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Question 1: **Question 1** [2 marks]

Show that the determinant of the $p \times p$ diagonal matrix $\mathbb{A} = (a_{ij})$ with $a_{ij} = 0$, $i \neq j$, is given by the product of the diagonal elements; that is, $|\mathbb{A}| = a_{11}a_{22}\cdots a_{pp}$.

Suppose A is our diagonal matrix, $A \in \mathbb{R}^{p \times p}$

$$A = \begin{bmatrix} a_{11} & & \cdots & 0 \\ \vdots & a_{22} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & a_{pp} \end{bmatrix}$$

Base case: $p=2$

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

$$\det(A) = a_{11}a_{22} - 0 \cdot 0 = a_{11} \cdot a_{22}$$

$\Rightarrow$ based case is true.

Inductive case:

If $p \times p$ matrix A and its determinatt $\det(A) = a_{11}a_{22}\cdots a_{pp}$,

Then for $(p+1) \times (p+1)$ matrix: $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{p \times 1}$

$$A' = \begin{bmatrix} A & \vec{0} \\ \vec{0}^T & a_{p+1,p+1} \end{bmatrix}$$

we introduce the Schur complement:

$$S = a_{p+1\,p+1} - \vec{0}^T A^{-1} \vec{0} = a_{p+1}\, a_{p+1}$$

By Schur formula:

$$\det(A') = \det(A) \cdot \det(S)$$

$$= a_{11}a_{22}\cdots a_{pp} \times a_{p+1\,p+1}$$

$\Rightarrow$ Inductive step is true

$\Rightarrow$ Therefore, the formula is true for all $p \geq 2$ cases.

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Question 2: **Question 2** [2 marks]

Show that the determinant of a square symmetric $p \times p$ matrix $\mathbb{A}$ can be expressed as the product of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$; that is, $|\mathbb{A}| = \prod_{i=1}^{p} \lambda_i$.

Suppose A is our square symmetric matrix, $A \in \mathbb{R}^{p \times p}$

To get the eigenvalue,

$$A x = \lambda x$$

$$\Rightarrow \det(A - \lambda I) = 0$$

Suppose the eigenvalue of A is $\lambda_1, \lambda_2, \ldots, \lambda_p$

After we set: $\det(A - \lambda I) = 0$,

then the characteristic function to solve is:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_p - \lambda) = 0$$

when $\lambda = 0$, we have:

$$p(0) = \lambda_1 \lambda_2 \cdots \lambda_p = \det(A - 0I) = \det(A)$$

Thus for any square matrix, the determinant equals the product of eigenvalues.

$\Rightarrow$ proofed.

STAT3017/6017 - Big Data Statistics - Assessment 1 2023

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```r
knitr::opts_chunk$set(echo = TRUE)
library(ggplot2)
```

Question 3 (a)

The n-dimensional volume of a Euclidean ball of radius R in a n-dimensional space is:

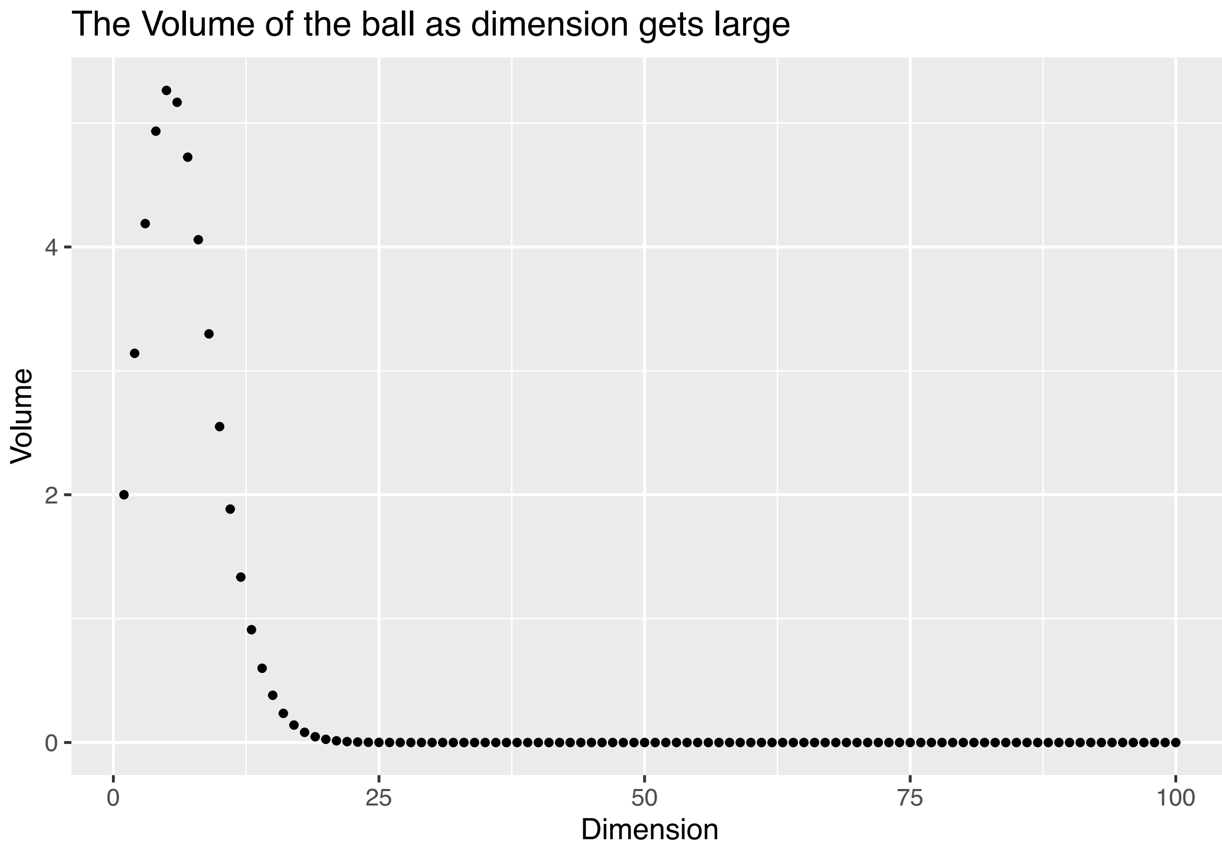
$$V_n(R) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n$$

The implementation is followed:

```r
vol_function <- function(R, n){
  pi^(n/2)*R^n/gamma(n/2+1)
}
```

(b)

```r
num_iteration <- 100
R<-1
vol <- data.frame(cbind(1:num_iteration, rep(0,100)))
names(vol) <- c("dim", "volume")
for(n in 1:num_iteration){
  vol[n,2] <- vol_function(R, n)
}
p1<-ggplot(vol, aes(x = dim, y = volume))+
  geom_point(size = 1)+
  labs(title = "The Volume of the ball as dimension gets large",
       x = "Dimension", y ="Volume")
p1
```



(c)

In Lecture 1, we discuss outliers detection. In the lecture case, all the mass lies in the tail when p approaches to infinity. This leads to almost all the points are outliers in a high-dimensional case.

Here, the increase of the number of dimension does not lead to a monotonic increase of the volume. However, the volume increases up to some dimension and then decrease to nearly 0 after it. The volume that a ball contains is less and less as the dimension goes to infinity.

The plot indicates a correspondence of the 2 cases. The fact that when the dimension gets large, the volume covered becomes less breaks we intuitive understanding for the high-dimensional objects. Thus, it leads to a further study.