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Surveillance of the covariance matrix based on the properties of the singular Wishart distribution

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ABSTRACT

A methodology which allows applying the standard monitoring techniques for the mean behaviour of Gaussian processes in the detection of shifts in the covariance matrix is developed. Moreover, the proposed methodology allows the use of an estimator of the covariance matrix based on a single observation. An extensive simulation study reveals the advantages of the considered approach.

1. Introduction

Statistical process control methods play an important role in quality improvement. They have been widely applied in Engineering for a long time. The aim is to detect a structural change in the process of interest as soon as possible after its occurrence. However, this problem setting is also important in other fields and recently several papers discussed the applications of sequential procedures in Economics, Medicine, Chemistry, and Finance (see Frisén (1992), Sonesson and Bock (2003), Lawson and Kleinman (2005), Schipper and Schmid (2001), Andersson et al. (2004), Schmid and Tzotchev (2004), Bodnar (2007) and Messaoud et al. (2008) among others). This leads to more advanced models with time dependent and multivariate processes.

The main tools of statistical process control are control charts. A control chart is characterized by a control statistic which is updated by using current information at each time point. When the control statistic exceeds the preselected value the chart signals an alarm, suggesting that a change in the parameters of the underlying process has occurred. The first control chart for the mean of an independent Gaussian process was proposed by Shewhart (1931). Later on, Page (1954) and Roberts (1959) derived control charts with memory, the so-called CUSUM and EWMA control charts respectively.

The extension of the univariate control charts to the multivariate case is not trivial and can be implemented in many alternative ways. The first multivariate control scheme for the mean was proposed by Hotelling (1947), which is a multivariate generalization of the univariate scheme of Shewhart (1931). The EWMA chart was extended by Lowry et al. (1992). Since the univariate CUSUM control scheme was derived from the sequential probability ratio test (SPRT) its generalization to the multivariate case is not straightforward. In the multivariate case the CUSUM chart depends on the size and direction of the expected shift. For that reason several authors proposed schemes for the mean which exclusively depend on the magnitude of the expected shift like, Crosier (1988), Pignatiello and Runger (1990) and Ngai and Zhang (2001).

Note that all the above mentioned charts are developed to detect shifts in the mean behaviour of the process. Application to other areas makes the monitoring of the variance or the covariance matrix increasingly important. In general we can apply the techniques of the mean charts to different volatility measures (Okhrin and Schmid, 2008a,b). Usually squared

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observations, their logarithms or other transformations are used. This leads, however, to non-standard distributions of the control statistics and substantially complicates the monitoring process. This fact is even more critical in the multivariate case due to a large number of components in the covariance matrix. Moreover, note that the discussion is most of the time restricted to EWMA schemes (see Yeh et al. (2005), Śliwa and Schmid (2005), Reynolds and Cho (2006) and Huwang et al. (2007)). The derivation of CUSUM charts for covariance matrices is technically difficult. For this reason, the few papers dealing with this type of charts translate the univariate schemes for the mean into multivariate schemes for variance measures (see Hawkins (1981, 1991)). Chan and Zhang (2001) suggested a CUSUM-type control chart for the covariance matrix of independent observations based on the projection pursuit method.

In this paper we introduce a new technique, which allows us to apply the standard EWMA or CUSUM control charts for the mean directly to monitor the covariance matrix of a Gaussian vector. Moreover, the method can be applied to the estimators of the covariance matrix based on a single recent observation. The idea of the approach is to transform the variance estimator to a set of Gaussian vectors. This can be done by using properties of the Wishart and singular Wishart distributions. The shift in the variance of the original process would cause a shift in the mean of the transformed quantities. This implies that monitoring the variance of the original process is equivalent to monitoring the mean of the Gaussian vectors. For the latter problem the classical control charts can be applied.

The rest of the paper is organized as follows. In the next section the change point model is presented. Then we establish the necessary properties of the singular Wishart distribution. Furthermore, we review the multivariate control charts used for monitoring. The simulation study in Section 4 compares the modified charts for the covariance matrix with the charts based on the transformed Gaussian quantities introduced in this paper. The proofs are given in the Appendices A and B.

2. Models for the target and the observed processes

In order to define the control problem we distinguish the target process and the observed process. The target process is the process that fulfills the quality requirements. Usually, it depends on some parameters, that can be set to corresponding target values (industry) or estimated using previous data (economics, finance). The observed process is the actual process that we observe in practice. Our task is to decide if the observed and the target processes coincide.

We denote the target process by $\{\mathbf{Y}_t\}$. It is assumed that $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ are identical and independently normally distributed with $\mathbf{Y}_i \sim \mathcal{N}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$, where the covariance matrix can be written as $\boldsymbol{\Sigma}_1 = \mathbf{D}_1\mathbf{R}_1\mathbf{D}_1$. The matrices $\mathbf{D}_1$ and $\mathbf{R}_1$ are the diagonal matrix of standard variances and the correlation matrix, respectively. All parameters of the target process are assumed to be known. In practice, however, especially in economic applications, the parameters of the target process are estimated. This causes an additional estimation risk. The impact of parameter uncertainty in the target process is beyond the scope of our paper. The interested reader should refer to Kramer and Schmid (2000) and Albers and Kallenberg (2004) where such problems are discussed. Without loss of generality we assume that $\boldsymbol{\mu}_1 = \mathbf{0}$.

We denote the observed process by $\{\mathbf{X}_t\}$. To model the observed process we apply the change point framework, i.e.

$$\mathbf{X}_t \sim \begin{cases} \mathcal{N}_p(\mathbf{0}, \mathbf{D}_1\mathbf{R}_1\mathbf{D}_1), & t < q \\ \mathcal{N}_p(\mathbf{0}, \mathbf{D}_\Delta\mathbf{R}_\Delta\mathbf{D}_\Delta), & t \geq q \end{cases}, \tag{1}$$

where $\mathbf{D}_\Delta$ and $\mathbf{R}_\Delta$ are the diagonal matrix of standard variances and the correlation matrix after the change. $\mathbf{D}_\Delta$, $\mathbf{R}_\Delta$, and $q \in N$ are unknown quantities. In the case $q < \infty$ we say that there is a change at the time point q. In the case of no change the target process coincides with the observed process. We say that the observed process is in control, i.e. $\mathbf{X}_t = \mathbf{Y}_t$. The in-control covariance matrix $\boldsymbol{\Sigma}_1$ is called the target covariance matrix. If the change occurs the process is said to be out of control. The change can be in the variances, in the correlations or in both quantities simultaneously. For changes in the variance we assume that $\mathbf{D}_\Delta = \boldsymbol{\Delta}\mathbf{D}_1$ with a diagonal matrix of shifts $\boldsymbol{\Delta}$, for changes in the correlations we set $\mathbf{R}_\Delta = \boldsymbol{\Delta}\mathbf{R}_1$ where $\boldsymbol{\Delta}$ is a positive definite correlation matrix. For changes in both quantities we assume that $\boldsymbol{\Sigma}_\Delta = \boldsymbol{\Delta}\boldsymbol{\Sigma}_1$ where $\boldsymbol{\Delta}$ is a symmetric positive definite matrix. If $\boldsymbol{\Delta}$ is an identity matrix we have no shifts. Note that $\mathbf{X}_t = \mathbf{Y}_t$ for $t \leq 0$, i.e. both processes are the same up to time point $q-1$. In the theoretical statements of Section 3 we assume that the shift occurs just before we collect the first sample, i.e. $q = 1$. In the simulation study in Section 4 we consider the maximum average delay as a performance measure of control carts. Therefore, here the shift is assumed to occur at an arbitrary time point q (see (20) and (21)).

3. Control charts for the covariance matrix

As an estimator of the covariance matrix we use the point estimator based on a single observation, i.e. at time point t the covariance matrix is estimated by $\mathbf{V}_t = \mathbf{X}_t\mathbf{X}_t'$. The matrix $\mathbf{V}_t$ follows a singular Wishart distributed $\mathbf{V}_t \sim W_p(1, \boldsymbol{\Sigma})$ (see, e.g. Srivastava (2003)), where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$ if the process is in control and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\Delta$ if the process is out of control. Its rank is equal to one with probability one in both cases. It is not new to exploit the unbiased point estimator $\mathbf{V}_t$ for monitoring purposes. Yeh et al. (2005) used $\mathbf{V}_t$ to update the matrix variate EWMA recursion.

However, the standard control schemes cannot be directly applied to the point estimator $\mathbf{V}_t$. For the EWMA chart, the variance of the control statistic has to be computed and this is a nontrivial task. Moreover, because the distribution of the control statistic is not symmetric, the two-sided EWMA chart for the variance depends on two critical values (see, e.g. Chan and Zhang (2001)). This generates additional computational difficulties. In this paper we use the properties of the singular Wishart distribution to transform $\mathbf{V}_t$ to a set of Gaussian vectors. Then the mean charts can be immediately applied to monitoring the shifts in the variance.

The important distributional properties of the singular Wishart distribution are summarized in Theorem 1. Let $\sigma_{1;ii}$ denotes the (i,i)th element of the matrix $\mathbf{\Sigma}_1$, $i = 1, \ldots, p$. By $\mathbf{\Sigma}_{1;21,i}$ we denote the ith column of the matrix $\mathbf{\Sigma}_1$ without $\sigma_{1;ii}$. Let $\mathbf{\Sigma}_{1;22,i}$ denote a quadratic matrix of order $p-1$, which is obtained from the matrix $\mathbf{\Sigma}_1$ by deleting the ith row and the ith column. Let $\mathbf{\Sigma}_{1;22\cdot1,i} = \mathbf{\Sigma}_{1;22,i} - \mathbf{\Sigma}_{1;21,i}\mathbf{\Sigma}'_{1;21,i}/\sigma_{1;ii}$. In the same way we define $v_{t;ii}$, $\mathbf{V}_{t;21,i}$, $\mathbf{V}_{t;22,i}$, $\mathbf{V}_{t;22\cdot1,i}$, $\sigma_{\Delta;ii}$, $\mathbf{\Sigma}_{\Delta;21,i}$, $\mathbf{\Sigma}_{\Delta;22,i}$, and $\mathbf{\Sigma}_{\Delta;22\cdot1,i}$ by splitting $\mathbf{V}_t$ and $\mathbf{\Sigma}_\Delta$ respectively.

Theorem 1. *Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be an identically independently distributed Gaussian process with $\mathbf{Y}_i \sim \mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma}_1)$. We assume that the observed process $\{\mathbf{X}_t\}$ is defined according to the model (1). Then*

(a) *in the in-control state*

$$\boldsymbol{\eta}_{i,t} = \mathbf{\Sigma}_{1;22\cdot1,i}^{-1/2}(\mathbf{V}_{t;21,i}/v_{t;ii} - \mathbf{\Sigma}_{1;21,i}/\sigma_{1;ii})v_{t;ii}^{1/2} \sim \mathcal{N}_{p-1}(\mathbf{0}_{p-1}, \mathbf{I}_{p-1})$$

and is independent of $v_{t;ii}$;

(b) *in the out-of-control state*

$$E(\boldsymbol{\eta}_{i,t}) = \mathbf{\Sigma}_{1;22\cdot1,i}^{-1/2}\mathbf{\Omega}_i\sigma_{1;ii}^{1/2}\frac{\sqrt{2}}{\sqrt{\pi}},$$

$$Var(\boldsymbol{\eta}_{i,t}) = \mathbf{\Sigma}_{1;22\cdot1,i}^{-1/2}\left(\mathbf{\Sigma}_{\Delta;22\cdot1,i} + \mathbf{\Omega}_i\sigma_{1;ii}(1-2\pi^{-1})\mathbf{\Omega}'_i\right)\mathbf{\Sigma}_{1;22\cdot1,i}^{-1/2},$$

where $\mathbf{\Omega}_i = \mathbf{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \mathbf{\Sigma}_{1;21,i}/\sigma_{1;ii}$.

Moreover, the process $\{\boldsymbol{\eta}_{i,t}\}$ is independent in time both in the in-control and in the out-of-control states.

For the proof of the first part of the theorem we refer to Bodnar and Okhrin (2008). The proof of the second part is given in the Appendix A.

In the first part of Theorem 1 it is shown that $\boldsymbol{\eta}_{i,t}$ follows standard normal distribution for each i and t. Furthermore, it holds that $\boldsymbol{\eta}_{i,t_1}$ and $\boldsymbol{\eta}_{i,t_2}$ are independent as long as $\mathbf{X}_{t_1}$ and $\mathbf{X}_{t_2}$ are independent. In the out-of-control state we observe a change in the mean vector of $\boldsymbol{\eta}_{i,t}$ if $\mathbf{\Omega}_i \neq \mathbf{0}$. This is, however, not the case if we consider the change point model as in (1). Thus, the transformation $\boldsymbol{\eta}_{i,t}$ translates the shift in the covariance matrix into a shift in the mean. Note that the covariance matrix and the distribution of $\boldsymbol{\eta}_{i,t}$ change in the out-of-control state too.

Remark 1. The distribution of $\boldsymbol{\eta}_{i,t}$ is no longer multivariate normal if we have a change in the mean vector of $\mathbf{Y}_t$, i.e. $\mathbf{X}_t = \boldsymbol{\mu} + \mathbf{Y}_t$. To show this point, we assume for simplicity, that there is only a change in the mean vector and the covariance matrix remains the same in the out-of-control state. Then, the density of $\boldsymbol{\eta}_{1;t} = \mathbf{\Sigma}_{22\cdot1}^{-1/2}(\mathbf{V}_{t;21}/v_{t;11} - \mathbf{\Sigma}_{21}/\sigma_{11})v_{t;11}^{1/2}$ is given by

$$f_{\boldsymbol{\eta}_1}(\mathbf{z}) = \frac{\pi^{-k/2}2^{-(k-2)/2}}{\sigma_{11}^{1/2}} \int_0^\infty \exp\left(-\frac{1}{2}(\sigma_{11}^{-1}t^2 + \mathbf{z}'\mathbf{z})\right)$$
$$\times \cosh\left(\sqrt{\boldsymbol{\mu}'\boldsymbol{\mu}}\sqrt{[t,\quad t\mathbf{\Sigma}'_{21}/\sigma_{11} + (\mathbf{\Sigma}_{22\cdot1}^{1/2}\mathbf{z})']\mathbf{\Sigma}^{-2}[t,\quad t\mathbf{\Sigma}_{21}/\sigma_{11} + \mathbf{\Sigma}_{22\cdot1}^{1/2}\mathbf{z}]'}\right) \mathrm{d}t,$$

where cosh(.) denotes the hyperbolic cosine function. The derivation of the density function is given in the Appendix B.

A similar result holds for other vectors $\boldsymbol{\eta}_{i,t}$. Hence, the control statistics, that are constructed using $\boldsymbol{\eta}_{i,t}$ will be sensitive to both shifts in the covariance matrix and shifts in the mean. In the multivariate setup we rarely have a justified reason to expect shifts only in the mean or only in the covariance matrix. This makes the diagnostic of the alarms generated by the schemes based on $\boldsymbol{\eta}_{i,t}$ particularly difficult. Nevertheless, note that a vast majority of the control charts for the variability are sensitive to the mean shifts too. This argument is supported by the results of the simulation study (see Section 4). The only exception is the MEWMV chart of Huwang et al. (2007). The chart successfully eliminates the impact of potential shifts in the mean by subtracting from the process the EWMA-type estimator of the mean. Unfortunately, due to technical difficulties we were unable to generalize the above theoretical results to the setup of Huwang et al. (2007), we provide some further discussion of this procedure in Section 4. Furthermore, note that using the developed methodology for detecting shifts only in the mean is not very promising, since there are numerous well established and much simpler charts developed for this type of changes. Therefore, in this paper we focus only on the shifts in the covariance matrix.

The normality, time independency and the shift in the mean caused by the shift in the covariance matrix allow us to use the standard multivariate EWMA and CUSUM control schemes of Crosier (1988), Pignatiello and Runger (1990), Lowry et al. (1992), and Ngai and Zhang (2001) to monitor changes in the means of the processes $\{\boldsymbol{\eta}_{i,t}\}$. Note that we have p vectors $\boldsymbol{\eta}_{i,t}$ and each of them should be monitored. To cope with this problem we follow the approach of Woodall and Ncube (1985). The simultaneous use of univariate control charts for each i is considered to be a single joint control chart. This control chart gives an alarm if at least one of the individual charts signals a shift. The Sections 3.2 and 3.3 contain technical details of the implementation of these charts.

As benchmark we consider the control charts which monitor the covariances of the process directly. A detailed discussion of the EWMA-type charts of Yeh et al. (2005), Śliwa and Schmid (2005), Reynolds and Cho (2006) and Huwang et al. (2007) as well as of the CUSUM-type chart of Chan and Zhang (2001) are given in Section 3.4. There are a few charts dealing with CUSUM procedures for the covariance matrices. This is due to the fact that the CUSUM schemes are either derived from the sequential probability ratio test or simply mimic the univariate charts. In the former case no theoretical results are available for monitoring the covariances. In the latter case, however, the problem of the direction of the shift is crucial. These drawbacks hinder the application of multivariate CUSUM charts to covariance monitoring. Moreover, since the chart of Yeh et al. (2005) outperforms the CUSUM scheme of Hawkins (1991), we concentrate exclusively on EWMA benchmarks with the only exception of the PPCUSUMv chart of Reynolds and Cho (2006).

3.1. T^2 control chart

The first multivariate control chart for independent observation was derived by Hotelling (1947). Here this control scheme can be applied to the processes $\{\boldsymbol{\eta}_{i,t}\}$. The control statistic is given by

$$T_t^2 = \max_{1\le i\le p}\{T_{i,t}^2\}, \quad \text{where} \quad T_{i,t}^2 = \boldsymbol{\eta}_{i,t}'\boldsymbol{\eta}_{i,t}. \tag{2}$$

Note that the individual control statistics $T_{i,t}^2$ are χ^2-distributed with p degrees of freedom in the in-control state. However, the distribution of the T_t^2 statistic is much more complicated. Unfortunately this classical control chart provided the worst results in the simulation study, and, therefore, we skip it from further discussion.

3.2. CUSUM control charts

Crosier (1988) considers a generalization of the univariate CUSUM chart of Crosier (1986) to the multivariate setting. In this approach the univariate quantities are replaced with vectors. Moreover, instead of shrinking against zero by a multiple of the standard deviation, the multivariate control statistic is scaled along some predetermined direction. Here we apply the CUSUM scheme of Crosier (1988) to the processes $\{\boldsymbol{\eta}_{i,t}\}$. Let $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_p^2}$ be the Euclidean norm of the p-dimensional vector $\mathbf{a}$. Further let $C_{i,t} = \|\mathbf{S}_{i,t-1} + \boldsymbol{\eta}_{i,t}\|$, where

$$\mathbf{S}_{i,t} = \begin{cases} \mathbf{0} & \text{if } C_{i,t} \le k \\ (\mathbf{S}_{i,t-1} + \boldsymbol{\eta}_{i,t})\left(1 - \dfrac{k}{C_{i,t}}\right) & \text{if } C_{i,t} > k \end{cases} \tag{3}$$

for $t \ge 1$ with $\mathbf{S}_{i,0} = \mathbf{0}$. $k > 0$ plays the role of a reference value. $MCUSUM_{i,t}$ is the length of the vector $\mathbf{S}_{i,t}$ defined by

$$MCUSUM_{i,t} = (\mathbf{S}_{i,t}'\mathbf{S}_{i,t})^{1/2} = \max\{0, C_{i,t} - k\}. \tag{4}$$

The individual control chart signals an alarm if $MCUSUM_{i,t}$ exceeds some preselected critical value. To have a unique critical value of the joint control chart we define the control statistic of the joint scheme by

$$MCUSUM_t = \max_{1\le i\le p}\{MCUSUM_{i,t}\}. \tag{5}$$

The scheme gives an out-of-control signal if $MCUSUM_t$ exceeds a preselected control limit h. The value of h is determined within a simulation study.

Pignatiello and Runger (1990) proposed two types of multivariate CUSUM charts, namely MC1 and MC2. Let $\mathbf{S}_{i;m,l} = \sum_{j=m+1}^{l}\boldsymbol{\eta}_{i,t}$ for $l, m \ge 0$. The control statistic of the individual control charts is defined as

$$MC1_{i,t} = \max\{\|\mathbf{S}_{i;t-n_{i,t},t}\| - kn_{i,t}, 0\}, \quad t \ge 1, \tag{6}$$

where

$$n_{i,t} = \begin{cases} n_{i,t-1} + 1 & \text{if } MC1_{i,t-1} > 0 \\ 1 & \text{if } MC1_{i,t-1} = 0. \end{cases} \tag{7}$$

The decision rule of the joint MC1 scheme is constructed by defining the $MC1$ statistic as follows

$$MC1_t = \max_{1\le i\le p}\{MC1_{i,t}\}. \tag{8}$$

The chart signals an alarm if $MC1_t > h$ for some fixed critical value h. The second chart considered by Pignatiello and Runger (1990) is based on

$$MC2_t = \max_{1\le i\le p}\{MC2_{i,t}\}, \tag{9}$$

where

$$MC2_{i,t} = \max\{0, MC2_{i,t-1} + D_{i,t}^2 - p - k\}, \quad t \geq 1 \tag{10}$$

with $D_{i,t}^2 = \boldsymbol{\eta}_{i,t}'\boldsymbol{\eta}_{i,t}$ and $MC2_{i,0} = 0$. Similarly as for the first chart we observe a signal if $MC2_t > h$.

Using the projection pursuit approach Ngai and Zhang (2001) derived the PPCUSUM chart. Applying this approach to the processes $\{\boldsymbol{\eta}_{i,t}\}$ we obtain the following CUSUM statistic

$$PPCUSUM_t = \max_{1\leq i\leq p}\{PPCUSUM_{i,t}\} \tag{11}$$

with

$$PPCUSUM_{i,t} = \max\{0, \|\mathbf{S}_{i;t-1,t}\| - k, \|\mathbf{S}_{i;t-2,t}\| - 2k, \ldots, \|\mathbf{S}_{i;0,t}\| - tk\} \tag{12}$$

for $t \geq 1$ with $\mathbf{S}_{i;t-v,t}$ defined above. The control scheme gives an alarm as soon as $PPCUSUM_t > h$.

3.3. MEWMA control charts

An alternative class of charts constitute the EWMA schemes. Lowry et al. (1992) proposed a control chart based on the multivariate EWMA recursion. As in the univariate case we define $\mathbf{Z}_{i,t} = r\boldsymbol{\eta}_{i,t} + (1-r)\mathbf{Z}_{i,t-1}$ with $\mathbf{Z}_{i,0} = \mathbf{0}$. We distinguish between the use of the exact or of the asymptotic variance of $\mathbf{Z}_{i,t}$. The control statistic of the individual charts is then defined by

$$Q_{i,t} = \frac{2-r}{r(1-(1-r)^{2t})}\mathbf{Z}_{i,t}'\mathbf{Z}_{i,t} \quad \text{and} \quad Qa_{i,t} = \frac{2-r}{r}\mathbf{Z}_{i,t}'\mathbf{Z}_{i,t} \tag{13}$$

respectively. The joint MEWMA statistic is then given by

$$MEWMA_t = \max_{1\leq i\leq p}\{Q_{i,t}\} \quad \text{and} \quad MEWMAa_t = \max_{1\leq i\leq p}\{Qa_{i,t}\}. \tag{14}$$

The second approach of deriving the EWMA chart is based on calculating the Mahalanobis distance of $\boldsymbol{\eta}_{i,t}$ and then applying the EWMA recursion to this distance. The individual control chart in this case is given by

$$QM_{i,t} = rD_{i,t}^2 + (1-r)QM_{i,t} \quad \text{with } QM_{i,0} = p-1. \tag{15}$$

The EWMA chart based on the Mahalanobis distance signals an alarm as soon as the control statistic

$$MEWMAM_t = \max_{1\leq i\leq p}\{QM_{i,t}\} \tag{16}$$

exceeds the preselected control limit.

Remark 2. Note that the design of the suggested control charts can be further simplified. Let $\tilde{\mathbf{X}}_t = \boldsymbol{\Sigma}_1^{-1/2}\mathbf{X}_t$. Then it holds that $\tilde{\mathbf{V}}_t = \tilde{\mathbf{X}}_t\tilde{\mathbf{X}}_t'$ is $W_p(1, \mathbf{0})$ in the in-control state and $W_p(1, \boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{\Sigma}_\Delta\boldsymbol{\Sigma}_1^{-1/2})$ in the out-of-control state. Furthermore, from Theorem 1 we obtain that

$$\tilde{\boldsymbol{\eta}}_{i,t} = \tilde{\mathbf{V}}_{t;21,i}/\tilde{v}_{t;ii}^{1/2} = (-1)^{\mathbf{1}_{\{\tilde{X}_i<0\}}}(\tilde{X}_1, \ldots, \tilde{X}_{i-1}, \tilde{X}_{i+1}, \ldots, \tilde{X}_p)'$$

follows the standard normal distribution, where the symbol $\mathbf{1}_A$ denotes the indicator function of the set A. Using $\tilde{\boldsymbol{\eta}}_{i,t}$ instead of $\boldsymbol{\eta}_{i,t}$ in the definition of the control charts simplifies the calculation of the control statistics. Later on, we refer to the control schemes based on $\tilde{\boldsymbol{\eta}}_{i,t}$ as standardized control charts.

3.4. Benchmark charts

In this section we review the control charts which we use as benchmarks in the simulation study. Śliwa and Schmid (2005) derived the EWMA control charts for detecting changes in the covariance matrix of the Gaussian vector autoregressive (VARMA) process. The Gaussian process with independent realizations is a special case of the VARMA process. Consequently, we can apply their approach to monitor the covariance matrix of $\mathbf{Y}_t$.

Let

$$\boldsymbol{\tau}_t = \left(X_{1,t}X_{1,t}, X_{1,t}X_{2,t}, \ldots, X_{p-1,t}X_{p,t}, X_{p,t}X_{p,t}\right)' = vech(\mathbf{V}_t).$$

It holds that

$$\boldsymbol{\mu}_\tau = E(vech(\mathbf{V}_t)) = \begin{cases} vech(\boldsymbol{\Sigma}_1), & t < q \\ vech(\boldsymbol{\Sigma}_\Delta), & t \geq q. \end{cases}$$

Moreover, Śliwa and Schmid (2005) provided the expression for the in-control covariance matrix of $\boldsymbol{\tau}_t$, which we denote by $Var_1(\boldsymbol{\tau}_t) = \boldsymbol{\Sigma}_\tau$. It is used in the computation of the control statistic.

The multivariate EWMA chart is given by

$$MEWMAo_t = (\tilde{\mathbf{Z}}_t - \boldsymbol{\mu}_\tau)' Cov_1(\tilde{\mathbf{Z}}_t)^{-1} (\tilde{\mathbf{Z}}_t - \boldsymbol{\mu}_\tau), \tag{17}$$

where $\tilde{\mathbf{Z}}_t = r\boldsymbol{\tau}_t + (1-r)\tilde{\mathbf{Z}}_{t-1}$ with $\tilde{\mathbf{Z}}_0 = \boldsymbol{\mu}_\tau$.

For the EWMA chart based on the Mahalanobis distance we consider

$$\tilde{D}_t^2 = (\boldsymbol{\tau}_t - \boldsymbol{\mu}_\tau)' \boldsymbol{\Sigma}_\tau^{-1} (\boldsymbol{\tau}_t - \boldsymbol{\mu}_\tau).$$

The control statistic is given by

$$MEWMAMo_t = r\tilde{D}_t^2 + (1-r)\, MEWMAMo_{t-1} \tag{18}$$

with $MEWMAMo_0 = p(p+1)/2$.

Yeh et al. (2005) applied the EWMA recursion to construct a weighted estimator of the covariance matrix of the normal distribution. The estimator is given by

$$\mathbf{S}_t = r\mathbf{X}_t\mathbf{X}_t' + (1-r)\mathbf{S}_{t-1}, \tag{19}$$

where $r \in (0, 1]$. Let $\mathbf{S}_{1;t}$ and $\mathbf{S}_{2;t}$ denote the stacked diagonal and nondiagonal elements of $\mathbf{S}_t$ respectively, i.e.

$$\mathbf{S}_{1;t} = (s_{11;t}, \ldots, s_{pp;t})', \qquad \mathbf{S}_{2;t} = (s_{12;t}, s_{13;t}, \ldots, s_{ij;t}, , \ldots, s_{p-1p;t})'.$$

The deviations from the target value are squared and cumulated as follows

$$D_{1;t} = \sum_{i=1}^{p} (s_{ii;t} - \sigma_{ii}^{(0)})^2, \qquad D_{2;t} = \sum_{1=i<j=p} (s_{ij;t} - \sigma_{ij}^{(0)})^2.$$

The control statistic in Yeh et al. (2005) is then given by the maximum of the standardized cumulated shift in the variances and in the covariances

$$MaxD_t = \max\left\{\frac{D_{1;t} - \mu_{D_1}}{\sigma_{D_1}}, \frac{D_{2;t} - \mu_{D_2}}{\sigma_{D_2}}\right\},$$

where the asymptotic moments μ_{D_1}, μ_{D_2}, σ_{D_1} and σ_{D_2} can be found in the original paper.

Huwang et al. (2007) suggest another EWMA chart, which detects shifts in the covariance matrix by monitoring its trace. The control statistic of this chart is given by

$$MEWMS_t = \frac{tr(\mathbf{S}_t) - \mu_{tr}}{\sigma_{tr}},$$

where

$$\mu_{tr} = p \quad \text{and} \quad \sigma_{tr}^2 = 2p\left((1-r)^{2(t-1)} + \sum_{i=2}^{t} r^2(1-r)^{2(t-i)}\right).$$

The quantity $\mathbf{S}_t$ is defined as in (19) and $tr(\cdot)$ denotes the trace of the matrix. The control chart signals an alarm if $MEWMS_t$ exceeds a preselected control limit.

While monitoring the variance of the process it is usually assumed that the mean stays constant. In practice it is difficult to verify this assumption. Huwang et al. (2007) suggest a chart for the covariance matrix which is insensitive to the shifts in the mean vector. The control statistics is given by

$$MEWMV_t = r(\mathbf{X}_t - \mathbf{Y}_t)(\mathbf{X}_t - \mathbf{Y}_t)' + (1-r)MEWMV_{t-1},$$

with $\mathbf{Y}_t = \tilde{r}\mathbf{X}_t + (1-\tilde{r})\mathbf{Y}_{t-1}$ and $MEWMV_0 = (\mathbf{X}_1 - \mathbf{Y}_1)(\mathbf{X}_1 - \mathbf{Y}_1)'$ and $\mathbf{Y}_0 = \mathbf{0}$. We can select either equal or different smoothing parameters in both the recursions. In the simulation study we set $\tilde{r} = 0.2$ and optimize the chart with respect to r. This makes the chart comparable with other charts, which depend on a single parameter too. The choice of $\tilde{r}$ is in line with the results of the simulation study by Huwang et al. (2007), where the best results are obtained for $\tilde{r} = 0.2$. The MEWMV chart gives an alarm if $MEWMV_t/\sqrt{Var(MEWMV_t)}$ exceeds some preselected level.

Reynolds and Cho (2006) suggest four control schemes for detecting changes in the covariance matrix that are based on the EWMA recursion. For the first two we define

$$EZ_{i,t} = (1-r)EZ_{i,t-1} + rX_{i,t}^2 \quad i = 1, \ldots, p$$

with $EZ_{i;0} = 1$. Let $\mathbf{\Sigma}_1^{(2)}$ denote the matrix with the elements that are squares of the corresponding elements of $\mathbf{\Sigma}_1$. Then the statistics of the control charts are expressed as

$$M1Z2_t = \frac{2-r}{2r}(EZ_{1,t}-1,\ldots,EZ_{p,t}-1)\left(\mathbf{\Sigma}_1^{(2)}\right)^{-1}(EZ_{1,t}-1,\ldots,EZ_{p,t}-1)',$$
$$M2Z2_t = \frac{2-r}{2r}(EZ_{1,t},\ldots,EZ_{p,t})\left(\mathbf{\Sigma}_1^{(2)}\right)^{-1}(EZ_{1,t},\ldots,EZ_{p,t})'.$$

These charts are computed as Mahalanobis distances of the EWMA recursions applied to the squared or demeaned squared observations.

The other two control charts of Reynolds and Cho (2006) are based on the squared deviations from the target of the regression-adjusted variables (see Hawkins (1991), Hawkins (1993)). The vector of the regression-adjusted variables is given by

$$\mathbf{A}_t = (diag\,\mathbf{\Sigma}_1^{-1})^{-1/2}\mathbf{\Sigma}_1^{-1}\mathbf{X}_t,$$

where $diag\,\mathbf{\Sigma}_1^{-1}$ is a diagonal matrix with the same diagonal elements as $\mathbf{\Sigma}_1^{-1}$. It holds $E(\mathbf{A}_t) = \mathbf{0}$, $\mathbf{\Sigma}_{A,\Delta} = (diag\,\mathbf{\Sigma}_1^{-1})^{-1/2}\mathbf{\Sigma}_1^{-1}\mathbf{D}_\Delta\mathbf{R}_\Delta\mathbf{D}_\Delta\mathbf{\Sigma}_1^{-1}(diag\,\mathbf{\Sigma}_1^{-1})^{-1/2}$ in the out-of-control state, and $\mathbf{\Sigma}_{A,1} = (diag\,\mathbf{\Sigma}_1^{-1})^{-1/2}\mathbf{\Sigma}_1^{-1}(diag\,\mathbf{\Sigma}_1^{-1})^{-1/2}$ in the in-control state. Then the control statistics are defined by

$$M1A2_t = \frac{2-r}{2r}(EA_{1,t}-1,\ldots,EA_{p,t}-1)\left(\mathbf{\Sigma}_{A,1}^{(2)}\right)^{-1}(EA_{1,t}-1,\ldots,EA_{p,t}-1)',$$
$$M2A2_t = \frac{2-r}{2r}(EA_{1,t},\ldots,EA_{p,t})\left(\mathbf{\Sigma}_{A,1}^{(2)}\right)^{-1}(EA_{1,t},\ldots,EA_{p,t})',$$

where $EA_{i,t}$ is defined similarly to $EZ_{i,t}$ by replacing $\mathbf{X}_t$ with $\mathbf{A}_t$.

Using the projection pursuit method, Chan and Zhang (2001) derived the PPCUSUMv control chart. Let λ_{ji}^u and λ_{ji}^l be the largest and the smallest eigenvalues of $\mathbf{X}_j\mathbf{X}_j' + \cdots + \mathbf{X}_i\mathbf{X}_i'$. We define

$$SU_{ji} = \lambda_{ji}^u - (i-j+1)k_u \quad \text{and} \quad SL_{ji} = \lambda_{ji}^l - (i-j+1)k_l.$$

Then the PPCUSUMv chart gives an out-of-control message as soon as

$$SU_i > h_u \quad \text{or} \quad SL_i < h_l,$$

where

$$SU_i = \max\{0, SU_{1i}, \ldots, SU_{ii}\} \quad \text{and} \quad SL_i = \min\{0, SL_{1i}, \ldots, SL_{ii}\}.$$

To simplify the calculation of the control statistic, Chan and Zhang (2001) suggested the use of $k_u = 1.5$, $k_l = 0.5$, and $h_l = h_u = h$.

4. Simulation study

The goal of this section is to compare the performance of the control charts derived in the previous section. To assess the performance it is necessary to define suitable performance measures. All the relevant ones are based on the run length of the control chart which is defined as

$$t_A = inf\{t \in \mathbb{N} : C_t \geq c\},$$

where C_t denotes the control statistic and c denotes the control limit. The run length is equal to the number of observations taken until the first alarm of the chart.

The most popular performance measure is the average run length (ARL). The ARL measures the average number of observations until the first alarm. All charts are calibrated to provide the same in-control average run length $ARL_1 = E(t_A)$. We chose $ARL_1 = 200$. This allows us to determine the control limit c for each chart.

The performance of each chart is then accessed on the basis of ARL and on the basis maximum expected delay (MED) (see Frisén (1992) and Knoth (2003)). For the ARL we always assume that the shift occurs at the first moment of time. This is, however, not the case in practical applications. A performance measure which allows for shifts at arbitrary time points is of interest. The expected delay of a stopping time t_A is defined as

$$ED_q(t_A) = E_q(t_A - q + 1 | t_A \geq q) \tag{20}$$

provided $E_q(t_A) < \infty$. ED measures the delay in detecting the shift if we know that the alarm is given after the time point q. Pollak and Siegmund (1975) proposed the use of the maximum expected delay

$$MED = \sup_{q\geq 1} ED_q(t_A) \tag{21}$$

which can be considered as the criterion for the worst-case scenario, i.e. it is the largest expected delay in detecting the shift, if it occurs at arbitrary time point q. The chart with the smallest MED is argued to be the best one. The problem is, however, that MED is very sensitive to the shift matrix $\mathbf{\Delta}$. Therefore, it is rarely possible to determine a chart which uniformly outperforms its alternatives. If there is no shift the MED coincides with the in-control ARL. If the shift occurs at $q = 1$ then the MED and the ARL are equal too. Moreover, the MED coincides with the out-of-control ARL for the Shewhart chart used for detecting changes in the mean of the univariate Gaussian process with independent observations. The application of the MED instead of the ARL is motivated by dealing with the inertia behaviour of a control chart, i.e. the tendency of the chart to detect a shift, if it is in control over a certain period of time (see, e.g., Woodall and Mahmoud (2005)). If a control chart can built a large amount of inertia, then the MED is maximized for larger values of q and it can be much higher than the corresponding ARL (e.g., the MC1 chart). For the control charts which are less inertial, the MED is usually obtained for $q = 1$ (e.g., the PPCUSUM chart). In this case the MED is close to the out-of-control ARL.

Because no explicit formulas for the ARL, the MED and the optimal control limit are available we use a Monte Carlo study. In our study 10^5 independent realizations of the target process are generated to estimate ARL_1. The control limits of all charts are determined from the regula falsi method (see, e.g., Conte and de Boor (1981)) to achieve $ARL_1 = 200$. Note that it is impossible to compute the maximum of ED over all possible values of q. For this reason we take only a bounded interval $1 \le q \le Q$. In our study Q is set to 30. For the estimation of MED we used 10^6 replications.

The target process is defined to be a four-dimensional Gaussian process with the mean vector $\boldsymbol{\mu} = \mathbf{0}$ and the covariance matrix $\mathbf{\Sigma}_1 = (0.3^{|i-j|})_{i,j=1,\dots,4}$. Consequently, it holds that $\mathbf{D}_1 = \mathbf{I}$ and $\mathbf{R}_1 = \mathbf{\Sigma}_1$. In order to compare the control charts two out-of-control situations are considered. The first one is given by generating changes in the variance of the first two variables. In this case it holds that $\sigma_{\mathbf{\Delta};11} = d_{11}\sigma_{1;11}$ and $\sigma_{\mathbf{\Delta};22} = d_{22}\sigma_{1;22}$ with $d_{11}, d_{22} \in \{0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0\}$. In the second out-of-control situation the changes in the correlation coefficient ρ_{21} are generated by taking $\rho_{\mathbf{\Delta};21} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ with $\rho_{1;21} = 0.3$ being the target value.

The purpose of process monitoring is not only to give an alarm if a shift occurs, but also to provide us with insights into the causes of the alarm. The control charts developed in this paper are aimed to detect shifts in the covariance matrix. However as many other schemes for monitoring variability they also react to shifts in the mean. This simplifies the monitoring, but complicates the diagnostics and the analysis of the causes of the alarm. To assess the impact of additional shifts in the mean vector we perform a study with the out-of-control mean given by $\boldsymbol{\mu}_1 = (0.5, 0.5, 0, 0)'$.

The control schemes depend on further design parameters. The CUSUM charts depend on the reference values k and the EWMA charts on the smoothing parameter r. In our study k takes values from the set $\{0.0, 0.1, \dots, 1.6, 1.7\}$. Larger values than 1.7 are not considered because it turned out that they lead to numerically instable results. The smoothing parameter of EWMA charts is taken equal to $r \in \{0.1, 0.2, \dots, 1.0\}$. Note that the Shewhart chart is a special case of the EWMA chart if $r = 1.0$. For each type of the chart we choose the optimal value of the design parameter (either k or r) which leads to the smallest out-of-control ARL or to the smallest MED.

The results of the simulation study are illustrated in Fig. 1 and in Tables 1–3. We do not provide the results for the standardized control charts discussed in Remark 2 (see Section 3.3), because their performance is very similar to the performance of the non-standardized counterparts. Moreover, we skipped the results for Hotelling's T^2 chart, since it provided the worst results and made the figures less readable. Fig. 1 shows the behaviour of the out-of-control ARL. The two upper figures show ARL as a function of the shift d_{11}. The shift d_{22} is set to 1.5 and there is no shift in the mean vector. In the lower figures the correlation coefficient ρ_{12} is shifted. The in-control value is 0.3. For this value the out-of-control ARL coincides with the in-control ARL of 200. The grey lines on the left-hand figures correspond to MEWMAo, MEWMAMo, MaxD, MEWMS, MEWMV. The grey lines on the right-hand side show the ARL for PPCUSUMv, M1A2, M1z2, M2A2, M2z2 schemes. The black lines correspond to the control charts based on the transformed quantities and are replicated on the left and on the right-hand sides for comparison purposes. MEWMAo, MaxD, M1A2 and M2z2 outperform the proposed charts for decreases in the variance. However, MEWMAM and MC2 show uniformly better performance for increases in variances. A similar situation is observed for shifts in the correlations, however, here the advantages of the MCUSUM and MC2 charts are much more evident. For example, for the out-of-control correlation of 0.6 the best proposed chart leads to the ARL of 78.85, while the best benchmark provides the ARL of 107.92.

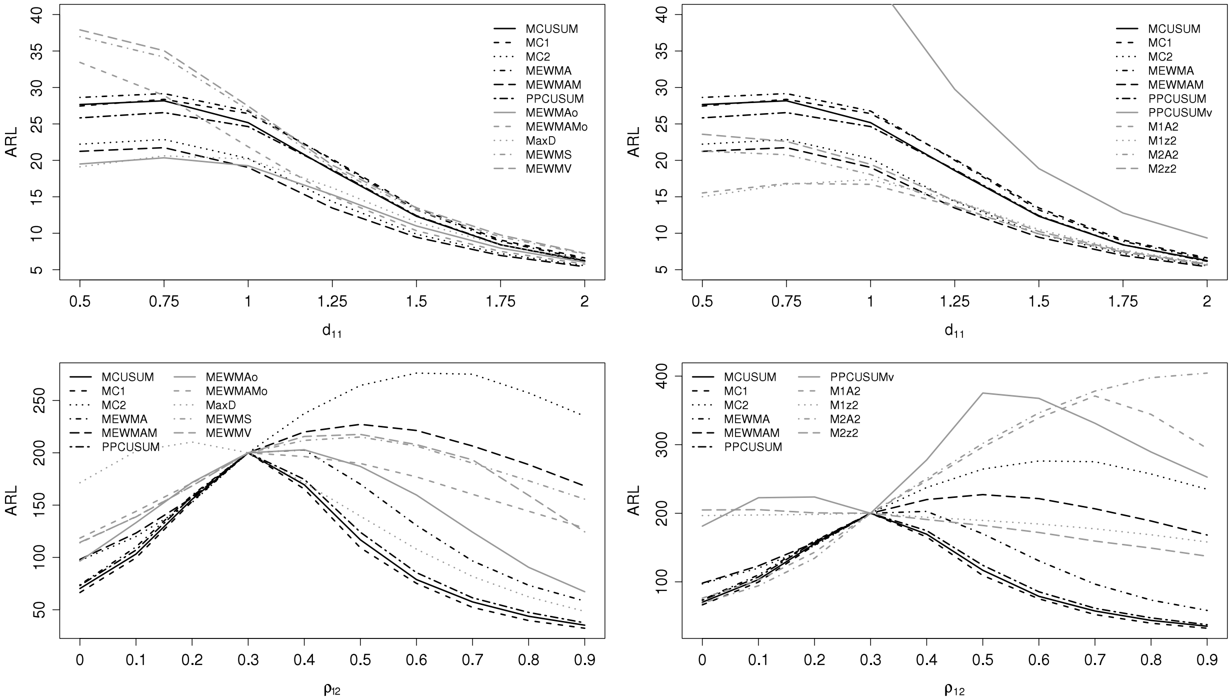
Note that all out-of-control ARLs for shifts in variances are lower than the in-control ARL of 200. This is consistent with the idea of the ARL as a performance measure. However, for the shift in correlations several charts lead to an out-of-control ARL which is higher than 200. Such behaviour of control charts is particularly dangerous. This occurs in our study for two proposed charts (MC2 and MEWMAM) and several benchmarks (M1A2, M1z2, MEWMS, MEWMV, PPCUSUMv). The control statistics reduce the dimensionality of the process and cause information loss, which leads to reduction in the sensitivity of the control statistics to shifts. In our case this is particularly important for the correlation coefficients which have a very complex functional impact on the behaviour of the control statistics. This shows the need for new types of control statistics which correctly react to arbitrary shifts in the model parameters.

Tables 1–3 summarize the results on MED. Table 1 refers to the case with shifts in the variances. In Table 2 we considered an additional shift in the mean vector. Both tables provide a similar evidence as the results based on the ARL. First, we observe a good performance of the MEWMA control charts suggested by Reynolds and Cho (2006). If the first variance decreases, then the M1A2 and M1z2 control schemes are the two best charts. On the third place we rank the MaxD scheme followed by the MEWMAM, MC2 and MEWMAo control charts. For positive shifts, the best performance is achieved by the MEWMAM control scheme, while the MC2 chart is the second best scheme. In this case the M1A2, M1z2, M2A2, and M2z2 approaches

Table 1
Maximum expected delay (MED) for different values of the shifts d_{11} and d_{22} and without shifts in the mean vector.

$d_{11} \setminus d_{22}$	0.5	0.75	1.0	1.25	1.5	1.75	2.0
	82.22 (*1.5*)	83.49 (*1.7*)	65.87 (*1.7*)	35.52 (*1.7*)	18.08 (*1.7*)	10.83 (*1.7*)	7.33 (*1.7*)
	79.16 (*1.3*)	82.50 (*1.3*)	69.23 (*1.4*)	38.98 (*1.6*)	19.82 (*1.6*)	11.56 (*1.6*)	7.78 (*1.7*)
	56.00 (*3.2*)	58.82 (*3.2*)	47.85 (*3.5*)	24.76 (*3.6*)	14.11 (*4.1*)	9.18 (*4.4*)	6.64 (*4.6*)
	81.27 (*0.2*)	85.87 (*0.5*)	69.94 (*0.5*)	38.66 (*0.6*)	19.65 (*0.6*)	11.45 (*0.6*)	7.71 (*0.6*)
	59.66 (*0.1*)	62.91 (*0.1*)	47.45 (*0.1*)	23.60 (*0.1*)	13.24 (*0.1*)	8.84 (*0.1*)	6.40 (*0.2*)
	75.26 (*1.0*)	79.28 (*1.4*)	65.62 (*1.5*)	36.12 (*1.5*)	18.28 (*1.7*)	10.69 (*1.6*)	7.27 (*1.7*)
	61.95 (*0.1*)	68.73 (*0.1*)	59.30 (*0.2*)	32.51 (*0.2*)	16.62 (*0.2*)	10.06 (*0.2*)	6.98 (*0.2*)
1.25	112.97 (*1.0*)	94.18 (*0.1*)	55.71 (*0.1*)	28.03 (*0.1*)	15.46 (*0.1*)	9.85 (*0.1*)	7.04 (*0.1*)
	48.45 (*0.1*)	57.35 (*0.1*)	53.15 (*0.1*)	30.50 (*0.1*)	16.23 (*0.1*)	10.22 (*0.1*)	7.33 (*0.1*)
	111.46 (*0.9*)	100.23 (*0.9*)	62.28 (*0.2*)	33.31 (*0.2*)	19.25 (*0.3*)	12.37 (*0.5*)	8.60 (*0.6*)
	112.24 (*1.0*)	100.83 (*0.6*)	63.11 (*0.2*)	34.02 (*0.2*)	19.66 (*0.3*)	12.68 (*0.4*)	8.87 (*0.6*)
	280.20	202.00	115.24	58.32	30.04	17.58	11.67
	36.65 (*0.1*)	44.17 (*0.1*)	43.68 (*0.1*)	25.46 (*0.1*)	13.86 (*0.1*)	8.97 (*0.1*)	6.45 (*0.2*)
	32.29 (*0.1*)	41.37 (*0.1*)	44.69 (*0.1*)	27.08 (*0.1*)	14.53 (*0.1*)	9.30 (*0.1*)	6.71 (*0.2*)
	68.10 (*0.2*)	63.35 (*0.1*)	46.23 (*0.1*)	25.40 (*0.1*)	14.24 (*0.1*)	9.32 (*0.1*)	6.70 (*0.2*)
	73.19 (*0.2*)	68.68 (*0.1*)	49.52 (*0.1*)	25.86 (*0.1*)	14.23 (*0.1*)	9.27 (*0.1*)	6.72 (*0.2*)
	27.66 (*1.7*)	28.17 (*1.7*)	25.17 (*1.7*)	18.60 (*1.7*)	12.33 (*1.7*)	8.41 (*1.7*)	6.25 (*1.7*)
	27.47 (*1.4*)	28.40 (*1.4*)	26.59 (*1.6*)	20.40 (*1.6*)	13.58 (*1.6*)	9.17 (*1.7*)	6.66 (*1.7*)
	22.22 (*3.6*)	22.86 (*3.6*)	20.26 (*3.7*)	14.34 (*4.1*)	9.88 (*4.1*)	7.21 (*4.7*)	5.55 (*4.7*)
	28.89 (*0.5*)	29.44 (*0.5*)	26.99 (*0.6*)	20.17 (*0.6*)	13.33 (*0.6*)	8.99 (*0.6*)	6.53 (*0.6*)
	21.24 (*0.1*)	21.74 (*0.1*)	19.03 (*0.1*)	13.46 (*0.1*)	9.45 (*0.1*)	6.95 (*0.2*)	5.41 (*0.2*)
	25.83 (*1.5*)	26.55 (*1.5*)	24.63 (*1.5*)	18.73 (*1.6*)	12.41 (*1.7*)	8.43 (*1.7*)	6.15 (*1.7*)
	21.23 (*0.1*)	22.08 (*0.1*)	20.96 (*0.1*)	16.31 (*0.2*)	11.13 (*0.2*)	7.85 (*0.2*)	5.92 (*0.2*)
1.5	33.52 (*0.1*)	29.05 (*0.1*)	22.09 (*0.1*)	15.25 (*0.1*)	10.42 (*0.1*)	7.59 (*0.1*)	5.82 (*0.1*)
	19.11 (*0.1*)	20.62 (*0.1*)	20.19 (*0.1*)	16.24 (*0.1*)	11.52 (*0.1*)	8.39 (*0.1*)	6.36 (*0.2*)
	37.35 (*0.9*)	34.27 (*0.5*)	26.84 (*0.3*)	18.99 (*0.3*)	13.22 (*0.4*)	9.48 (*0.5*)	7.13 (*0.7*)
	38.23 (*0.6*)	35.04 (*0.4*)	27.44 (*0.3*)	19.35 (*0.3*)	13.49 (*0.4*)	9.76 (*0.5*)	7.26 (*0.6*)
	79.07	64.10	45.42	29.75	18.87	12.78	9.35
	15.53 (*0.1*)	16.83 (*0.1*)	16.71 (*0.1*)	13.68 (*0.1*)	9.90 (*0.1*)	7.40 (*0.1*)	5.64 (*0.2*)
	15.01 (*0.1*)	16.70 (*0.1*)	17.37 (*0.1*)	14.56 (*0.1*)	10.51 (*0.1*)	7.70 (*0.1*)	5.85 (*0.2*)
	21.32 (*0.1*)	20.77 (*0.1*)	18.06 (*0.1*)	13.76 (*0.1*)	9.95 (*0.1*)	7.42 (*0.1*)	5.72 (*0.2*)
	23.58 (*0.1*)	22.60 (*0.1*)	19.45 (*0.1*)	14.47 (*0.1*)	10.24 (*0.1*)	7.55 (*0.1*)	5.78 (*0.2*)
	13.82 (*1.7*)	13.92 (*1.7*)	13.07 (*1.7*)	10.97 (*1.7*)	8.48 (*1.7*)	6.52 (*1.7*)	5.16 (*1.7*)
	13.94 (*1.6*)	14.18 (*1.6*)	13.68 (*1.6*)	11.84 (*1.7*)	9.23 (*1.7*)	7.05 (*1.7*)	5.51 (*1.7*)
	12.18 (*4.1*)	12.43 (*4.1*)	11.53 (*4.1*)	9.35 (*4.3*)	7.24 (*4.4*)	5.73 (*4.7*)	4.67 (*4.7*)
	14.46 (*0.6*)	14.60 (*0.6*)	13.86 (*0.6*)	11.73 (*0.6*)	9.05 (*0.6*)	6.89 (*0.6*)	5.38 (*0.7*)
	11.74 (*0.1*)	11.94 (*0.1*)	11.02 (*0.1*)	8.97 (*0.1*)	7.00 (*0.2*)	5.54 (*0.2*)	4.55 (*0.2*)
	13.04 (*1.5*)	13.21 (*1.5*)	12.73 (*1.5*)	10.93 (*1.7*)	8.48 (*1.7*)	6.48 (*1.7*)	5.08 (*1.7*)
	11.77 (*0.1*)	11.92 (*0.1*)	11.37 (*0.2*)	9.76 (*0.2*)	7.76 (*0.2*)	6.08 (*0.2*)	4.93 (*0.2*)
1.75	15.44 (*0.1*)	14.15 (*0.1*)	12.11 (*0.1*)	9.61 (*0.1*)	7.50 (*0.1*)	5.93 (*0.1*)	4.84 (*0.1*)
	11.21 (*0.1*)	11.64 (*0.1*)	11.51 (*0.1*)	10.27 (*0.1*)	8.39 (*0.1*)	6.66 (*0.2*)	5.32 (*0.2*)
	17.78 (*0.9*)	16.92 (*0.7*)	14.85 (*0.5*)	12.02 (*0.5*)	9.37 (*0.5*)	7.33 (*0.6*)	5.79 (*0.9*)
	18.16 (*0.8*)	17.23 (*0.6*)	15.16 (*0.6*)	12.32 (*0.4*)	9.64 (*0.6*)	7.49 (*0.6*)	5.98 (*0.8*)
	33.20	28.70	22.85	17.12	12.68	9.52	7.50
	9.33 (*0.1*)	9.66 (*0.1*)	9.64 (*0.1*)	8.80 (*0.1*)	7.28 (*0.2*)	5.83 (*0.2*)	4.79 (*0.2*)
	9.42 (*0.1*)	9.95 (*0.1*)	10.15 (*0.1*)	9.32 (*0.1*)	7.69 (*0.1*)	6.13 (*0.2*)	4.96 (*0.2*)
	11.09 (*0.2*)	10.99 (*0.2*)	10.33 (*0.1*)	8.90 (*0.1*)	7.31 (*0.1*)	5.85 (*0.2*)	4.80 (*0.2*)
	12.23 (*0.2*)	12.00 (*0.2*)	11.28 (*0.1*)	9.50 (*0.1*)	7.61 (*0.1*)	6.02 (*0.2*)	4.89 (*0.2*)
	8.71 (*1.7*)	8.73 (*1.7*)	8.39 (*1.7*)	7.50 (*1.7*)	6.28 (*1.7*)	5.18 (*1.7*)	4.33 (*1.7*)
	8.76 (*1.6*)	8.87 (*1.7*)	8.65 (*1.6*)	7.90 (*1.7*)	6.69 (*1.7*)	5.51 (*1.7*)	4.59 (*1.7*)
	8.05 (*4.5*)	8.14 (*4.4*)	7.73 (*4.4*)	6.69 (*4.7*)	5.60 (*4.7*)	4.68 (*4.7*)	4.01 (*4.6*)
	8.99 (*0.6*)	9.04 (*0.6*)	8.73 (*0.6*)	7.84 (*0.6*)	6.58 (*0.6*)	5.39 (*0.7*)	4.44 (*0.8*)
	7.85 (*0.2*)	7.92 (*0.2*)	7.50 (*0.2*)	6.51 (*0.2*)	5.46 (*0.2*)	4.58 (*0.2*)	3.89 (*0.3*)
	8.25 (*1.5*)	8.33 (*1.7*)	8.14 (*1.7*)	7.35 (*1.7*)	6.21 (*1.7*)	5.12 (*1.7*)	4.26 (*1.7*)
	7.87 (*0.2*)	7.81 (*0.2*)	7.49 (*0.2*)	6.78 (*0.2*)	5.84 (*0.2*)	4.89 (*0.2*)	4.16 (*0.2*)
2.0	9.44 (*0.1*)	8.91 (*0.1*)	8.00 (*0.1*)	6.88 (*0.1*)	5.75 (*0.1*)	4.81 (*0.1*)	4.08 (*0.1*)
	7.83 (*0.1*)	7.98 (*0.1*)	7.91 (*0.1*)	7.38 (*0.1*)	6.38 (*0.2*)	5.35 (*0.2*)	4.53 (*0.2*)
	10.66 (*0.9*)	10.32 (*0.7*)	9.55 (*0.6*)	8.32 (*0.7*)	6.98 (*0.7*)	5.75 (*0.8*)	4.75 (*1.0*)
	11.00 (*0.8*)	10.61 (*0.8*)	9.85 (*0.6*)	8.59 (*0.6*)	7.20 (*0.8*)	5.92 (*0.8*)	4.94 (*0.8*)
	18.31	16.39	13.91	11.39	9.18	7.43	6.18
	6.60 (*0.2*)	6.77 (*0.2*)	6.70 (*0.2*)	6.30 (*0.2*)	5.54 (*0.2*)	4.72 (*0.2*)	4.05 (*0.2*)
	6.83 (*0.1*)	7.07 (*0.1*)	7.10 (*0.2*)	6.72 (*0.2*)	5.83 (*0.2*)	4.98 (*0.2*)	4.20 (*0.2*)
	7.22 (*0.2*)	7.24 (*0.2*)	6.94 (*0.2*)	6.36 (*0.2*)	5.56 (*0.2*)	4.74 (*0.2*)	4.07 (*0.2*)
	7.96 (*0.2*)	7.87 (*0.2*)	7.57 (*0.2*)	6.86 (*0.2*)	5.84 (*0.2*)	4.92 (*0.2*)	4.20 (*0.2*)

Note: In the in-control state $d_{11} = d_{22} = 1$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, PPCUSUM, MEWMAo, MEWMAMo, MaxD, MEWMS, MEWMV, PPCUSUMv, M1A2, M1z2, M2A2, M2z2. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.



Note: Out-of-control ARL of the charts from the Section 3.2-3.4 as a function of the shift d_{11} (upper plots, $d_{22} = 1.5$) and the shifted correlation coefficient ρ_{21} (lower plots, the target value of ρ_{21} is 0.3). The proposed strategies MCUSUM, MC1, MC2, MEWMA, MEWMAM, PPCUSUM are replicated both on the left- and right-hand sides for comparison purposes. The benchmark charts MEWMAo, MEWMAMo, MaxD, MEWMS, MEWMV are given in left column, while PPCUSUMv, M1A2, M1z2, M2A2, M2z2 on the right-hand side. The in-control ARL is 200. 10^6 replications are used in the simulation study.

Fig. 1. Out-of-control ARL for changes in the variances or in the correlations.

have a similar performance. The same holds for the results obtained by the MEWMAo scheme of Śliwa and Schmid (2005). In Table 2, we considered the same scenario for the shifts in the variances by adding the shift of size 0.5 in the first two components of the mean vector, i.e. $\boldsymbol{\mu}_{\Delta} = (0.5, 0.5, 0, 0)'$. We observed that the ranking of the control charts does not change dramatically. For negative shifts in the first variance, the best results are reached by the M1A2 and M1z2 control schemes, while for positive shifts the MC2 and MEWMAM control charts outperform the remaining competitors.

Table 3 even gives a stronger evidence in favor of the control charts suggested in the paper. Here the MED is given as a function of the correlation coefficient ρ_{21} without a shift in the mean (upper block of the table) and with shifted means (lower block of the table). The in-control correlation is equal to 0.3. For decreases in the correlation, the best performance is achieved by the MC1, MCUSUM, and M2A2 control charts. For positive shifts, we observe a poor performance of the M1A2, M1z2, M2A2, and M2z2 control charts. On the first two places we ranked the MCUSUM and MC1 schemes which clearly dominate other alternatives. The third best chart is the PPCUSUM scheme, followed by the MaxD chart. The fifth and the sixth places are given to the MEWMA and MEWMAo control charts correspondingly. The remaining competitors perform much worse. The situation changes if the shifts in the means are added. For negative shifts, the best performance is obtained by the MC2, MEWMAM, M1A2, and M2A2 charts, while for positive shifts the MC1, MCUSUM, PPCUSUM, and MaxD control charts are the best ones.

By comparing the values in Table 1 with their counterparts in Table 2 we observe that the out-of-control ARLs with the shifts in the mean are smaller. This is the case for all of the charts, with the exception of MEWMV, where the decrease is minor. This implies that the charts for monitoring the volatility react to shifts in the mean. From the point of view of diagnostics this can be seen as a drawback of the procedure. However, note that the ranks of the charts stay almost unchanged. This implies that the presence or absence of the shifts in the mean influences only the frequency of misleading signals, but keeps the relative performance of the charts constant. The forth column of Table 3 contains the MEDs for all charts in the presence of the shift in the mean only. MEWMV is the only chart which manages to keep the ARL at the level of the in-control ARL. This is achieved by subtracting from the process the EWMA-type estimator of the mean. Despite having this unique property, the MEWMV chart shows a very modest performance for various shift in the variances. For the shifts in the correlation this chart even leads to the MEDs higher than 200. The PPCUSUMv and M1A2 have also high out-of-control ARL for mean shifts. However, this should not be linked to the ability of the charts to neglect the mean shifts and is more due to the fact that these charts lead generally to high out-of-control ARLs. The rest of the charts show a relatively similar

Table 2
Maximum expected delay (MED) for different values of the shifts d_{11} and d_{22} and with shifts in the mean vector.

$d_{11} \setminus d_{22}$	0.5	0.75	1.0	1.25	1.5	1.75	2.0
	45.35 (*0.5*)	51.47 (*0.9*)	40.50 (*1.5*)	24.31 (*1.5*)	14.25 (*1.7*)	9.26 (*1.7*)	6.66 (*1.7*)
	44.42 (*0.4*)	52.44 (*0.8*)	43.84 (*1.4*)	26.75 (*1.6*)	15.49 (*1.6*)	9.89 (*1.7*)	6.97 (*1.7*)
	43.63 (*3.4*)	42.66 (*3.5*)	29.42 (*3.6*)	17.53 (*3.9*)	11.24 (*4.0*)	7.89 (*4.6*)	5.97 (*4.6*)
	43.16 (*0.1*)	51.08 (*0.1*)	44.07 (*0.5*)	26.72 (*0.5*)	15.52 (*0.6*)	9.85 (*0.6*)	6.94 (*0.7*)
	44.14 (*0.1*)	42.14 (*0.1*)	28.05 (*0.1*)	16.41 (*0.1*)	10.68 (*0.1*)	7.65 (*0.2*)	5.78 (*0.2*)
	44.14 (*0.6*)	49.33 (*0.9*)	40.17 (*1.4*)	24.36 (*1.4*)	14.19 (*1.6*)	9.11 (*1.7*)	6.46 (*1.7*)
	30.68 (*0.1*)	30.10 (*0.1*)	25.45 (*0.1*)	17.48 (*0.2*)	11.35 (*0.2*)	7.89 (*0.2*)	5.95 (*0.2*)
1.25	65.65 (*0.1*)	49.66 (*0.1*)	31.83 (*0.1*)	19.00 (*0.1*)	12.02 (*0.1*)	8.35 (*0.1*)	6.28 (*0.1*)
	35.25 (*0.1*)	35.71 (*0.1*)	29.33 (*0.1*)	19.26 (*0.1*)	12.37 (*0.1*)	8.70 (*0.1*)	6.52 (*0.2*)
	74.60 (*0.8*)	59.81 (*0.2*)	38.35 (*0.2*)	23.79 (*0.3*)	15.24 (*0.4*)	10.48 (*0.5*)	7.65 (*0.6*)
	110.71 (*1.0*)	99.71 (*0.9*)	62.96 (*0.2*)	33.74 (*0.2*)	19.38 (*0.4*)	12.38 (*0.5*)	8.58 (*0.7*)
	148.45	108.48	68.06	39.02	22.78	14.60	10.22
	29.96 (*0.1*)	33.46 (*0.1*)	30.36 (*0.1*)	18.99 (*0.1*)	11.73 (*0.1*)	8.07 (*0.1*)	6.02 (*0.2*)
	26.57 (*0.1*)	29.31 (*0.1*)	26.41 (*0.1*)	17.50 (*0.1*)	11.37 (*0.1*)	8.01 (*0.1*)	6.03 (*0.2*)
	46.10 (*0.1*)	42.24 (*0.1*)	31.13 (*0.1*)	19.02 (*0.1*)	11.91 (*0.1*)	8.31 (*0.1*)	6.22 (*0.2*)
	41.94 (*0.1*)	36.95 (*0.1*)	26.77 (*0.1*)	16.81 (*0.1*)	10.94 (*0.1*)	7.88 (*0.1*)	6.00 (*0.2*)
	22.00 (*1.5*)	21.99 (*1.5*)	19.31 (*1.5*)	14.41 (*1.7*)	10.18 (*1.7*)	7.42 (*1.7*)	5.69 (*1.7*)
	22.39 (*1.4*)	22.62 (*1.4*)	20.53 (*1.6*)	15.79 (*1.6*)	11.13 (*1.7*)	7.97 (*1.7*)	6.02 (*1.7*)
	19.47 (*3.9*)	19.03 (*3.9*)	15.55 (*4.0*)	11.46 (*4.2*)	8.35 (*4.4*)	6.36 (*4.7*)	5.08 (*4.7*)
	22.88 (*0.4*)	23.11 (*0.5*)	20.75 (*0.5*)	15.78 (*0.6*)	11.01 (*0.6*)	7.85 (*0.7*)	5.93 (*0.7*)
	18.47 (*0.1*)	18.04 (*0.1*)	14.61 (*0.1*)	10.82 (*0.1*)	8.08 (*0.1*)	6.17 (*0.2*)	4.95 (*0.2*)
	20.89 (*1.4*)	20.92 (*1.4*)	18.86 (*1.4*)	14.38 (*1.5*)	10.16 (*1.7*)	7.32 (*1.7*)	5.55 (*1.7*)
	15.75 (*0.1*)	15.55 (*0.1*)	14.12 (*0.2*)	11.18 (*0.2*)	8.39 (*0.2*)	6.44 (*0.2*)	5.11 (*0.2*)
1.5	24.08 (*0.1*)	20.82 (*0.1*)	16.15 (*0.1*)	11.89 (*0.1*)	8.73 (*0.1*)	6.65 (*0.1*)	5.30 (*0.1*)
	16.41 (*0.1*)	16.58 (*0.1*)	15.33 (*0.1*)	12.40 (*0.1*)	9.39 (*0.1*)	7.28 (*0.2*)	5.69 (*0.2*)
	28.83 (*0.5*)	25.57 (*0.4*)	20.37 (*0.4*)	15.09 (*0.4*)	11.05 (*0.5*)	8.28 (*0.6*)	6.45 (*0.8*)
	37.52 (*0.9*)	34.44 (*0.5*)	27.11 (*0.3*)	19.07 (*0.3*)	13.23 (*0.4*)	9.43 (*0.6*)	7.11 (*0.8*)
	53.07	43.28	31.95	22.22	15.29	11.01	8.40
	14.05 (*0.1*)	14.76 (*0.1*)	14.21 (*0.1*)	11.55 (*0.1*)	8.72 (*0.1*)	6.70 (*0.2*)	5.28 (*0.2*)
	13.55 (*0.1*)	14.20 (*0.1*)	13.69 (*0.1*)	11.31 (*0.1*)	8.70 (*0.1*)	6.69 (*0.2*)	5.29 (*0.2*)
	17.80 (*0.1*)	17.11 (*0.1*)	14.85 (*0.1*)	11.59 (*0.1*)	8.72 (*0.1*)	6.78 (*0.2*)	5.33 (*0.2*)
	18.03 (*0.1*)	16.89 (*0.1*)	14.35 (*0.1*)	11.16 (*0.1*)	8.49 (*0.1*)	6.57 (*0.2*)	5.24 (*0.2*)
	12.28 (*1.7*)	12.18 (*1.7*)	11.18 (*1.7*)	9.34 (*1.7*)	7.46 (*1.7*)	5.90 (*1.7*)	4.79 (*1.7*)
	12.43 (*1.5*)	12.48 (*1.5*)	11.79 (*1.7*)	10.07 (*1.6*)	8.01 (*1.7*)	6.31 (*1.7*)	5.07 (*1.7*)
	11.25 (*4.3*)	11.10 (*4.5*)	9.81 (*4.4*)	8.02 (*4.4*)	6.41 (*4.7*)	5.22 (*4.7*)	4.37 (*4.7*)
	12.77 (*0.5*)	12.73 (*0.5*)	11.93 (*0.6*)	10.04 (*0.6*)	7.88 (*0.7*)	6.16 (*0.7*)	4.94 (*0.8*)
	10.88 (*0.1*)	10.69 (*0.1*)	9.49 (*0.1*)	7.74 (*0.2*)	6.23 (*0.2*)	5.06 (*0.2*)	4.25 (*0.2*)
	11.66 (*1.4*)	11.69 (*1.6*)	10.93 (*1.6*)	9.23 (*1.7*)	7.35 (*1.6*)	5.78 (*1.7*)	4.71 (*1.7*)
	9.90 (*0.2*)	9.68 (*0.2*)	8.96 (*0.2*)	7.71 (*0.2*)	6.35 (*0.2*)	5.22 (*0.2*)	4.37 (*0.2*)
1.75	12.80 (*0.1*)	11.72 (*0.1*)	10.02 (*0.1*)	8.20 (*0.1*)	6.61 (*0.1*)	5.37 (*0.1*)	4.48 (*0.1*)
	10.25 (*0.1*)	10.33 (*0.1*)	9.91 (*0.1*)	8.76 (*0.1*)	7.23 (*0.2*)	5.93 (*0.2*)	4.91 (*0.2*)
	15.24 (*0.8*)	14.24 (*0.5*)	12.40 (*0.5*)	10.27 (*0.5*)	8.24 (*0.6*)	6.57 (*0.6*)	5.32 (*1.0*)
	17.87 (*0.9*)	17.02 (*0.7*)	14.88 (*0.5*)	12.00 (*0.5*)	9.38 (*0.6*)	7.26 (*0.6*)	5.80 (*0.8*)
	25.58	22.35	18.15	14.11	10.88	8.49	6.86
	8.84 (*0.1*)	9.04 (*0.1*)	8.84 (*0.1*)	7.92 (*0.1*)	6.63 (*0.2*)	5.41 (*0.2*)	4.51 (*0.2*)
	8.83 (*0.1*)	9.05 (*0.1*)	8.89 (*0.1*)	8.01 (*0.1*)	6.65 (*0.2*)	5.47 (*0.2*)	4.55 (*0.2*)
	10.09 (*0.2*)	9.99 (*0.1*)	9.22 (*0.1*)	7.99 (*0.1*)	6.64 (*0.1*)	5.44 (*0.2*)	4.53 (*0.2*)
	10.54 (*0.2*)	10.26 (*0.2*)	9.44 (*0.1*)	8.07 (*0.1*)	6.60 (*0.2*)	5.42 (*0.2*)	4.51 (*0.2*)
	8.13 (*1.7*)	8.00 (*1.7*)	7.56 (*1.7*)	6.72 (*1.7*)	5.71 (*1.7*)	4.80 (*1.7*)	4.09 (*1.7*)
	8.18 (*1.6*)	8.18 (*1.7*)	7.87 (*1.7*)	7.07 (*1.7*)	6.05 (*1.7*)	5.07 (*1.7*)	4.29 (*1.7*)
	7.64 (*4.7*)	7.55 (*4.5*)	6.96 (*4.6*)	6.02 (*4.7*)	5.13 (*4.7*)	4.34 (*4.7*)	3.79 (*4.7*)
	8.40 (*0.6*)	8.32 (*0.6*)	7.93 (*0.6*)	7.06 (*0.7*)	5.96 (*0.7*)	4.95 (*0.7*)	4.15 (*0.8*)
	7.43 (*0.2*)	7.35 (*0.2*)	6.77 (*0.2*)	5.89 (*0.2*)	4.98 (*0.2*)	4.26 (*0.3*)	3.65 (*0.3*)
	7.74 (*1.6*)	7.70 (*1.7*)	7.34 (*1.7*)	6.61 (*1.7*)	5.59 (*1.7*)	4.70 (*1.7*)	3.99 (*1.7*)
	6.93 (*0.2*)	6.78 (*0.2*)	6.43 (*0.2*)	5.80 (*0.2*)	5.04 (*0.2*)	4.33 (*0.2*)	3.76 (*0.2*)
2.0	8.36 (*0.1*)	7.87 (*0.1*)	7.09 (*0.1*)	6.15 (*0.1*)	5.22 (*0.1*)	4.45 (*0.1*)	3.84 (*0.1*)
	7.41 (*0.1*)	7.42 (*0.1*)	7.18 (*0.2*)	6.54 (*0.2*)	5.73 (*0.2*)	4.90 (*0.2*)	4.21 (*0.2*)
	9.61 (*0.9*)	9.25 (*0.9*)	8.56 (*0.6*)	7.48 (*0.6*)	6.31 (*0.9*)	5.26 (*1.0*)	4.40 (*1.0*)
	10.72 (*0.9*)	10.39 (*0.7*)	9.54 (*0.7*)	8.35 (*0.6*)	6.96 (*0.8*)	5.73 (*0.9*)	4.76 (*0.9*)
	15.27	13.84	11.89	10.00	8.23	6.83	5.78
	6.35 (*0.2*)	6.42 (*0.2*)	6.32 (*0.2*)	5.85 (*0.2*)	5.17 (*0.2*)	4.47 (*0.2*)	3.87 (*0.2*)
	6.53 (*0.2*)	6.61 (*0.2*)	6.53 (*0.2*)	6.02 (*0.2*)	5.28 (*0.2*)	4.55 (*0.2*)	3.96 (*0.2*)
	6.84 (*0.2*)	6.78 (*0.2*)	6.51 (*0.2*)	5.92 (*0.2*)	5.19 (*0.2*)	4.48 (*0.2*)	3.89 (*0.2*)
	7.29 (*0.2*)	7.17 (*0.2*)	6.79 (*0.2*)	6.11 (*0.2*)	5.27 (*0.2*)	4.53 (*0.2*)	3.92 (*0.2*)

Note: The out-of-control mean vector is given by $\mu_\Delta = (0.5, 0.5, 0, 0)'$. In the in-control state $d_{11} = d_{22} = 1$. The order of the charts is MCUSUM, MC1, MC2, MEWMA, MEWMAM, PPCUSUM, MEWMAo, MEWMAMo, MaxD, MEWMS, MEWMV, PPCUSUMv, M1A2, M1z2, M2A2, M2z2. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

Table 3
Maximum expected delay (MED) for different shifts in the correlation coefficient ρ_{12}.

$\rho_{12} = 0.0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
No shift in the means									
70.26 (*0.3*)	103.21 (*0.2*)	155.53 (*0.2*)	MCUSUM	169.44 (*0.1*)	116.85 (*0.2*)	78.85 (*0.2*)	57.49 (*0.2*)	43.77 (*0.3*)	35.26 (*0.3*)
66.43 (*0.2*)	99.62 (*0.2*)	153.53 (*0.1*)	MC1	165.37 (*0.1*)	109.22 (*0.1*)	75.25 (*0.1*)	52.43 (*0.2*)	40.00 (*0.2*)	32.58 (*0.2*)
96.96 (*3.4*)	120.02 (*3.2*)	152.62 (*3.2*)	MC2	237.56 (*4.7*)	264.39 (*4.7*)	276.82 (*4.7*)	275.11 (*4.7*)	258.27 (*4.7*)	235.00 (*4.7*)
76.28 (*0.1*)	113.12 (*0.1*)	163.00 (*0.1*)	MEWMA	206.62 (*0.1*)	173.65 (*0.1*)	132.35 (*0.1*)	99.12 (*0.1*)	74.85 (*0.1*)	59.37 (*0.1*)
98.13 (*0.1*)	123.28 (*0.1*)	156.98 (*0.1*)	MEWMAM	221.38 (*1.0*)	228.94 (*1.0*)	222.72 (*1.0*)	207.71 (*1.0*)	189.17 (*1.0*)	168.41 (*1.0*)
73.12 (*0.3*)	107.04 (*0.2*)	158.33 (*0.3*)	PPCUSUM	174.62 (*0.1*)	124.10 (*0.1*)	85.48 (*0.2*)	61.40 (*0.2*)	47.45 (*0.2*)	37.49 (*0.3*)
105.58 (*0.2*)	140.82 (*0.2*)	179.51 (*0.2*)	MEWMAo	206.20 (*0.7*)	195.32 (*0.2*)	168.65 (*0.2*)	136.98 (*0.1*)	101.18 (*0.1*)	74.89 (*0.1*)
118.77 (*0.1*)	144.91 (*0.1*)	171.11 (*0.1*)	MEWWMAMo	198.00 (*1.0*)	191.46 (*1.0*)	177.87 (*1.0*)	161.73 (*0.8*)	145.12 (*1.0*)	129.91 (*0.9*)
172.41 (*0.1*)	201.61 (*0.1*)	210.28 (*0.1*)	MaxD	171.66 (*0.1*)	139.43 (*0.1*)	107.92 (*0.1*)	81.80 (*0.1*)	62.33 (*0.1*)	48.44 (*0.1*)
113.95 (*0.4*)	139.11 (*0.2*)	168.48 (*0.2*)	MEWMS	216.01 (*0.9*)	218.74 (*1.0*)	209.86 (*1.0*)	191.42 (*0.1*)	155.37 (*0.1*)	120.99 (*0.1*)
114.38 (*0.3*)	138.76 (*0.2*)	168.35 (*0.2*)	MEWMV	215.63 (*0.8*)	219.21 (*1.0*)	209.90 (*1.0*)	193.76 (*0.1*)	159.29 (*0.1*)	124.43 (*0.1*)
181.21	222.61	223.71	PPCUSUMv	277.77	375.27	367.51	331.21	289.30	252.69
81.54 (*0.1*)	108.45 (*0.1*)	146.19 (*0.1*)	M1A2	244.53 (*0.9*)	289.50 (*0.9*)	332.08 (*0.9*)	340.96 (*0.1*)	317.00 (*0.1*)	276.93 (*0.1*)
196.67 (*0.1*)	197.46 (*0.1*)	198.32 (*0.1*)	M1z2	195.16 (*0.5*)	190.61 (*0.5*)	184.63 (*0.5*)	177.73 (*0.5*)	168.61 (*0.5*)	159.41 (*0.5*)
69.80 (*0.1*)	94.57 (*0.1*)	134.06 (*0.1*)	M2A2	249.32 (*0.9*)	297.67 (*1.0*)	341.04 (*1.0*)	375.94 (*1.0*)	400.28 (*1.0*)	416.22 (*0.9*)
206.52 (*0.8*)	205.75 (*0.7*)	202.82 (*0.7*)	M2z2	192.13 (*0.7*)	183.10 (*0.2*)	171.97 (*0.2*)	159.60 (*0.2*)	149.20 (*0.2*)	137.51 (*0.2*)
With shifts in the means ($\mu_\Delta = (0.5, 0.5, 0, 0)'$)									
69.48 (*1.4*)	80.33 (*1.4*)	89.61 (*1.4*)	94.50 (*0.8*)	77.04 (*0.3*)	59.75 (*0.3*)	47.11 (*0.3*)	38.34 (*0.3*)	32.17 (*0.4*)	27.57 (*0.4*)
74.98 (*1.4*)	87.15 (*1.4*)	96.61 (*1.4*)	96.81 (*0.3*)	73.62 (*0.2*)	56.16 (*0.2*)	44.21 (*0.2*)	35.71 (*0.3*)	29.42 (*0.3*)	25.09 (*0.3*)
47.83 (*3.5*)	53.73 (*3.3*)	60.93 (*3.3*)	70.08 (*3.3*)	80.53 (*3.3*)	92.71 (*3.3*)	104.27 (*3.7*)	103.10 (*4.7*)	98.44 (*4.7*)	92.50 (*4.7*)
74.07 (*0.4*)	85.82 (*0.3*)	94.18 (*0.3*)	95.49 (*0.1*)	83.05 (*0.1*)	69.02 (*0.1*)	56.37 (*0.1*)	46.47 (*0.1*)	39.14 (*0.1*)	34.02 (*0.1*)
48.08 (*0.1*)	55.24 (*0.1*)	63.62 (*0.1*)	73.22 (*0.1*)	83.39 (*0.1*)	94.30 (*0.1*)	104.61 (*0.2*)	100.17 (*0.6*)	92.06 (*1.0*)	83.88 (*1.0*)
70.11 (*1.4*)	81.16 (*1.4*)	90.29 (*1.4*)	93.97 (*1.0*)	79.79 (*0.3*)	62.91 (*0.3*)	49.82 (*0.3*)	40.22 (*0.3*)	33.47 (*0.3*)	28.97 (*0.3*)
70.39 (*0.2*)	81.20 (*0.2*)	87.81 (*0.2*)	88.22 (*0.2*)	83.15 (*0.2*)	72.79 (*0.1*)	60.25 (*0.1*)	50.14 (*0.1*)	41.75 (*0.1*)	35.08 (*0.1*)
61.34 (*0.1*)	67.95 (*0.1*)	73.51 (*0.1*)	77.61 (*0.1*)	79.75 (*0.1*)	79.42 (*0.1*)	77.20 (*0.1*)	73.61 (*0.1*)	69.59 (*0.1*)	64.81 (*0.1*)
82.76 (*0.1*)	81.31 (*0.1*)	76.63 (*0.1*)	69.13 (*0.1*)	61.15 (*0.1*)	52.12 (*0.1*)	43.78 (*0.1*)	36.58 (*0.1*)	30.71 (*0.1*)	26.17 (*0.1*)
62.99 (*0.2*)	70.52 (*0.2*)	78.87 (*0.2*)	86.34 (*0.2*)	93.28 (*0.2*)	99.65 (*0.2*)	101.68 (*0.5*)	95.49 (*0.8*)	88.51 (*0.8*)	81.12 (*1.0*)
115.04 (*0.2*)	138.28 (*0.2*)	168.04 (*0.2*)	196.77 (*0.5*)	213.19 (*0.9*)	215.93 (*0.9*)	207.76 (*0.9*)	192.86 (*1.0*)	160.88 (*0.1*)	125.84 (*0.1*)
118.92	132.24	147.07	161.78	169.29	163.20	151.20	137.72	125.28	112.17
45.83 (*0.1*)	57.88 (*0.1*)	75.37 (*0.1*)	101.86 (*0.1*)	139.15 (*0.1*)	189.20 (*0.1*)	241.76 (*0.3*)	273.01 (*1.0*)	294.83 (*1.0*)	307.00 (*0.9*)
69.66 (*0.1*)	69.61 (*0.1*)	70.08 (*0.1*)	69.63 (*0.1*)	69.92 (*0.1*)	69.82 (*0.1*)	69.40 (*0.1*)	68.77 (*0.1*)	68.00 (*0.1*)	67.08 (*0.1*)
43.33 (*0.1*)	54.69 (*0.1*)	71.45 (*0.1*)	96.09 (*0.1*)	133.56 (*0.1*)	186.60 (*0.1*)	237.96 (*0.8*)	265.42 (*0.8*)	284.81 (*1.0*)	293.67 (*1.0*)
67.56 (*0.1*)	66.54 (*0.1*)	65.28 (*0.1*)	63.84 (*0.1*)	62.53 (*0.1*)	60.57 (*0.1*)	58.81 (*0.1*)	56.98 (*0.1*)	55.56 (*0.1*)	53.55 (*0.1*)

Note: MED for the charts from Sections 3.2–3.4 as a function of the shift in ρ_{21} without (upper block) and with (lower block) shifts in the mean vector. The in-control value of ρ_{21} is 0.3. The in-control ARL is 200. The optimal parameters of the charts are given in parenthesis. 10^6 replications are used in the simulation study.

behaviour. Another issues related to the sensitivity of charts for variability to shifts in the mean is the choice of performance measures. Note that the considered schemes are matched to lead to the same in-control ARL. A small out-of-control ARL is a sign of good performance of the scheme. This approach, however, does not and cannot penalize the schemes for producing misleading signals. A possible solution to this problem is considering a joint scheme which monitors mean and variability by two different charts. Development of joint schemes and schemes which react to shifts only in particular parameters of the model is a very important issue, but goes beyond the scope of this paper.

5. Summary

In this paper we develop a methodology for sequentially monitoring the shifts in the covariance matrix of an uncorrelated Gaussian process. The method allows us to use a point estimator of the covariance matrix based on a single observation. Up until now only EWMA-type charts were almost exclusively applied for this purpose with the exception of the PPCUSUMV chart suggested by Chan and Zhang (2001). We transform the point estimator of the covariance matrix into a set of Gaussian vectors. Then both standard multivariate EWMA and CUSUM charts of all types can be applied for monitoring covariance matrices. The simulation study shows that the proposed technique has a good performance. However, it is not possible to provide a unique ranking of the control procedures. While for positive changes in the variances the best results are reached for the MC2 and the MEWMA control charts, for negative shifts the M1A2 and M1z2 are the best charts. For positive changes in the correlation the best results are obtained by the MC1 and the MCUSUM schemes.

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Appendix A. Proof of Theorem 1(b)

It holds that

$$\begin{aligned}\boldsymbol{\eta}_{i,t} &= \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}(\mathbf{V}_{t;21,i}/v_{ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{ii})v_{t;ii}^{1/2}\\ &= \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{-1/2}(\mathbf{V}_{t;21,i}/v_{ii} - \boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} + \boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{1;ii})v_{t;ii}^{1/2}\\ &= \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{-1/2}(\mathbf{V}_{t;21,i}/v_{t;ii} - \boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii})v_{t;ii}^{1/2}\\ &\quad + \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{-1/2}(\boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{1;ii})v_{t;ii}^{1/2}.\end{aligned}$$

Hence, from Theorem 1(a) we get

$$E(\boldsymbol{\eta}_{i,t}) = \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}(\boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{1;ii})E(v_{t;ii}^{1/2})$$

and

$$\begin{aligned}Var(\boldsymbol{\eta}_{i,t}) &= \boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{-1/2}(\boldsymbol{\Sigma}_{\Delta;22\cdot1,i} + (\boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{1;ii})Var(v_{t;ii}^{1/2})\\ &\quad \times (\boldsymbol{\Sigma}_{\Delta;21,i}/\sigma_{\Delta;ii} - \boldsymbol{\Sigma}_{1;21,i}/\sigma_{1;ii})')\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{-1/2}\boldsymbol{\Sigma}_{\Delta;22\cdot1,i}^{1/2}\boldsymbol{\Sigma}_{1;22\cdot1,i}^{-1/2}.\end{aligned}$$

From $v_{t;ii} \sim W_1(1, \sigma_{ii})$ we get that $E(v_{t;ii}^{1/2}) = \sigma_{ii}^{1/2}\sqrt{2}/\sqrt{\pi}$ and $Var(v_{t;ii}^{1/2}) = E(v_{t;ii}) - E(v_{t;ii}^{1/2})^2 = \sigma_{ii}(1 - 2/\pi)$.

Appendix B. Proof of Remark 1

In order to simplify the notations we omit the index t in $\mathbf{V}_t$ and $\boldsymbol{\eta}_{1;t}$. It holds that

$$tr(\boldsymbol{\Sigma}^{-1}\mathbf{V}) = tr((\sigma_{11}^{-1} + (\mathbf{V}_{21}/V_{11} - \boldsymbol{\Sigma}_{21}/\sigma_{11})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\mathbf{V}_{21}/V_{11} - \boldsymbol{\Sigma}_{21}/\sigma_{11}))V_{11}).$$

Since $\mathbf{V}$ consists only of two functionally independent components V_{11} and $\mathbf{V}_{21}$ we can write that

$$\mathbf{V} = \Big[1 \;\; (\mathbf{V}_{21}/V_{11})'\Big]' V_{11}\Big[1 \;\; (\mathbf{V}_{21}/V_{11})'\Big].$$

The joint density of V_{11} and $\mathbf{V}_{21}$ is then given by

$$\begin{aligned}f_{V_{11},\mathbf{V}_{21}}(c_{11}, \mathbf{C}_{21}) &= \frac{\pi^{(1-k)/2}2^{-k/2}}{\sqrt{\pi}|\boldsymbol{\Sigma}|^{1/2}}c_{11}^{-\frac{k}{2}}\,{}_0F_1\left(\frac{1}{2}; \frac{1}{4}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\left[1 \;\; (\mathbf{C}_{21}/c_{11})'\right]' c_{11}\left[1 \;\; (\mathbf{C}_{21}/c_{11})'\right]\right)\\ &\quad \times \exp\Big(-\frac{1}{2}(\sigma_{11}^{-1} + (\mathbf{C}_{21}/c_{11} - \boldsymbol{\Sigma}_{21}/\sigma_{11})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\mathbf{C}_{21}/c_{11} - \boldsymbol{\Sigma}_{21}/\sigma_{11}))c_{11}\Big).\end{aligned}$$

Making the transformation

$$\mathbf{x} = \mathbf{C}_{21}c_{11}^{-1} \quad \text{and} \quad y = c_{11}$$

with the Jacobian c_{11}^{k-1} yields

$$\begin{aligned}f_{V_{11},\mathbf{V}_{21}/V_{11}}(y, \mathbf{x}) &= \frac{\pi^{-k/2}2^{-k/2}}{|\boldsymbol{\Sigma}|^{1/2}}y^{\frac{k-1-1}{2}}\,{}_0F_1\left(\frac{1}{2}; \frac{1}{4}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}[1 \;\; \mathbf{x}']'y[1 \;\; \mathbf{x}']\right)\\ &\quad \times \exp\left(-\frac{1}{2}(\sigma_{11}^{-1} + (\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}^{-1})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}))y\right).\end{aligned}$$

Using the definition of the matrix hypergeometric function and the fact that the non-negative eigenvalues of the matrices

$$\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}[1\mathbf{x}']'y[1 \;\; \mathbf{x}'] \quad \text{and} \quad [1 \;\; \mathbf{x}']\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}[1\mathbf{x}']'y$$

coincide, we get

$$\begin{aligned}f_{V_{11},\mathbf{V}_{21}/V_{11}}(y, \mathbf{x}) &= \frac{\pi^{-k/2}2^{-k/2}}{|\boldsymbol{\Sigma}|^{1/2}}y^{\frac{k-1-1}{2}}\,{}_0F_1\left(\frac{1}{2}; \frac{1}{4}[1 \;\; \mathbf{x}']\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}[1 \;\; \mathbf{x}']'y\right)\\ &\quad \times \exp\Big(-\frac{1}{2}(\sigma_{11}^{-1} + (\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}^{-1})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}))y\Big).\end{aligned}$$

Since ${}_0F_1(\frac{1}{2}; z) = (\exp(2\sqrt{z}) + \exp(-2\sqrt{z}))/2 = \cosh(2\sqrt{z})$ and $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}'\boldsymbol{\mu}$, it follows that

$$\begin{aligned}f_{V_{11},\mathbf{V}_{21}/V_{11}}(y, \mathbf{x}) &= \frac{\pi^{-k/2}2^{-k/2}}{|\boldsymbol{\Sigma}|^{1/2}}y^{\frac{k-1-1}{2}}\,cosh\left(\sqrt{\boldsymbol{\mu}'\boldsymbol{\mu}}\sqrt{[1 \;\; \mathbf{x}']\boldsymbol{\Sigma}^{-2}[1 \;\; \mathbf{x}']'y}\right)\\ &\quad \times \exp\Big(-\frac{1}{2}(\sigma_{11}^{-1} + (\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}^{-1})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11}))y\Big).\end{aligned}$$

Next we make the transformation $\mathbf{z} = \boldsymbol{\Sigma}_{22\cdot1}^{-1/2}(\mathbf{x} - \boldsymbol{\Sigma}_{21}/\sigma_{11})\sqrt{y}$ with the Jacobian $|\boldsymbol{\Sigma}_{22\cdot1}|^{1/2}y^{-(k-1)/2}$. Using the fact that $|\boldsymbol{\Sigma}| = \sigma_{11}|\boldsymbol{\Sigma}_{22\cdot1}|$ and integrating over y leads to the statement of the theorem.

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