Homework 1 – Math 440

Turn-in Problems. The following problems will be due on **Friday, January 30**.

1. **p2.5.i** Find the limit inferior and limit superior of each of the following sequences $\{s_n\}$. You need not give full proofs that your answers are correct, just brief explanations.

   (a) $s_n = 2 - \dfrac{1}{n}$

   (b) $s_n = n \mod 4$

   (c) $s_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} + \cos(\pi n) & \text{if } n \text{ is odd} \end{cases}$

   (d) $s_n = f(n)$, where $f : \mathbb{N} \longrightarrow \mathbb{Q} \cap (0,1)$ is a bijection

2. **p2.5.ii** Give an example of a sequence $\{s_n\}$ and a real number s such that $s_n < s$ for all n, but $\overline{\lim}\, s_n \geq s$.

3. **p2.5.iv** Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of nonnegative real numbers.

   (a) Prove that $\overline{\lim}(a_n b_n) \leq (\overline{\lim}\, a_n)(\overline{\lim}\, b_n)$.

   (b) Give an example to show that equality need not hold in the inequality from part (a), and demonstrate that your example works.

Note: On Problem 2-7-iii below, be sure to use only results from Section 2.7 or before. The point is to prove convergence or divergence without using the p-series test.

4. **p2.7.iii** Use induction to show that $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$ for all $n \geq 1$. Use this fact to prove or disprove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

5. **p2.7.iv** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Prove that $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ converges if and only if $\{a_n\}_{n=1}^{\infty}$ converges. If $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ converges, what is the sum?

6. **p2.7.v** Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ converges, and find the sum. $\left(\underline{\text{Hint}}: \dfrac{2}{k(k+1)} = \dfrac{2}{k} - \dfrac{2}{k+1}\right)$

Practice Problems. The following problems will not be collected. However, doing and understanding them is important for your success in the course.

7. **p2.7.ii** Determine whether the series $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ converges or diverges. Justify your answer.

Homework 2 – Math 440

Turn-in Problems. The following problems will be due on **Friday, February 6**.

Though Problem 7.1.ix below will involve calculations, keep in mind that your solutions should still read as proofs with complete sentences. For part (b), it is your goal to determine which values of p yield a convergent series and which yield a divergent series.

8. **p7.1.ix** Test the following series for convergence, and justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{2k^3-k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{p^n}{n!}, \quad p \in \mathbb{R}$

9. **p7.1.iv** Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum a_k < \infty$. For each of the following, either prove that the given series converges, or provide an example for which the series diverges. If you give an example to illustrate divergence, make sure to justify why your example works.

(a) $\displaystyle\sum_{k=1}^{\infty} a_k^2$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{a_k^2}{1+a_k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \sqrt{\frac{a_k}{1+a_k}}$

10. **p7.1.vi** Prove part (a) of Corollary 7.1.3, the Limit Comparison Test.

On Problem 7.1.viii below, keep in mind that we proved the forward direction in class. Therefore, you can simply cite our proof and then prove the other direction to complete the problem.

11. **p7.1.viii** Suppose that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. This result is called the *Cauchy Condensation Test*.

12. **p7.2.3** If $\sum a_k$ converges, does $\sum a_k^2$ always converge? Justify your answer.

13. **p7.3.i** Suppose $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\{b_n\}_{n=1}^{\infty}$ is bounded. Prove that $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

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14. **p7.1.iii** Determine all values of p for which $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^p \ln k}$ converges. Justify your answer.

Homework 3 – Math 440

Turn-in Problems. The following problems will be due on **Friday, February 13**.

15. **p7.2.i** Define $b_k = \begin{cases} \frac{1}{k^2} & \text{if } k \text{ is odd} \\ \frac{1}{k^3} & \text{if } k \text{ is even} \end{cases}$.

(a) Does the series $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$ satisfy the hypotheses of the Alternating Series Test? Explain.

(b) Does $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$ converge or diverge? Prove your answer.

For Problem 7.3.2 below, a very useful starting hint is given in the back of the textbook.

16. **p7.3.2** Suppose that $\sum a_k^2 < \infty$ and $\sum b_k^2 < \infty$. Prove that the series $\sum a_k b_k$ converges absolutely.

Hint: For part (a) of Problem 7.3.v below, the hard part is showing that $b_k = (\ln k)/k$ is decreasing. Do this by showing that the function $f(x) = (\ln x)/x$ is decreasing on an appropriate interval.

17. **p7.3.v** Determine whether each of the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers. You may assume the usual calculus facts about the exponential and logarithmic functions.

(a) $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k \ln k}{k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+(-2)^k}$

18. **p8.2.1** Prove Theorem 8.2.5.

19. **p8.2.iv** For each $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \frac{1}{n}\lfloor nx \rfloor$.

(a) Sketch the graph of f_n for $n = 2, 3$ and 4.

(b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f, and clearly state what that function is.

(c) Does there exist a value of n for which f_n is continuous on $[0,1]$? Is f continuous on $[0,1]$. Feel free to state your answers without proof.

(d) Explain why the results of parts (b) and (c) above do not contradict Corollary 8.3.2.

20. **p8.2.v** Define $f_k(x) = \dfrac{x^2-2}{\sqrt{1+k^3x^2}}$.

(a) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1,3]$.

(b) Prove that $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[1,3]$.

Practice Problems. The following problems will not be collected. However, doing and understanding them is important for your success in the course.

21. **p8.1.1** Find the pointwise limits of each of the following sequences of functions on the given set.

(a) $\left\{\dfrac{nx}{1+nx}\right\}$, $x \in [0,\infty)$

(b) $\left\{\dfrac{\sin(nx)}{1+nx}\right\}$, $x \in [0,\infty)$

(c) $\left\{(\cos x)^{2n}\right\}$, $x \in \mathbb{R}$

(d) $\left\{nxe^{-nx^2}\right\}$, $x \in \mathbb{R}$