$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

January 4th PDEs / ODE review

say we have equation $y'' - y' - 6y = 0$

$y(0) = 2 \quad y'(0) = -1$

$\mathcal{L}\{f(t)\} = F(s)$ input a function $f(t)$ output $F(s)$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$\mathcal{L}\{e^{at}\},\ t \geq 0 \quad F(s) = \int_0^\infty e^{(a-s)t} = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$

If $a < s$ $F(s) = \frac{1}{s-a}$ else diverges

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$\mathcal{L}\{u(t-a)\} = \int_a^\infty e^{-st} \cdot dt = \frac{e^{-sa}}{s}$$

because $< a$ ~~if~~ $u(t-a) = 0$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$$

$$\Gamma(n+1) = n! = \int_0^\infty \underbrace{e^{-t}}_{dv} \cdot \underbrace{t^n}_{u} \, dt$$

$$= -e^{-t} \cdot t^n \Big|_0^\infty - \int_0^\infty -e^{-t} \cdot n \cdot t^{n-1} \, dt$$

$$= n \cdot \Gamma(n)$$

ex: $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n$ where $u = st$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} \cdot u^n \, du$$

$$= \frac{1}{s^{n+1}} \cdot \Gamma(n+1) = \frac{n!}{s^{n+1}}$$

$\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,\mathcal{L}\{a\} + b\,\mathcal{L}\{b\}$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$

$f'(t) + 2f(t) = 3, \quad f(0) = 4$ apply laplace

$\mathcal{L}\{f'(t)\} + 2\mathcal{L}\{f(t)\} = \mathcal{L}\{3\}$

$s \cdot F(s) - f(0) + 2F(s) = \frac{3}{s}$

$F(s) = \frac{4 + 3/s}{s+2} = \frac{4s+3}{s(s+2)} = \frac{3}{2s} + \frac{5}{2s+4} = F(s)$

applying inverse laplace using table

$f(t) = \frac{3}{2} + \frac{5}{2}e^{-2t}$