(d) Derive a formula for the first moment of the Fisher LSD in terms of its parameters s and t. You can assume that $h > t$ as this always holds by the definition of h.

$$\int_a^b x\, f_{s,t}(x)\, dx$$

$s \in (0, +\infty)$

$t \in [0, 1]$

$h > t$

$$= \int_a^b x \frac{1-t}{2\pi x(s+xt)} \sqrt{(x-a)(b-x)}\, dx$$

$$= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{\frac{|1+hz|^2}{(1-t)^2}(1-z^2)^2}{z(1+hz)(z+h)(tz+h)(t+hz)} dz$$

$$= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{|1+hz|^2(1-z^2)^2(1-t)^{-2}}{z(1+hz)(z+h)(tz+h)(t+hz)} dz$$

$$= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{\cancel{(1+hz)}\cancel{(1+hz^{-1})}(1-z^2)^2(1-t)^{-2}}{z^2\cancel{(1+hz)}\cancel{(1+hz^{-1})}(tz+h)(t+hz)} dz$$

$$= -\frac{h^2}{4\pi i(1-t)} \oint_{|z|=1} \frac{(1+z^2)^2}{z^2(tz+h)(t+hz)}$$

The integrand function has 3 simple poles at

$$z_0 = 0$$

$$z_1 = -\frac{h}{t}$$

$$z_2 = -\frac{t}{h}$$

As we know $h > t$, $z_1 = -\frac{h}{t} < -1$, for $t \in [0,1]$

Thus it is not a pole in the unit circle.

By the residue theorm:

$$\oint_\ell f(z)\, dz = 2\pi i \sum_{j=1}^{N} a_j$$

At $z_0 = 0$ with order 2,

Define $g(z) = (z - z_0)^n f(z)$ then

$$\operatorname{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} g^{(n-1)}(z)$$

$$g(z) = \cancel{z^2} \cdot \frac{(1-z^2)^2}{\cancel{z^2}\,(tz+h)(t+hz)}$$

$$= \frac{(1-z^2)^2}{(tz+h)(t+hz)}$$

$$g'(z) = \frac{-4z(1-z^2)(tz+h)(t+hz) - (1-z^2)^2(h^2+t^2+2htz)}{[(tz+h)(t+hz)]^2}$$

$$\lim_{z \to 0} g'(z) = \frac{-(h^2+t^2)}{(h \cdot t)^2}$$

$$= -\frac{1}{t^2} - \frac{1}{h^2}$$

$$\operatorname{Res}(f; z_0) = \frac{1}{(2-1)!} \cdot \left(-\frac{1}{t^2} - \frac{1}{h^2}\right) = -\frac{1}{t^2} - \frac{1}{h^2}$$

At $z_3 = -\frac{t}{h}$ with order 1,

Define $g(z) = (z - z_0)^n f(z)$ then

$$\text{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} g^{(n-1)}(z)$$

$$g(z) = \left[z - \left(-\frac{t}{h}\right)\right]^1 \cdot \frac{(1-z^2)^2}{z^2 \, (tz+h)(t+hz)} = \left(z + \frac{t}{h}\right) \cdot \frac{(1-z^2)^2}{z^2(tz+h)(t+hz)}$$

$$= \frac{(1-z^2)^2}{h\,(tz+h)\, z^2}$$

$$\lim_{z \to \frac{t}{h}} g(z) = \frac{\left(1 - \left(\frac{t}{h}\right)^2\right)^2}{h\left(-\frac{t^2}{h} + h\right) t^2/h^2} = \frac{\left(1 - \frac{t^2}{h^2}\right)^2}{(-t^2 + h^2)\, t^2/h^2} = \frac{\left(\frac{1}{h^2}(h^2 - t^2)\right)^2}{(h^2 - t^2)\, t^2/h^2}$$

$$= \frac{(h^2 - t^2)^{\cancel{2}}}{h^2 \cancel{(h^2 - t^2)}\, t^2} = \frac{h^2 - t^2}{h^2 t^2}$$

$$\text{Res}(f; z_3) = \frac{1}{(1-1)!} \cdot \frac{h^2 - t^2}{h^2 t^2} = \frac{h^2 - t^2}{h^2 t^2}$$

$$\int_a^b x\, f_{s,t}(x)\, dx = -\frac{h^2}{4\pi i(1-t)}\left[2\pi i\left(-\frac{t^2+h^2}{t^2h^2} + \frac{h^2-t^2}{t^2h^2}\right)\right]$$

$$= -\frac{h^2}{4\pi i(1-t)}\left[2\pi i\left(\frac{\cancel{h^2}-t^2-t^2-\cancel{h^2}}{t^2h^2}\right)\right]$$

$$= -\frac{h^2}{4\pi i(1-t)}\left[2\pi i\left(\frac{-2\cancel{t^2}}{\cancel{t^2}h^2}\right)\right]$$

$$= \cancel{-}\frac{\cancel{h^2}}{\cancel{4\pi i}(1-t)}\left[\cancel{2\pi i}\left(\cancel{-}\frac{2}{\cancel{h^2}}\right)\right]$$

$$= \frac{1}{1-t}$$

Question 2 [2 marks]
Show that the second moment

$$\int x^2\, p_{s,t}(x)\, dx = \frac{h^2+1-t}{(1-t)^3}.$$

$$\int_a^b x^2 f_{s,t}(x)\, dx$$

$s \in (0, +\infty)$

$t \in [0, 1]$

$h > t$

$$= \int_a^b x^2 \frac{1-t}{2\pi x(s+xt)} \sqrt{(x-a)(b-x)}\, dx$$

$$= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{\left(\frac{|1+hz|^2}{(1-t)^2}\right)^2 (1-z^2)^2}{z(1+hz)(z+h)(tz+h)(t+hz)}\, dz$$

$$= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{\left(\frac{|1+hz|^2}{(1-t)^2}\right)^2 (1-z^2)^2}{z(1+hz)(z+h)(tz+h)(t+hz)}\, dz$$

$$= -\frac{h^2}{4\pi i(1-t)^3} \oint_{|z|=1} \frac{(1+hz)^2(1+hz^{-1})^2(1-z^2)^2}{z(1+hz)(z+h)(tz+h)(t+hz)}\, dz$$

$$= -\frac{h^2}{4\pi i(1-t)^3} \oint_{|z|=1} \frac{(1+hz)^{\cancel{2}}(1+hz^{-1})^{\cancel{2}}(1-z^2)^2}{z^2\cancel{(1+hz)}\cancel{(1+hz^{-1})}(tz+h)(t+hz)}\, dz$$

$$= -\frac{h^2}{4\pi i(1-t)^3} \oint_{|z|=1} \frac{(1+hz)(1+hz^{-1})\ (1-z^2)^2}{z^2\ (tz+h)(t+hz)}\, dz$$

$$= -\frac{h^2}{4\pi i(1-t)^3 th} \oint_{|z|=1} \frac{(1+hz)(z+h)\ (1-z^2)^2}{z^3\ (z+h/t)(z+t/h)}\, dz$$

The integrand function has 3 simple poles at

$z_0 = 0$

$z_1 = -\frac{h}{t}$

$z_2 = -\frac{t}{h}$

As we know $h > t$, $z_1 = -\frac{h}{t} < -1$, for $t \in [0,1]$

Thus it is not a pole in the unit circle.

By the residue theorm: $\oint_\ell f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_j$

At $z_0 = 0$ with order 3,

Define $g(z) = (z-z_0)^n f(z)$ then $Res(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} g^{(n-1)}(z)$

$$g(z) = \cancel{(z-0)^3} \cdot \frac{(1+hz)(z+h)(1-z^2)^2}{\cancel{z^3}(z+h/t)(z+t/h)}$$

$$= \frac{(1+hz)(z+h)(1-z^2)^2}{(z+h/t)(z+t/h)}, \quad h(z) = (1+hz)(z+h)(1-z^2)^2$$

$$g'(z) = \frac{h'(z)(z+h/t)(z+t/h) - h(z)\cdot(2z+h/t+t/h)}{[(z+h/t)(z+t/h)]^2}$$

$$= \underbrace{\frac{h'(z)}{(z+h/t)(z+t/h)}}_{=: I_1} - \underbrace{\frac{h(z)\cdot(2z+h/t+t/h)}{[(z+h/t)(z+t/h)]^2}}_{=: I_2}$$

$$h(z) = (z+h+hz^2+h^2z)(1-2z^2+z^4)$$

$$= z + h + hz^2 + h^2 z - 2z^3 - 2hz^2 - 2hz^4 - 2hz^3 + z^5 + hz^4 + hz^6 + h^2z^5$$

$$= h + (1+h^2)z - hz^2 - (2+2h)z^3 - hz^4 + (1+h^2)z^5 + hz^6$$

$\lim_{z\to 0} h(z) = h$, $\lim_{z \to 0} h'(z) = 1+h^2$, $\lim_{h \to 0} h''(z) = -2h$.

$$g''(z) = I_1' - I_2'$$

$$I_1' = \frac{h''(z)\cdot(z+h/t)(z+t/h) - h'(z)\cdot(2z+h/t+t/h)}{[(z+h/t)(z+t/h)]^2}$$

$$\lim_{z\to 0} I_1' = \frac{-2h\cdot 1 - (1+h^2)(h/t+t/h)}{1}$$

$$= -2h - (1+h^2)(h/t+t/h)$$

$$-I_2' = \frac{\{h'(z)(2z+h/t+t/h)+h(z)\cdot 2\}[(z+h/t)(z+t/h)]^2}{[(z+h/t)(z+t/h)]^4}$$

$$- \frac{h(z)\cdot(2z+h/t+t/h)\cdot 2\cdot(z+h/t)(z+t/h)\cdot(2z+h/t+t/h)}{[(z+h/t)(z+t/h)]^4}$$

$$-\lim_{z\to 0} I_2' = -\{(1+h^2)(h/t+t/h)+2h\}\cdot 1 + h\cdot(h/t+t/h)\cdot 2\cdot(h/t+t/h)$$

$$= -2h - [1+h^2-2h(h/t+t/h)](h/t+t/h)$$

$$\lim_{z\to 0} g''(t) = \lim_{z\to 0} I_1 - \lim_{z\to 0} I_2$$

$$= -2h - (1+h^2)(h/t+t/h) - 2h -$$

$$[1+h^2-2h(h/t+t/h)](h/t+t/h)$$

$$= -4h - [2(1+h^2)-2h[h/t+t/h]](h/t+t/h)$$

$$\mathrm{Res}(f, z_0) = \frac{1}{(3-1)!}\left\{-4h - \left[2(1+h^2) - 2h[h/t + t/h)\right](h/t + t/h)\right\}$$

$$= -2h - \left(1 + h^2 - h^2/t - t\right) \cdot \frac{h^2+t^2}{th}$$

At $z_2 = -\frac{t}{h}$ with order 1,

$$g(z) = \cancel{\left[z - \left(-\frac{t}{h}\right)\right]^1} \cdot \frac{(1+hz)(z+h)(1-z^2)^2}{z^3(z+h/t)\cancel{(z+t/h)}}$$

$$= \frac{(1+hz)(z+h)(1-z^2)^2}{z^3(z+h/t)}$$

$$\lim_{z \to \frac{t}{h}} g(z) = \frac{(1-t)(-t/h+h)(1-t^2/h^2)^2}{-t^3/h^3(-t/h+h/t)}$$

$$= \frac{h^3(1-t)(h-t/h)(1-t^2/h^2)(1-t^2/h^2)}{-t^3(h^2-t^2)/ht}$$

$$= \frac{(1-t)(h^2-t)\cdot\cancel{(h^2-t^2)}(1-t^2/h^2)}{-t^{\cancel{3}2}\cancel{(h^2-t^2)}/h\cancel{t}}$$

$$= \frac{(1-t)(h^2-t)(h^2-t^2)}{-ht^2}$$

$$= \frac{(1-t)(h^2-t)(t^2-h^2)}{ht^2}$$

$$= \mathrm{Res}(f, z_2)$$

$$\mathrm{Res}(f, z_2) + \mathrm{Res}(f, z_0)$$

$$= \frac{(1-t)(h^2-t)(t^2-h^2)}{ht^2} - 2h - \left(1+h^2 - h^2/t - t\right) \cdot \frac{h^2+t^2}{th}$$

$$= \frac{(h^2-t-th^2+t^2)(t^2-h^2)}{ht^2} - 2h - \frac{(h^2+h^4-h^4/t-th^2+t^2+t^2h^2-th^2-t^3)}{th}$$

$$= -2h + \frac{-h^4+th^2+th^4-h^2t^2+t^2h^2-t^3-t^3h^2+t^4-(h^2t+h^4t-h^4-th^2+t^3+t^3h^2-t^2h^2-t^4)}{ht^2}$$

$$= \frac{\cancel{-2h^2t^2}\cancel{-h^4}\cancel{+th^2}\cancel{+th^4}\cancel{-h^2t^2}\cancel{+t^2h^2}-t^3-t^3h^2+t^4\cancel{-h^2t}\cancel{-h^4t}\cancel{+h^4}\cancel{+th^2}-t^3-t^3h^2\cancel{+t^2h^2}+t^4}{ht^2}$$

$$= \frac{2t^4-2h^2t^3-2t^3}{ht^2}$$

$$= \frac{2t^2-2h^2t-2t}{h}$$

$$\int_a^b x^2 P_{s,t}(x)\,dx = -\frac{h^2}{4\pi i(1-t)^3 th}\oint_{|z|=1} \frac{(1+hz)(z+h)(1-z^2)^2}{z^3(z+h/t)(z+t/h)}\,dz$$

$$= -\frac{h^2}{4\pi i(1-t)^3 th}\left\{2\pi i\left[\mathrm{Res}(f, z_0) + \mathrm{Res}(f, z_2)\right]\right\}$$

$$= -\frac{\cancel{h^2}}{\cancel{4\pi i}(1-t)^3 t\cancel{h}} \cdot \left[\cancel{2\pi i}\left(\frac{\cancel{2}t^2-\cancel{2}h^2\cancel{t}-\cancel{2t}}{\cancel{h}}\right)\right]$$

$$= \frac{-(t-h^2-1)}{(1-t)^3}$$

$$= \frac{h^2+1-t}{(1-t)^3}$$

Question 3 [2 marks]
Show that the variance equals $h^2/(1-t)^3$.

Observe that

1st moment: $\int_a^b x \, P_{s,t}(x) \, dx = \frac{1}{1-t}$

2nd moment: $\int_a^b x^2 \, P_{s,t}(x) \, dx = \frac{h^2+1-t}{(1-t)^3}$

Denote the eigenvalue as $\lambda \sim P_{s,t}(x)$

Easy to show:

$$\mathrm{Var}(\lambda) = \mathbb{E}(\lambda^2) - [\mathbb{E}(\lambda)]^2$$

$$= \frac{h^2+1-t}{(1-t)^3} - \left[\frac{1}{1-t}\right]^2$$

$$= \frac{h^2+1-t-(1-t)}{(1-t)^3}$$

$$= \frac{h^2}{(1-t)^3}$$