i = \frac{\hat{\tau}\hat{\mu} - \hat{\sigma}\hat{\eta}}{\sqrt{\hat{\theta}}} \tag{12}$$

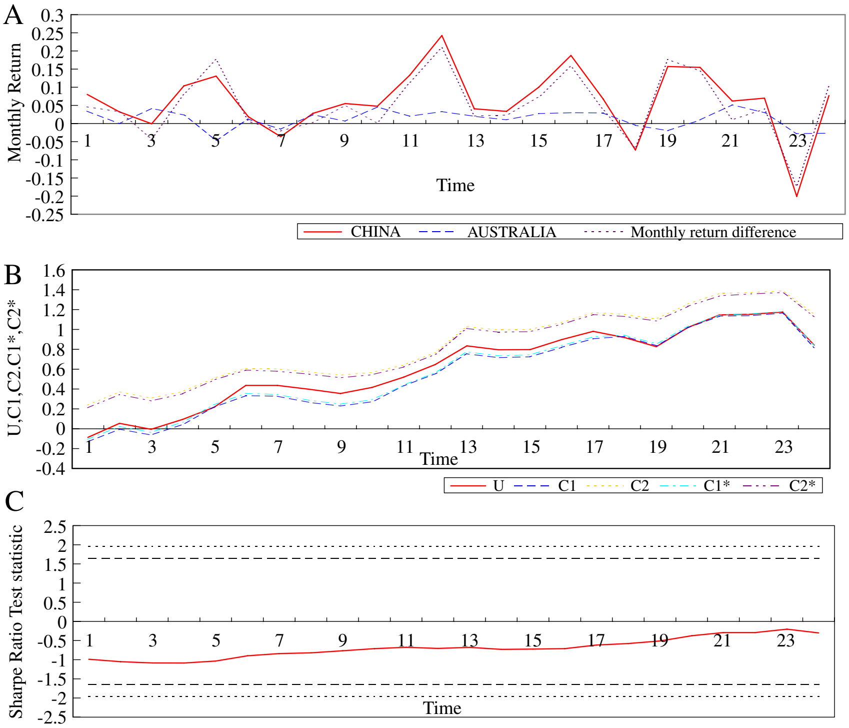
which follows standard normal distribution asymptotically with

$$\theta = \frac{1}{T}\left[2\sigma^2\tau^2 - 2\sigma\tau Cov(X_t, Y_t) + \frac{1}{2}\mu^2\tau^2 + \frac{1}{2}\eta^2\sigma^2 - \frac{\mu\eta}{\sigma\tau}Cov(X_t, Y_t)^2\right]$$

to test the equality of the Sharpe ratios with the hypothesis in (5). Rather than using one-year data to compute the values of our proposed statistic, we use the overlapping three-year data to compute the Sharpe ratio statistic as discussed before. The results are also reported in Table 1 next to the results for our proposed statistic, while their plots are depicted in Fig. 1(C).

As shown in Table 1 and Fig. 1(C), we cannot detect any significant difference between the Sharpe ratios, implying that the performances of these two indices are indistinguishable. We note that the three-year monthly data being used for computing the Sharpe ratio statistic could be covering too short a period to satisfy the asymptotic statistical properties for the test but, still, we cannot find any significant difference between the performances of these two indices. If we use any longer period, the result is expected to be insignificant as the high means in some sub-periods could be offset by the low means in other

[3] For this period, the p-values of the Jarque–Bera test are 0.4426 and 0.0117 for China and Australia, respectively, and the p-values of the Ljung–Box test for lags 6 and 12 are 0.5330 and 0.6147 for China, and 0.0780 and 0.1252 for Australia, respectively. The results lead us to reject the hypothesis that the China data are drawn from a normal distribution at the 5% level and to reject the hypothesis that the Australia data are independent at the 10% level. These findings support our claim as discussed in the introduction and conclusion sections that large samples may not be suitable for use.



The dotted lines in Figures 1B and 1C are critical values.

Note: The mean-variance ratio test statistic U is defined in Theorem 2 with C_1 C_1^*, C_2, and C_2^* defined in (9) and the Sharpe ratio test statistic Z is defined in (12). C_1 and C_2 are critical values of the 5% level while C_1^* and C_2^* are critical values of the 10% level.

Fig. 1. Plots of the monthly returns for China and Australia and the corresponding mean–variance ratio test U and Sharpe ratio test statistic Z.

sub-periods. Thus, a possible limitation of applying the Sharpe ratio test is that it would usually reach the conclusion of indistinguishable performances among the indices, which may not be the situation in reality. In this respect, looking for a statistic for evaluating the performances among assets for short periods is essential. In this paper, we adopt our proposed statistic to conduct the analysis. As shown in Table 1 and Fig. 1(B), we find that our proposed statistic does not disappoint us, as it does show some significant differences in performance between these two indices. This information could be useful to investors in their decision making process.

4. Concluding remarks

The limitation of the MVR test provided in this paper is that the test requires iid and normality assumptions for the underlying variables being examined. The MVR test could be used in medical sciences, physical sciences, social sciences, finance, and many other areas. Since it is not difficult to obtain iid observations following normality for data in medical sciences, physical sciences, social sciences, and many other areas, it is not a problem to apply the MVR test proposed in this paper to these areas.

Nonetheless, it is common that observations from economics and finance are not iid and they do not satisfy the normality assumption either. Thus, one may doubt the applicability of the MVR test in economics and finance. Nevertheless, we note that it is still possible to obtain iid observations following the normality assumption in economics and finance, especially for weekly data, monthly data, and annual data. For example, it is well-known that some daily (stock or exchange rate) returns may not be white noise but contain a weak lag-1 autocorrelation and thus they may not be iid. In addition, it is well-known that most daily (stock or exchange rate) returns are not normally distributed. However, in many situations, daily observations are not available and only weekly, monthly, or annual observations are available; see Wong et al. (2006) for examples of these kinds of data. In addition, we note that academics recommend not using daily data, to avoid day-of-the-week effects, nonsynchronous trading, and noises that can result from using daily data; see, for example, Chen et al. (forthcoming) and the references therein for more information. In this situation, the autocorrelations become insignificant and thus, very likely, the hypothesis of being independent should not be rejected. In addition, by the central limit theorem, weekly, monthly, and annual observations could follow the normality assumption. Thus, it is possible to get financial data satisfying the iid requirement and with the hypothesis of being normally distributed not being rejected.[4] This implies that the MVR test could still be used in economics and finance.

We also note that the MVR test statistic recommended in this paper is suitable not only for small samples but also for large samples. On the other hand, the CV and SR tests are only suitable for large samples and may not be suitable for small samples in many situations. In this sense, our proposed MVR test is better. We note that it is not uncommon to find that getting large samples in medical sciences, physical sciences, and social sciences could be expensive or even impossible and thus academics and practitioners could consider applying the MVR test instead of employing the CV or SR tests.

However, it is not difficult and usually not expensive to obtain large samples in finance. One may thus argue that it is not necessary to apply the MVR test in finance. We note that in many situations, though it is possible to obtain large samples, one could use just small samples in the analysis. A simple example is that during the financial crisis in 2008, many fund managers and policy makers wanted to analyze how serious the market crash was. In this situation, data from before 2008 are available but cannot be used as the nature of the data changed at the crisis. Thus, only observations starting from the downturn could then be used in the analysis and this was a small sample. In this situation, the CV and SR tests are not recommended. Fund managers and policy makers may consider applying our proposed MVR test for their analysis.

In addition, we note that there are two basic approaches to the problem of portfolio selection under uncertainty. One approach is based on the concept of expected utility maximization (see, e.g., Egozcue and Wong (2010)) and stochastic dominance (SD) test statistics (see, e.g., Linton et al. (2005)). The other is the mean-risk (MR) analysis in which the portfolio choice is made with respect to two measures — the expected portfolio mean return and the portfolio risk. A disadvantage of the latter approach is that it is derived by assuming the Von Neumann–Morgenstern quadratic utility function and that the returns are normally distributed (Hanoch and Levy, 1969). However, Wong (2007), Wong and Ma (2008) and others have shown that, under some regularity conditions, the conclusion drawn from the MR comparison could be equivalent to the comparison of expected utility maximization for any risk-averse investor, not necessarily with only a quadratic utility function.

In addition, the major limitation of the MR and SD statistics is that up to now academics could only develop their asymptotic distributions, and not their distributions for small samples. Nonetheless, the MVR test developed in this paper circumvents their limitations. Thus, the test developed in this paper sets a milestone in the literature of financial economics, as our test is the first test making such a comparison possible. Further study may extend the test developed in our paper to non-normal distributions; see, for example, Bai and Guo (1999), Bai and Chow (2000), and Wong and Bian (2005) for more information on non-normality.

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Appendix

Proof of Theorem 1. Let

$$\theta = \frac{\mu}{\sigma^2} - \frac{\eta}{\tau^2}, \qquad \vartheta_1 = \frac{\eta}{\tau^2}, \qquad \vartheta_2 = -\frac{1}{2\sigma^2}, \quad \text{and} \quad \vartheta_3 = -\frac{1}{2\tau^2}. \tag{13}$$

The joint density function of (X, Y) in (1) becomes

$$p(x, y) = (2\pi\sigma^2)^{-n/2}(2\pi\tau^2)^{-n/2} \exp\left(-\frac{n\mu^2}{2\sigma^2}\right) \exp\left(-\frac{n\eta^2}{2\tau^2}\right) \exp(\theta u + \vartheta_1 t_1 + \vartheta_2 t_2 + \vartheta_3 t_3),$$

and the hypotheses in (4) become $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$. Applying the theorem from Lehmann (1986, pp. 145–149), the critical function

$$\phi(u, t) = \begin{cases} 1, & \text{when } u \ge C_0(t) \\ 0, & \text{when } u < C_0(t) \end{cases} \tag{14}$$

is the UMPU test where C_0 is determined by

$$\int_{C_0(t)}^{\infty} f(u|t, \theta = 0)\, \mathrm{d}u = \alpha. \tag{15}$$

[4] We have found many observations in finance for which the hypothesis of being iid is not rejected and the hypothesis of being normally distributed is not rejected either. We have displayed one example, shown in our illustration section. Other results are available on request.

The value C_0 can be obtained and thereafter the critical function can be obtained provided that we know the conditional distribution $f(u|t)$. Thus, we have to construct the conditional distribution. To do that, we first let $Z_1 = \frac{1}{n}\sum X_i$, $Z_2 = \frac{1}{n}\sum Y_i$, $Z_3 = \frac{\sum(X_i-\bar{X})^2}{\sigma^2}$, and $Z_4 = \frac{\sum(Y_i-\bar{Y})^2}{\tau^2}$, and define $U = nZ_1$, $T_1 = nZ_1 + nZ_2$, $T_2 = \sigma^2 Z_3 + nZ_1^2$, and $T_3 = \tau^2 Z_4 + nZ_2^2$; one can easily show that the joint density function of (U, T_1, T_2, T_3) can be expressed as

$$f(u,t) = \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{n}}\sqrt{2\pi}\frac{\tau}{\sqrt{n}}2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{t_2-\frac{u^2}{n}}{\sigma^2}\right)^{\frac{n-1}{2}-1}\frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{t_3-\frac{(t_1-u)^2}{n}}{\tau^2}\right)^{\frac{n-1}{2}-1}\frac{1}{n^2\sigma^2\tau^2}$$
$$\times\, \mathrm{e}^{-\frac{n\mu^2}{2\sigma^2}-\frac{n\eta^2}{2\tau^2}}\mathrm{e}^{\theta u+\vartheta_1 t_1+\vartheta_2 t_2+\vartheta_3 t_3} \tag{16}$$

with $(-\sqrt{nt_2} \le u \le \sqrt{nt_2})$ and $[(t_1 - \sqrt{nt_3}) \le u \le (t_1 + \sqrt{nt_3})]$.

Since

$$f(u|t) = \frac{f(u,t)}{f(t)} \tag{17}$$

where $f(t) = \int_\Omega f(u,t)\,\mathrm{d}u$, Eq. (15) is equivalent to

$$\frac{\int_{C_0(t)}^{\infty} f(u,t|\theta=0)\,\mathrm{d}u}{\int_\Omega f(u,t|\theta=0)\mathrm{d}u} = \alpha. \tag{18}$$

Eq. (18) can further be simplified to (7) in which for every fixed t and n, the value of C_0 can then be determined by $\int_{C_0}^{\infty} f_{n,t}^*(u)\,\mathrm{d}u = K_1$ where $f_{n,t}^*(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$ and $K_1 = \alpha\int_\Omega f_{n,t}^*(u)\,\mathrm{d}u$. □

Proof of Theorem 2. Using the transformation in (13), the hypotheses set up in (5) become $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Applying the theorem from Lehmann (1986, pp. 145–149), we obtain the critical function

$$\phi(u,t) = \begin{cases} 1 & \text{when } u \le C_1(t) \text{ or } \ge C_2(t) \\ 0 & \text{when } C_1(t) < u < C_2(t) \end{cases}$$

which is the UMPU test with the C's determined by

$$E_{\theta=0}[\phi(U,T) \mid t] = \alpha \quad \text{and} \quad E_{\theta=0}[U\phi(U,T) \mid t] = \alpha E_{\theta=0}[U \mid t].$$

Making use of the joint density function $f(u,t)$ in (16) and the conditional density function $f(u|t)$ in (17) derived in the proof of Theorem 1, the critical function then becomes

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \le C_1(t) \text{ or } \ge C_2(t) \\ 0, & \text{when } C_1(t) < u < C_2(t) \end{cases}$$

where

$$\int_{C_1(t)}^{C_2(t)} f_{H_0}(u|t)\,\mathrm{d}u = 1-\alpha, \quad \text{and} \quad \int_{C_1(t)}^{C_2(t)} uf_{H_0}(u|t)\,\mathrm{d}u = (1-\alpha)\int_\Omega uf_{H_0}(u|t)\,\mathrm{d}u. \tag{19}$$

In addition, the equations in (19) are equivalent to the following:

$$\frac{\int_{C_1(t)}^{C_2(t)} f_{H_0}(u,t)\,\mathrm{d}u}{\int_\Omega f_{H_0}(u,t)\,\mathrm{d}u} = 1-\alpha, \quad \text{and} \quad \int_{C_1(t)}^{C_2(t)} uf_{H_0}(u,t)\,\mathrm{d}u = (1-\alpha)\int_\Omega uf_{H_0}(u,t)\,\mathrm{d}u. \tag{20}$$

Under H_0 (that is $\theta = 0$), the equations in (20) can be further simplified as follows: for every fixed t and for each n, the values of (C_1, C_2) can be determined from

$$\begin{cases} \int_{C_1}^{C_2} f_{n,t}^*(u)\,\mathrm{d}u = K_2 \\ \int_{C_1}^{C_2} uf_{n,t}^*(u)\,\mathrm{d}u = K_3 \end{cases} \tag{21}$$

where $K_2 = (1-\alpha)\int_\Omega f_{n,t}^*(u)\,\mathrm{d}u$ and $K_3 = (1-\alpha)\int_\Omega uf_{n,t}^*(u)\,\mathrm{d}u$. □

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