Math 322 – Linear Algebra

Homework 2

Amandeep Gill

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Problem 1 If V, W are vector spaces over a field F, with V_1 and W_1 respective subspaces of V and W, and with a linear function $T : V \to W$, then $T(V_1)$ is a subspace of W and $\{\vec{x} : T(\vec{x}) \in W_1\}$ is a subspace of V.

Proof: Let V, W be vector spaces over a field F, with V_1 and W_1 respective subspaces of V and W, and let $T : V \to W$ be a linear function.

(a) $T(V_1)$ is a subspace of W.

Since V_1 is a subspace of V, for all $\vec{x} \in V_1$ there exists $\vec{x_1}, \vec{x_2} \in V_1$ and $c \in F$ such that $\vec{x} = c\vec{x_1} + \vec{x_2}$. Using the definition of linearity of T, $T(\vec{x}) = T(c\vec{x_1} + \vec{x_2}) = cT(\vec{x_1}) + T(\vec{x_2})$. Therefore $T(V_1)$ is closed for vector addition and scalar multiplication, and is thus a subspace of W.

(b) $\{\vec{x} : T(\vec{x}) \in W_1\}$ is a subspace of V.

Let $V_T = \{\vec{x} : T(\vec{x}) \in W_1\}$. For all $\vec{x} \in V_T$, there exists $\vec{x_1}, \vec{x_2} \in V_T$ and $c \in F$ such that $T(\vec{x}) = cT(\vec{x_1}) + T(\vec{x_2})$ as $T(V_T)$ is a subspace of W. Since T is linear, $T(\vec{x}) = T(c\vec{x_1} + \vec{x_2})$ and $\vec{x} = c\vec{x_1} + \vec{x_2}$. V_T is thus closed under scalar multiplication and vector addition and is a subspace of V.

$\blacksquare$

Problem 2 If $B \in M_{n\times n}(F)$ such that B is an invertible matrix, then the function $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ such that $\Phi(A) = B^{-1}AB$, then Φ is an isomorphism.

Proof: Let $c \in F$ and $A_1, A_2 \in M_{n\times n}(F)$ such that for all $A \in M_{n\times n}$, we have $A = cA_1 + A_2$, then $\Phi(cA_1 + A_2) = B^{-1}(cA_1 + A_2)B$. By the distributive and scalar multiplicative laws for $n \times n$ matrices, we have $\Phi(cA_1 + A_2) = cB^{-1}A_1B + B^{-1}A_2B = c\Phi(A_1) + \Phi(A_2)$. So Φ is linear. Further, for all A if $\Phi(A) = 0_n$ then $B^{-1}AB = 0_n$ and $B(B^{-1}AB)B^{-1} = 0_n$, so by associativity, $A = 0_n$ and $\text{null}(\Phi) = \{0_n\}$. Therefore Φ is an isomorphism.

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Problem 3 Let V, W be vector spaces over F and $T, U : V \to W$ be linear.

(a) $R(T+U) = R(T) + R(U)$

Proof: Using the definition of function addition, $R(T+U) = (T+U)(V)$. By the same, $(T+U)(V) = T(V) + U(V)$. Since $R(T) = T(V)$ and $R(U) = U(V)$. Thus $R(T+U) = R(T) + R(U)$.

$\blacksquare$

(b) If $\dim(W) \in \mathbb{N}$ then $\operatorname{rank}(T+U) \leqslant \operatorname{rank}(T) + \operatorname{rank}(U)$

Proof: By part (a), $R(T+U) \subseteq R(T)+R(U)$. $\operatorname{rank}(T+U) = \dim(R(T+U))$. Using the Dimension Theorem for finite vector spaces, we have that $\dim(R(T+U)) = \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U))$. Since $\dim(R(T) \cap R(U)) \geqslant 0$, $\operatorname{rank}(T+U) \leqslant \dim(R(T)) + \dim(R(U)) = \operatorname{rank}(T) + \operatorname{rank}(U)$.

$\blacksquare$

(c) Deduce that $\operatorname{rank}(A+B) \leqslant \operatorname{rank}(A) + \operatorname{rank}(B)$ for $A, B \in M_{m\times n}(F)$

Proof: Let $\dim(V) = n$, $\dim(W) = m$ and $A, B \in M_{m\times n}(F)$, such that $L_A = T$, and $L_B = U$. Then $\operatorname{rank}(T) = \operatorname{rank}(L_A) = \operatorname{rank}(A)$ and $\operatorname{rank}(U) = \operatorname{rank}(L_B) = \operatorname{rank}(B)$. Using the result obtained from part (b), $\operatorname{rank}(T+U) \leqslant \operatorname{rank}(T) + \operatorname{rank}(U) = \operatorname{rank}(A) + \operatorname{rank}(B)$. Since $\operatorname{rank}(A+B) = \operatorname{rank}(T+U)$, $\operatorname{rank}(A+B) \leqslant \operatorname{rank}(A) + \operatorname{rank}(B)$.

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