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Miscellanea

Some tests for correlation matrices

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SUMMARY

Tests for certain properties of correlation matrices proposed by Bartlett & Rajalakshman (1953) and Kullback (1959, 1967) do not have the asymptotic distributions ascribed to them.

1. INTRODUCTION

A test for a specified correlation matrix was proposed by Bartlett & Rajalakshman (1953). Given a p-variate normal sample of size N for which the correlation matrix is R, to test the hypothesis that the population correlation matrix is $P = P_0$, completely specified, against a general alternative, they proposed the statistic

$$T_1 = N\left\{\log \frac{|P_0|}{|R|} - p + \operatorname{tr}(P_0^{-1}R)\right\}$$

and gave its asymptotic distribution as $\chi^2_{\frac{1}{2}p(p-1)}$. In a subsequent paper, Bartlett (1954) noted that this was not entirely correct, but that the statistic T_1 could be used as χ^2 unless the apparent significance of the result was on the borderline. This qualification seems to have escaped notice as in subsequent applications T_1 has been used as $\chi^2_{\frac{1}{2}p(p-1)}$ without comment (Kullback, 1967), and Kullback (1959) has even 'proved' that T_1 has this asymptotic χ^2 distribution under the null hypothesis. It therefore seems worth while to derive the asymptotic distribution of T_1, which is shown to be a linear form in $\frac{1}{2}p(p-1)$ independent χ^2_1 variables, and not in general $\chi^2_{\frac{1}{2}p(p-1)}$ unless $P_0 = I$, the identity matrix.

In practice, the construction of simultaneous confidence bounds on the elements of P_0 would be more useful than a test of the null hypothesis. Confidence bounds based on the statistic T_1 are not readily obtainable, but Roy's simultaneous confidence bounds for the elements of a covariance matrix are easily converted to bounds on the elements of the correlation matrix.

2. DISTRIBUTION OF T_1

We find the asymptotic distribution of T_1 by the delta method. As originally proposed, T_1 contained the factor $N-(2p+11)/6$ rather than N but this does not affect the asymptotic distribution. Now

$$\frac{\partial T_1}{\partial r_{jk}} = N(-2r^{jk} + 2\rho^{jk})$$

and evaluating this at $r_{jk} = \rho_{jk}$, we find that all first derivatives are zero. Then

$$\frac{\partial^2 T_1}{\partial r_{jk}\,\partial r_{lm}} = 2N(R^{-1}J_{jk}R^{-1})_{lm},$$

where J_{jk} is the $p\times p$ matrix whose (j,k)th and (k,j)th elements are 1 and whose other elements are zero. Evaluating at $r_{jk} = \rho_{jk}$, we find

$$T_1 \doteq N \sum_{j<k} \sum_{l<m} (\rho^{jl}\rho^{km} + \rho^{jm}\rho^{kl})(r_{jk}-\rho_{jk})(r_{lm}-\rho_{lm}).$$

Now the $(r_{jk}-\rho_{jk})$ are asymptotically multivariate normal with means 0 and covariance matrix Ψ/n, where Ψ may be determined as by Anderson (1958, p. 76), and is given explicitly by its elements

$$\psi_{jk,jk} = (1-\rho_{jk}^2)^2,$$

$$\psi_{jk,jl} = -\tfrac{1}{2}\rho_{jk}\rho_{jl}(1-\rho_{jk}^2-\rho_{jl}^2-\rho_{kl}^2) + \rho_{kl}(1-\rho_{jk}^2-\rho_{jl}^2),$$

$$\psi_{jk,lm} = \tfrac{1}{2}\rho_{jk}\rho_{lm}(\rho_{jl}^2+\rho_{jm}^2+\rho_{kl}^2+\rho_{km}^2) - \rho_{jk}\rho_{jl}\rho_{kl} - \rho_{jk}\rho_{jm}\rho_{km} - \rho_{jl}\rho_{jm}\rho_{lm} - \rho_{kl}\rho_{km}\rho_{lm}.$$

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Now T_1 is distributed asymptotically under H_0 as a quadratic form in normal variables with zero means. Let the matrix of the form be A, so that $a_{jk,lm} = \rho^{jl}\rho^{km} + \rho^{jm}\rho^{kl}$. Let the eigenvalues of $\Psi^{\frac{1}{2}} A \Psi^{\frac{1}{2}}$ be $\lambda_1, \ldots, \lambda_\nu$, with corresponding eigenvectors $\mathbf{l}_1, \ldots, \mathbf{l}_\nu$, where $\nu = \frac{1}{2}p(p-1)$. Then

$$T_1 \simeq N\mathbf{r}' A\mathbf{r} = \sum_{i=1}^{\nu} \lambda_i (\mathbf{l}_i' \mathbf{x}_i)^2,$$

where the $\mathbf{x}_i$ are independent $N_\nu(\mathbf{0}, I)$ variables. Thus T_1 is distributed asymptotically as a linear form in independent χ_1^2 variables, the coefficients of the form being the eigenvalues of $A\Psi$. These cannot easily be determined except in special cases.

3. Case $p = 2$

Here $$T_1 \simeq N\{\rho^{11}\rho^{22} + (\rho^{12})^2\}(r-\rho)^2 = \frac{N(1+\rho^2)}{(1-\rho^2)^2}(r-\rho)^2,$$

while $$\psi_{12,12} = (1-\rho^2)^2. \quad \text{Hence} \quad T_1 \simeq (1+\rho^2)\,\chi_1^2,$$

and T_1 does not have an asymptotic χ_1^2 distribution unless $\rho = 0$. The use of T_1 as χ_1^2 would seriously overestimate the significance of an observed result if ρ is large. Thus if $\rho = 0{\cdot}8$, a test using χ_1^2 at the nominal 5 % level would have true size 0·126, asymptotically, and a nominal 1 % level test would have true size 0·044. In practice of course Fisher's z transformation would be used here.

4. Case $p = 3$

If any two of the correlations in P_0 are equal, it may be shown that both A and Ψ have the structure of compound symmetry (Votaw, 1948), and the eigenvalues of $A\Psi$ may be easily obtained. Thus, for example, if P_0 is a matrix of equal correlations, we find

$$T_1 \simeq \lambda_1 \chi_1^2 + \lambda_2 \chi_2^2,$$

where $\lambda_1 = 1 + 2\rho^2$, $\lambda_2 = \{\frac{1}{2}(1+\rho)(2+2\rho-\rho^2)\}/(1+2\rho)$, ρ being the common correlation. Thus T_1 will not have an asymptotic χ_3^2 distribution unless $\rho = 0$.

If P_0 is an autocorrelation matrix, we have

$$T_1 \simeq \chi_1^2 + \lambda_1 \chi_1^2 + \lambda_2 \chi_1^2,$$

where $\lambda_1 = 1 + \frac{1}{2}\rho^2(1-\rho^2)$, $\lambda_2 = 1 + \frac{1}{2}\rho^2(3+\rho^2)$. Again T_1 will not have an asymptotic χ_3^2 distribution unless $\rho = 0$.

In the first case, if $\rho = -\frac{1}{3}$ we have $\lambda_1 = \lambda_2 = \frac{11}{9}$ and $T_1 \simeq \frac{11}{9}\chi_3^2$ under the null hypothesis. Thus a test using χ_3^2 at the nominal 5 % level would have true size 0·094, asymptotically. If $\rho = 0{\cdot}8$, then

$$T_1 \simeq 2{\cdot}28\chi_1^2 + 1{\cdot}025\chi_2^2.$$

Approximating the distribution of T_1 by either $a\chi_\nu^2$ or $a + b\{(\chi_\nu^2 - \nu)/\sqrt{(2\nu)}\}$, we find that a test using χ_3^2 at the nominal 5 % level would have true size 0·150 asymptotically.

In the second case, if $\rho = 1/\sqrt{2} = 0{\cdot}707$, then

$$T_1 \simeq \chi_1^2 + 1{\cdot}125\chi_1^2 + 1{\cdot}875\chi_1^2$$

and a test using χ_3^2 at the nominal 5 % level would have true size 0·127 asymptotically.

These results can be extended to any specified matrix P_0, though in general the eigenvalues must be obtained numerically. Suppose for example we wish to test the hypothesis

$$P_0 = \begin{bmatrix} 1 & 0{\cdot}8 & 0 \\ 0{\cdot}8 & 1 & 0{\cdot}2 \\ 0 & 0{\cdot}2 & 1 \end{bmatrix}.$$

Then $$P_0^{-1} = \begin{bmatrix} 3{\cdot}000 & -2{\cdot}500 & 0{\cdot}500 \\ -2{\cdot}500 & 3{\cdot}125 & -0{\cdot}625 \\ 0{\cdot}500 & -0{\cdot}625 & 1{\cdot}125 \end{bmatrix}.$$

The matrix A of the quadratic form in the r_{jk} is

$$A = \begin{bmatrix} 15\cdot625 & -3\cdot125 & 3\cdot125 \\ -3\cdot125 & 3\cdot625 & -3\cdot125 \\ 3\cdot125 & -3\cdot125 & 3\cdot90625 \end{bmatrix},$$

while their covariance matrix is

$$\Psi = \begin{bmatrix} 0\cdot1296 & 0\cdot072 & -0\cdot0256 \\ 0\cdot072 & 1 & 0\cdot768 \\ -0\cdot0256 & 0\cdot768 & 0\cdot9216 \end{bmatrix}.$$

The matrix $A\Psi$ is

$$A\Psi = \begin{bmatrix} 1\cdot720 & 0\cdot400 & 0\cdot080 \\ 0\cdot064 & 1\cdot000 & 0\cdot016 \\ 0\cdot080 & 0\cdot100 & 1\cdot120 \end{bmatrix}.$$

The characteristic equation of $A\Psi$ is $\lambda^3 - 3\cdot84\lambda^2 + 4\cdot7328\lambda - 1\cdot8896 = 0$ whose roots are $\lambda_1 = 0\cdot9624$, $\lambda_2 = 1\cdot1121$, $\lambda_3 = 1\cdot7655$. Then $T_1 \simeq 0\cdot9624\chi_1^2 + 1\cdot1121\chi_1^2 + 1\cdot7655\chi_1^2$ under the null hypothesis. This may be approximated by either $T_1 \simeq 1\cdot375\chi^2_{2\cdot79}$ or $T_1 \simeq 0\cdot252 + 1\cdot472\chi^2_{2\cdot44}$, or, rather more crudely, by $T_1 \simeq 1\cdot28\chi_3^2$. It is then clear whether a sample value of T_1 leads to rejection of the null hypothesis.

5. Two-sample test

A two-sample test for the equality of two correlation matrices has been proposed by Kullback (1967). Given two independent p-variate normal samples of sizes $N_1 + 1$ and $N_2 + 1$, for which the sample correlation matrices are R_1 and R_2 respectively, and the population correlation matrices are P_1 and P_2 respectively, to test the hypothesis that $P_1 = P_2$, unspecified, against a general alternative, Kullback proposes the statistic

$$T_2 = -2\log\{(|R_1|^{\frac{1}{2}N_1}|R_2|^{\frac{1}{2}N_2})/|\bar{R}|^{\frac{1}{2}N}\},$$

where $N = N_1 + N_2$, $N\bar{R} = N_1R_1 + N_2R_2$. Kullback gives the asymptotic distribution of T_2 as $\chi^2_{\frac{1}{2}p(p-1)}$.

Application of the delta method as above shows that

$$T_2 \simeq \frac{N_1N_2}{N}\sum_{j<k}\sum_{l<m}(\rho^{jl}\rho^{km} + \rho^{jm}\rho^{kl})\{r_{jk}^{(1)} - r_{jk}^{(2)}\}\{r_{lm}^{(1)} - r_{lm}^{(2)}\}$$

under H_0, where $R_i = \{r_{jk}^{(i)}\}\,(i = 1, 2)$. Thus exactly the same results hold for T_2 as for T_1, provided that as $N \to \infty$, $\lim(N_1/N_2) = 1$, with the exception that the eigenvalues of $A\Psi$ are now unknown, since the common P is not specified by H_0. This, however, does not affect the asymptotic distribution.

The test may be carried out in practice as above. Since P is not specified by H_0 it is estimated from the data by $\bar{R}$. Inversion of $\bar{R}$ is then carried out, and the covariance matrix is also estimated using the elements of $\bar{R}$.

Similar remarks apply to the k-sample test, an obvious generalization of the two-sample test, proposed by Kullback (1967).

6. Simultaneous confidence bounds

It appears very difficult to obtain simultaneous confidence bounds on all the ρ_{jk} by using the statistic T_1. Such bounds may be obtained quite easily from Roy's (1957) simultaneous confidence bounds on the elements of the covariance matrix.

Let $S \sim W_p(k, \Sigma)$. Then simultaneous $100(1-\alpha)$ % confidence bounds on all quadratic forms in the elements of Σ are given by

$$\lambda_1 \leqslant \frac{\mathbf{1}'S\mathbf{1}}{\mathbf{1}'\Sigma\mathbf{1}} \leqslant \lambda_2,$$

where λ_1 and λ_2 are such that

$$P(\lambda_1 < c_1 < \ldots < c_p < \lambda_2) = 1 - \alpha;$$

$c_1, \ldots, c_p$ are the ordered characteristic roots of $S\Sigma^{-1}$. Hanumara & Thompson (1968) have tabulated λ_1 and λ_2 for various values of α.

We may rewrite the double inequality as

$$\frac{1}{\lambda_2}\mathbf{1}'S\mathbf{1} \leqslant \mathbf{1}'\Sigma\mathbf{1} \leqslant \frac{1}{\lambda_1}\mathbf{1}'S\mathbf{1}.$$

Taking $\mathbf{l}'$ successively as $[1, 0, 0, \ldots, 0]$, $[0, 1, 0, \ldots, 0]$, $[l_1, l_2, 0, \ldots, 0]$, we may eliminate the variances and by choosing l_1, l_2 to minimize the length of the confidence intervals we obtain simultaneous confidence intervals for all ρ_{jk} of the form

$$\max\left(\frac{\lambda_1}{\lambda_2}r_{jk}+\frac{\lambda_1}{\lambda_2}-1, \frac{\lambda_2}{\lambda_1}r_{jk}-\frac{\lambda_2}{\lambda_1}+1\right) \leqslant \rho_{jk} \leqslant \min\left(\frac{\lambda_2}{\lambda_1}r_{jk}+\frac{\lambda_2}{\lambda_1}-1, \frac{\lambda_1}{\lambda_2}r_{jk}-\frac{\lambda_1}{\lambda_2}+1\right).$$

These intervals have a joint confidence coefficient of at least $100(1-\alpha)$ %. In fact the confidence coefficient will be considerably greater than this, especially for small p, as unwanted confidence intervals for the variances are also obtained.

Tables of percentage points of the extreme roots of a single Wishart matrix have been given by Hanumara & Thompson (1968). It is of interest to compare the confidence interval for $p = 2$ with that obtained by Fisher's transformation. Suppose that a correlation of $r = 0{\cdot}6$ is observed in a sample of $n = 31$. Then a 95 % confidence interval for ρ from Fisher's z transformation is (0·312, 0·787). From Hanumara & Thompson (1968), the upper and lower $2\frac{1}{2}$ % points are $\lambda_2 = 53{\cdot}39$, $\lambda_1 = 13{\cdot}49$. This gives the confidence interval (−0·583, 0·899) which is very much inferior to the Z interval.

As a further example, suppose the correlation matrix

$$R = \begin{bmatrix} 1 & 0{\cdot}6 & 0{\cdot}1 \\ 0{\cdot}6 & 1 & 0{\cdot}3 \\ 0{\cdot}1 & 0{\cdot}3 & 1 \end{bmatrix}$$

is observed in a sample of $n = 101$. Then simultaneous 95 % confidence intervals for the three population correlations, if normality is assumed, are $0{\cdot}04 \leqslant \rho_{12} \leqslant 0{\cdot}83$, $-0{\cdot}54 \leqslant \rho_{13} \leqslant 0{\cdot}62$ and $-0{\cdot}45 \leqslant \rho_{23} \leqslant 0{\cdot}71$.

References

Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. New York: Wiley.

Bartlett, M. S. (1954). A note on the multiplying factors for various χ^2 approximations. *J. R. Statist. Soc.* B **16**, 296–98.

Bartlett, M. S. & Rajalakshman, D. V. (1953). Goodness of fit tests for simultaneous autoregressive series. *J. R. Statist. Soc.* B **15**, 107–24.

Hanumara, R. C. & Thompson, W. A. Jr. (1968). Percentage points of extreme roots of a Wishart matrix. *Biometrika* **55**, 505–12.

Kullback, S. (1959). *Information Theory and Statistics*. New York: Wiley.

Kullback, S. (1967). On testing correlation matrices. *Appl. Statist.* **16**, 80–5.

Roy, S. N. (1957). *Some Aspects of Multivariate Analysis*. New York: Wiley.

Votaw, D. F. (1948). Testing compound symmetry in a normal multivariate distribution. *Ann. Math. Statist.* **19**, 447–73.

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On Hodges's bivariate sign test and a test for uniformity of a circular distribution

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Summary

Ajne (1968) studied a test N for uniformity of a circular distribution. Although his problem was apparently quite different from testing about location of a bivariate distribution to which Hodges's sign test applies, the two tests are, in fact, identical. The null distribution and tables for the N test developed by Ajne in 1968 are essentially a duplication of those for Hodges's test given earlier (1955–1962). Also it is shown that the Rayleigh test is locally most powerful invariant for the alternatives of non-uniformity on a circle generated by projecting a bivariate normal probability mass.