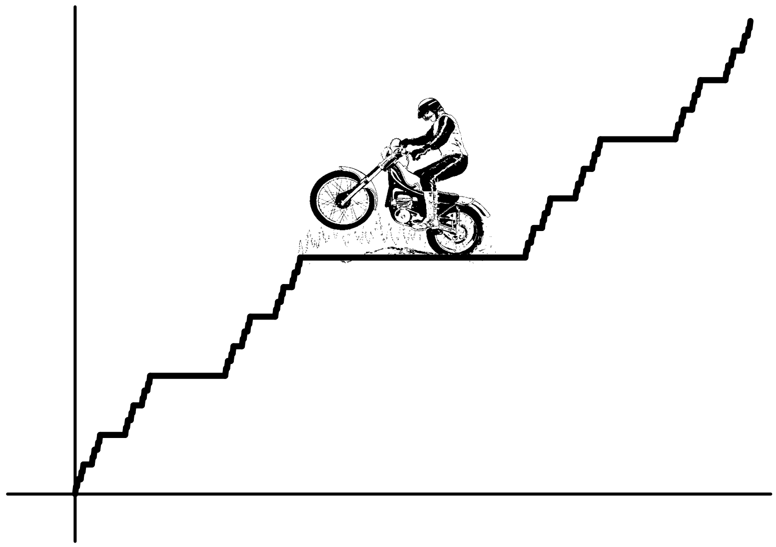
Real Analysis II
Workbook



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Note to Students (Please Read): This workbook contains examples and exercises that will be referred to regularly during class. Please purchase or print out the rest of the workbook before our next class and bring it to class with you every day.

1. **To Print Out the Workbook.** Go to the web address below

   http://www.sonoma.edu/users/m/morrisj/m440/frame.html

   and click on the link "Math 440 Workbook", which will open the workbook as a .pdf file. BE FOREWARNED THAT THERE ARE LOTS OF PICTURES AND MATH FONTS IN THE WORKBOOK, SO SOME PRINTERS MAY NOT ACCURATELY PRINT PORTIONS OF THE WORKBOOK. If you do choose to try to print it, please leave yourself enough time to purchase the workbook before our next class class in case your printing attempt is unsuccessful.

2. **To Purchase the Workbook.** Go to *Digi-Type*, the print shop at 1726 E. Cotati Avenue (across from campus, in the strip mall behind the Seven-Eleven). Ask for the workbook for Math 340. The copying charge will probably be between \$10.00 and \$20.00. You can also visit the *Digi-Type* webpage (http://www.digi-type.com/) to order your workbook ahead of time for pick-up.

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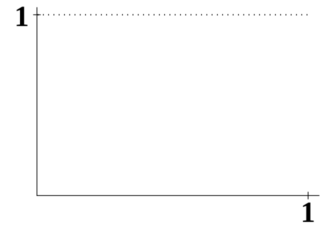
Introductory Examples

Example 1. Consider the sequence

$$\{a_n\}_{n=1}^{\infty} = 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ -\frac{1}{6},\ \cdots.$$

Compare and contrast the following two infinite series:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots \\ a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \cdots &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots \end{aligned}$$



Example 2. Let $f : [0,1] \longrightarrow \mathbb{R}$ denote the characteristic function of the rationals on $[0,1]$, which is defined by

$$f(x) \;=\; \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases} .$$

Discuss the integrability of f.

Comments About Proofs

As with any upper division math class, reading and writing mathematical proofs are very important components of the course. As graduates of Math 340 and likely other upper division classes, you already possess extensive experience and significant abilities regarding these two skills. Therefore, I think you have progressed to the point where you can largely trust your own judgment when it comes to what details to include in a proof and which to exclude. I will, however, make a couple of comments and offer a little advice when it comes to writing proofs for this class.

- You may assume that your audience is well-acquainted with results from Math 340, which means that these can be used freely in any arguments you make without specific citations. If you feel that you're using a Math 340 result in a particularly unusual way, or if you think it would make your proof significantly more clear to cite a Math 340 result by name, by all means feel free to do so. There will likely be many occasions, though, where it would be tedious to have to cite a 340 result by name every time you use it in a proof, so don't feel obligated to do so.

- Overall, the amount of detail you put into a proof will depend on your level of sophistication and the assumed level of sophistication of your audience. This means that, as a general rule, when a concept is new, you will probably put more detail and explanation into proofs relating to that concept than you would if it's a concept that has become familiar and revisited many times. For example, in Math 340, you probably took great pains early on in the course to carefully choose your δ and construct a detailed argument to get a quantity less than ϵ when proving things about limits of functions, but later in the course, you felt comfortable leaving out some of these details.

- One general principle that I always like to keep in mind is this: the more elementary the result that you're being asked to prove, the more detail that's probably appropriate to show. This doesn't mean that proofs of elementary results will always be longer, just that it generally makes sense to include more detail when you're proving an elementary result. As a simple illustration of this fact, suppose that you were asked to <u>prove</u> the very elementary result that the sum of two odd integers is even. The fact that you're being asked to prove this is indicative of the fact that your assumed audience is not very sophisticated, meaning that you'd probably go back to the definition odd integers, writing your odd integers as $2m + 1$ and $2n + 1$ for some $m, n \in \mathbb{Z}$ and going from there, showing all of your algebra steps, and demonstrating that the sum of $2m + 1$ and $2n + 1$ is a multiple of 2 by factoring out the 2 explicitly. Contrast that with a situation in which you are proving a much more advanced result in number theory, and you find yourself in the situation where, in the midst of a much longer proof, you have to add two odd numbers together. Here, you'd likely just state that the sum of two odd integers is even without proof. It would be tedious (and somewhat silly) to waste time proving that the sum of two odd integers is even in that situation since the level of sophistication of the proof you're writing is so far beyond that level.

- For the starred problems in our textbook, partial solutions or outlines of solution strategies are given in the back of the book, often containing statements of fact without justification. Keep in mind that in these instances, it is up to you to decide which statements require further justification and to provide this justification so that the final version of your proof is complete and clear.

- Finally, remember that proof writing is not a black and white thing; we each have our own style and will undoubtedly make different judgment calls about what needs to be explained and what is clear without explanation in a written proof. This is a part of mathematics, and the best we can do in these situations is to communicate with one another and ask for clarification when necessary.

Introductory Notation and Terminology

Definition 2.1.7. A sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to *converge* if there exists $p \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a positive integer n_0 such that

$$|p_n - p| < \epsilon \quad \text{for all } n \geq n_0.$$

If this is the case, we say that $\{p_n\}$ *converges to* p, or that p is the *limit* of the sequence $\{p_n\}$ and we write

$$\lim_{n\to\infty} p_n = p \qquad \text{or} \qquad p_n \to p.$$

If $\{p_n\}$ does not converge, then $\{p_n\}$ is said to *diverge*.

Some Notes.

1. When the indexing of the sequence is not important, our text often abbreviates the sequence notation by writing $\{p_n\}$ instead of $\{p_n\}_{n=1}^{\infty}$.
2. If $\lim\limits_{n\to\infty} p_n = \infty$, then $\{p_n\}$ obviously diverges, but does so in a special way. To distinguish this type of divergence from the divergence of, say $\{(-1)^n\}$, we will say that $\{p_n\}$ *diverges to infinity* if $\lim\limits_{n\to\infty} p_n = \infty$.

Definition 2.5.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. The *limit superior* of $\{s_n\}$, denoted by $\overline{\lim\limits_{n\to\infty}}\, s_n$ (or $\overline{\lim}\, s_n$) is defined as

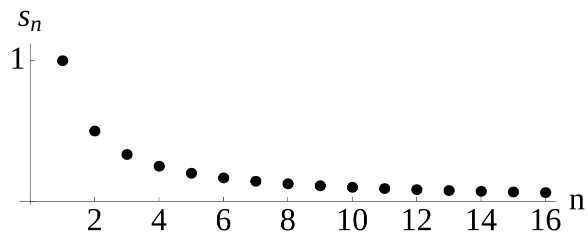
$$\overline{\lim_{n\to\infty}}\, s_n = \lim_{k\to\infty} \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \inf_{k\in\mathbb{N}} \sup\{s_n : n \geq k\}.$$

The *limit inferior* of $\{s_n\}$, denoted by $\underline{\lim\limits_{n\to\infty}}\, s_n$ (or $\underline{\lim}\, s_n$) is defined as

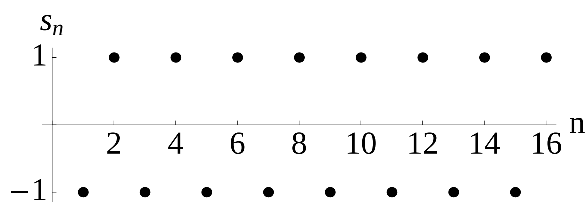
$$\underline{\lim_{n\to\infty}}\, s_n = \lim_{k\to\infty} \inf\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \sup_{k\in\mathbb{N}} \inf\{s_n : n \geq k\}.$$

Example 1. For each of the following sequences $\{s_n\}$, find $\varliminf_{n\to\infty} s_n$ and $\varlimsup_{n\to\infty} s_n$.

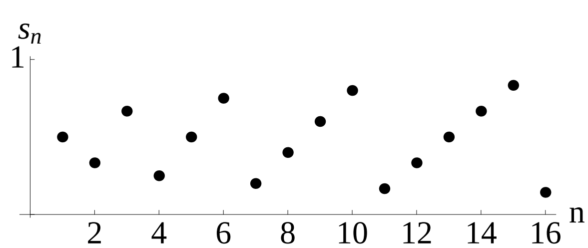
(a) $s_n = 1/n$



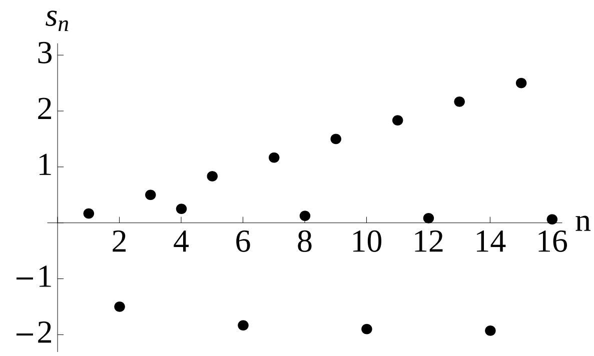
(b) $s_n = (-1)^n$



(c) $\{s_n\} = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \ldots$



(d) $s_n = \begin{cases} \frac{n}{6} & \text{if} \quad n \text{ is odd} \\ (-1)^{n/2} + \frac{1-n}{n} & \text{if} \quad n \text{ is even} \end{cases}$



Some Observations:

1. A sequence $\{s_n\}$ converges to a real number s if and only if $\overline{\lim_{n\to\infty}} s_n$ and $\underline{\lim_{n\to\infty}} s_n$ both equal ______________ .

2. **Alternate Definition of the limit inferior and superior**: Let

$$E = \{\text{limits of subsequences of } \{s_n\}\}.$$

Then $\overline{\lim_{n\to\infty}} s_n =$ ______________ and $\underline{\lim_{n\to\infty}} s_n =$ ______________.

Section 2.7 – Series of Real Numbers

Definition 2.7.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence obtained from $\{a_n\}$, where for each $n \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k$. The sequence $\{s_n\}$ is called an *infinite series*, or *series*, and is denoted either as

$$\sum_{k=1}^{\infty} a_k \qquad \text{or as} \qquad a_1 + a_2 + \cdots + a_n + \cdots.$$

For each $n \in \mathbb{N}$, the expression s_n is called the nth *partial sum* of the series and a_n is called the nth *term* of the series.

The series $\sum_{k=1}^{\infty} a_k$ *converges* if and only if the sequence $\{s_n\}$ of nth partial sums converges in $\mathbb{R}$. If $\lim_{n\to\infty} s_n = s$, then s is called the *sum* of the series, and we write

$$s = \sum_{k=1}^{\infty} a_k.$$

If the sequence $\{s_n\}$ diverges, then the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is said to *diverge*.

Example 1. Discuss the convergence of each series.

(a) $\displaystyle\sum_{k=1}^{\infty} (k+3)$ (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

Theorem 2.7.3. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon \qquad \text{for all } m > n \geq n_0.$$

Note. The preceding theorem says that whether a series converges or diverges is entirely determined by the "end" of the series.

- Convergence of a series is unaffected by changing finitely many terms of the series, though of course the sum may change.
- If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then for any $\epsilon > 0$, there exists n_0 such that $\displaystyle\left|\sum_{k=n_0}^{\infty} a_k\right| < \epsilon$.
- **Corollary 2.7.5**. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim\limits_{k\to\infty} a_k = 0$.

Example 2. If r is a real number, then $\sum_{k=0}^{\infty} r^k$ is called a *geometric* series. For what values of r does this series converge?

Examples and Exercises

1. Decide whether each of the following series converge or diverge, and justify your answers. If a series converges, find its sum.

   (a) $\displaystyle\sum_{k=0}^{\infty} 3^{-k}$ (b) $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

Sections 7.1-7.3 – Convergence Tests

Theorem 7.1.1. If $\sum_{k=1}^{\infty} a_k = \alpha$ and $\sum_{k=1}^{\infty} b_k = \beta$, then

(a) $\sum_{k=1}^{\infty} ca_k = c\alpha$, for any $c \in \mathbb{R}$, and

(b) $\sum_{k=1}^{\infty} (a_k + b_k) = \alpha + \beta$.

Theorem 7.1.2 (Comparison Test). Suppose $\sum a_k$ and $\sum b_k$ are series of nonnegative real numbers satisfying $0 \le a_k \le Mb_k$ for some $M > 0$ and all integers $k \ge k_0$, for some fixed $k_0 \in \mathbb{N}$.

(a) If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$

(b) If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$

Corollary 7.1.3 (Limit Comparison Test). Suppose $\sum a_k$ and $\sum b_k$ are series of positive real numbers.

(a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\sum a_k$ converges if and only if $\sum b_k$ converges.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.

Theorem 7.1.5 (Integral Test). Let $\{a_k\}_{k=1}^{\infty}$ be a decreasing sequence of nonnegative real numbers, and let f be a nonnegative monotone decreasing function on $[1, \infty)$ satisfying $f(k) = a_k$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} a_k < \infty \qquad \text{if and only if} \qquad \int_1^{\infty} f(x)\, dx < \infty.$$

Example 1. Suppose that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. This result is called the *Cauchy Condensation Test*.

Example 2. Show that if $p \in \mathbb{R}$, then the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges if $p \leq 1$ and converges if $p > 1$.

Definition 7.3.1. A series $\sum a_k$ of real numbers is said to be *absolutely convergent* (or converges absolutely) if $\sum |a_k|$ converges. The series is said to be *conditionally convergent* if it is convergent but not absolutely convergent.

Theorem 7.3.3. If $\sum a_k$ converges absolutely, then $\sum a_k$ converges and $\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$.

Theorem 7.3.4 (Ratio and Root Test). Let $\sum a_k$ be a series of real numbers, and let $\alpha = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}$. Also, if $a_k \neq 0$ for all $k \in \mathbb{N}$, let $R = \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$ and $r = \underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$.

(a) If $\alpha < 1$ or $R < 1$, then the series $\sum a_k$ is absolutely convergent (and hence convergent).

(b) If $\alpha > 1$ or $r > 1$, then the series $\sum a_k$ is divergent.

(c) If $\alpha = 1$ or $r \leq 1 \leq R$, then the test is inconclusive.

Theorem 7.3.4 Lite (Weak Ratio and Root Test). Let $\sum a_k$ be a series of real numbers, and let $\alpha = \lim_{k\to\infty} \sqrt[k]{|a_k|}$. Also, if $a_k \neq 0$ for all $k \in \mathbb{N}$, let $R = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$.

(a) If $\alpha < 1$ or $R < 1$, then the series $\sum a_k$ is absolutely convergent (and hence convergent).

(b) If $\alpha > 1$ or $R > 1$, then the series $\sum a_k$ is divergent.

(c) If $\alpha = 1$ or $R = 1$, then the test is inconclusive.

Theorem 7.2.1 Let $\{a_k\}$ and $\{b_k\}$ be sequences of real numbers. Set $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$ if $n \geq 1$. Then if $1 \leq p \leq q$,

$$\sum_{k=p}^{q} a_k b_k \;=\; \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) \;+\; A_q b_q \;-\; A_{p-1} b_p.$$

Theorem 7.2.2 (Dirichlet Test) Suppose $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers satisfying the following:

(a) the partial sums $A_n = \sum_{k=1}^{n} a_k$ form a bounded sequence,

(b) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and

(c) $\lim_{k\to\infty} b_k = 0$.

Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Theorem 7.2.3 (Alternating Series Test). If $\{b_k\}$ is a sequence of real numbers satisfying

(a) $b_1 \geq b_2 \geq \cdots \geq 0$, and
(b) $\lim_{k\to\infty} b_k = 0$,

then $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$ converges.

Example 3. Decide whether each of the following series converge absolutely, converge conditionally, or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}(k-1)}{k^2\sqrt{k}+1}$

Rearrangements

Definition 7.3.6 A series $\sum a'_k$ is a *rearrangement* of the series $\sum a_k$ if there is a one-to-one function j from $\mathbb{N}$ onto $\mathbb{N}$ such that $a'_k = a_{j(k)}$ for all $k \in \mathbb{N}$.

Example 4. Describe how to construct a rearrangement of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ that converges to 0.

Theorem 7.3.8 If the series $\sum a_k$ converges absolutely, then every rearrangement of $\sum a_k$ converges to the same sum.

Theorem 7.3.9 Let $\sum a_k$ be a conditionally convergent series of real numbers. Suppose that $\alpha \in \mathbb{R}$. Then there exists a rearrangement $\sum a'_k$ of $\sum a_k$ which converges to α.

Examples and Exercises

1. Test the following series for convergence or divergence. You may assume the continuity and differentiability of the exponential and logarithmic functions.

   (a) $\displaystyle\sum_{k=1}^{\infty} e^{1/k^2}$ (b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+1}{k^4-2k}$

2. Suppose that $\sum_{k=1}^{\infty} a_k$ converges and that $a_k \geq 0$ for all k.

   (a) If $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$, prove that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

   (b) Is the result of (a) still true if we drop the assumption that $a_k \geq 0$ for all k? Justify your answer.

3. Determine whether the following series converge absolutely, converge conditionally, or diverge.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{\sqrt[3]{k^2}}$ (b) $\displaystyle\sum_{k=1}^{\infty} k^3 e^{-k}$

4. Give an example of an absolutely convergent series in which infinitely many terms are positive and infinitely many terms are negative.

Sections 8.1 & 8.2 – Pointwise and Uniform Convergence

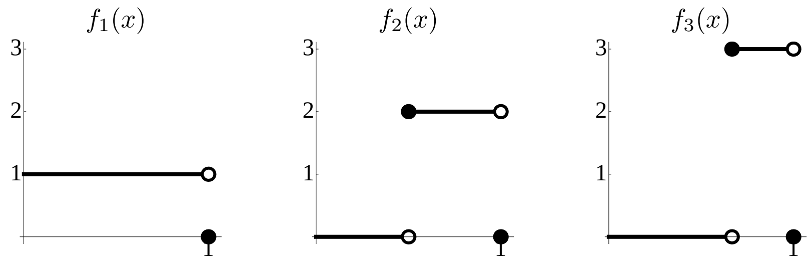
Definition 8.1.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions defined on a set E. The sequence $\{f_n\}$ *converges pointwise* on E if $\{f_n(x)\}_{n=1}^{\infty}$ converges for every $x \in E$. If this is the case, then f defined by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in E,$$

defines a function on E. The function f is called the *limit of the sequence* $\{f_n\}$. A series $\sum f_n$ of functions converges pointwise on E to a function g if its sequence of partial sums converges pointwise on E to g.

Example 1. For each $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < \frac{n-1}{n} \\ n & \text{if} \quad \frac{n-1}{n} \le x < 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$$



Show that $\{f_n\}$ converges pointwise on $[0,1]$. Then, calculate $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx$ and $\int_0^1 \lim\limits_{n\to\infty} f_n(x)\,dx$.

Some Observations: As the previous example shows, even if a sequence $\{f_n\}$ of integrable functions converges pointwise to an integrable function f, it can happen that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$. Other textbook examples and exercises illustrate other bad behavior of pointwise convergence, such as:

1. A sequence of continuous (and hence integrable) functions that converges pointwise to a function that is not only not continuous, but not integrable as well.

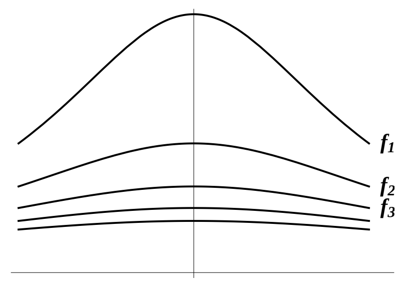
2. A sequence of differentiable functions $\{h_n\}$ that converges pointwise to a differentiable function h, but such that $\lim_{n\to\infty} h'_n(x) \neq h'(x)$.

Definition 8.2.1. A sequence of real-valued functions $\{f_n\}$ defined on a set E converges *uniformly* to f on E, if for every $\epsilon > 0$, there exists a positive integer n_0 such that

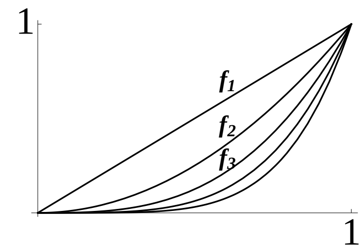
$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in E \text{ and all } n \geq n_0.$$

Similarly, a series $\sum_{k=1}^{\infty} f_k$ of real-valued functions converges uniformly on a set E if and only if the sequence $\{S_n\}$ of partial sums converges uniformly on E.

Example 2. Let $f_n(x) = \dfrac{1}{n + x^2}$. Prove that $\{f_n\}$ converges uniformly on all of $\mathbb{R}$.



Example 3. Let $f_n(x) = x^n$. Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$.



Example 4. Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+x)^2 + x}$ converges uniformly on $[0, \infty)$.

Theorem 8.2.3 (Cauchy Criterion). A sequence $\{f_n\}$ of real-valued functions defined on a set E converges uniformly on E if and only if for every $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \qquad \text{for all } x \in E \text{ and all } n, m \geq n_0.$$

Corollary 8.2.4. The series $\sum_{k=1}^{\infty} f_k$ of real-valued functions on E converges uniformly on E if and only if given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\left|\sum_{k=n+1}^{m} f_k(x)\right| < \epsilon \qquad \text{for all } x \in E \text{ and all } m > n \geq n_0.$$

Theorem 8.2.5. Suppose the sequence $\{f_n\}$ of real-valued functions on the set E converges pointwise to f on E. For each $n \in \mathbb{N}$, set

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $\{f_n\}$ converges uniformly to f on E if and only if $\lim_{n\to\infty} M_n = 0$.

Theorem 8.2.7 (Weierstrass M-Test). Suppose $\{f_k\}$ is a sequence of real-valued functions defined on a set E, and $\{M_k\}$ is a sequence of real numbers satisfying

$$|f_k(x)| \leq M_k \qquad \text{for all } x \in E \text{ and } k \in \mathbb{N}.$$

If $\displaystyle\sum_{k=1}^{\infty} M_k < \infty$, then $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly and absolutely on E.

Examples and Exercises

1. Let $f_n(x) = 1 + \frac{x}{n}$. Find the pointwise limit of $\{f_n(x)\}_{n=1}^{\infty}$. Does $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $[0, 2]$? on $[0, \infty)$? Justify your answers.

2. Prove that $\sum_{k=1}^{\infty} \frac{x}{(1+kx)^2}$ converges uniformly on $[a, \infty)$ for any fixed $a > 0$. How about on $(0, \infty)$?

Sections 8.3-8.5 – Properties of Uniform Convergence

Theorem 8.3.1. Suppose $\{f_n\}$ is a sequence of real-valued functions that converges uniformly to a function f on a subset E of $\mathbb{R}$. Let p be a limit point of E, and suppose that for each $n \in \mathbb{N}$, we have $\lim_{x\to p} f_n(x) = A_n$. Then the sequence $\{A_n\}$ converges, and

$$\lim_{x\to p}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\lim_{x\to p} f_n(x)\right).$$

Corollary 8.3.2 Let E be a subset of $\mathbb{R}$, and let $\{f_n\}$ be a sequence of continuous real-valued functions on E.

(a) If $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.

(b) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, then $S(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous on E.

Theorem 8.4.1 Suppose $f_n \in \mathcal{R}[a,b]$ for all $n \in \mathbb{N}$, and suppose that the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$. Then $f \in \mathcal{R}[a,b]$, and

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

Corollary 8.4.2 If $f_k \in \mathcal{R}[a,b]$ for all $k \in \mathbb{N}$, and if $f(x) = \sum_{k=1}^{\infty} f_k(x),\ \ x \in [a,b]$, where the series converges uniformly on $[a,b]$, then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f(x)\,dx \;=\; \sum_{k=1}^{\infty}\int_a^b f_k(x)\,dx.$$

Theorem 8.5.1 Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$. If

(a) $\{f_n'\}$ converges uniformly on $[a, b]$, and

(b) $f_n(x_0)$ converges for some $x_0 \in [a, b]$,

then $\{f_n\}$ converges uniformly to a function f on $[a, b]$, with $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Examples and Exercises

1. Define $f : [0, 1] \longrightarrow \mathbb{R}$ by $f(x) = \sum_{k=0}^{\infty} x(1 - x)^k$.

   (a) Show that $f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \ \text{ or } \ x = 1 \\ 1 & \text{if} \quad 0 < x < 1 \end{cases}$.

   (b) Use Corollary 8.3.2 to explain why the series defining f cannot be uniformly convergent on $[0, 1]$.

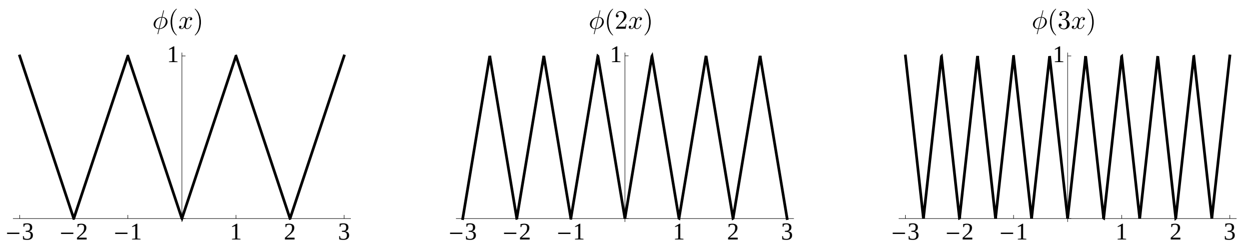
2. For each $n \in \mathbb{N}$, let $f_n(x) = x + n$. Explain why Theorem 8.5.1 doesn't apply to the sequence $\{f_n\}$. Does the conclusion still hold?

3. It can be shown that $\sum_{k=0}^{\infty} (-1)^k x^{4k}$ converges to $f(x) = 1/(1 + x^4)$ on $[0, 1/2]$. Prove that this convergence is uniform, and then evaluate $\int_0^{1/2} \frac{1}{1+x^4}\, dx$ as a power series. Justify your calculations.

4. Show that $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ is everywhere differentiable, and find $f'(x)$.

5. In this problem,we will explore whether or not it is possible to construct a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous everywhere but nowhere differentiable.

    (a) Define $\phi : [-1, 1) \longrightarrow$ by $\phi(x) = |x|$, and extend the definition of ϕ to all of $\mathbb{R}$ by letting $\phi(x + 2) = \phi(x)$ for all $x \in \mathbb{R}$. The graphs of $\phi(nx)$ for $n = 1, 2, 3$ are shown below:



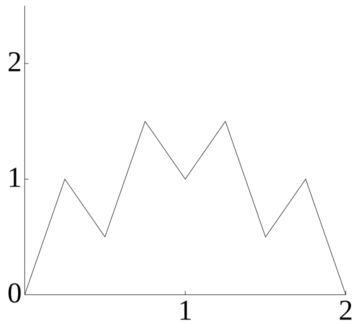
    For the graphs shown above, where is $\phi(nx)$ continuous? Where does it fail to be differentiable?

(b) Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

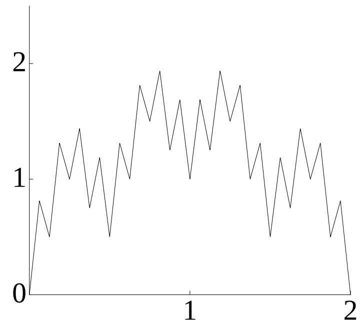
$$f(x) \;=\; \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \phi(4^k x).$$

Prove that f is well-defined and continuous.

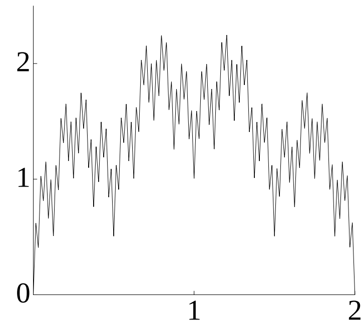
$$S_1(x) = \sum_{k=0}^{1} \left(\frac{3}{4}\right)^k \phi(4^k x)$$



$$S_2(x) = \sum_{k=0}^{2} \left(\frac{3}{4}\right)^k \phi(4^k x)$$



$$S_3(x) = \sum_{k=0}^{3} \left(\frac{3}{4}\right)^k \phi(4^k x)$$



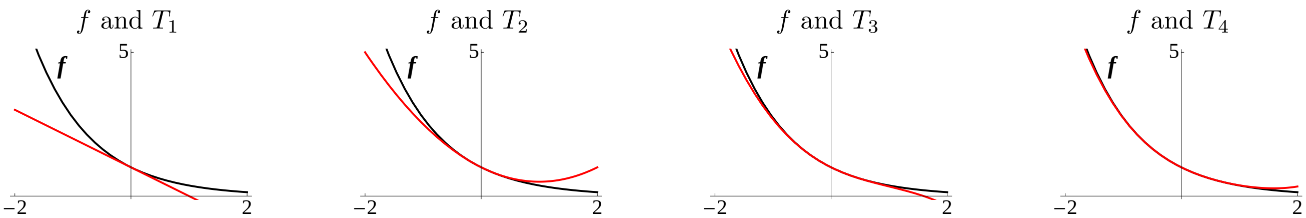
(c) Fix any $x_0 \in \mathbb{R}$ and any $m \in \mathbb{N}$, and choose $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is chosen so that no integer lies strictly between the numbers $4^m x_0$ and $4^m(x_0 + \delta_m)$. Show that

$$\left| \frac{f(x_0 + \delta_m) - f(x_0)}{\delta_m} \right| \to \infty \qquad \text{as} \qquad m \to \infty.$$

What does this tell you about f?

Section 8.7 – Power Series

Example 1. Let $f(x) = e^{-x}$, and for each n, let T_n denote the nth degree polynomial that satisfies $f(0) = T_n(0),\ f'(0) = T_n'(0),\ f''(0) = T_n''(0),\ \ldots,\quad$ and $\ f^{(n)}(0) = T_n^{(n)}(0)$. (In other words, the functions themselves and the first n derivatives of f and T_n are equal at $x = 0$.) Find formulas for T_1, T_2, T_3, and T_4.



Some Questions

- What types of functions can be represented by infinite series?
- How do we find a series representation for a function? How "quickly" does the series converge?

Definition 8.7.1. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$. A series of the form

$$\sum_{k=0}^{\infty} a_k(x-c)^k \;=\; a_0 \;+\; a_1(x-c) \;+\; a_2(x-c)^2 \;+\; a_3(x-c)^3 \;+\; \cdots$$

is called a *power series* in $(x-c)$. When $c=0$, the series is called a power series in x. The numbers a_k are called the *coefficients* of the power series.

Example 2. For what values of x does the power series $\displaystyle\sum_{k=0}^{\infty} x^k$ converge?

Definition 8.7.2. Given a power series $\sum a_k(x-c)^k$, the *radius of convergence* R is defined by

$$\frac{1}{R} \;=\; \overline{\lim_{n\to\infty}}\; \sqrt[n]{|a_n|}.$$

If $\overline{\lim}\; \sqrt[n]{|a_n|} = \infty$ we take $R=0$, and if $\lim\; \sqrt[n]{|a_n|} = 0$ we set $R=\infty$.

Note. We also have $\displaystyle\frac{1}{R} = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ if the previous limit exists and $a_k \neq 0$ for all k.

Example 3. Find the radius of convergence of the following: (a) $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{2^k}$ (b) $\displaystyle\sum_{k=0}^{\infty}\frac{(x+1)^k}{k!}$

Theorem 8.7.3. Given a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ with radius of convergence R and $0 < R \leq \infty$, then the series

(a) converges absolutely for all x with $|x - c| < R$, and

(b) diverges for all x with $|x - c| > R$.

(c) Furthermore, if $0 < \rho < R$, then the series converges uniformly for all x with $|x - c| \leq \rho$.

Some Summarizing Notes on the Power Series $\displaystyle\sum_{k=0}^{\infty} a_k(x-c)^k$:

1. The power series must satisfy one of the following three behaviors:
   (i) The series converges for all x.
   (ii) The series converges only when $x = c$.
   (iii) There exists $R > 0$ such that the series converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$.

2. The interval of values on which a power series converges is called the interval of convergence of the power series. Either the Root or Ratio Test generally works to determine the radius of convergence, but separate tests are needed to determine convergence at the endpoints.

3. The convergence is absolute within the radius of convergence but may be conditional at the endpoints (see Theorem 8.7.3).

4. If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|L$, so $R = \infty$ if $L = 0$ and $R = \dfrac{1}{L}$ if $0 < L < \infty$.

5. If $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, then $f'(x)$ and $\int_c^x f(t)\,dt$ can be obtained using term-by-term differentiation and antidifferentiation of the power series, respectively, within the radius of convergence. Specifically, if $|x-c| < R$, we have

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots &&= \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} \\
\int_c^x f(t)\,dt &= a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots &&= \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1}
\end{aligned}
$$

Taylor Series

Definition 8.7.9. A real-valued function f defined on an open interval I is said to be *infinitely differentiable* on I if $f^{(n)}(x)$ exists on I for all $n \in \mathbb{N}$. The set of infinitely differentiable functions on an open interval I is denoted by $C^\infty(I)$.

Definition 8.7.13. Let f be a real-valued function defined on an open interval I, and let $c \in I$ and $n \in \mathbb{N}$. Suppose $f^{(n)}(x)$ exists for all $x \in I$. The polynomial

$$T_n(f,c)(x) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the *Taylor polynomial of order* n of f at the point c. If f is infinitely differentiable on I, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the *Taylor series* of f at c. The function

$$R_n(x) \;=\; R_n(f,c)(x) \;=\; f(x) - T_n(f,c)(x)$$

is called the *remainder* or *error function* between f and $T_n(f,c)$.

Example 4. Find the Taylor series for $f(x) = e^x$ at $x = 0$. What is the radius of convergence of this series? Also, use your series to derive a series for $g(x) = e^{-x^2}$.

Theorem 8.7.16. Suppose f is a real-valued function on an open interval I with $c \in I$ and $n \in \mathbb{N}$. If $f^{(n+1)}(t)$ exists for every $t \in I$, then for any $x \in I$, there exists a ξ between x and c such that

$$R_n(x) \;=\; R_n(f,c)(x) \;=\; \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

Examples and Exercises

1. Show that $f(x) = \cos x$ is equal to its Taylor series at $c = 0$ for all $x \in \mathbb{R}$. Then, use an appropriate Taylor polynomial to estimate $\cos(1.5)$ to within 0.001 of its true value, making it clear why you chose the number of terms that you did.

2. Evaluate $\int_0^{1/5} \arctan(x^2)\,dx$ as a power sereis, and justify your conclusions.

Sections 7.4 – Normed Linear Spaces

Definition 7.4.7. A set X with two operations "+", vector addition, and "$\cdot$", scalar multiplication, satisfying

$$\begin{aligned} \mathbf{x}+\mathbf{y} \in X \qquad & \text{for all } \mathbf{x},\mathbf{y} \in X, \qquad \text{and} \\ c\cdot\mathbf{x} \in X \qquad & \text{for all } \mathbf{x} \in X,\ c \in \mathbb{R} \end{aligned}$$

is a *vector space* over $\mathbb{R}$ if the following are satisfied for all $\mathbf{x},\mathbf{y} \in X$ and all $a,b \in \mathbb{R}$:

(a) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

(b) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$.

(c) There is a unique element in X called the zero element, denoted $\mathbf{0}$, such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for all $\mathbf{x} \in X$.

(d) There exists a unique element $-\mathbf{x} \in X$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(e) $(ab)\cdot\mathbf{x} = a\cdot(b\cdot\mathbf{x})$.

(f) $a\cdot(\mathbf{x}+\mathbf{y}) = a\cdot\mathbf{x} + a\cdot\mathbf{y}$.

(g) $(a+b)\cdot\mathbf{x} = a\cdot\mathbf{x} + b\cdot\mathbf{x}$.

(h) $1\cdot\mathbf{x} = \mathbf{x}$.

Example 1. $\mathbb{R}^n = \{(x_1,x_2,\ldots,x_n) : x_i \in \mathbb{R} \quad \text{for all } i=1,2,\ldots,n\}$ is a vector space.

Example 2. $V = \{\text{continuous functions } f:\mathbb{R} \longrightarrow \mathbb{R}\}$ is a vector space.

Definition 7.4.1. A sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers is said to be in l^2, or to be *square summable*, if $\sum_{k=1}^{\infty} a_k^2 < \infty$.

Example 3. l^2 is a vector space.

Definition 7.4.8. Let X be a vector space over $\mathbb{R}$. A function $\| \, \| : X \longrightarrow \mathbb{R}$ satisfying

(a) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$,

(b) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(c) $\|c \cdot \mathbf{x}\| = |c| \, \|\mathbf{x}\|$ for all $c \in \mathbb{R}$ and all $\mathbf{x} \in X$, and,

(d) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in X$

is called a *norm* on X. The pair $(X, \| \, \|)$ is called a *normed linear space*.

Note. For a normed linear space X and $\mathbf{x}, \mathbf{y} \in X$, the quantity $\|\mathbf{x}\|$ can be viewed as the "length" of $\mathbf{x}$, while the quantity $\|\mathbf{x} - \mathbf{y}\|$ can be viewed as the "distance" between $\mathbf{x}$ and $\mathbf{y}$.

Example 4. The absolute value function is a norm on the vector space $\mathbb{R}$.

Example 5. For each $\mathbf{x} = \{x_i\}_{i=1}^{\infty} \in l^2$, define $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{\infty} x_i^2 \right)^{1/2}$. Then $\| \, \|_2 : l^2 \longrightarrow \mathbb{R}$ is a norm.

Examples and Exercises

1. Prove that $\|\ \| : \mathbb{R}^n \longrightarrow \mathbb{R}$ as defined to the right is <u>not</u> a norm on $\mathbb{R}^n$.

$$\|\mathbf{x}\| = \begin{cases} 0 & \text{if} \quad \mathbf{x} = \mathbf{0} \\ 1 & \text{if} \quad \mathbf{x} \neq \mathbf{0} \end{cases}$$

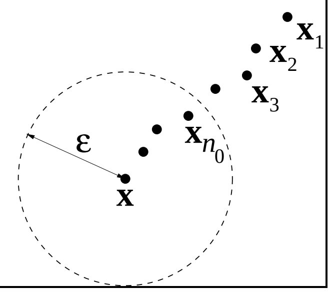
2. Given $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, define two norms $\|\ \|_a$ and $\|\ \|_b$ on $\mathbb{R}^2$ as shown to the right. (You need not prove that these functions are norms.) Describe and sketch the following two subsets of $\mathbb{R}^2$.

$$\begin{aligned} \|\mathbf{x}\|_a &= \sqrt{x_1^2 + x_2^2} \\ \|\mathbf{x}\|_b &= \max\{|x_1|, |x_2|\} \end{aligned}$$

(a) $\left\{\mathbf{x} \in \mathbb{R}^2 \ : \ \|\mathbf{x}\|_a < 1\right\}$ (b) $\left\{\mathbf{x} \in \mathbb{R}^2 \ : \ \|\mathbf{x}\|_b < 1\right\}$

Definition. Let X be a normed linear space with norm $\| \, \|$. We say that a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of vectors in X *converges in the norm* to a vector $\mathbf{x} \in X$ if and only if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| < \epsilon \qquad \text{for all } n \geq n_0.$$



3. Consider the vector $\mathbf{x} = \left\{2^{-i}\right\}_{i=0}^{\infty} \in l^2$. Prove that the sequence described by

$$\begin{aligned} \mathbf{x}_0 &= 0,\ 0,\ 0,\ 0,\ 0,\ \cdots \\ \mathbf{x}_1 &= 2^0,\ 0,\ 0,\ 0,\ 0,\ \cdots \\ \mathbf{x}_2 &= 2^0,\ 2^{-1},\ 0,\ 0,\ 0,\ \cdots \\ \mathbf{x}_3 &= 2^0,\ 2^{-1},\ 2^{-2},\ 0,\ 0,\ \cdots \\ &\vdots \end{aligned}$$

converges in the norm to $\mathbf{x}$.

Section 9.1 – Orthogonal Functions

Definition 9.0.0. If X is a vector space over $\mathbb{R}$, a function $\langle\, ,\, \rangle : X \times X \longrightarrow \mathbb{R}$ is an *inner product* on X if

(a) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in X$.

(b) $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in X$, and

(d) $\langle a\mathbf{x} + b\mathbf{y},\, z \rangle \;=\; a\,\langle \mathbf{x}, \mathbf{z} \rangle + b\,\langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ and $a, b \in \mathbb{R}$.

Note. Two nonzero vectors $\mathbf{x}, \mathbf{y} \in X$ are called ______________________________ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Example 1. For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$, define $\langle \mathbf{a}, \mathbf{b} \rangle \;=\; \displaystyle\sum_{j=1}^{n} a_j b_j$. Show that this is an inner product on $\mathbb{R}^n$.

Definition. Let $f, g \in \mathcal{R}[a,b]$, define

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

Note: The above integral does not "quite" define an inner product on $\mathcal{R}[a,b]$, but it <u>does</u> define an inner product on $\mathcal{C}[a,b]$, the set of all continuous functions on $[a,b]$.

Definition 9.1.1. A finite or countable collection of Riemann integrable functions $\{\phi_n\}$ on $[a,b]$ satisfying $\int_a^b \phi_n^2\,dx \neq 0$ is *orthogonal* on $[a,b]$ if

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{for all} \quad n \neq m.$$

Example 2. Let $f_1(x) = 1$, $f_2(x) = 1 - 2x$, and $f_3(x) = 1 - 6x + 6x^2$. Show that $\{f_1, f_2, f_3\}$ is orthogonal on $[0, 1]$.

Definition 9.1.3. A finite or countable collection of Riemann integrable functions $\{\phi_n\}$ is *orthonormal* on $[a, b]$ if

$$\int_a^b \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}.$$

Example 3. Find real numbers α_1 and β_1 so that $\{\alpha_1, \, \alpha_2(1-2x)\}$ is orthonormal on $[0,1]$.

Definition. Let X be a vector space with inner product $\langle \, , \, \rangle$. If $\mathbf{x} \in X$, we define $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Note. The function $\| \ \|$ is a norm on X.

Theorem 9.0.1 (Cauchy-Schwarz Inequality). Let X be a vector space with inner product $\langle\, , \rangle$. Then, if we define $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for each $\mathbf{x} \in X$, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\|\|\mathbf{y}\| \qquad \text{for all } \mathbf{x}, \mathbf{y} \in X.$$

Corollary 9.0.2 (Cauchy-Schwarz Inequality in $\mathbb{R}^n$). If $n \in \mathbb{N}$, and $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then $$\sum_{k=1}^{n} |a_k b_k| \le \sqrt{\sum_{k=1}^{n} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}.$$

Corollary 9.0.3 (Cauchy-Schwarz Inequality in l^2). If $\{a_k\}, \{b_k\} \in l^2$, then $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent and $$\sum_{k=1}^{\infty} |a_k b_k| \le \|\{a_k\}\|_2 \, \|\{b_k\}\|_2.$$

Corollary 9.0.4 (Minkowski's Inequality). If $\{a_k\}$ and $\{b_k\}$ are in l^2, then $\{a_k + b_k\}_{n=1}^{\infty}$ is in l^2 and $\|\{a_k + b_k\}\|_2 \le \|\{a_k\}\|_2 + \|\{b_k\}\|_2$.

Approximation in the Mean

Goal: Given a function $f \in \mathcal{R}[a,b]$ and a family $\{\phi_n\}$ of orthogonal functions, we want to choose constants c_n so that

(1) $$S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$$

gives the "best possible" approximation of f on $[a,b]$.

Theorem 9.1.4. Let $f \in \mathcal{R}[a,b]$ and let $\{\phi_n\}$ be a finite or countable collection of orthogonal functions on $[a,b]$. For $N \in \mathbb{N}$, let S_N be defined by (1). Then the quantity

$$\int_a^b \big(f(x) - S_N(x)\big)^2 \, dx$$

is minimal if and only if

(2) $$c_n = \qquad\qquad\qquad , \qquad n = 1, 2, \ldots, N.$$

Furthermore, for this choice of c_n,

(3) $$\int_a^b \big(f(x) - S_N(x)\big)^2 \, dx = \int_a^b f^2(x)\, dx - \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 \, dx.$$

Theorem 9.1.4 Comments: Let $\{\phi_n\}$ be a sequence of orthogonal functions.

- For each $n \in \mathbb{N}$, the number c_n defined in (2) is called the ________________________ of f with respect to the system $\{\phi_n\}$.

- With c_n defined as in equation (2), the series $\sum_{n=1}^{\infty} c_n\phi_n(x)$ is called the ________________________ of f with respect to the system $\{\phi_n\}$. This is denoted by

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x).$$

- Let $f \in \mathcal{R}[a,b]$. For each $n \in \mathbb{N}$, let S_N denote the Nth partial sum of the Fourier series of f; i.e., $S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$. By equation (3), we have:

Example 4. Using the fact that $\{\sin(nx)\}_{n=1}^{\infty}$ is orthogonal on $[-\pi, \pi]$, find the Fourier series for $f(x) = x^3$ on $[-\pi, \pi]$. You may want to use *Mathematica* to calculate the involved integrals.

Section 9.2 – Completeness and Parseval's Equality

Definition 9.2.1. A sequence $\{f_n\}$ of Riemann integrable functions on $[a, b]$ *converges in the mean* to $f \in \mathcal{R}[a, b]$ if

$$\lim_{n\to\infty} \int_a^b \big(f(x) - f_n(x)\big)^2 \, dx \;=\; 0.$$

Comments:

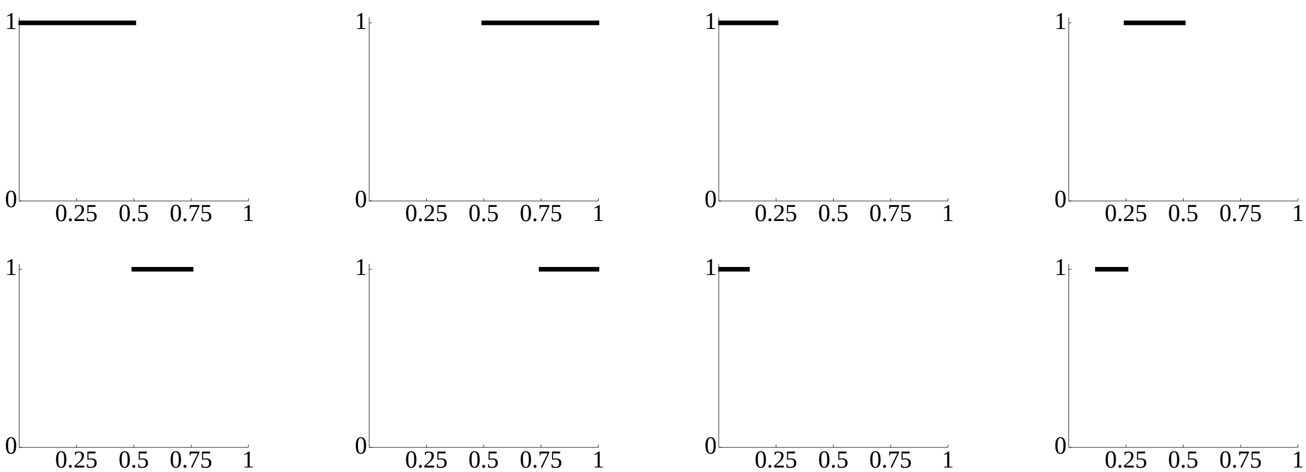
1. Using norm notation, $\{f_n\}$ converges in the mean to f if and only if ________________________.

   Convergence in the mean is sometimes referred to as ________________________________.

2. Uniform convergence of a sequence of Riemann integrable functions implies mean-square convergence (Theorem 9.2.2).

Example 1. For each $n \in \mathbb{N}$, write $n = 2^k + j$, where $k = 0, 1, 2, \ldots,$ and $0 \le j < 2^k$, define f_n as shown:

$$f_n(x) \;=\; \begin{cases} 1, & \frac{j}{2^k} \le x \le \frac{j+1}{2^k} \\ 0, & \text{otherwise} \end{cases}$$



Theorem 9.2.4. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of orthogonal functions on $[a,b]$. Then the following are equivalent:

(a) For every $f \in \mathcal{R}[a,b]$,
$$\lim_{N\to\infty} \int_a^b \big(f(x) - S_N(x)\big)^2 \, dx \;=\; 0,$$
where S_N is the Nth partial sum of the Fourier series of f.

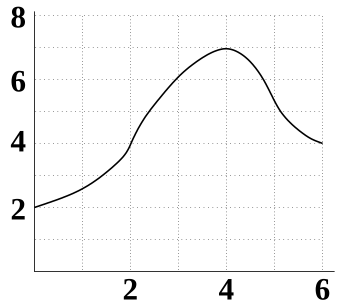
(b) For every $f \in \mathcal{R}[a,b]$,
$$\sum_{n=1}^{\infty} c_n^2 \int_a^b \phi_n^2(x)\, dx \;=\; \int_a^b f^2(x)\, dx,$$
where the c_n are the Fourier coefficients of f.

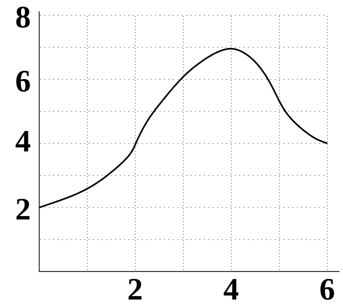
Definition 9.2.5. A sequence $\{\phi_n\}_{n=1}^{\infty}$ of orthogonal functions on $[a,b]$ is said to be *complete* if for every $f \in \mathcal{R}[a,b]$, we have
$$\sum_{n=1}^{\infty} c_n^2 \int_a^b \phi_n^2(x)\, dx \;=\; \int_a^b f^2(x)\, dx.$$

Section 10.1 – Introduction to Lebesgue Measure

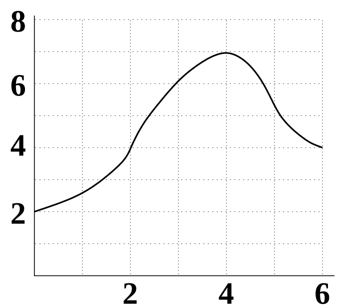
Example 1.

(a) Let f be the function whose graph is given below, and consider the partition $\mathcal{P} = \{0, 2, 4, 6\}$ of $[0, 6]$. Estimate the upper and lower Riemann sums, $\mathcal{U}(\mathcal{P}, f)$ and $\mathcal{L}(\mathcal{P}, f)$, and draw in the associated rectangles.



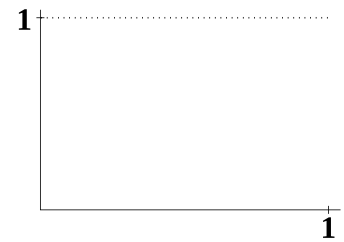


(b) Illustrate the idea of Lebesgue integration for the same function f using the "measurable partition" $\mathcal{P} = \{E_1, E_2, E_3\}$, where $E_i = f^{-1}\big([2i, 2i+2)\big)$ for $i = 1, 2, 3$.



Example 2. Let f denote the characteristic function of the irrational numbers on the interval $[0,1]$, which is defined as shown to the right. Does the Riemann integral $\int_0^1 f(x)\,dx$ exist? Also, speculate about the value of the upper and lower Lebesgue sums associated with the partition $\mathcal{P} = \left\{ f^{-1}(\{0\}),\ f^{-1}(\{1\}) \right\}$.

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if} \quad x \notin \mathbb{Q} \cap [0,1] \end{cases}$$



Sections 3.1 & 3.2 – Review of the Topology of $\mathbb{R}$

Definition 3.1.3.

(a) A subset O of $\mathbb{R}$ is *open* if every point of O is an interior point of O; that is, if for every $x \in O$, there exists $\epsilon > 0$ such that $(x - \epsilon,\, x + \epsilon) \subset O$.

(b) A subset F of $\mathbb{R}$ is *closed* if $F^c = \mathbb{R} \backslash F$ is open.

Note. A subset F of $\mathbb{R}$ is closed if and only if it contains all of its limit points. (**Theorem 3.1.9**)

Example 1. Which of the following sets are open? Which are closed? In each case, give a brief reason/ explanation for your answer.

(a) $(0, 1)$

(b) $[0, 1]$

(c) $\mathbb{R}$

(d) $[0, 1)$

(e) $\left\{\frac{1}{n} : n \in J\right\}$

(f) $\mathbb{Z}$

(g) $\{1, 2, 3, 4, 5\}$

Some Facts About Open and Closed Sets

1. **Theorem 3.1.6**. Unions of open sets are open, intersections of *finitely many* open sets are open.
2. **Theorem 3.1.7**. Intersections of closed sets are closed, unions of *finitely many* closed sets are closed.

Compact Sets

Definition 3.2.1. Let $E \subset \mathbb{R}$. A collection $\{O_\alpha\}_{\alpha \in A}$ of open subsets of $\mathbb{R}$ is an *open cover* of E if

$$E \subset \bigcup_{\alpha \in A} O_\alpha.$$

Note. An open cover is a set of open sets.

Definition. Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of E. If there exist finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that $E \subset \cup_{i=1}^n O_{\alpha_i}$, then we say that $\{O_{\alpha_i}\}_{i=1}^n$ is a *finite subcover* of E.

Notes.

1. The term "finite subcover" refers back to a previously chosen open cover; the sets in a finite subcover must come from this open cover.

2. The word "finite" in the term "finite subcover" refers to the number of sets in the subcover, not to the number of elements in the sets themselves.

Definition 3.2.3. A set K is called *compact* iff, for every open cover $\{O_\alpha\}_{\alpha \in A}$ of K, there exist finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that $K \subset \cup_{i=1}^{n} O_{\alpha_i}$. In other words, K is compact iff every open cover of K has a finite subcover.

Example 2. Use the definition of compactness to decide which of the following sets are compact.

(a) $\mathbb{R}$

(b) $[0, 1)$

(c) $\{1, 2, 3\}$

(d) $E = \left\{\frac{1}{n} : n \in J\right\} \cup \{0\}$

Theorem 3.2.9 (Heine-Borel-Bolzano-Weierstrass). Let K be a subset of $\mathbb{R}$. Then the following are equivalent:

(a) K is closed and bounded.

(b) K is compact.

(c) Every infinite subset of K has a limit point in K.

Theorem 4.3.4. If $K \subset \mathbb{R}$ is compact and $f : K \longrightarrow \mathbb{R}$ is continuous on K, then f is uniformly continuous on K.

Example 3. Determine whether the following sets are compact, and justify your answer.

(a) $\left\{\frac{1}{n} : n \in J\right\}$

(b) $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$.

(c) $\mathrm{im}(f)$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = x^2$.

Section 10.2 – Measure of Open and Compact Sets

Definition 10.2.1. If J is an interval, we define the *measure* of J, denoted $m(J)$, to be the length of J.

Definition 10.2.2. If U is an open subset of $\mathbb{R}$ with $U = \bigcup_n I_n$, where $\{I_n\}$ is a finite or countable collection of pairwise disjoint open intervals, we defined the *measure* of U, denoted $m(U)$, by $m(U) = \sum_n m(I_n)$.

Notes.

1. This definition works because every open set in $\mathbb{R}$ can be written uniquely as an at most countable union of disjoint open intervals.
2. By convention, we let $m(\emptyset) = 0$.

Example 1. Find $m(U)$, where $U = \bigcup_{n=0}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} \right)$.

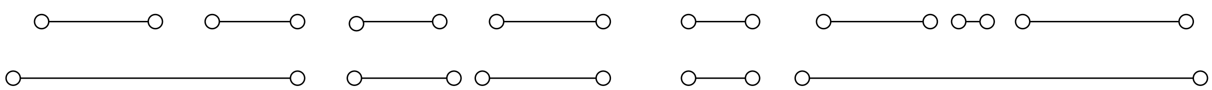
Theorem 10.2.4. If U and V are open subsets of $\mathbb{R}$ with $U \subset V$, then $m(U) \leq m(V)$.

Proof. Let U and V be open sets with $U \subset V$, and write

$$U = \bigcup_n I_n \qquad \text{and} \qquad V = \bigcup_m J_m,$$

where $\{I_n\}_n$ and $\{J_m\}_m$ are finite or countable collections of pairwise disjoint open intervals.

U
V



Definition 10.2.8. If E is a subset of $\mathbb{R}$, the *characteristic function* of E, denoted χ_E, is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases}.$$

Lemma 10.2.7. If $\{I_n\}_{n=1}^N$ is a finite collection of bounded open intervals, then

$$m\left(\bigcup_{n=1}^{N} I_n\right) \leq \sum_{n=1}^{N} m(I_n).$$

Theorem 10.2.6. If $\{U_n\}_n$ is a finite or countable collection of open subsets of $\mathbb{R}$, then

$$m\left(\bigcup_{n} U_n\right) \leq \sum_{n} m(U_n).$$

Theorem 10.2.9. If U and V are open subsets of $\mathbb{R}$, then $m(U) + m(V) = m(U \cup V) + m(U \cap V)$.

Definition 10.2.10. Let K be a compact subset of $\mathbb{R}$. The *measure* of K, denoted $m(K)$, is defined by

$$m(K) = m(U) - m(U \backslash K),$$

where U is any bounded open subset of $\mathbb{R}$ containing K.

Theorem 10.2.11. If K is compact, then $m(K)$ is well-defined.

Theorem 10.2.13

(a) If K is compact and U is open with $K \subset U$, then $m(K) \leq m(U)$.

(b) If K_1 and K_2 are compact with $K_1 \subset K_2$, then $m(K_1) \leq m(K_2)$.

Definition 10.2.14 If U is an open subset of $\mathbb{R}$ with $U = \cup_n I_n$, where $\{I_n\}_n$ is a finite or countable collection of pairwise disjoint open intervals, and $a, b \in \mathbb{R}$, we define

$$m(U \cap [a, b]) = \sum_n m(I_n \cap [a, b]).$$

Theorem 10.2.15. If U is an open subset of $\mathbb{R}$ and $a, b \in \mathbb{R}$, then

$$m(U \cap [a,b]) + m(U^c \cap [a,b]) \;=\; b - a.$$

Section 10.3 – Inner and Outer Measure and Measurable Sets

Definition 10.3.1. Let E be a subset of $\mathbb{R}$. The Lebesgue *outer measure* and *inner measure* of E, denoted by $\lambda^*(E)$ and $\lambda_*(E)$, respectively, are defined as follows:

$$\begin{aligned}\lambda^*(E) &= \inf\{m(U) \;:\; U \text{ is open with } E \subset U\}\\ \lambda_*(E) &= \sup\{m(K) \;:\; K \text{ is compact with } K \subset E\}\end{aligned}$$

Theorem 10.3.2

(a) For any subset E of $\mathbb{R}$, we have $0 \le \lambda_*(E) \le \lambda^*(E)$.

(b) If $E_1 \subset E_2 \subset \mathbb{R}$, then $\lambda_*(E_1) \le \lambda_*(E_2)$ and $\lambda^*(E_1) \le \lambda^*(E_2)$.

Example 1. If $E = [0, 2)$, calculate $\lambda_*(E)$ and $\lambda^*(E)$.

Example 2. Prove that, for any open set U in $\mathbb{R}$, we have $\lambda_*(U) = \lambda^*(U) = m(U)$.

Definition 10.3.4

(a) A bounded subset E of $\mathbb{R}$ is said to be *Lebesgue measurable* or *measurable* if $\lambda_*(E) = \lambda^*(E)$. If this is the case, then the *measure* of E, denoted $\lambda(E)$, is defined as

$$\lambda(E) = \lambda_*(E) = \lambda^*(E).$$

(b) An unbounded set E is *measurable* if $E \cap [a, b]$ is measurable for every closed and bounded interval $[a, b]$. If this is the case, we define

$$\lambda(E) = \lim_{k\to\infty} \lambda(E \cap [-k, k]).$$

Which Sets Are Measurable?

Theorem 10.3.5. Every set E of outer measure zero is measurable with $\lambda(E) = 0$.

Theorem 10.3.6. Every interval I is measurable with $\lambda(I) = m(I)$.

Goal. Develop a theorem that sounds like $\quad \text{measure}(E \cap [a,b]) + \text{measure}(E^c \cap [a,b]) = b - a.$

Theorem 10.3.7. For any $a, b \in \mathbb{R}$ and $E \subset \mathbb{R}$,

$$\lambda^*(E \cap [a,b]) \; + \; \lambda_*(E^c \cap [a,b]) \; = \; b - a.$$

Theorem 10.3.9. Suppose E_1, E_2 are subsets of $\mathbb{R}$. Then

(a) $\lambda^*(E_1 \cup E_2) + \lambda^*(E_1 \cap E_2) \; \leq \; \lambda^*(E_1) + \lambda^*(E_2)$, and

(b) $\lambda_*(E_1 \cup E_2) + \lambda_*(E_1 \cap E_2) \; \geq \; \lambda_*(E_1) + \lambda_*(E_2)$.

Section 10.4 – Properties of Measurable Sets

Example 1. Let $E = \bigcup_{n=1}^{\infty} [n,\, n+2^{-n}]$. Can we measure E?

Theorem 10.4.1. If E_1 and E_2 are measurable subsets of $\mathbb{R}$, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable with

$$\lambda(E_1) + \lambda(E_2) = \lambda(E_1 \cup E_2) + \lambda(E_1 \cap E_2).$$

Corollary 10.4.3. A set E is measurable if and only if E^c is measurable.

Theorem 10.4.4.

(a) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}$, then $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$.

(b) If $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of $\mathbb{R}$, then $\lambda_*\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \sum_{n=1}^{\infty} \lambda_*(E_n)$.

Theorem 10.4.5. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets. Then $\cup_{n=1}^{\infty} E_n$ and $\cap_{n=1}^{\infty} E_n$ are measurable with

(a) $$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda(E_n).$$

If, in addition, the sets $E_1, E_2, \ldots$ are pairwise disjoint, then

(b) $$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

Example 2. Show that $\mathbb{Q}^c \cap [0,1]$ is measurable, and find its measure.

Example 3. Let $\{K_n\}$ be a sequence of compact sets such that $K_1 \supset K_2 \supset K_3 \supset \cdots$, and such that $\lambda(K_n) = \frac{3n-2}{2n}$ for each n. Is the set $K = \bigcap_{n=1}^{\infty} K_n$ measurable? If so, what do you think $\lambda(K)$ is?

Theorem 10.4.6. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets.

(a) If $E_1 \subset E_2 \subset \cdots$, then $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.

(b) If $E_1 \supset E_2 \supset \cdots$ and $\lambda(E_1) < \infty$, then $\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.

Examples and Exercises

1. The Smith-Volterra Cantor set, K, is constructed as follows. Start with the set $K_0 = [0, 1]$. Obtain K_1 by removing an open interval of length 1/4 from the middle of K_0, so that

$$K_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right].$$

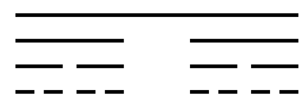


Figure 1. First four stages in the construction of K.

To obtain K_2, remove open intervals of length 1/16 from the middle of the 2 closed intervals comprising K_1, so that $K_2 = [0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$. Continuing this process recursively, we obtain K_n by removing open intervals of length $1/4^n$ from the middle of each of the 2^{n-1} closed intervals at the previous stage. The final Cantor set K is defined by $K = \cap_{n=1}^{\infty} K_n$. Show that K is measurable and find $\lambda(K)$.

Section 10.5 – Measurable Functions

Definition 10.5.1. Let f be a real-valued function defined on $[a, b]$. The function f is said to be *measurable* if for every $s \in \mathbb{R}$, the set $\{x \in [a, b] \ : \ f(x) > s\}$ is measurable. More generally, if E is a measurable subset of $\mathbb{R}$, a function $f : E \longrightarrow R$ is *measurable* if

$$\{x \in E \ : \ f(x) > s\}$$

is measurable for every $s \in \mathbb{R}$.

Theorem 10.5.3. Let f be a real-valued function defined on a measurable set E. Then f is measurable if and only if any of the following hold:

(a) $\{x \ : \ f(x) > s\}$ is measurable for every $s \in \mathbb{R}$.

(b) $\{x \ : \ f(x) \geq s\}$ is measurable for every $s \in \mathbb{R}$.

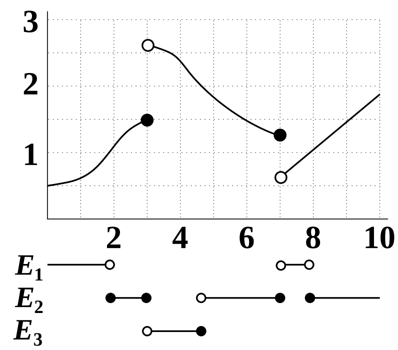
(c) $\{x \ : \ f(x) < s\}$ is measurable for every $s \in \mathbb{R}$.

(d) $\{x \ : \ f(x) \leq s\}$ is measurable for every $s \in \mathbb{R}$.

Example 1. Let $f : [0, 10] \longrightarrow \mathbb{R}$ be the function pictured to the right, and define $E_i = \{x \ : \ i - 1 \leq f(x) < i\}$ for $i = 1, 2, 3$. Consider the partition

$$\mathcal{P} \ = \ \{E_1, \, E_2, \, E_3\}$$

of $[0, 10]$. Also define $m_i = i - 1$ and $M_i = i$ for $i = 1, 2, 3$. Write out the form of $\mathcal{L}_L(\mathcal{P}, f)$ and $\mathcal{U}_L(\mathcal{P}, f)$, the lower and upper Lebesgue sums of f associated with the partition $\mathcal{P}$.



Example 2. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is measurable.

Example 3. Recall that the characteristic function of a set $A \subset \mathbb{R}$ is defined as shown to the right. Give a necessary and sufficient condition for the function χ_A to be measurable.

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Theorem 10.5.4. Suppose f, g are measurable real-valued functions defined on a measurable set E. Then

(a) $f + c$ and cf are measurable for every $c \in \mathbb{R}$.

(b) $f + g$ is measurable.

(c) fg is measurable, and

(d) $1/g$ is measurable provided that $g(x) \neq 0$ for all $x \in E$.

Definition 10.5.6. A property P is said to hold *almost everywhere* (abbreviated a.e.) if the set of points where P does not hold has measure zero; that is, $\lambda(\{x : P \text{ does not hold}\}) = 0$.

Theorem 10.5.5. Every continuous real-valued function on $[a, b]$ is measurable.

Theorem 10.5.8. Suppose f and g are real-valued functions defined on a measurable set A. If f is measurable and $g = f$ a.e., then g is measurable on A.

Theorem 10.5.9. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions defined on a measurable set A such that $\{f_n(x)\}_{n=1}^{\infty}$ is bounded for every $x \in A$. Let

$$\phi(x) = \sup\{f_n(x) : n \in \mathbb{N}\} \qquad \text{and} \qquad \psi(x) = \inf\{f_n(x) : n \in \mathbb{N}\}.$$

Then ϕ and ψ are measurable on A.

Corollary 10.5.10. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions defined on a measurable set A, and let f be a real-valued function on A. If $f_n \to f$ a.e. on A, then f is measurable on A.

Section 10.6 – The Lebesgue Integral of a Bounded Function

Definition 10.6.1. Let E be a measurable subset of $\mathbb{R}$. A *measurable partition* of E is a finite collection $\mathcal{P} = \{E_1, \ldots, E_n\}$ of pairwise disjoint measurable subsets of E such that $\bigcup_{k=1}^{n} E_k = E$. Given the measurable partition $\mathcal{P}$, we define the *lower* and *upper Lebesgue sums* of f with respect to $\mathcal{P}$, denoted $\mathcal{L}_L(\mathcal{P}, f)$ and $\mathcal{U}_L(\mathcal{P}, f)$ respectively, by

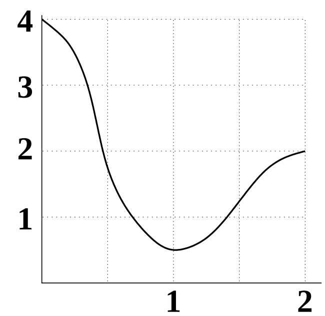
$$\mathcal{L}_L(\mathcal{P}, f) = \sum_{k=1}^{n} m_k \lambda(E_k) \qquad \text{and} \qquad \mathcal{U}_L(\mathcal{P}, f) = \sum_{k=1}^{n} M_k \lambda(E_k),$$

where

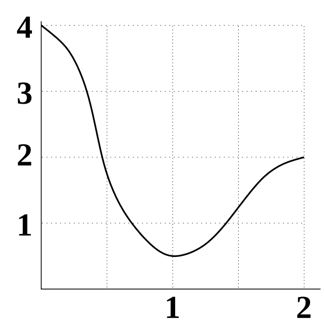
$$m_k = \inf\{f(x) : x \in E_k\} \qquad \text{and} \qquad M_k = \sup\{f(x) : x \in E_k\}.$$

Example 1. Given below is the graph of a measurable function $f : [0, 2] \longrightarrow \mathbb{R}$. For each measurable partition $\mathcal{P} = \{E_1, E_2, \ldots, E_n\}$, estimate $\mathcal{L}_L(\mathcal{P}, f)$ and $\mathcal{U}_L(\mathcal{P}, f)$.

(a) $E_1 = f^{-1}([0, 2))$, $\quad E_2 = f^{-1}([2, 4])$



(b) $E_1 = f^{-1}([0, 1))$, $E_2 = f^{-1}([1, 2))$, $E_3 = f^{-1}([2, 3))$, $E_4 = f^{-1}([3, 4])$



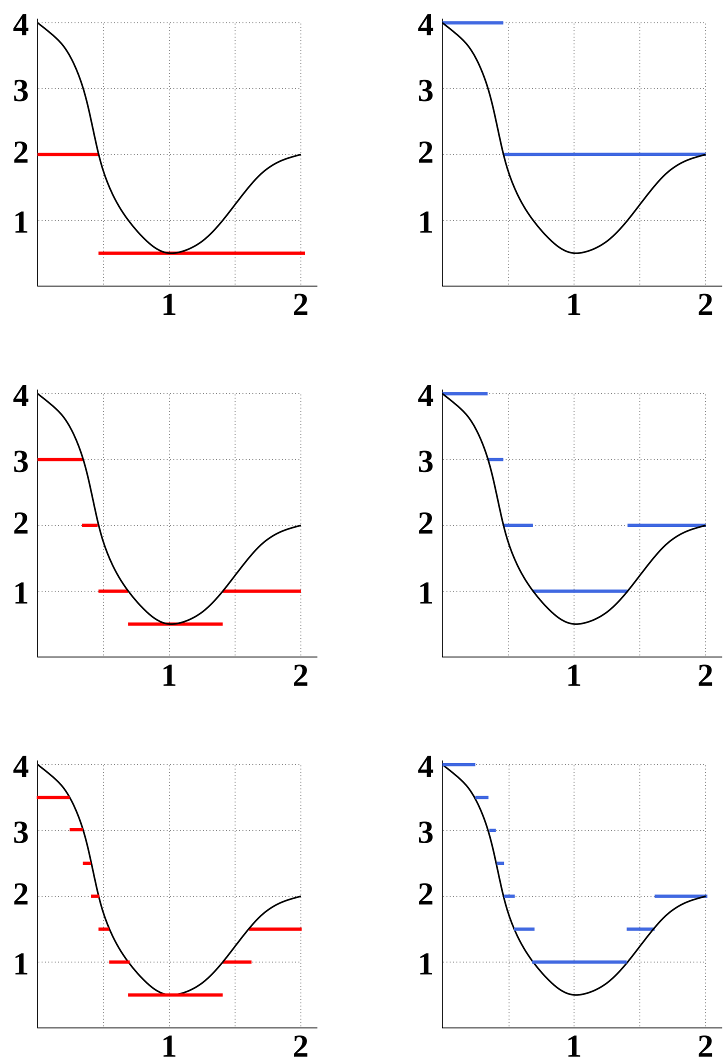
Definition 10.6.4. Let E be a measurable set and let $\mathcal{P}$ be a measurable partition of E. A measurable partition $\mathcal{Q}$ of E is a *refinement* of $\mathcal{P}$ if every set in $\mathcal{Q}$ is a subset of some set in $\mathcal{P}$.

Lemma 10.6.5. If $\mathcal{P}$, $\mathcal{Q}$ are measurable partitions of $[a,b]$ such that $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then

$$\mathcal{L}_L(\mathcal{P},f) \le \mathcal{L}_L(\mathcal{Q},f) \le \mathcal{U}_L(\mathcal{Q},f) \le \mathcal{U}_L(\mathcal{P},f).$$

As a consequence,

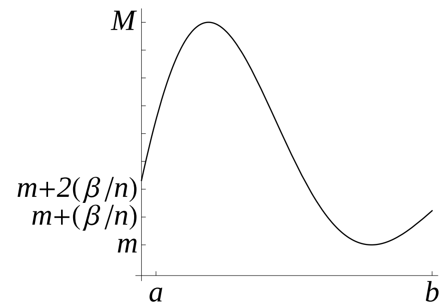
$$\sup_{\mathcal{P}} \mathcal{L}_L(\mathcal{P},f) \le \inf_{\mathcal{Q}} \mathcal{U}_L(\mathcal{Q},f).$$



Theorem 10.6.2. Let f be a bounded real-valued function $[a, b]$. Then

$$\sup_{\mathcal{P}} \mathcal{L}_L(\mathcal{P}, f) = \inf_{\mathcal{Q}} \mathcal{U}_L(\mathcal{Q}, f),$$

where the infimum and supremum are taken over all measurable partitions $\mathcal{Q}$ and $\mathcal{P}$ of $[a, b]$, if and only if f is measurable on $[a, b]$.



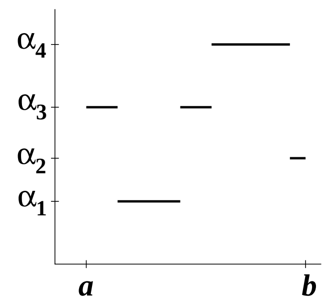
Definition 10.6.3. If f is a bounded real-valued measurable function on $[a, b]$, the *Lebesgue integral* of f over $[a, b]$, denoted $\int_{[a,b]} f\, d\lambda$ (or $\int_a^b f\, d\lambda$) is defined by

$$\int_{[a,b]} f\, d\lambda \;=\; \sup_{\mathcal{P}} \mathcal{L}_L(\mathcal{P}, f),$$

where the supremum is taken over all measurable partitions of $[a, b]$. If A is a measurable subset of $[a, b]$, the Lebesgue integral over A, denoted $\int_A f\, d\lambda$, is defined by

$$\int_A f\, d\lambda \;=\; \int_{[a,b]} f\chi_A\, d\lambda.$$

Definition. A *simple function* on $[a, b]$ is a measurable real-valued function on $[a, b]$ that assumes only a finite number of values.



Example 2. Let $\mathcal{P} = \{E_1, E_2, \ldots, E_n\}$ be a measurable partition of $[a, b]$, and consider the simple function $s : [a, b] \longrightarrow \mathbb{R}$ defined by

$$s \;=\; \sum_{i=1}^{n} \alpha_i \chi_{E_i}.$$

Show that s is Lebesgue integrable on $[a, b]$ and calculate the integral.

Corollary 10.6.8. If f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable on $[a,b]$, and

$$\int_{[a,b]} f\,d\lambda \;=\; \int_a^b f(x)\,dx.$$

Theorem 10.6.10. Suppose f, g are bounded, real-valued measurable functions on $[a,b]$ and $\alpha, \beta \in \mathbb{R}$.

(a) $\displaystyle\int_{[a,b]} (\alpha f \,+\, \beta g)\,d\lambda \;=\; \alpha \int_{[a,b]} f\,d\lambda \,+\, \beta \int_{[a,b]} g\,d\lambda.$

(b) If A_1, A_2 are disjoint measurable subsets of $[a,b]$, then $\displaystyle\int_{A_1 \cup A_2} f\,d\lambda \;=\; \int_{A_1} f\,d\lambda + \int_{A_2} f\,d\lambda.$

(c) If $f \geq g$ a.e. on $[a,b]$, then $\displaystyle\int_{[a,b]} f\,d\lambda \geq \int_{[a,b]} g\,d\lambda.$

(d) If $f = g$ a.e. on $[a,b]$, then $\displaystyle\int_{[a,b]} f\,d\lambda = \int_{[a,b]} g\,d\lambda.$

(e) $\displaystyle\left|\int_{[a,b]} f\,d\lambda\right| \leq \int_{[a,b]} |f|\,d\lambda.$

Example 3. Explain, in the context of the above theorems, why $\int_{[0,1]} \chi_{\mathbb{Q}}\,d\lambda = 0$.

Theorem 10.6.11 (Bounded Convergence Theorem). Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued measurable functions on $[a, b]$ for which there exists a positive constant M such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [a, b]$. If

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e. on } [a, b],$$

then f is Lebesgue integrable on $[a, b]$ and

$$\int_{[a,b]} f \, d\lambda = \lim_{n\to\infty} \int_{[a,b]} f_n \, d\lambda.$$

Examples and Exercises

1. Consider the function f defined to the right. Is f Lebesgue integrable on $[0,1]$? If so, calculate $\int_{[0,1]} f \, d\lambda$. If not, explain why not.

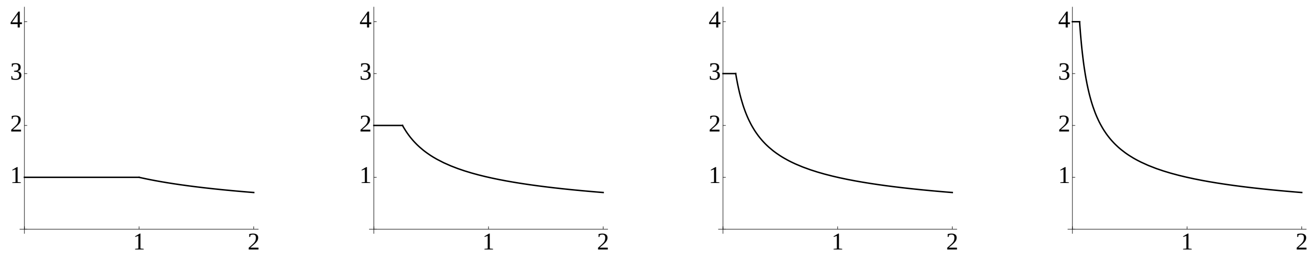
$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ x^2 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

2. Suppose that $f : [0,3] \longrightarrow [0,\infty)$ is bounded and measurable, and that $\lambda(\{x : f(x) \geq 3\}) = 2$. What is the smallest possible value that $\int_{[0,3]} f \, d\lambda$ could have? Justify your answer.

Section 10.7 – The General Lebesgue Integral

Example 1. Define $f : (0, 2] \longrightarrow \mathbb{R}$ by $f(x) = 1/\sqrt{x}$. Also, for each $n \in \mathbb{N}$, define $f_n : (0, 2] \longrightarrow \mathbb{R}$ as shown to the right. The graphs of f_n for several values of n are shown below:

$$f_n(x) = \min\{f(x), n\} = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$$



Observe that, for each fixed $n \in \mathbb{N}$, the function f_n is bounded.

(a) Show that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ pointwise on $(0, 2]$.

(b) Explain why, for each $n \in \mathbb{N}$, the Lebesgue integral $\int_{(0,2]} f_n \, d\lambda$ exists, and find its value. What can you say about the sequence $\left\{\int_{(0,2]} f_n \, d\lambda\right\}_{n=1}^{\infty}$ of Lebesgue integrals?

Definition 10.7.1. Let f be a nonnegative measurable function defined on a measurable subset A of $\mathbb{R}$. The *Lebesgue integral* of f over A, denoted $\int_A f\,d\lambda$, is defined as follows:

1. If A is bounded, then
$$\int_A f\,d\lambda \;=\; \lim_{n\to\infty}\int_A f_n\,d\lambda \;=\; \sup_n \int_A \min\{f,n\}\,d\lambda.$$

2. If A is unbounded, then
$$\int_A f\,d\lambda \;=\; \lim_{n\to\infty}\int_{A\cap[-n,n]} f\,d\lambda.$$

Example 2. Use the above definition to calculate $\displaystyle\int_1^\infty \frac{1}{x^2}\,d\lambda$.

Definition 10.7.2. A nonnegative measurable function f defined on a measurable subset A of $\mathbb{R}$ is said to be (Lebesgue) *integrable* on A if $\int_A f\,d\lambda < \infty$.

Example 3. Discuss the integrability of $f(x) = x^2$ on the set $\mathbb{R}$.

Theorem 10.7.4. Let f, g be nonnegative measurable functions defined on a measurable set A, and let c be any positive real number. Then

1. $\displaystyle\int_A (f+g)\,d\lambda \;=\; \int_A f\,d\lambda + \int_A g\,d\lambda$ and $\displaystyle\int_A cf\,d\lambda \;=\; c\int_A f\,d\lambda.$

2. If A_1, A_2 are disjoint measurable subsets of A, then

$$\int_{A_1 \cup A_2} f\,d\lambda \;=\; \int_{A_1} f\,d\lambda + \int_{A_2} f\,d\lambda.$$

3. If $f \le g$ a.e. on A, then $\displaystyle\int_A f\,d\lambda \le \int_A g\,d\lambda$ with equality if $f = g$ a.e. on A.

Example 4. For each of the following sequences $\{f_n\}$, find the pointwise limit f of the sequence. Then, confirm that the statement

$$\int_A f\,d\lambda \;=\; \lim_{n\to\infty}\int_A f_n\,d\lambda$$

does <u>not</u> hold (i.e., that the conclusion of the Bounded Convergence Theorem does not hold). Finally, explain why the Bounded Convergence Theorem does not apply to the sequence.

(a) $f_n \;=\; \begin{cases} n & \text{if } \; 0 < x \le \frac{1}{n} \\ 0 & \text{if } \; \frac{1}{n} < x \le 1 \end{cases}$ on the set $A = (0, 1]$.

(b) $f_n \;=\; \begin{cases} 2 + (-1)^n & \text{if } \; x \in [n-1,\, n] \\ 0 & \text{if } \; x \notin [n-1,\, n] \end{cases}$ on the set $A = [0, \infty)$.

Theorem 10.7.5 (Fatou's Lemma). If $\{f_n\}$ is a sequence of nonnegative measurable functions on a measurable set A, and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on A, then

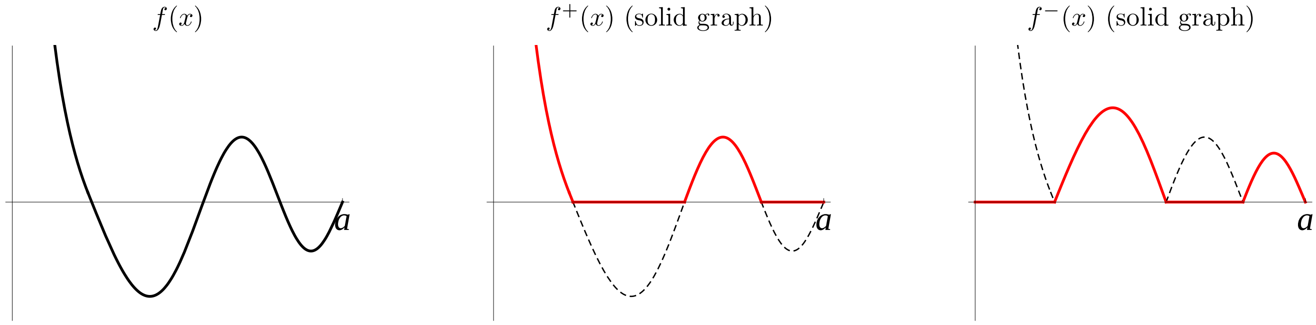
$$\int_A f\, d\lambda \;\leq\; \varliminf_{n\to\infty} \int_A f_n\, d\lambda.$$

Example 5. Apply Fatou's Lemma to the sequence $\{f_n\}$ from Example 4(b) on page 101.

Example 6. Consider a general measurable function $f : [0, a] \longrightarrow \mathbb{R}$, and define

$$f^+(x) = \max\{f(x), 0\} \qquad \text{and} \qquad f^-(x) = \max\{-f(x), 0\}.$$

Discuss the Lebesgue integrability of f on $A = [0, a]$.



Definition 10.7.7. Let f be a measurable real-valued function defined on a measurable subset A of $\mathbb{R}$. The function f is said to be (Lebesgue) *integrable* on A if $|f|$ is integrable on A. The set of Lebesgue integrable functions on A is denoted by $\mathcal{L}(A)$. For $f \in \mathcal{L}(A)$, the *Lebesgue integral* of f on A is defined by

$$\int_A f\,d\lambda \;=\; \int_A f^+\,d\lambda - \int_A f^-\,d\lambda.$$

Theorem 10.7.8. Suppose f and g are Lebesgue integrable functions on the measurable set A. Then

1. For any $c \in \mathbb{R}$, the functions $f+g$ and cf are integrable on A with
$$\int_A (f+g)\,d\lambda \;=\; \int_A f\,d\lambda + \int_A g\,d\lambda \qquad \text{and} \qquad \int_A cf\,d\lambda \;=\; c\int_A f\,d\lambda.$$
2. If $f \le g$ a.e. on A, then
$$\int_A f\,d\lambda \;\le\; \int_A g\,d\lambda, \quad \text{with equality if } f = g \text{ a.e.}$$
3. If A_1 and A_2 are disjoint measurable subsets of A, then
$$\int_{A_1 \cup A_2} f\,d\lambda \;=\; \int_{A_1} f\,d\lambda + \int_{A_2} f\,d\lambda.$$

Theorem 10.7.9 (Lebesgue's Dominated Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set A such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists a.e. on A. Suppose there exists a nonnegative integrable function g on A such that $|f_n(x)| \leq g(x)$ a.e. on A. Then f is integrable on A and

$$\int_A f\, d\lambda = \lim_{n\to\infty} \int_A f_n\, d\lambda.$$