In GTSAM, the partial derivatives of the error function for each optimization variable need to be given when customizing a factor. The procedure for deriving the partial derivatives of binary factors for optimization variables of type SE(3) is given in [1], based on this, the procedure for deriving the partial derivatives of ternary factor error functions for variables of type SE(3) is given below.

The derivation of SE(3) type variables requires the introduction of Lie algebra, and the definitions in Lie algebra relevant to this paper are given below: each Lie algebra element denoted as $\boldsymbol{\xi}$， $\boldsymbol{\xi}$ is a six-dimensional vector. $\boldsymbol{\xi}=[\boldsymbol{\rho},\boldsymbol{\phi}]^{\mathrm{T}}$, $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ are both three-dimensional vectors, the former stands for translation, and the latter stands for rotation. The ∧ and ∨ operations are defined as follows:

$$\boldsymbol{\phi}^{\wedge}=\begin{pmatrix}0 & -\phi_3 & \phi_2\\ \phi_3 & 0 & -\phi_1\\ -\phi_2 & \phi_1 & 0\end{pmatrix} \tag{1}$$

$$\boldsymbol{\xi}^{\wedge}=\begin{pmatrix}\boldsymbol{\phi}^{\wedge} & \boldsymbol{\rho}\\ \boldsymbol{0}^{\boldsymbol{T}} & 0\end{pmatrix} \tag{2}$$

$$\ln(T)^{\vee}=\boldsymbol{\xi} \tag{3}$$

For the sake of simplicity, T_i、T_j and T_k are used to represent the three variables connected by the ternary factor, $T_i,T_j,T_k\in\mathrm{SE}(3)$. The error function of the ternary factor used in LIMOT [2] in the Lie algebra form is:

$$\boldsymbol{e}=\ln\left(T_i^{-1}T_jT_k^{-1}\right)^{\vee} \tag{4}$$

According to the Li algebra derivation, T_i , T_j and T_k are given a left perturbation $\delta\boldsymbol{\xi}_i$、$\delta\boldsymbol{\xi}_j$ and $\delta\boldsymbol{\xi}_k$, respectively, and then the error function becomes:

$$\hat{\boldsymbol{e}}=\ln\left(T_i^{-1}\exp\left((-\delta\boldsymbol{\xi}_i)^{\wedge}\right)\exp\left((\delta\boldsymbol{\xi}_j)^{\wedge}\right)T_jT_k^{-1}\exp\left((-\delta\boldsymbol{\xi}_k)^{\wedge}\right)\right)^{\vee} \tag{5}$$

To continue the derivation, the Baker-Campbell-Hausdorff (BCH) expression [3], Eq. (6), and the accompanying property on SE(3), Eq. (7) is introduced.

$$\ln\left(\exp\left(\boldsymbol{\xi}^{\wedge}\right)\exp\left(\delta\boldsymbol{\xi}^{\wedge}\right)\right)^{\vee}\approx\boldsymbol{\mathcal{T}}_r^{-1}\delta\boldsymbol{\xi}+\boldsymbol{\xi} \tag{6}$$

$$T\exp(\boldsymbol{\xi}^{\wedge})\,T^{-1}=\exp\left((\mathrm{Ad}(\mathrm{T})\boldsymbol{\xi})^{\wedge}\right) \tag{7}$$

where ξ is the Lie algebra $T\in\mathrm{SE}(3)$, $\boldsymbol{\mathcal{T}}_r$ is the Jacobi matrix, the form of which is given later, and $\mathrm{Ad}(T)$ is the adjoint matrix of T , which is defined as:

$$\mathrm{Ad}(T)=\begin{pmatrix}R & \boldsymbol{t}^{\wedge}R\\ 0 & R\end{pmatrix} \tag{8}$$

where R， $\boldsymbol{t}$ are the rotation matrix and translation vector corresponding to the transformation matrix T , respectively.

In order to use the BCH expression, i.e., Eq. (6), it is necessary to move all of the perturbation terms in Eq. (5) to the rightmost end. This requires utilizing the concomitant property on SE3. Transforming the form of (7), we have:

$$\exp(\boldsymbol{\xi}^\wedge)\, T = T\exp\left((Ad(T^{-1})\boldsymbol{\xi})^\wedge\right) \tag{9}$$

Eq. (9) shows that by introducing a concomitant term, it is possible to exchange the T on the left and right sides of the perturbation, based on which Eq. (5) can be further simplified:

$$\begin{aligned}
\hat{\boldsymbol{e}} &= \ln\left(T_i^{-1}\exp\left((-\delta\boldsymbol{\xi}_i)^\wedge\right)\exp\left(\delta\boldsymbol{\xi}_j{}^\wedge\right)T_j T_k^{-1}\exp\left((-\delta\boldsymbol{\xi}_k)^\wedge\right)\right)^\vee \\
&= \ln\left(T_i^{-1}\exp\left((-\delta\boldsymbol{\xi}_i)^\wedge\right)T_j T_k^{-1}\exp\left((\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_j)^\wedge\right)\exp\left((-\delta\boldsymbol{\xi}_k)^\wedge\right)\right)^\vee \\
&= \ln\left(T_i^{-1}T_j T_k^{-1}\exp\left(-\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_i)^\wedge\right)\exp\left((\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_j)^\wedge\right)\exp\left((-\delta\boldsymbol{\xi}_k)^\wedge\right)\right)^\vee \\
&\approx \ln\left(T_i^{-1}T_j T_k^{-1}\left[\boldsymbol{E} - \left(\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_j\right)^\wedge - (\delta\boldsymbol{\xi}_k)^\wedge\right]\right)^\vee
\end{aligned} \tag{10}$$

where $\boldsymbol{E}$ is the identity matrix. Using the BCH expression, we can obtain:

$$\begin{aligned}
\hat{\boldsymbol{e}} &\approx \ln\left(T_i^{-1}T_j T_k^{-1}\left[\boldsymbol{E} - \left(\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_i\right)^\wedge + \left(\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right)\delta\boldsymbol{\xi}_j\right)^\wedge - (\delta\boldsymbol{\xi}_k)^\wedge\right]\right)^\vee \\
&\approx \boldsymbol{e} + \frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_i} + \frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_j} + \frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_k}
\end{aligned} \tag{11}$$

where the partial derivative of the ternary factor with respect to T_i is:

$$\frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_i} = -\boldsymbol{\mathcal{J}}_r^{-1}(\boldsymbol{e})\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right) \tag{12}$$

the partial derivative with respect to T_j is:

$$\frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_j} = \boldsymbol{\mathcal{J}}_r^{-1}(\boldsymbol{e})\mathrm{Ad}\left(\left(T_j T_k^{-1}\right)^{-1}\right) \tag{13}$$

the partial derivative with respect to T_k is:

$$\frac{\partial\boldsymbol{e}}{\partial\delta\boldsymbol{\xi}_k} = -\boldsymbol{\mathcal{J}}_r^{-1}(\boldsymbol{e}) \tag{14}$$

The form of $\boldsymbol{\mathcal{J}}_r^{-1}(\boldsymbol{e})$ is too complicated and can be approximated using the following equation when the error is small [3]:

$$\boldsymbol{\mathcal{J}}_r^{-1}(\boldsymbol{e}) \approx \boldsymbol{E} + \frac{1}{2}\begin{pmatrix}\boldsymbol{\phi}_e^\wedge & \boldsymbol{\rho}_e^\wedge \\ 0 & \boldsymbol{\phi}_e^\wedge\end{pmatrix} \tag{15}$$

Where $\boldsymbol{\rho_e}$ and $\boldsymbol{\phi_e}$ are the translation, and rotation components corresponding to the Lie algebra e, respectively. The derivation of the partial derivatives of the ternary factor error function with respect to each optimization variable is completed.

References

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