Math 440 – Real Analysis II

Homework 10

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Problem 58 Let f and g be nonnegative, real-valued functions defined on a measurable set A with $n \in \mathbb{N}$, then for all $x \in A$

$$\min\{f(x)+g(x), n\} \leqslant \min\{f(x), n\}+\min\{g(x), n\} \leqslant \min\{f(x)+g(x), 2n\}$$

Proof: Let $f, g : A \to \mathbb{R}^{0+}$

case let $f(x) > n$ and $g(x) > n$:
Then the inequality holds trivially.

case wolog let $f(x) > n$ and $g(x) \leqslant n$:
Then $n < n + g(x) \leqslant f(x) + g(x)$ and $2n$, so the inequality holds.

case let $f(x) \leqslant n$, $g(x) \leqslant n$, and $f(x) + g(x) > n$:
Then $f(x) + g(x) \leqslant 2n$, so the inequality holds.

case let $f(x) + g(x) \leqslant n$:
Then the inequality holds trivially.

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Problem 59 If $f : (0, \infty) \to \mathbb{R}$ such that for each $n \in \mathbb{N}$, $f(x) = (-1/2)^n$ for $n - 1 \leqslant x < n$, then f is Lebesgue integrable and $\int\limits_{(0,\infty)} f d\lambda = -\frac{1}{3}$.

Proof: If $\lfloor x \rfloor$ is even then $f(x) \leqslant 0$, and if $\lfloor x \rfloor$ is odd then $f(x) < 0$. So let $f^-(x) = (1/2)^{2n-1}$ for $2n-2 \leqslant x < 2n-1$ and let $f^+(x) = (1/2)^{2n}$ for $2n - 1 \leqslant x < 2n$. Thus f^+ and f^- are nonnegative and bounded, with $f(x) = f^+(x) - f^-(x)$. Additionally, both functions are continuous on a countable union of disjoint intervals, so by Theorems 10.5.5 and 10.4.5 the functions f^- and f^+ are measurable. Hence, by Theorem 10.7.1, f^- and f^+ are Lebesgue integrable, and by 10.7.4 f is as well.

Therefore:

$$\begin{aligned}
\int\limits_{(0,\infty)} f d\lambda &= \int\limits_{(0,\infty)} f^+ d\lambda - \int\limits_{(0,\infty)} f^- d\lambda \\
&= \lim_{n\to\infty} \int\limits_{(0,n]} f^+ d\lambda - \lim_{n\to\infty} \int\limits_{(0,n]} f^- d\lambda \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (\tfrac{1}{2^{2k}})\lambda[2k-2, 2k-1) - \lim_{n\to\infty} \sum_{k=1}^{n} (\tfrac{1}{2^{2k-1}})\lambda[2k-1, 2k) \\
&= \sum_{k=1}^{\infty} (\tfrac{1}{2^{2k}}) - \sum_{k=1}^{\infty} (\tfrac{1}{2^{2k-1}}) = -\tfrac{1}{3}
\end{aligned}$$

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Problem 60 Let $f, g : A \to \mathbb{R}^{0+}$ be measurable

(b) If A_1, A_2 are measurable, disjoint subsets of A, then

$$\int\limits_{A_1 \cup A_2} f d\lambda = \int\limits_{A_1} f d\lambda + \int\limits_{A_2} f d\lambda$$

Proof: Let $f_1 = f\chi_{A_1}$ and $f_2 = f\chi_{A_2}$. Then $\int\limits_{A_1} f d\lambda = \int\limits_{A_1} f_1 d\lambda$ and $\int\limits_{A_2} f d\lambda = \int\limits_{A_2} f_2 d\lambda$, and since $f_1(x) = 0$ for all $x \notin A_1$ and $f_2(x) = 0$ for all $x \notin A_2$, $\int\limits_{A_1} f d\lambda = \int\limits_{A_1 \cup A_2} f_1 d\lambda$ and $\int\limits_{A_2} f d\lambda = \int\limits_{A_1 \cup A_2} f_2 d\lambda$. As A_1, A_2 are disjoint, $(f_1 + f_2)(x) = f(x), \forall x \in A_1 \cup A_2$, thus

$$\begin{aligned} \int\limits_{A_1} f d\lambda + \int\limits_{A_2} f d\lambda &= \int\limits_{A_1 \cup A_2} f_1 d\lambda + \int\limits_{A_1 \cup A_2} f_2 d\lambda \\ &= \int\limits_{A_1 \cup A_2} (f_1 + f_2) d\lambda && \text{By Thm 10.7.4(a)} \\ &= \int\limits_{A_1 \cup A_2} f d\lambda \end{aligned}$$

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(c) If $f \leqslant g$ a.e. on A, then $\int\limits_A f d\lambda \leqslant \int\limits_A g d\lambda$ with equality if $f = g$ a.e.

Proof: Let $E = \{x \in A : f(x) > g(x)\}$, $E_1 = \{x \in A : f(x) = g(x)\}$, and $E_2 = \{x \in A : f(x) < g(x)\}$. Because $\lambda(E) = 0$ by assumption, $\int\limits_A f d\lambda = \int\limits_{E_1 \cup E_2} f d\lambda$ by Theorem 10.7.4(b), and similarly for g. Let $h(x) = (g - f)(x)$ for all $x \in E_1 \cup E_2$. As h is nonnegative,

$$\begin{aligned} \int\limits_A g d\lambda &= \int\limits_{E_1 \cup E_2} g d\lambda \\ &= \int\limits_{E_1 \cup E_2} (f + h) d\lambda \\ &= \int\limits_{E_1 \cup E_2} f d\lambda + \int\limits_{E_1 \cup E_2} h d\lambda && \text{By Thm 10.7.4(a)} \\ &= \int\limits_A f d\lambda + \int\limits_{E_1} h d\lambda + \int\limits_{E_2} h d\lambda && \text{By Thm 10.7.4(b)} \\ &= \int\limits_A f d\lambda + \int\limits_{E_2} h d\lambda && \text{As } h(x) = 0,\ \forall x \in E_1 \end{aligned}$$

Therefore if $g = f$ a.e., then $\int\limits_A f d\lambda = \int\limits_A g d\lambda$, otherwise $\int\limits_A f d\lambda < \int\limits_A g d\lambda$.

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Problem 61 Let f be a nonnegative integrable function on $[a, b]$. For each $n \geqslant 0$, if $E_n = \{x : n \leqslant f(x) < n + 1\}$ then $\sum_{n=0}^{\infty} n\lambda(E_n) < \infty$.

Proof: Let f_k be defined such that $f_k(x) = f(x)$ if $f(x) < k + 1$ and 0 otherwise, then $f_k(x) = f(x)$ for all $x \in E_k$ and $f_k(x) = 0$ for all $x \in E_k^c$ where $A_k = \bigcup_{n=0}^{k} E_n$ and $A_k^c = [a, b] \backslash A_k$. By Theorem 10.7.4, $\int_{[a,b]} f d\lambda = \int_{A_k} f d\lambda + \int_{A_k^c} f d\lambda$, and because f is integrable, $\int_{A_k^c} f d\lambda \geqslant 0$, and $f_k(x) = f(x)$ for all $x \in A_k$, $\int_{A_k} f_k d\lambda \leqslant \int_{[a,b]} f d\lambda < \infty$ for all $k \geqslant 0$. Since f_k is bounded on $[a, b]$ and $\mathcal{P}_k = \{A_k^c, E_0, E_1, \ldots, E_k\}$ partitions $[a, b]$, $\mathcal{L}_L(\mathcal{P}_k, f_k) \leqslant \int_{[a,b]} f_k d\lambda$ by Definition 10.6.3 for all k, and so $\mathcal{L}_L(\mathcal{P}_k, f_k) = 0\lambda(A_k^c) + \sum_{n=0}^{k} m_n \lambda(E_n)$. Because for all $n \leqslant k$, $n \leqslant m_n = \inf\{f_k(x) \text{ for all } x \in E_n\}$, $\sum_{n=0}^{k} m_n \lambda(E_n) \leqslant \mathcal{L}_L(\mathcal{P}_k, f_k)$. Therefore $\sum_{n=0}^{k} m_n \lambda(E_n) \leqslant \int_{[a,b]} f d\lambda < \infty$ for all $k \geqslant 0$, so $\sum_{n=0}^{\infty} m_n \lambda(E_n) < \infty$.

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Problem 62 Let f be a nonnegative measurable function on a measurable set A. If $\int\limits_A f d\lambda = 0$ then $f = 0$ almost everywhere on A.

Proof:

(a) For each $n \in \mathbb{N}$ let $f_n : A \to [0, \infty)$ be defined as $f_n(x) = \min\{f(x), n\}$, then $f_n(x) \leqslant f(x)$ for all $n \in \mathbb{N}$ and $x \in A$ since $\min\{f(x), n\} \leqslant f(x)$.

(b) Let $n, m \in \mathbb{N}$ be given and $A_m = A \cap [-m, m]$ so that, by Theorem 10.7.4(b), $\int\limits_A f d\lambda = \int\limits_{A \cap A_M} f d\lambda + \int\limits_{A \setminus A_M} f d\lambda$. As the Lebesgue integral is nonnegative and $A \cap A_m = A_m$, $\int\limits_{A_m} f d\lambda = 0$. Hence $\int\limits_{A_m} f_n d\lambda = 0$ given that $f_n(x) \leqslant f(x)$ for all $x \in A_m$.

(c) Let $E_{m,n} = \{x \in A_m : f_n(x) \neq 0\}$ such that $\int\limits_{A_m} f_n d\lambda = \int\limits_{E_{m,n}} f_n d\lambda + \int\limits_{A_m \setminus E_{m,n}} f_n d\lambda$. By previous argument $\int\limits_{E_{m,n}} f_n d\lambda = 0$, and by Definition 10.6.3 $\int\limits_{E_{m,n}} f_n d\lambda \geqslant i\lambda(E_m)$, where $i = \inf\{f_n(x) : x \in E_m\}$. Therefore $i\lambda(E_{m,n}) = 0$, so $\lambda(E_{m,n}) = 0$ as $i \geqslant 0$. Because n, m are arbitrary, $\lambda(E_{m,n}) = 0$ for all $m, n \in \mathbb{N}$

(d) Let $E = \{x \in A : f(x) \neq 0\}$. Then since $n > 0$ for all $n \in \mathbb{N}$ and $f(x) > 0$ for all $x \in E$, this gives $f_n(x) \neq 0$ for all $x \in E$ and all $n \in \mathbb{N}$. Additionally, for all fixed $x \in E$ there exists an $m \in \mathbb{N}$ such that $x < m$. Therefore, for every $x \in E$, $x \in E_{m,n}$ for some $m, n \in \mathbb{N}$ and thus $E \subset \bigcup\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} E_{m,n}$.

(e) By Theorem 10.4.5, $\lambda(E) \leqslant \lambda\left(\bigcup\limits_{m=1}^{\infty} \bigcup\limits_{n=1}^{\infty} E_{m,n}\right) \leqslant \sum\limits_{m=1}^{\infty} \sum\limits_{m=1}^{\infty} \lambda(E_{m,n})$. Since $\lambda(E_{m,n}) = 0$ for all $m, n \in \mathbb{N}$, $\lambda(E) = 0$, so $f = 0$ almost everywhere.

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