Exercises

Book: Elementary Analysis The Theory Of Calculus

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Lesson: Continuity of Function

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Topic: ≪18≫ Properties of Continuous Functions

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Question No. 18.1:- Let f be a continuous real-valued function on a closed interval $[a, b]$.Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exist x_0, y_0 in $[a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$. Show that if f assumes its maximum at $x_0 \in [a, b]$, then f assumes its minimum at x_0.

Proof:- Consider the continuous function f on $[a, b]$. Then it is bounded.Let M and m be the superimum and infimum of f in $[a, b]$ respectively.Given that $-f$ assumes its maximum at $x_0 \in [a, b]$, $\exists$ a positive integer M such that

$$-f(x) \leq M, x \in [a, b]$$

. Here $M = f(x_0)$ i.e.,

$$-f(x) \leq f(x_0), x \in [a, b] \ldots (i)$$

since the function under consideration is $f(x)$, therefore multiply eq^n (i) by -1

$$\Rightarrow f(x) \geq -f(x_0)$$

$$-f(x_0) \leq f(x)$$

$\Rightarrow \exists$ positive integer m such that

$$m \leq f(x)$$

also,

$$inf(f(x)) = -sup(-f(x))$$

$$i.e., inf(f(x)) = -sup(-f(x))$$
$$m = -f(x_0)$$

This gives ,

$$m \leq f(x), x \in [a, b]$$

Hence , by definition f assume its minimum at$x_0 \in [a, b]$.

Question No. 18.2:- Reread the proof of Theorem 18.1 with [a, b] replaced by (a, b). Where does it break down? Discuss.

Solution:- By the proof of theorem 18.1, we assume that$f$ is not bounded on (a, b). Then to each $n \in \mathbb{N}$, these corresponds an $x_n \in (0, b)$ such that $|f(x_n)| > n$. By Bolzano-Weirstrass theorem,(x_n) has a subsequence (x_{n_k}) that converges to some real number x_0.If the interval is not closed these is possibility that a function may be either unbounded or may not have maximum and minimum value.The proof breaks down at the assertion that x_0, which is defined as the limit of the convergent subsequence (x_{n_k}) is in the interval (a, b). Indeed, we would only know that this point is in the closed interval$[a, b]$. S_n is a convergent sequence with $a<S_n<b$ then $a \leq \lim_{n\to\infty} \leq b$.But since (a, b) is an open interval a, b are not the end points included in the interval.Hence limit of S_n cannot be founded.

Question18.3:- Not in syllabus.

Question18.4:- Let $\mathcal{S} \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in $\mathcal{S}$ converging to a number $x_0 \notin \mathcal{S}$. Show there exists an unbounded continuous function on $\mathcal{S}$.

Solution:- Let $\mathcal{S} \subset \mathbb{R}$. Suppose there exists a sequence (x_n) in $\mathcal{S}$ such that $x_n \to x_0 \notin \mathcal{S}$. Let$f(x) = \frac{1}{x-x_0}$.Then f is well-defined on $\mathcal{S}$ since $x_0 \in \mathcal{S}$, and is continuous since it is a quotient of continuous functions such that the denominator is nonzero. Now for any $\mathcal{M} > 0$, choose $\mathcal{N}$such that $n > \mathcal{N}$ implies $|x_n - x_0| < \frac{1}{\mathcal{M}}$. Then for $n > \mathcal{N}$, $|f(x_n| = \frac{1}{|x_n - x_0|} > \mathcal{M}$. Since $\mathcal{M}$ was arbitrary, f is unbounded on $\mathcal{S}$.

Question-18.5:-(a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(x) \leq g(x)$. Prove that $f(x) = g(x)$ for at least one x in $[a, b]$.

Solution:-Define $h : [a, b] \to R$ by $h(x) := f(x) - g(x)$. h is continuous on $[a, b]$.

Furthermore,

$$h(a) = f(a)g(a) \geq 0,$$

$$h(b) = f(b)g(b) \leq 0.$$

So $h(a) \geq 0 \geq h(b)$, so we can apply the intermediate value theorem to find an $x_0 \in [a, b]$ satisfying $h(x_0) = 0. Thus, f(x_0) = g(x_0)$.

(b) Show that Example 1 in the text is a special case of part (a).

Proof : We take f defined on $[0, 1]$ as in the example, and $g(x) = x$. The fact that $f(x) \in [0, 1]$ for all x implies, in particular, that

$$f(0) \geq 0 = g(0),$$

$$f(1) \leq 1 = g(1),$$

and we are looking for a point x_0 where $f(x_0) = x_0 = g(x_0)$.

Hence proved

Question 18.6:- Prove $x = cosx$ for some x in $(0, \frac{\pi}{2})$.

Proof:- Consider the function $f(X) = cos(x) - x$ which is a continuous function since both $cosx$ and x are continuous. Then, $f(0) = 1$ and $f(\frac{\pi}{2}) = f(\frac{-\pi}{2}) = 0$ Thus, by **Intermediate Value Theorem**, there is some $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$. This means exactly that $cos(x) = x$ has a solution in the interval $(0, \frac{\pi}{2})$.

Question 18. 7:- Prove $xe^x = 2$ for some x in $(0, 1)$.

Solution:- Let $f(x) = xe^x$ implying $f(0) = 0 < 2$ and $f(1) = e > 2$ and by the intermediate value theorem there must exist some $x \epsilon [0, 1]$ such that $f(x) = 2$.
But this is for a closed interval $[0, 1]$. For open interval $(0, 1)$,if $f(x) = 2$ is some point in the interval $[0, 1]$,and we know for a fact that its not 0 or 1,we therefore know that $f(x) = 2$ is in the interval $(0, 1)$.

Question 18.8 :- Suppose f is a real valued continous function $\mathbb{R}$ and $f(a)f(b) < 0$, for some a, b $\in \mathbb{R}$.Prove there exists x between such that $f(x) = 0$

Solution :- f is a real valued continous function on $\mathbb{R}$ and given $f(a)f(b) < 0$.

$$\Rightarrow f(a)f(b) < 0$$

$$f(a) < 0$$

$$\Rightarrow f(a)f(b) < 0$$

$$f(a) < 0$$

$$f(a) \in I^-$$

Let $f(a) = -m$ then,

$$\begin{aligned} &= -mf(b) < 0 \\ &= mf(b) > 0 \\ &= f(b) > 0, \end{aligned}$$

$$f(b) \in I^{+}$$

Let $f(b) = j$,
Hence,$f(x)$ is continous over (-m,j).
Hence,$f(x)$ follows IMV property on [-m,j].
Since $0 \in$ (-m,j),
then,by IMV,$\exists\ x_0 \in \mathbb{R}$
such that $f(x_0) = 0$

Hence Proved.

Question 18.9:- Prove that a polynomial function f of odd degree has at least one real root.

Solution:- Let $p(x)$ be a polynomial of odd degree
then

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

for m is odd.
Assume $a_m = 1$,

$$\lim_{x \to -\infty} p(x) = -\infty$$

$\Rightarrow p(x) < 0$ for some x and $p(x) > 0$ for some other x.

$$p(x) = x^n \left[1 + \frac{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}{x^n} \right] \tag{1}$$

Let $c = 1 + |a_0| + |a_1| + \cdots + \||a_{n-1}|$.
If $|x| > c$, then
$|a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}| \leq (|a_0| + |a_1| + \cdots + |a_{n-1}|)\,|x| < |x|^n$
So,the number in brackets in (1) is positive . Now if $x > c$, then $x^n > 0$. So, $f(x) > 0$.
And if $x < -c$, then $x^n < 0$. So, $f(x) < 0$.

Question 18.10:- Suppose that f is continuous on $[0, 2]$and $f(0) = f(2)$.Prove that there exist x, y in [0,2] such that $|y - x|$ and $f(x) = f(y)$.

Hint:Consider $g(x) = f(x + 1) - f(x)$on [0,1]

Proof:- Given:f is continuous on [0,2]. $f(0) = f(2)$

To prove that $f(x) = f(y)$, where $|y - x| = 1$

Consider $g(x) = f(x + 1) - f(x)$ on[0,1]

Then g(0)=$f(1) - f(0) = f(1) - f(2)$

$g(1) = f(2) - f(1)$

$g(0) = -g(1)$

$\Rightarrow$ at same point $g(x) = 0$

$\Rightarrow f(x + 1) = f(x)$

Take $y = x + 1$

$f(y) = f(x)$

Thus, g is a continuous function on [0,1] is satisfy Intermediate Value Theorem on [0,1].

Question 18.11:- State and prove Theorem 18.5 for strictly decreasing functions.

Solution:-Theorem 18.5: **Statement** *Let g be a strictly decreasing function on an interval J such that g(J) is an interval I.Then g is continuous on J.*

Proof:- Consider x_0.We assume x_0 is not an endpoint of J; Then $g(x_0)$ is not an endpoint of I, So there exists $\epsilon_0 > 0$ such that $(g(x_0) - \epsilon_0, g(x_0) + \epsilon_0) \subseteq I$.
Let $\epsilon > 0$.Assuming $\epsilon < \epsilon_0$.Then there exists $x1, x2 \epsilon J$ such that $x1 < x2$ and $g(x1) = g(x_0) - \epsilon$ and $g(x2) = g(x_0) + \epsilon$.For a strictly decreasing function if x1¡x2 then $g(x1) > g(x2)$.So,$x1 < x_0 < x2$.Also,if $x1 < x < x2$,then $g(x1) > g(x) > g(x2)$,hence $g(x_0) - \epsilon > g(x) > g(x_0) + \epsilon$, and hence $|g(x_0) - g(x)| < \epsilon$.Now if

δ =min$(x2 - x_0, x_0 - x1)$,then $|x_0 - x| < \delta$ implies $x1 > x > x2$ and hence $|g(x_0) - g(x)| < \epsilon$.

<u>Question 18.12</u>:- Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and let $f(0) = 0$.

(a) *Observe that f is discontinuous at* 0.

(b) *Show f has the intermediate value property on R.*

<u>Solution</u>:-

$$f(x) = \left\{ \begin{array}{lll} \sin\left(\frac{1}{x}\right) & , & x \neq 0 \\ 0 & , & x = 0 \end{array} \right\}$$

(a) To prove discontinuity of f at 0,we find a sequence (x_n) converging to 0 such that $f(x_n)$ does not converge to $f(0) = 0$.
So we will arrange for

$$\sin\left(\frac{1}{x_n}\right) = \frac{1}{x_n}$$

where x_n converges to 0.
Thus,we want

$$\sin\left(\frac{1}{x_n}\right) = 1,$$

$\Rightarrow$ $\frac{1}{x_n} = (2n\pi + \pi/2)$

$$x_n = \frac{1}{(2n\pi + \pi/2)}$$

Then,$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{(2n\pi+\pi/2)} = 0$
while

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right)$$

$$= 1$$

$\Rightarrow$ $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} f(x_n)$

Since,for every sequence x_n converging to 0,$\lim f(x_n) \neq f(0)$.Hence f is discontinuous at 0.

(b) Now if we take the interval

$$I = [-2/\pi, 2/\pi]$$

then f is discontinuous at $0 \in I$,but

$$f\,(-2/\pi) = -1$$

$$f\,(2/\pi) = 1$$

We get $1/\pi \in [-2/\pi, 2/\pi]$
where

$$\sin\left(\frac{1}{x}\right) = 0$$

Therefore,this function satisfies intermediate value property although it is discontinuous.

$$\ldots$$