Math 440 – Real Analysis II

Homework 1

Amandeep Gill

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Problem 1 Find the limit inferior and limit superior of each of the following sequences $\{s_n\}$.

(a) $s_n = 2 - \frac{1}{n}$

Proof:

$$\overline{\lim}\{s_n\} = \underline{\lim}\{s_n\} = \lim_{n\to\infty} s_n = 2$$

$\blacksquare$

(b) $s_n = n \bmod 4$

Proof: For all $k \in \mathbb{N}$ the sequence $\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ contains only the values $\{0, 1, 2, 3\}$. Thus $\forall k$, $0 \leqslant s_k \leqslant 3$, so

$$\overline{\lim}\{s_n\} = 3$$

And

$$\underline{\lim}\{s_n\} = 0$$

$\blacksquare$

(c) $s_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} + \cos \pi n & \text{if } n \text{ is odd} \end{cases}$

Proof: For all $n \in \mathbf{N}$, $\frac{1}{n} + \cos \pi n \leq n$, so the $\sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ is dominated by the even terms of s_n. Therefore

$$\overline{\lim}\{s_n\} = \lim_{n\to\infty} 2n = \infty$$

By the same argument, the $\inf\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ is dominated by the odd terms of s_n, which is split between two cases: $s_k = \frac{1}{k} + 1$ and $s_k = \frac{1}{k} - 1$. Thus

$$\underline{\lim}\{s_n\} = \lim_{n\to\infty} \frac{1}{4n+3} - 1 = -1$$

$\blacksquare$

(d) $s_n = f(n)$, where $f\colon \mathbb{N} \to \mathbb{Q} \cap (0,1)$ is a bijection

Proof: Let $\epsilon > 0$ and $\overline{S} = \{s_n\colon 1 - s_n < \epsilon\}$
By Math340 results we know that there exists an infinite number of rationals between any two distinct reals (in this case 1 and $1-\epsilon$). Therefore $\overline{S}$ is a countably infinite set, and $\overline{S} \setminus \{s_1, s_2, \ldots, s_{k-1}\}$ yields a similarly infinite set.
From this,

$$\overline{\lim}\{s_n\} = \sup \overline{S} = 1$$

Similarly, with $\underline{S} = \{s_n\colon s_n < \epsilon\}$ we have

$$\underline{\lim}\{s_n\} = \inf \underline{S} = 0$$

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Problem 2 Give an example of a sequence $\{s_n\}$ and a real number s such that $s_n < s$ for all n, but $\overline{\lim}\, s_n \geqslant s$

Proof: Let $s_n = 1 - \frac{1}{n}$ and $s = 1$, then $\forall n, s_n < s$ and $\overline{\lim}\, s_n = s$

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Problem 3 Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of nonnegative real numbers.

Proof: $\overline{\lim}(a_n b_n) \leqslant (\overline{\lim}\, a_n)(\overline{\lim}\, b_n)$

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Since a_n, b_n are bounded and nonnegative, let $A, B \in \mathbb{R}^+ \cup \{0\}$ such that $\overline{\lim}\{a_n\} = A$ and $\overline{\lim}\{b_n\} = B$
By definition, $\forall n, a_n \leqslant A$ and $b_n \leqslant B$, so $a_n b_n \leqslant AB$
AB is thus an upper bound on the sequence $\{a_n b_n\}$
Therefore $\overline{\lim}\{a_n b_n\} \leqslant AB$

Problem 4 The series $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges

Proof: Let $s_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$
Then $s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$
Since $\frac{1}{\sqrt{k}}$ is monotone decreasing, we have

$$s_n \geqslant \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

So for all $n \geqslant 1$, the partial sum $s_n \geqslant \sqrt{n}$
Therefore, $\overline{\lim}\{s_n\} \geqslant \overline{\lim}\{\sqrt{n}\} = \infty$

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Problem 5 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real values.

The series $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges if and only if $\{a_n\}$ converges.

Proof: Let $s_n = \sum_{k=1}^{n}(a_k - a_{k+1})$

The partial sums s_n are telescoping sums, so $\forall n, s_n = a_1 - a_{n+1}$

Therefore, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_1 - a_n = a_1 - \lim_{n\to\infty} a_n$

Since $\lim_{n\to\infty} a_n$ exists if and only if $\exists A \in \mathbb{R}$ such that $\{a_n\} \to A$

$\{s_n\}$ converges to $a_1 - A$ if and only if $\{a_n\}_{n\in\mathbb{N}}$ converges to A

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Problem 6 $\sum_{k=1}^{\infty} \frac{2}{k(k+1)} = 2$

Proof: By partial fraction decomposition, $\frac{2}{2(k+1)} = \frac{2}{k} - \frac{2}{k+1}$

Let $a_k = \frac{2}{k}$, then $a_1 = 2$ and $\lim_{k\to\infty} a_k = 0$

Therefore by previous work,

$\sum_{k=1}^{\infty}(a_k - a_{k+1}) = a_1 - \lim_{n\to\infty} a_n = 2 - 0 = 2$

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Problem 7 $\sum_{k=1}^{\infty}(\sqrt{k+1} - \sqrt{k})$ diverges

Proof: Let $a_k = -\sqrt{k}$

$\sum_{k=1}^{\infty}(\sqrt{k+1} - \sqrt{k}) = \sum_{k=1}^{\infty}(a_k - a_{k+1})$

Since $\{a_k\}$ diverges, the series diverges by previous results.

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