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Задание №1

$$A = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$$

1. $$\det(A-\lambda E) = \begin{vmatrix} (\cos\theta - \lambda) & -i\sin\theta \\ i\sin\theta & -\cos\theta - \lambda \end{vmatrix} = -(\cos^2\theta - \lambda^2) + i^2\sin^2\theta =$$

$$= \lambda^2 - 1 = 0$$

$$\left[\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \end{aligned}\right.$$

$\lambda = 1$

$$\begin{pmatrix} \cos\theta - 1 & -i\sin\theta \\ i\sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \psi_1(\cos\theta - 1) - i\sin\theta\,\psi_2 = 0 \\ i\psi_1\sin\theta + \psi_2(\cos\theta + 1) = 0 \end{cases} \qquad \psi_1 = \frac{i\sin\theta}{\cos\theta - 1}\psi_2$$

$$\bar{f}_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \frac{i\sin\theta}{\cos\theta - 1}\psi_2 \\ \psi_2 \end{pmatrix} = \psi_2 \begin{pmatrix} \frac{i\sin\theta}{\cos\theta - 1} \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} \cos\theta + 1 & -i\sin\theta \\ i\sin\theta & 1 - \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \psi_1(\cos\theta + 1) - i\sin\theta\,\psi_2 = 0 \\ i\psi_1\sin\theta + \psi_2(1 - \cos\theta) = 0 \end{cases} \qquad \psi_1 = \frac{i\sin\theta}{\cos\theta + 1}\psi_2$$

$$\bar{f}_2 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_2 \begin{pmatrix} \frac{i\sin\theta}{\cos\theta + 1} \\ 1 \end{pmatrix}$$

Пусть $\bar{g}_1 = \bar{f}_1$

$$\bar{g}_2 = \bar{f}_2 - \lambda_{21}\bar{g}_1, \text{ где } \lambda_{21} = \frac{(\bar{f}_2, \bar{g}_1)}{(\bar{g}_1, \bar{g}_1)}$$

$$\lambda_{21} = 1 + \frac{1}{\cos\theta}$$

$$\bar{g}_2 = \bar{f}_2 - \left(1 + \frac{1}{\cos\theta}\right)\bar{g}_1 = \bar{f}_2 - \left(1 + \frac{1}{\cos\theta}\right)\bar{f}_1$$

Заметим, что

$$\hat{A}\bar{g}_2 = \hat{A}\bar{f}_2 - \hat{A}\left(1 + \frac{1}{\cos\theta}\right)\bar{f}_1 = 0\bar{f}_2 - \left(1 + \frac{1}{\cos\theta}\right)\cdot 0\bar{f}_1 = 0\cdot\bar{g}_2$$

т.е $g_2 \perp g_1$

ч.т.д

2. $e^{i\alpha A}$ – ?

$$e^{i\alpha A} = \sum_{k=1}^{\infty} \frac{(i\alpha A)^k}{k!} = 1 + \frac{i\alpha A}{1} - \frac{\alpha^2 A^2}{2} - \frac{i\alpha^3 A^3}{6} + \ldots = 1 + i\sum_{j=1}^{\infty} \frac{(-1)^{j+1}\alpha^{2j-1} A^{2j+1}}{(2j-1)!} + \sum_{k=1}^{\infty} \frac{(-1)^k (\alpha^{2k}) A^{2k}}{(2k)!}$$

$$A^2|f\rangle = A|A|f\rangle = A\left|\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right. = A\left|\begin{pmatrix} f_1\cos\theta - f_2 i\sin\theta \\ f_1 i\sin\theta - f_2\cos\theta \end{pmatrix}\right. =$$

$$= \begin{pmatrix} f_1\cos^2\theta - if_2\sin\theta\cos\theta + f_1\sin^2\theta + if_2\sin\theta\cos\theta \\ if_1\sin\theta\cos\theta + f_2\sin^2\theta - if_1\cos\theta\sin\theta + f_2\cos^2\theta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = |f\rangle$$

$$A^{2k} = 1, \quad A^{2j-1} = A$$

$$e^{i\alpha A} = 1 + iA\sum_{j=1}^{\infty} \frac{(-1)^{j+1}\alpha^{2j-1}}{(2j-1)!} + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{2k}}{(2k)!} = \underline{iA\sin\alpha + \cos\alpha}$$

Задание №2

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

1. $$\det(A - \lambda E) = \frac{1}{\sqrt{2}} \begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = \frac{1}{\sqrt{2}}\left(-\lambda(\lambda^2 + i^2) + i(-i\lambda)\right) =$$

$$= \frac{1}{\sqrt{2}}\left(-\lambda^3 + 2\lambda\right) = 0$$

$$\left[\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = \sqrt{2} \\ \lambda_3 = -\sqrt{2} \end{array}\right.$$

- $\lambda_1 = 0$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -i\psi_2 = 0 \\ \psi_1 i - i\psi_3 = 0 \\ i\psi_3 = 0 \end{cases} \Rightarrow \begin{cases} \psi_1 = \psi_3 \\ \psi_2 = 0 \end{cases} \Rightarrow f_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \psi_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- $\lambda_2 = \sqrt{2}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{2} & -i & 0 \\ i & -\sqrt{2} & -i \\ 0 & i & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\psi_1\sqrt{2} - i\psi_2 = 0 \\ i\psi_1 - \sqrt{2}\psi_2 - i\psi_3 = 0 \\ i\psi_2 - \sqrt{2}\psi_3 = 0 \end{cases} \Rightarrow \begin{cases} \psi_1 = \frac{-i}{\sqrt{2}}\psi_2 \\ \psi_3 = \frac{i}{\sqrt{2}}\psi_2 \end{cases} \Rightarrow f_2 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{2}\psi_2 \begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} = \frac{1}{2}\psi_2 \begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix}$$

- $\lambda_3 = -\sqrt{2}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & -i & 0 \\ i & \sqrt{2} & -i \\ 0 & i & \sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \sqrt{2}\,\psi_1 - i\psi_2 = 0 \\ i(\psi_1 - \psi_3) + \sqrt{2}\,\psi_2 = 0 \\ i\psi_2 + \sqrt{2}\,\psi_3 = 0 \end{cases} \Rightarrow f_3 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{2}\psi_2 \begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix} = \frac{1}{2}\psi_2 \begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$$

$$f_1 \cdot f_2 = 1\cdot(-i) + 0 + 1\cdot i = 0 \Rightarrow \bar{f}_1 \perp \bar{f}_2$$

$$f_1 \cdot f_3 = 1\cdot i + 0 + 1\cdot(-i) = 0 \Rightarrow \bar{f}_1 \perp \bar{f}_3$$

Пусть $\bar{g}_1 = \bar{f}_1$, $\bar{g}_2 = \bar{f}_2$

$$\bar{g}_3 = \bar{f}_3 - \lambda_{31}\bar{g}_1 - \lambda_{32}\bar{g}_2$$

$$\lambda_{31} = \frac{(\bar{f}_3, \bar{g}_1)}{(\bar{g}_1, \bar{g}_1)} = 0 \; ; \; \lambda_{32} = \frac{(\bar{f}_3, \bar{g}_2)}{(\bar{g}_2, \bar{g}_2)} = 1$$

$$\bar{g}_3 = \bar{f}_3 - \bar{g}_2 = \bar{f}_3 - \bar{f}_2$$

Т.к. $\bar{g}_3\hat{A} = \hat{A}\bar{f}_3 - \hat{A}\bar{f}_2 = 0f_3 - 0f_2 = 0\cdot \bar{g}_3$

то $\bar{g}_3 \perp \bar{g}_2$

3.

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} + C_3 \begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$$

$$\begin{cases} C_1 - iC_2 + iC_3 = 1 \\ \sqrt{2}C_2 + \sqrt{2}C_3 = 0 \\ C_1 + iC_2 - iC_3 = 0 \end{cases} \Rightarrow \quad \begin{aligned} C_1 &= \frac{1}{2} \\ C_2 &= i\frac{1}{4} \\ C_3 &= -i\frac{1}{4} \end{aligned}$$

$$|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{i}{4}\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} - \frac{i}{4}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$$

2. $$e^{i\theta A} = 1 + \frac{i\theta A}{1} - \frac{\theta^2 A^2}{2} - \frac{i\theta^3 A^3}{6} + \ldots = \sum_{j=0}^{\infty} \frac{(\theta A)^{2j}(-1)^j}{(2j)!} + i\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\theta^{2k+1}A^{2k+1}}{(2k+1)!}$$

$$A|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -if_2 \\ f_1 i - if_3 \\ if_2 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} -f_2 \\ f_1 - f_3 \\ f_2 \end{pmatrix}$$

$$A^2|f\rangle = A|A|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{i}{\sqrt{2}}\begin{pmatrix} -f_2 \\ f_1 - f_3 \\ f_2 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} i(f_3 - f_1) \\ -if_2 - if_2 \\ i(f_1 - f_3) \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} f_3 - f_1 \\ -2f_2 \\ f_1 - f_3 \end{pmatrix}$$

$$A^3|f\rangle = \frac{-1}{2\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} f_3 - f_1 \\ -2f_2 \\ f_1 - f_3 \end{pmatrix} = \frac{-1}{2\sqrt{2}}\begin{pmatrix} 2if_2 \\ (f_3 - f_1)i - i(f_1 - f_3) \\ -2if_2 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} -f_2 \\ f_3 - f_1 \\ f_2 \end{pmatrix}$$

$$A^{2n+1} = A, \quad A^{2n} = A^2$$

$$e^{i\theta A} = A^2 \sum_{j=0} \frac{(-1)^j \theta^{2j}}{(2j)!} + iA \sum_{k=1} \frac{(-1)^{k+1}\theta^{2k-1}}{(2k-1)!} = \underline{A^2 \cos\theta + iA\sin\theta}$$

Задание №3

$$\hat{T}_a \psi(x) = \psi(x+a)$$

1. $\langle \psi | \hat{T}_a \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, \psi(x+a) |x\rangle\, dx = \int_{-\infty}^{+\infty} \psi^*(x'-a)\, \psi(x') |x'\rangle dx' =$

$x' = x + a$

$= \langle \hat{T}_{-a} \psi | \psi \rangle \Rightarrow \underline{\hat{T}_a^+ = \hat{T}_{-a}} \Rightarrow$ <u>не эрмитов</u>

3. $\hat{T}_a^+ = \hat{T}_{-a}$

$\hat{T}_a^+ \hat{T}_a |\psi\rangle = \hat{T}_{-a} \psi(x+a) = \psi(x) \Rightarrow \underline{\hat{T}_a^+ \hat{T}_a = 1}$

4. $\hat{T}_a \psi(x) = \int_{-\infty}^{+\infty} T_a(x, x')\, \psi(x')\, dx' = \psi(x+a)$

$$\underline{\hat{T}_a(x, x') = \delta(x' - (x+a))}$$

Задание №4

$$\hat{S} = \alpha\left(x^2 \frac{d}{dx} - \frac{d}{dx}\, x^2\right)$$

1. $$\hat{S}|\psi\rangle = \alpha\left(x^2 \frac{d\psi}{dx} - 2x\psi - x^2\frac{d\psi}{dx}\right) = -2x\alpha\psi$$

$$\langle\psi|\hat{S}|\psi\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\cdot(-2\alpha x)\psi(x)\,dx = \langle\hat{S}^+\psi|\psi\rangle = \int_{-\infty}^{+\infty}(\hat{S}^+\psi)^*\psi(x)\,dx =$$

$$= \int_{-\infty}^{+\infty}(-2\alpha x\cdot\psi(x))^*\cdot\psi\,dx$$ (это только для $\alpha \in Re$)

2. Для $\alpha \in Re$ оператор $\hat{S}$ эрмитов

1) $\alpha \in Re$: $\hat{S}^+ = \hat{S}$

2) $\alpha \notin Re$: $(\hat{S}^+\psi)^* = -2\alpha x\psi^* = -2\underbrace{\alpha e^{i\pi}}_{\alpha} x\psi^*$

$$\hat{S}^+ = -2\alpha^* x$$

3. $$\hat{S}|\psi\rangle = \int_{-\infty}^{+\infty} S(x,x')\psi(x')\,dx' = -2\alpha x\,\psi(x)$$

$$S(x,x') = -2\alpha x\,\delta(x'-x)$$

Задание №5

$\cdot$ $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Д-ть: $B^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$

1. $BX = X'$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2\end{pmatrix} = \begin{pmatrix} x_1' \\ x_2'\end{pmatrix}$$

$B^{-1}X' = X$

$$\begin{pmatrix} a^{-1} & b^{-1} \\ c^{-1} & d^{-1}\end{pmatrix}\begin{pmatrix} x_1' \\ x_2'\end{pmatrix} = \begin{pmatrix} x_1 \\ x_2\end{pmatrix}$$

$$\begin{pmatrix} a^{-1}(ax_1+bx_2) + b^{-1}(cx_1+dx_2) \\ c^{-1}(ax_1+bx_2) + d^{-1}(cx_1+dx_2)\end{pmatrix} = \begin{pmatrix} x_1 \\ x_2\end{pmatrix}$$

$$\begin{array}{cc} a^{-1} & b^{-1} \end{array}$$
$$\left(\begin{array}{cc|c} a & c & 1 \\ b & d & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c} ad & cd & d \\ bc & cd & 0\end{array}\right) \to (ad-bc)a^{-1} = d \Rightarrow a^{-1} = \frac{d}{ad-bc}$$

$$\searrow \left(\begin{array}{cc|c} ab & cb & b \\ ab & ad & 0\end{array}\right) \to (bc-ad)\,b^{-1} = b \Rightarrow b^{-1} = \frac{-b}{ad-bc}$$

$$\left(\begin{array}{cc|c} \overset{c^{-1}}{a} & \overset{d^{-1}}{c} & 0 \\ b & d & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} ad & cd & 0 \\ bc & cd & c \end{array}\right) \Rightarrow (bc-ad)\cancel{b}^{-1} = c \Rightarrow c^{-1} = \frac{-c}{ad-bc}$$

$$\searrow \left(\begin{array}{cc|c} ab & cb & 0 \\ ab & ad & a \end{array}\right) \Rightarrow (ad-bc)d^{-1} = a \Rightarrow d^{-1} = \frac{a}{ad-bc}$$

$$\begin{pmatrix} a^{-1} & b^{-1} \\ c^{-1} & d^{-1} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

ч.т.д

2. $$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\langle\varphi|O\psi\rangle = \langle O^{+}\varphi|\psi\rangle$$

$$|O\psi\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1\cos\theta - \psi_2\sin\theta \\ \psi_1\sin\theta + \psi_2\cos\theta \end{pmatrix}$$

$$\langle\varphi|O\psi\rangle = \varphi_1^{*}(\psi_1\cos\theta - \psi_2\sin\theta) + \varphi_2^{*}(\psi_1\sin\theta + \psi_2\cos\theta)$$

$$\langle\varphi|O\psi\rangle = \psi_1(\varphi_1^{*}\cos\theta + \varphi_2^{*}\sin\theta) + \psi_2(\varphi_2^{*}\cos\theta - \varphi_1^{*}\sin\theta) = \langle O^{+}\varphi|\psi\rangle$$

$$\langle O^{+}\varphi| = \begin{pmatrix} \varphi_1^{*}\cos\theta + \varphi_2^{*}\sin\theta \\ \varphi_2^{*}\cos\theta - \varphi_1^{*}\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \varphi_1^{*} \\ \varphi_2^{*} \end{pmatrix}$$

$$O^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

3. $$O^{-1} = \frac{1}{\det O}\cdot O^{+} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Задание №6

$$\langle(\hat{A}\hat{B})^{+}\varphi|\psi\rangle = \langle\varphi|\hat{A}\hat{B}\psi\rangle = \langle\hat{A}^{+}\varphi|\hat{B}\psi\rangle = \langle\hat{A}^{+}\cdot\hat{B}^{+}\varphi|\psi\rangle$$

$$(\hat{A}\hat{B})^{+} = \hat{A}^{+}\cdot\hat{B}^{+}$$

ч.т.д.

Задание №7

$$\langle\psi|\hat{C}^{+}\hat{C}|\psi\rangle = \langle\hat{C}\psi|\hat{C}\psi\rangle = \int_{-\infty}^{+\infty}|\hat{C}\psi|^2dx \geq 0$$

ч.т.д.