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Theory and Methods Section

An Asymptotic χ^2 Test for the Equality of Two Correlation Matrices

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An asymptotic χ^2 test for the equality of two correlation matrices is derived. The key result is a simple representation for the inverse of the asymptotic covariance matrix of a sample correlation matrix. The test statistic has the form of a standard normal theory statistic for testing the equality of two covariance matrices with a correction term added. The applicability of asymptotic theory is demonstrated by two simulation studies and the statistic is used to test the difference in the factor patterns resulting from a set of tests given to retarded and non-retarded children. Two related tests are presented: a test for a specified correlation matrix and a test for equality of correlation matrices in two or more populations.

1. INTRODUCTION

Given samples from two p-variate normal populations we seek a test of the hypothesis that the two populations have the same correlation matrix without assuming that they have equal standard deviations or means. One possible test is the likelihood ratio test. Unfortunately, it is difficult to maximize the required likelihood function under the assumption of equal correlation. General maximization techniques quickly get out of hand since the number of parameters involved is on the order of $p^2/2$. When p is large, say 100, the number of parameters exceeds 5,000.

Another approach might proceed as follows. Let d_{ij} denote an arbitrary element in the difference $R_1 - R_2$ of two sample correlation matrices obtained from independent samples of sizes n_1 and n_2 from two p-variate normal populations. Let $x_{ij} = (n_1 n_2)^{1/2}(n_1+n_2)^{-1/2} d_{ij}$, $i<j$, and let $\mathbf{x}$ denote the column vector of the $p(p-1)/2$ variables x_{ij} in lexographic order. If the original population correlation matrices are equal and non-singular, then $\mathbf{x}$ has an asymptotic normal distribution with mean zero and non-singular covariance matrix Γ. It is not difficult to find a consistent estimator $\hat{\Gamma}$ of Γ and when this is done the statistic

$$\chi^2 = \mathbf{x}'\hat{\Gamma}^{-1}\mathbf{x} \tag{1.1}$$

has an asymptotic χ^2 distribution with $p(p-1)/2$ degrees of freedom. Since this statistic is a positive definite quadratic form in the differences d_{ij}, it should be sensitive to departures from the equal correlation assumption.[1] The principal difficulty with it is that $\hat{\Gamma}$ is a $p(p-1)/2$ by $p(p-1)/2$ matrix and consequently, unless p is fairly modest, the inversion of $\hat{\Gamma}$ puts severe demands on computer storage and precision. If $p=100$, a value not unusual for factor analysis appli-

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[1] Hsu [6, p. 400] and others have suggested this approach.

cations of the type considered in Section 6, $\hat{\Gamma}$ has over 24 million components. If the appropriate estimate of $\hat{\Gamma}$ is used, the statistic proposed next is mathematically identical to that given by (1.1). Because of its form, however, it has the advantage that the principal part of its computation involves the inversion of a $p \times p$ matrix and the multiplication of two such matrices. On present day computers this is a modest computation even for $p = 100$.

There is a fair amount of literature on testing properties of correlation matrices. Most authors [2, 4, 5, 8], consider testing the equality of coefficients within a single correlation matrix. Closer to the topic considered here, Aitkin, *et al.* [2] give a likelihood ratio test for a specified correlation matrix. An asymptotic χ^2 test for the equality of an arbitrary number of population correlation matrices in the bivariate case was given by Pearson and Wilks [10, p. 374]. Kullback [7, p. 80] has remarked that this test does not appear to generalize to larger correlation matrices. In 1964 Seal [12, p. 124] noted that, unfortunately, there is no criterion available to test the hypothesis that correlation matrices are equal though the hypothesis of equality of covariance matrices can be tested relatively easily. In 1967 Kullback [7] proposed a computationally feasible asymptotic χ^2 test for the equality of an arbitrary number of population correlation matrices which is a generalization of the problem considered here. We will show, however, that Kullback's statistic does not, in general, have an asymptotic χ^2 distribution under his null hypothesis.

2. KULLBACK'S TEST

The test statistic proposed by Kullback [7, p. 83] has the form

$$\chi_K^2 = \sum_{i=1}^{k} N_i \log \frac{|R|}{|R_i|} \tag{2.1}$$

where $R_1, \cdots, R_k$ are sample correlation matrices based on independent samples of sizes $n_1, \cdots, n_k$ from p-variate normal populations. Each $N_i = n_i - 1$ and $R = (N_1 R_1 + \cdots + N_k R_k)/N$ where $N = N_1 + \cdots + N_k$. Kullback asserts that if all k populations have the same non-singular correlation matrix, then the test statistic has an asymptotic χ^2 distribution with $(k-1)p(p-1)/2$ degrees of freedom. To see that this is in general not the case, let $k = p = 2$, $N_1 = N_2 = n$, and

$$R_1 = \begin{pmatrix} 1 & r_1 \\ r_1 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & r_2 \\ r_2 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then, from (2.1),

$$\chi_K^2 = n \log \frac{1 - r^2}{1 - r_1^2} + n \log \frac{1 - r^2}{1 - r_2^2} \cdot \tag{2.2}$$

Expanding the first logarithm about r gives

$$\log \frac{1 - r^2}{1 - r_1^2} = 2r(1 - r^2)^{-1}(r_1 - r) + (1 + \bar{r}^2)(1 - \bar{r}^2)^{-2}(r_1 - r)^2$$

where $\bar{r} \in (r_1, r)$. Similarly,

$$\log \frac{1-r^2}{1-r_2^2} = 2r(1-r^2)^{-1}(r_2-r) + (1+\tilde{r}^2)(1-\tilde{r}^2)^{-2}(r_2-r)^2$$

where $\tilde{r} \in (r_2, r)$. Since $r_1 - r = r - r_2 = (r_1 - r_2)/2$, it follows from (2.2) that

$$\chi_K^2 = \frac{n}{4}\left((1+\bar{r}^2)(1-\bar{r}^2)^{-2} + (1+\tilde{r}^2)(1-\tilde{r}^2)^{-2}\right)(r_1-r_2)^2.$$

If the two populations have a common correlation ρ, then[2] as $n \rightarrow \infty$,

$$\sqrt{n}(r_i - \rho) \xrightarrow{L} N(0, (1-\rho^2)^2)$$

for $i = 1, 2$ and

$$n(r_1 - r_2)^2 \xrightarrow{L} 2(1-\rho^2)^2\chi^2(1).$$

Here and elsewhere $\xrightarrow{L}$ denotes convergence in law. Since $\bar{r}$ and $\tilde{r}$ are consistent estimators of ρ,

$$\chi_K^2 \xrightarrow{L} (1+\rho^2)\chi^2(1)$$

as $n \rightarrow \infty$. A slightly weaker form of this result has been asserted, without proof, by Aitkin [1, p. 445] in which a related single sample result is proved. Kullback's statistic fails, by a factor of $1+\rho^2$, to have its asserted asymptotic distribution. The error is in the non-conservative direction. Critical values may be too small by a factor of nearly 2. In this example the difficulty is easy to correct. An appropriate correction for larger values of p will be obtained in Section 4.

3. FORMULA FOR Γ^{-1}

Let φ be the logarithm of the density function of a p-variate normal population with mean vector μ and non-singular covariance matrix Σ. Then

$$\varphi = -\frac{p}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)$$

and

$$\varphi_\alpha = -\tfrac{1}{2}(\log|\Sigma|)_\alpha + \mu_\alpha'\Sigma^{-1}(x-\mu) - \tfrac{1}{2}(x-\mu)'(\Sigma^{-1})_\alpha(x-\mu)$$

where the subscript "α" denotes partial differentiation with respect to α which at this point denotes an arbitrary parameter. The population information matrix [11, p. 270] is

$$\mathcal{I}(\alpha, \beta) = \operatorname{cov}(\varphi_\alpha, \varphi_\beta) = \mu_\alpha'\Sigma^{-1}\mu_\beta + \tfrac{1}{2}\operatorname{tr}((\Sigma^{-1})_\alpha\Sigma(\Sigma^{-1})_\beta\Sigma). \tag{3.1}$$

The last term follows from the fact that for jointly normal variables x, y, z, w with zero means,

$$\operatorname{cov}(xy, zw) = \operatorname{cov}(x, z)\operatorname{cov}(y, w) + \operatorname{cov}(x, w)\operatorname{cov}(y, z).$$

[2] See [14, p. 276].

Making use of the fact that $(\Sigma^{-1})_\alpha = -\Sigma^{-1}\Sigma_\alpha\Sigma^{-1}$, (3.1) becomes

$$\mathcal{I}(\alpha, \beta) = \mu_\alpha'\Sigma^{-1}\mu_\beta + \tfrac{1}{2}\,\mathrm{tr}(\Sigma_\alpha\Sigma^{-1}\Sigma_\beta\Sigma^{-1}). \tag{3.2}$$

Our first task will be to express the information matrix $\mathcal{I}$ of our normal population in terms of means, μ_i, standard deviations, σ_i, and correlations, ρ_{ij}. This matrix is $q\times q$ when $q=p+p(p+1)/2$. The matrix and its inverse may be partitioned as follows

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_{11} & 0 & 0 \\ 0 & \mathcal{I}_{22} & \mathcal{I}_{23} \\ 0 & \mathcal{I}_{32} & \mathcal{I}_{33} \end{bmatrix}, \qquad \mathcal{I}^{-1} = \begin{bmatrix} \mathcal{I}^{11} & 0 & 0 \\ 0 & \mathcal{I}^{22} & \mathcal{I}^{23} \\ 0 & \mathcal{I}^{32} & \mathcal{I}^{33} \end{bmatrix} \tag{3.3}$$

where the subscripts and superscripts 1, 2, 3 are associated with means, standard deviations, and correlations respectively. The zeros in (3.3) are a consequence of the form of (3.2). The matrix $\mathcal{I}^{33}=\Gamma$ is the asymptotic covariance matrix for the maximum likelihood estimates of the correlation coefficients ρ_{ij}. Unfortunately $\mathcal{I}_{33}$ is not the inverse of this matrix, but because of the form of (3.3)[3]

$$\Gamma^{-1} = \mathcal{I}_{33} - \mathcal{I}_{32}(\mathcal{I}_{22})^{-1}\mathcal{I}_{23}. \tag{3.4}$$

Using (3.4), we seek an explicit expression for Γ^{-1}. Letting $\Sigma=(\sigma_{ij})$, $P=(\rho_{ij})$, and $\Delta=(\sigma_i)$, a diagonal matrix, then $\Sigma=\Delta P\Delta$ and

$$\Sigma_\alpha = \Delta_\alpha P\Delta + \Delta P_\alpha\Delta + \Delta P\Delta_\alpha. \tag{3.5}$$

Since the maximum likelihood estimator of P is invariant under change of scale, we may assume without loss of generality that all $\sigma_i=1$. Then $\Sigma=P$ and from (3.5)

$$\Sigma_\alpha = \Delta_\alpha P + P_\alpha + P\Delta_\alpha. \tag{3.6}$$

Using (3.2) and (3.6) and letting δ_{ij} denote the Kronecker delta,

$$\begin{aligned} \mathcal{I}_{22}(\sigma_i, \sigma_j) &= \delta_{ij} + \rho_{ij}\rho^{ij} \\ \mathcal{I}_{32}(\rho_{ij}, \sigma_k) &= \rho^{ij}(\delta_{ik} + \delta_{jk}) \\ \mathcal{I}_{33}(\rho_{ij}, \rho_{kl}) &= \rho^{ik}\rho^{jl} + \rho^{il}\rho^{jk}. \end{aligned} \tag{3.7}$$

Let $t_{ij}=\delta_{ij}+\rho_{ij}\rho^{ij}$, $T=(t_{ij})$ and $T^{-1}=(t^{ij})$. Then $T=\mathcal{I}_{22}$ and from (3.4) and (3.7),

$$\Gamma^{-1}(i, j; k, l) = \rho^{ik}\rho^{jl} + \rho^{il}\rho^{jk} - \rho^{ij}(t^{ik} + t^{jk} + t^{il} + t^{jl})\rho^{kl}. \tag{3.8}$$

Equation (3.8) defines the inverse of the asymptotic covariance matrix of the maximum likelihood estimators for normal population correlation coefficients. We are actually more interested in the quadratic form this matrix defines. Let $Y=(y_{ij})$ be any symmetric zero-diagonal matrix, then

$$\begin{aligned} \chi^2(Y, P) &= \sum_{i<j}\sum_{k<l} y_{ij}\Gamma^{-1}(i, j; k, l)y_{kl} \\ &= \tfrac{1}{2}\,\mathrm{tr}(YP^{-1}YP^{-1}) - dg'(P^{-1}Y)T^{-1}dg(P^{-1}Y) \end{aligned} \tag{3.9}$$

[3] See [13, p. 210].

where $\mathrm{dg}(A)$ denotes the diagonal of a square matrix A written as a column vector.

4. DERIVATION OF THE TEST

Let R_1 and R_2 be sample correlation matrices based on independent samples of sizes n_1 and n_2 from two p-variate normal populations with correlation matrices P_1 and P_2. If P_1 and P_2 have a common non-singular value P, then as $n_1 \to \infty$ and $n_2 \to \infty$,

$$\sqrt{n_1}(R_1 - P) \xrightarrow{L} Y \quad \text{and} \quad \sqrt{n_2}(R_2 - P) \xrightarrow{L} Y \tag{4.1}$$

where Y is a symmetric zero diagonal random matrix whose super-diagonal components have a multivariate normal distribution with mean zero and non-singular covariance matrix Γ. It follows from the independence of the samples that

$$X = \left(\frac{n_1 n_2}{n_1 + n_2}\right)^{1/2} (R_1 - R_2) \xrightarrow{L} Y \tag{4.2}$$

as $n_1, n_2 \to \infty$. Moreover,

$$\overline{R} = (n_1 R_1 + n_2 R_2)/(n_1 + n_2) \xrightarrow{L} P \tag{4.3}$$

and

$$(X, \overline{R}) \xrightarrow{L} (Y, P). \tag{4.4}$$

Since $\chi^2(X, \overline{R})$ is a continuous function of $(X, \overline{R})$, at least for $\overline{R}$ non-singular,

$$\chi^2 = \chi^2(X, \overline{R}) \xrightarrow{L} \chi^2(Y, P). \tag{4.5}$$

It follows from (3.9) that χ^2 has an asymptotic χ^2-distribution with $p(p-1)/2$ degrees of freedom whenever $P_1 = P_2 = P$. If $P_1 \neq P_2$ then X, and hence χ^2, approach infinity as $n_1, n_2 \to \infty$. Thus a test based on χ^2 is sensitive, asymptotically at least, to all departures from the null hypothesis $P_1 = P_2$.

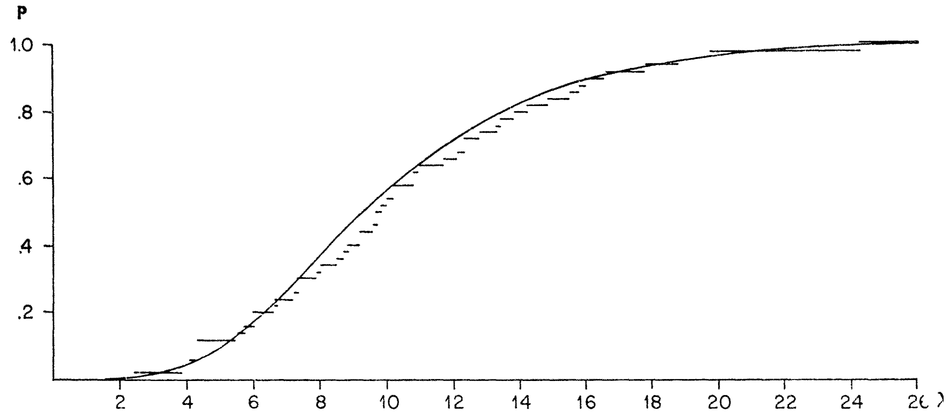
Summarizing the computation of χ^2, let $\overline{R} = (\bar{r}_{ij})$, $S = (\delta_{ij} + \bar{r}_{ij}\bar{r}^{ij})$, $c = n_1 n_2/(n_1 + n_2)$ and $Z = c^{1/2}\overline{R}^{-1}(R_1 - R_2)$. Then

$$\chi^2 = \tfrac{1}{2}\,\mathrm{tr}(Z^2) - \mathrm{dg}'(Z)S^{-1}\mathrm{dg}(Z). \tag{4.6}$$

If the sample correlations in (4.6) are replaced by sample covariances, the first term on the right is a standard asymptotic χ^2 statistic for testing the equality of two covariance matrices. Thus the second term may be viewed as a correction employed when testing correlation matrices. It is also the case that the first term on the right in (4.6) is asymptotically equal to Kullback's statistic, and again the second term may be viewed as a correction.

To get some idea of the applicability of our asymptotic theory to moderate-size samples we will look at two simulation examples. To obtain some feeling for its power we will use it to seek a significant difference between two real-data correlation matrices.

Figure 1. EXACT χ^2 DISTRIBUTION WITH 10 df AND SAMPLE DISTRIBUTION BASED ON A SAMPLE SIZE 50, $n_1=n_2=100, p=5$



5. SIMULATION OF THE χ^2 DISTRIBUTION

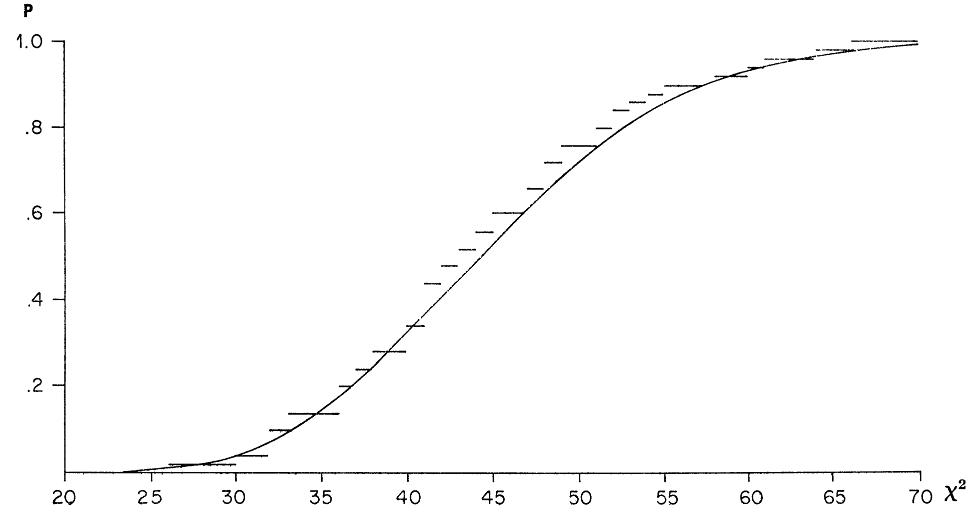
Two samples of 100 pseudo-random vectors were drawn from a normal population with mean zero and a 5×5 covariance matrix, Σ. The components σ_{ij} of Σ were equal to 1 for $i=j$ and 1/2 otherwise. The samples were used to construct sample correlation matrices R_1 and R_2 and these in turn were used to compute χ^2 as given by (4.6). The entire process was repeated 50 times giving 50 sample values of χ^2. The sample distribution and the theoretical distribution of a χ^2 variable with 10 degrees of freedom are plotted in Figure 1. The maximum deviation, $D=.05$, is well below the 20 percent point, $D_{.20}=.15$, of the Kolomogorov-Smirnov statistic for a sample of size 50. To the eye, the fit looks quite good.

The distribution of the first term in (4.6) is invariant under rotations of the two sample spaces. If the second term were also invariant (it is not), the distribution of χ^2 would be totally determined by n_1, n_2, and the eigen values of Σ which in the previous example were 3, 1/2, 1/2, 1/2, 1/2. This suggests looking at an example with these parameters altered. This was done by letting $n_1=50$, $n_2=150$ and Σ be the 10 by 10 matrix, shown in Figure 2, which has eigen values 5/10, 6/10, 7/10, 8/10, 9/10, 11/10, 12/10, 13/10, 14/10, 15/10. The resulting sample distribution of χ^2 and the theoretical distribution of a χ^2 variable with 45 degrees of freedom are given in Figure 3. Again the maximum deviation D is well below $D_{.20}=.15$ and the fit looks good.

Figure 2. COVARIANCE MATRIX FOR THE SECOND SIMULATION STUDY

$$\begin{bmatrix}
1 & .1 & & & & & & & & \\
.1 & 1 & & & & & & & & \\
& & 1 & .2 & & & & & & \\
& & .2 & 1 & & & & & & \\
& & & & 1 & .3 & & & & \\
& & & & .3 & 1 & & & & \\
& & & & & & 1 & .4 & & \\
& & & & & & .4 & 1 & & \\
& & & & & & & & 1 & .5 \\
& & & & & & & & .5 & 1
\end{bmatrix}$$

Figure 3. EXACT χ^2 DISTRIBUTION WITH 45 df AND SAMPLE DISTRIBUTION BASED ON A SAMPLE OF SIZE 50, $n_1=50$, $n_2=150$, $p=10$



6. TESTING THE EQUALITY OF TWO-FACTOR PATTERNS

The initial stimulus for this work came from the problem of comparing two factor patterns from factor analyses of data from retarded and normal children. The problem was to determine if observed differences in the sample factor patterns were significant. When the population factor patterns are identifiable in the sense of Anderson and Rubin [3], this problem is equivalent to that of determining when differences in observed correlation matrices are significant.

The table gives sample correlation matrices obtained from a group of 40 retarded and 89 normal children reported in [9]. The twelve variables included a variety of tests of physical and mental ability. Evaluating the χ^2-statistic

CORRELATION MATRICES FOR RETARDED AND NORMAL CHILDREN[a]

Test→ ↓	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
1	...	465	069	403	483	298	243	203	118	414	192	245
2	543	...	153	383	428	232	166	243	001	193	192	314
3	560	684	...	178	176	236	104	125	−028	190	081	184
4	532	414	484	...	793	638	471	429	280	348	261	391
5	526	485	503	870	...	561	494	433	407	296	335	445
6	498	602	614	755	763	...	426	425	215	287	266	395
7	444	354	521	500	572	576	...	660	484	177	301	267
8	336	323	264	372	423	433	611	...	423	276	363	314
9	399	427	514	493	601	612	793	542	...	107	309	092
10	398	648	491	560	662	680	466	288	549	...	225	343
11	568	527	546	619	672	650	503	520	646	651	...	280
12	654	674	712	527	616	651	570	508	663	561	636	...

[a] Intercorrelations among the 12 tests are found above the diagonal for the normal group, below the diagonal for the retarded group. Decimal points are omitted.

given in (4.6) gives a value of 74.8 which, with 66 degrees of freedom, is not significant at the 5 percent level. The lack of statistical significance appears to be due in part to the similarity of the observed correlation matrices but perhaps more importantly to the relatively small size, 40, of the sample of retarded children. Had this sample been the same size as that of the normal group, the same correlation matrices would have yielded a χ^2 value of 120 which is easily significant at the 1 percent level.

7. RELATED TESTS

Kullback has suggested that the author use the results of Section 3 to obtain asymptotic χ^2 tests for two additional and related hypotheses. The first is the hypothesis that a single population has a specified correlation matrix and the second, the hypothesis that two or more populations have equal correlation matrices.

Let R be the sample correlation matrix of a sample of size n from a p-variate normal population with population correlation matrix $P=(\rho_{ij})$. Using (3.9) and simple asymptotic arguments similar to those of Section 4, it is easy to show that

$$\chi^2 = \tfrac{1}{2}\,\mathrm{tr}(Z^2) - \mathrm{dg}'(Z)T^{-1}\mathrm{dg}(Z) \tag{7.1}$$

when $Z=\sqrt{n}P^{-1}(R-P)$ and $T=(\delta_{ij}+\rho_{ij}\rho^{ij})$ has an asymptotic χ^2 distribution with $p(p-1)/2$ degrees of freedom. Large observed values of this statistic suggest rejection of the hypothesis that the sampled population has P as its correlation matrix.

To obtain a test for the homogeneity of a set of correlation matrices, let $R_1, \cdots, R_k$ be sample correlation matrices based on samples of size $n_1, \cdots, n_k$ from k, p-variate, normal populations which have the same, but unknown, correlation matrix. Let $n=n_1+\cdots+n_k$, $\bar{R}=(n_1R_1+\cdots+n_kR_k)/n=(\bar{r}_{ij})$, $S=(\delta_{ij}+\bar{r}_{ij}\bar{r}^{ij})$ and $Z_i=\sqrt{n_i}\bar{R}^{-1}(R_i-\bar{R})$. Then,

$$\chi^2 = \sum_{i=1}^{k} \left(\tfrac{1}{2}\,\mathrm{tr}(Z_i^2) - \mathrm{dg}'(Z_i)S^{-1}\,\mathrm{dg}(Z_i)\right) \tag{7.2}$$

has an asymptotic χ^2 distribution with $(k-1)p(p-1)/2$ degrees of freedom. Large values of χ^2 suggest rejection of the hypothesis that all k populations have the same correlation matrix. The assertion that the statistic has an asymptotic χ^2 distribution is true in the strongest sense. As in Section 4, the statement holds as long as each $n_i \to \infty$. No assumptions are made about the relative rates of increase of the n_i. An interesting consequence of this is that although the sum in equation (7.2) converges in distribution to an χ^2 variable, the terms in the sum need not converge to χ^2 variables. Indeed, they need not converge at all.

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