Name: Songze Yang
Uni ID: u7192786
Question 1:

Question 1 [2 marks]

Let $\mathbb{x}$ be a random vector with multivariate Gaussian distribution $N_p(0, \Gamma)$. Show that if $\text{rank}(\Gamma) = p$ then

$$\mathbb{x}'\Gamma^{-1}\mathbb{x} \sim \chi^2(p),$$

where $\chi^2(p)$ denotes the chi-squared distribution with p degrees of freedom.

The covariance matrix is Γ positive definite or semidefinite, $\Gamma \succeq 0$

When the $\text{rank}(\Gamma) = p$ then Γ matrix is invertible and

the eignevalue of Γ : $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_p > 0$ to make Γ full rank.

By spectral decomposition: $\Gamma e = \lambda e \Rightarrow \Gamma e_i = \lambda_i e_i$

From the fact that $\Gamma^{-1} = \sum_{i=1}^{p} \lambda_i^{-1} \underset{(p\times 1)}{e_i^T} \underset{(1\times p)}{e_i}$, we know $\Gamma^{-1} e_i = \lambda_i^{-1} e_i$

Thus $(x-0)^T \Gamma^{-1}(x-0) = x^T \Gamma^{-1} x = \sum_{i=1}^{p} \frac{1}{\lambda_i} \underset{(p\times 1)}{x} \underset{(1\times p)}{e_i^T} \underset{(p\times 1)}{e_i} \underset{(1\times p)}{x^T}$

$$= \sum_{i=1}^{p} \left(1/\sqrt{\lambda_i}\, e_i\, x\right)^2$$

Denote $\underset{(p\times p)}{z_i} = 1/\sqrt{\lambda_i}\, \underset{p\times 1}{e_i} \underset{1\times p}{x}$, thus the above is $\sum_{i=1}^{p} z_i^2$

In vector form we get $\mathbb{Z} = \underset{p\times p}{A^T} \underset{p\times 1}{X}$, $\mathbb{Z} \in \mathbb{R}^{p\times 1}$

$$\underset{(p\times p)}{A} = \begin{bmatrix} 1/\sqrt{\lambda_1}\, e_1 \\ 1/\sqrt{\lambda_2}\, e_2 \\ \vdots \\ 1/\sqrt{\lambda_n}\, e_p \end{bmatrix}$$

We know $x \sim N(0, \Gamma)$, thus $\mathbb{Z} = Ax \sim N_p(0, A^T \Gamma A)$

$$\underset{(p\times p)}{A^T} \underset{(p\times p)}{\Gamma} \underset{(p\times p)}{A}$$

$$= \left[1/\sqrt{\lambda_1}\, e_1, 1/\sqrt{\lambda_2}\, e_2, \dots, 1/\sqrt{\lambda_p}\, e_p\right] \Gamma \begin{bmatrix} 1/\sqrt{\lambda_1}\, e_1 \\ 1/\sqrt{\lambda_2}\, e_2 \\ \vdots \\ 1/\sqrt{\lambda_p}\, e_p \end{bmatrix}$$

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$$= \underbrace{\left[1/\sqrt{\lambda_1}\, e_1, 1/\sqrt{\lambda_2}\, e_2, \ldots, 1/\sqrt{\lambda_p}\, e_p \right]}_{(p \times p)} \left[\sum_{i=1}^{p} \lambda_i e_i^T e_i \right] \underbrace{\begin{bmatrix} 1/\sqrt{\lambda_1}\, e_1 \\ 1/\sqrt{\lambda_2}\, e_2 \\ \vdots \\ 1/\sqrt{\lambda_p}\, e_p \end{bmatrix}}_{(p \times p)}$$

$$= \left[\sqrt{\lambda_1}\, e_1, \sqrt{\lambda_2}\, e_2, \ldots, \sqrt{\lambda_p}\, e_p \right] \cdot \underbrace{\begin{bmatrix} 1/\sqrt{\lambda_1}\, e_1 \\ 1/\sqrt{\lambda_2}\, e_2 \\ \vdots \\ 1/\sqrt{\lambda_p}\, e_p \end{bmatrix}}_{(p \times p)}$$

$$= I$$

Thus $Z \sim N(0, I)$ and

$Z_1, Z_2, \ldots Z_p$ are independent standard normal variables.

Thus by definite of chi-squared distribution;

$$\sum_{i=1}^{p} Z_p^2 \sim \chi^2(p)$$

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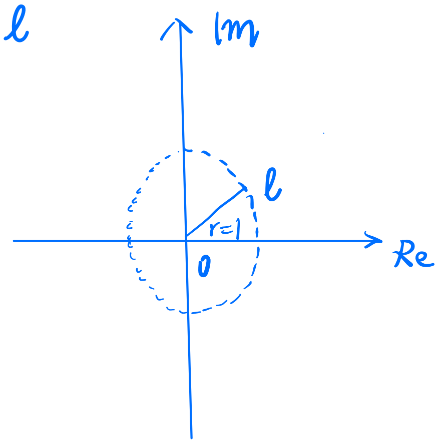
Question 3:

Question 3 [2 marks]
Using contour integration, calculate the following integral

$$\oint_{|z|=1} \frac{4+z}{z+z^2}\,dz.$$

Denote the unit circle $|z|=1$ as a contour ℓ



$z(t) = e^{it}$, $t \in [0, 2\pi]$, then $\frac{dz}{dt} = ie^{it}$

$$\oint_{\ell} \frac{4+z}{z+z^2}\,dz = \int_0^{2\pi} \frac{4+e^{it}}{e^{it}+(e^{it})^2} \cdot ie^{it}\,dt$$

$$= i \int_0^{2\pi} \frac{4+e^{it}}{1+e^{it}}\,dt$$

$$= i \int_0^{2\pi} \frac{3}{1+e^{it}}\,dt + i \int_0^{2\pi} 1\,dt$$

$$= 3i \int_0^{2\pi} \frac{1}{1+e^{it}}\,dt + it\Big|_0^{2\pi}$$

$$\frac{1}{1+e^{it}} - e^{it}$$

$$= 3i \int_0^{2\pi} \frac{1+e^{it}-e^{it}}{1+e^{it}}\,dt + (2\pi i - 0 \cdot i)$$

$$= 3i \int_0^{2\pi} 1\,dt - 3i \int_0^{2\pi} \frac{e^{it}}{1+e^{it}}\,dt + 2\pi i$$

$$= 3it\Big|_0^{2\pi} + 2\pi i - 3 \int_0^{2\pi} \frac{ie^{it}}{1+e^{it}}\,dt$$

$$= 3i(2\pi - 0) + 2\pi i - 3 \ln(1+e^{it})\Big|_0^{2\pi}$$

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$$= 8\pi i - \left[3\ln(1+e^{2\pi i}) - 3\ln(1+e^{i0})\right]$$

$$= 8\pi i - \left[3\ln(1+e^{2\pi i}) - 3\ln 2\right]$$

$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi)$$

$$= 1 + i \cdot 0$$

$$= 1$$

$$= 8\pi i - [3\ln(1+1) - 3\ln 2]$$

$$= 8\pi i$$

Method 2: Contour by Residuals (Cannot apply, see the last page)

Denote the unit circle $|z| = 1$ as a contour ℓ

Denote $f(z) = \dfrac{4+z}{z+z^2}$

$$\oint_\ell f(x)\,dz = 2\pi i \sum_{a \in \ell} \text{Res}(f; a)$$

Singularities a: $f(z) = \dfrac{4+z}{z+z^2} = \dfrac{4+z}{z(z+1)}$

$z = 0$ and $z = -1$ both with order $n=1$

Define $g(z) = (z-a)^n f(z)$ then

$$\text{Res}(f;a) = \frac{1}{(n-1)!} \lim_{z \to a} g^{(n-1)}(z)$$

For residual $a = 0$:

$$g(z) = (z-0)^1 f(z) = z \cdot \frac{4+z}{z+z^2}$$

$$\frac{1}{(n-1)!} \lim_{z \to 0} g^{(n-1)}(z) = \frac{1}{(1-1)!} \lim_{z \to 0} g(z) = 1 \cdot \lim_{z \to 0} \frac{4+z}{z+1}$$

$$= \frac{4}{1} = 4$$

$$\text{Res}(f(x); 0) = 4$$

For residual $a = -1$:

$$g(z) = [z-(-1)]^1 f(z) = (z+1) \frac{4+z}{z(z+1)} = \frac{4+z}{z}$$

$$\frac{1}{(n-1)!} \lim_{z \to -1} g^{(n-1)}(z) = 1 \cdot \lim_{z \to -1} g(z) = \lim_{z \to -1} \frac{4+z}{z} = \frac{4+(-1)}{-1}$$

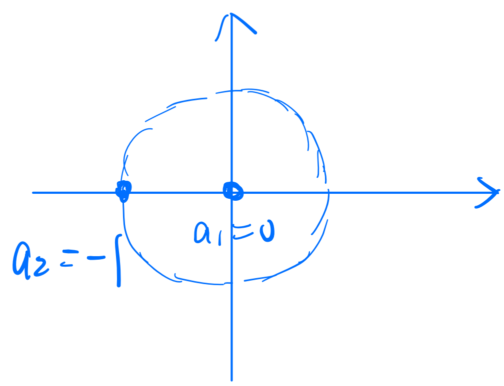
$$= -3$$

$\mathrm{Res}(f(x), -1) = -3$

$$\oint_{\ell} f(x)\, dz = 2\pi i \sum_{a \in \ell} \mathrm{Res}(f; a) = 2\pi i\,(4-3) = 2\pi i$$

We can see the answer does not match

This is simply because we cannot apply the residual theorm as our residuals:



When the residual $a_2 = -1$ lies a the boundary, the integral is the undefined.