ДЗ 3 „Квантовая механика"

Шлейхер

БФЗ-19-1

Задание 1

$$S = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

1) $$\det(S - \lambda \hat{I}) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta - \lambda \end{vmatrix} = -(\cos^2\theta - \lambda^2) - \sin^2\theta = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

• $\lambda = 1$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \psi_1(\cos\theta - 1) + \psi_2 \sin\theta e^{-i\varphi} = 0 \\ \psi_1 \sin\theta e^{i\varphi} - \psi_2(\cos\theta + 1) = 0 \end{cases}$$

$$\psi_1 = \frac{\sin\theta}{1-\cos\theta} e^{-i\varphi} \psi_2 = ctg\frac{\theta}{2} e^{-i\varphi}\psi_2$$

$$f_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_2 \begin{pmatrix} ctg\frac{\theta}{2} e^{-i\varphi} \\ 1 \end{pmatrix}$$

• $\lambda = -1$

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \psi_1(\cos\theta + 1) + \psi_2 \sin\theta e^{-i\varphi} = 0 \\ \psi_1 \sin\theta e^{i\varphi} + \psi_2(1 - \cos\theta) = 0 \end{cases}$$

$$\psi_1 = \frac{-\sin\theta}{1+\cos\theta} e^{-i\varphi}\psi_2 = -tg\frac{\theta}{2} e^{-i\varphi}\psi_2$$

$$f_2 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_2 \begin{pmatrix} -tg\frac{\theta}{2} e^{-i\varphi} \\ 1 \end{pmatrix}$$

$$\langle f_1 | f_2 \rangle = ctg\left(\frac{\theta}{2}\right) e^{-i\varphi} (-tg\theta) e^{i\varphi} + 1 \cdot 1 = 0$$

ч.т.д.

Задание №2

$U(x,y) = \frac{m\omega^2}{2}(x^2+y^2)$ $\qquad x(0)=a \qquad y(0)=0$

$\qquad\qquad\qquad\qquad\qquad\quad \dot{x}(0)=0 \qquad \dot{y}(0)=v_0$

1) $L = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} - \frac{m\omega^2}{2}(x^2+y^2)$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$-m\omega^2 x = m\ddot{x}$$

$$\ddot{x} + \omega^2 x = 0$$

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$\begin{cases} x(0) = C_1 + C_2 = a \\ \dot{x}(0) = i\omega(C_1 - C_2) = 0 \end{cases}$$

$$C_2 = C_1 = \frac{a}{2}$$

$$x(t) = \frac{a}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = a\cos\omega t$$

$$x^2 = a^2\cos^2\omega t$$

$$\frac{x^2}{a^2} = \cos^2\omega t$$

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

$$-m\omega^2 y = m\ddot{y}$$

$$\ddot{y} + \omega^2 y = 0$$

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$$y(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ \dot{y}(0) = i\omega(C_1 - C_2) = v_0 \end{cases}$$

$$C_1 = \frac{v_0}{2i\omega}, \quad C_2 = \frac{-v_0}{2i\omega}$$

$$y(t) = \frac{v_0}{2i\omega}\left(e^{i\omega t} - e^{-i\omega t}\right) = \frac{v_0}{\omega}\sin\omega t$$

$$y^2 = \frac{v_0^2}{\omega^2}\sin^2\omega t$$

$$\left(\frac{\omega y}{v_0}\right)^2 = \sin^2\omega t$$

$$\frac{x^2}{a^2} + \left(\frac{\omega y}{v_0}\right)^2 = 1$$

$$\frac{\omega y}{v_0} = \sqrt{-\frac{x^2}{a^2} + 1} \Rightarrow y = \frac{v_0}{\omega}\sqrt{-\frac{x^2}{a^2} + 1}$$

$$\boxed{y = \frac{v_0}{\omega}\sqrt{1 - \frac{x^2}{a^2}}}$$

2) $E = T + U = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} + \frac{m\omega^2}{2}(x^2+y^2)$

$$E = \frac{m}{2}\frac{dx^2+dy^2}{dt^2} + \frac{m\omega^2}{2}(x^2+y^2)$$

$$dt = \frac{\sqrt{dx^2+dy^2}}{\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} = \frac{m\,dx}{dt} = m\frac{dx}{\sqrt{dx^2+dy^2}}\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$$

$$p_y = m\dot{y} = m\frac{dy}{\sqrt{dx^2+dy^2}}\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$$

$$S_0 = \int p_x dx + p_y dy$$

$$S_0 = \int m\sqrt{dx^2+dy^2}\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)} = \int m\underbrace{\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\sqrt{1+\dot{x}^2}}_{L(x,\dot{x})}\,dy$$

где $\dot{x} = \frac{dx}{dy}$

Запишем E через начальные условия

$$E = \frac{mv_0^2}{2} + \frac{ma^2\omega^2}{2} = const$$

$$L(x,\dot{x}) = \sqrt{\omega^2(a^2-x^2-y^2)+v_0^2}\sqrt{1+\dot{x}^2}$$

$$\frac{\partial L}{\partial x} = \frac{d}{dy}\frac{\partial L}{\partial \dot{x}}$$

$$\frac{-\omega^2 x\sqrt{1+\dot{x}^2}}{\sqrt{\omega^2(a^2-x^2-y^2)+v_0^2}} = \frac{d}{dy}\Big(\underbrace{\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\sqrt{\omega^2(a^2-x^2-y^2)+v_0^2}}_{\varphi(x)}\Big)$$

$$\int -\omega^2 x\cdot\dot{x} = \int \varphi(x)\,\varphi'(x)$$

$$-\omega^2x^2 = \varphi^2(x) + C$$

$$\left.\begin{array}{l}\frac{dx}{dy} = 0\\ x(0)=0\\ y(0)=0\end{array}\right\} \quad C = -\frac{0}{1}\left(\sqrt{v_0^2}\right)^2 + \omega^2a^2 \Rightarrow C = -\omega^2a^2$$

$$-\omega^2(x^2 + a^2) = \frac{\dot{x}^2}{1+\dot{x}^2}\left(\omega^2(a^2-x^2-y^2)+v_0^2\right)$$

$$\dot{x}^2\left(\omega^2(a^2-x^2-y^2)+v_0^2+\omega^2(x^2-a^2)\right) = -\omega^2(x^2-a^2)$$

$$\dot{x}^2 = \frac{\omega^2(a^2-x^2)}{v_0^2-\omega^2y^2} \;\Big|:\omega^2 \Rightarrow \ddot{x} = \sqrt{\frac{a^2-x^2}{\left(\frac{v_0}{\omega}\right)^2-y^2}}$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{dy}{\sqrt{\left(\frac{v_0}{\omega}\right)^2-y^2}}$$

$$\arcsin\frac{x}{a} = \arcsin\left(\frac{y\omega}{v_0}\right)+C$$

$$\arcsin\left(\frac{x}{a}\right) - \arcsin\left(\frac{y\omega}{v_0}\right) = C$$

$$\sin\left(\arcsin\frac{x}{a} - \arcsin\left(\frac{y\omega}{v_0}\right)\right) = \sin C$$

$$\left.\begin{array}{l} \frac{x}{a}\sqrt{1-\left(\frac{y\omega}{v_0}\right)^2} - \frac{y\omega}{v_0}\sqrt{1-\frac{x^2}{a^2}} = \frac{a}{a}\sqrt{1-0}-0 = \sin C \Rightarrow \sin C = 1 \\ \cos\left(\arcsin\frac{x}{a} - \arcsin\left(\frac{y\omega}{v_0}\right)\right) = \cos C \\ \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\left(\frac{y\omega}{v_0}\right)^2} + \frac{x}{a}\cdot\frac{y\omega}{v_0} = 0 \Rightarrow \cos C = 0 \end{array}\right\} C = \frac{\pi}{2}$$

$$\sin\left(\arcsin\frac{x}{a}\right) = \sin\left(\arcsin\left(\frac{y\omega}{v_0}\right)+\frac{\pi}{2}\right) = \cos\left(\arcsin\left(\frac{y\omega}{v_0}\right)\right)$$

$$\frac{x}{a} = \sqrt{1-\left(\frac{y\omega}{v_0}\right)^2}$$

$$\frac{x^2}{a}+\left(\frac{y\omega}{v_0}\right)^2 = 1 \rightarrow$$ эллипс.

Задание №3

$$\hat{H} = -\mu B S \,, \quad |\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

$$\psi = e^{-\frac{\hat{H}t}{i\hbar}}$$

$$|\psi(t)\rangle = e^{-\frac{\hat{H}t}{i\hbar}} |\psi(0)\rangle = e^{-\frac{\hat{H}t}{i\hbar}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e^{-\frac{\hat{H}t}{i\hbar}} = e^{-\frac{\mu B S t}{i\hbar}} = e^{\frac{i\mu B S t}{\hbar}} = e^{iaS} = 1 + \frac{iaS}{1} - \frac{(aS)^2}{2} - \frac{i(aS)^3}{6} + \dots =$$

$$\frac{\mu B t}{\hbar} = a$$

$$= \sum_{j=0}^{\infty} \frac{(aS)^{2j}(-1)^j}{(2j)!} + i \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^{2k+1} S^{2k+1}}{(2k+1)!}$$

$$S|f\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \cos\theta + f_2 \sin\theta e^{-i\varphi} \\ f_1 \sin\theta e^{i\varphi} - f_2 \cos\theta \end{pmatrix}$$

$$S^2|f\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} f_1 \cos\theta + f_2 \sin\theta e^{-i\varphi} \\ f_1 \sin\theta e^{i\varphi} - f_2 \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} f_1 \cos^2\theta + f_2 \cos\theta \sin\theta e^{-i\varphi} + f_1 \sin^2\theta - f_2 \sin\theta \cos\theta e^{-i\varphi} \\ f_1 \sin\theta \cos\theta e^{i\varphi} + f_2 \sin^2\theta - f_1 \cos\theta \sin\theta e^{i\varphi} + f_2 \cos^2\theta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$S^{2n} = 1 \,, \quad S^{2n+1} = S$$

$$e^{iaS} = 1 \cdot \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j}}{(2j)!} + iS \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^{2k+1}}{(2k+1)!} = \cos a + iS \sin a = \cos\left(\frac{\mu B t}{\hbar}\right) + iS \sin\left(\frac{\mu B t}{\hbar}\right)$$

$$\underline{|\psi(t)\rangle = \left(\cos\left(\frac{\mu B t}{\hbar}\right) + iS \sin\left(\frac{\mu B t}{\hbar}\right)\right) |\psi(0)\rangle}$$

Задание №4

$$\psi(x) = const\, x^2 e^{-\frac{x}{\lambda}} \quad , x \geq 0$$

1) $$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

$$const^2 \int_{0}^{+\infty} x^4 e^{-\frac{2x}{\lambda}} = const^2 \left(-\frac{\lambda x^4}{2} e^{-\frac{2x}{\lambda}} \Big|_{0}^{+\infty} + 2\lambda \int_0^{+\infty} x^3 e^{-\frac{2x}{\lambda}} dx = \ldots\ldots \right.$$

$$= (const)^2 e^{-\frac{2x}{\lambda}} \left(-\frac{(2\lambda x^4 + 4\lambda^2 x^3 + 6\lambda^3 x^2 + 6\lambda^4 x + 3\lambda^5)}{4} \right) \Big|_0^{+\infty} = (const)^2 \frac{3\lambda^5}{4} = 1$$

$$\underbrace{const = \frac{2\sqrt{3\lambda}}{3\lambda^3}}$$

2) $$w = \int_x^{x+dx} |\psi(x')|^2 dx' = \int_x^{x+dx} \frac{4}{3\lambda^5} x'^4 e^{-\frac{2x'}{\lambda}} dx'$$

3) $$\langle x \rangle = \int_0^{+\infty} x |\psi(x)|^2 dx = \frac{4}{3\lambda^5} \int_0^{+\infty} x^5 e^{-\frac{2x}{\lambda}} dx = \frac{4}{3\lambda^5} \left(\overset{0}{\cancel{-\frac{\lambda}{2} x^5 e^{-\frac{2x}{\lambda}}}} \Big|_0^{+\infty} + \frac{5\lambda}{2} \int_0^{+\infty} e^{-\frac{2x}{\lambda}} x^4 dx \right) =$$

$$= \frac{10}{3\lambda^4} \int_0^{+\infty} e^{-\frac{2x}{\lambda}} x^4 dx = \frac{10}{3\lambda^4} \cdot \frac{1}{\left(\frac{2\sqrt{3\lambda}}{3\lambda^3}\right)^2} = \underline{2{,}5\lambda}$$