Assessment 2

Due by Tuesday 29 August 2023 09:00

Question 1 [2 marks]

Let $\mathbb{x}$ be a random vector with multivariate Gaussian distribution $N_p(0, \Gamma)$. Show that if $\text{rank}(\Gamma) = p$ then

$$\mathbb{x}'\Gamma^{-1}\mathbb{x} \sim \chi^2(p),$$

where $\chi^2(p)$ denotes the chi-squared distribution with p degrees of freedom.

Question 2 [6 marks]

In this question we consider the empirical spectral density (ESD) of large matrices with entries that are given by independent identically distributed random variables with zero mean and variance equal to 1. The steps you will take are:

[1] (a) Simulate $n = 500$ matrices $\mathbb{X}_n$ of size $p \times p$ where $p = 100$ with entries sampled from the standard Normal distribution. Calculate the eigenvalues of each of the matrices $\mathbb{X}_n/\sqrt{n}$ and plot the location of the eigenvalues together (i.e., obtained from all the matrices) on the complex plane.

[1] (b) Comment on the distribution of the eigenvalues.

[1] (c) Repeat part (a), this time with variance larger than 1, what do you notice?

[1] (d) Repeat the experiment three times with p and n larger and larger with the ratio p/n fixed. What do you observe?

[1] (e) Now, repeat the experiment with the entries sampled from a Student-T distribution with parameter $1 < \nu \leq 2$. What do you observe and what can you conclude?

[1] (f) Why don't you get a Wigner semi-circle distribution? Why don't you get a Marchenko-Pastur distribution for the limiting spectral distribution?

Question 3 [2 marks]

Using contour integration, calculate the following integral

$$\oint_{|z|=1} \frac{4+z}{z+z^2}\,\mathrm{d}z.$$