Math 322 – Linear Algebra

Homework 1

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Problem 1 Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ with the operations component-wise addition and $\forall c \in \mathbb{R}$,

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$$

Show whether V is a vector space over $\mathbb{R}$

Proof: V is not a vector space since it fails the distributive property of vector over scalars
$(1+1)\vec{a} = ((1+1)a_1, \frac{a_2}{1+1}) = (2a_1, \frac{a_2}{2}) \neq (a_1 + a_1, a_2 + a_2) = (2a_1, 2a_2)$

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Problem 2 Show that if $W \subseteq V$ and V is a vector space over some field F, W is a vector space if and only if $span(W) = W$

Proof:

($\Rightarrow$) Let W be a subspace of V, then by the definition of vector spaces $\forall \vec{w_1}, \vec{w_2} \in W$ and $\forall c \in F$, $c\vec{w_1} + \vec{w_2} \in W$. Since $W \subseteq span(W)$ and $span(W) \subseteq W$ as the span consists of all linear combination of elements of W, $span(W) = W$.

($\Leftarrow$) Let $span(W) = W$. Since $W \subseteq V$ and because $span(W)$ denotes all possible linear combinations of elements from W, $\forall \vec{w_1}, \vec{w_2} \in W$ and $\forall c \in F$, $c\vec{w_1} + \vec{w_2} \in W$. Therefore W is a vector space.

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Problem 3 Let V be a vector space over some field F such that $dim(V) = n$, and let $S \subseteq V$ such that $span(S) = V$, then $\exists \beta \subseteq S$ such that $span(\beta) = V$, $|\beta| = n$, and $|S| \geqslant n$.

Proof: Since V is a vector space, V has a basis β_1 such that $span(\beta_1) = V$ and $|\beta| = n$. If $\beta_1 \subseteq S$, then the hypothesis holds. If $\beta_1 \not\subseteq S$, then $\beta_1 \subseteq span(S)$. This means that $\forall b_i \in \beta$, $\exists s_1, s_2, \ldots, s_m \in S$ and $c_1, c_2, \ldots, c_m \in F$ such that $b_i = c_1 s_1 + c_2 s_2 + \cdots + c_m s_m$. Let B be the set containing every s_i from S where the scalar multiplier is not the 0 in F, then $|B| \leqslant |\beta| \cdot m$ and $span(B) = S$. Because B is finite set that spans S, $\exists \beta \subseteq B$ such that $\beta \subseteq S$ and $span(\beta) = V$, and $|\beta| = n$. It also follows that $|S| \geqslant |\beta| = n$.

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