Math 440 – Real Analysis II

Exam 1

Amandeep Gill

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Problem 1 Given the series $\sum_{k=1}^{\infty} p^{-k}k^p$, for what values of p does it:

(a) diverges when $-1 < p \leqslant 1$

Proof: If $p = 0$ then $p^{-k}k^p = \frac{1}{0}$, which is unbounded and therefore divergent. If $p = 1$ then $p^{-k}k^p = k$, which is unbounded and therefore divergent. If $0 < p < 1$ then $p = 1/r$ for some $r > 1$ and $p^{-k}k^p = r^k \sqrt[r]{k}$, which is unbounded and therefore divergent.
If $-1 < p < 0$ then $p = -1/r$ for some $r > 1$ and $p^{-k}k^p = (-1)^k \frac{r^k}{\sqrt[r]{k}}$. Since r^k grows without bounds, and for all r, there exists some $k_\circ \in \mathbb{N}$ such that $r^{k_\circ} > \sqrt[r]{k_\circ}$, $(-1)^k \frac{r^k}{k^p}$ is unbounded and therefore divergent.

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(b) converge conditionally when $p = -1$

Proof: If $p = -1$ then $p^{-k}k^p = \frac{(-1)^k}{k}$, which is the oscillating harmonic series, and is conditionally convergent.

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(c) converge absolutely when $|p| > 1$

Proof: If $p < -1$ then $p^{-k}k^p = \frac{(-1)^k}{p^k k^p}$. Since $\left|\frac{(-1)^k}{p^k k^p}\right| = \frac{1}{p^k k^p} \leqslant \frac{1}{k^p}$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{p^k k^p}$ converges absolutely by the Comparison Test against a P-series where $P > 1$.

If $p > 1$ then using the Ratio Test leads to $\lim_{k\to\infty} \frac{p^{-k-1}(k+1)^p}{p^k k^p} = \frac{1}{p} < 1$, meaning the series converges. Because $|p^k k^p| = p^k k^p$, the series is absolutely convergent.

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Problem 2 If $\{a_n\}$ is a sequence with $a_n > 0$ for all n then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Proof: Let $\{a_n\}$ be a sequence with $a_n > 0$ for all n

$(\implies)$ Assume $\sum\limits_{n=1}^{\infty} a_n$ converges. Since $a_n > 0$, $a_n + 1 > 1$ which means that $\frac{a_n}{1+a_n} < a_n$ for all n. Therefore, by the Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is a convergent series.

$(\impliedby)$ Assume $\sum\limits_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. This implies that $\lim\limits_{n\to\infty} \frac{a_n}{1+a_n} = 0$, so $\lim\limits_{n\to\infty} 1 - \frac{1}{1+a_n} = 0$, which means $\lim\limits_{n\to\infty} a_n = 0$. $\{a_n\}$ is therefore a bounded sequence so there exists $m \in \mathbb{R}^+$ such that $m = \max\{a_n\}$. Then for all n, $\frac{a_n}{1+m} \leqslant \frac{a_n}{1+a_n}$ and by the Comparison Test $\sum\limits_{n=1}^{\infty} \frac{a_n}{1+m}$ converges, and therefore so does $\sum\limits_{n=1}^{\infty} a_n$.

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Problem 3 Let $\{a_n\}$ be a sequence $a_n \neq 0$ for all n.

(a) Disprove: if $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all n then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Proof: Let $a_n = \frac{1}{n}$. For all n, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n}{n+1}\right| < 1$, however $\sum\limits_{n=1}^{\infty} a_n$ is the harmonic series and therefore does not converge.

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(b) If there exists a convergent subsequence of $\{b_n\} = \left\{\sqrt[n]{|a_n|}\right\}_{n=1}^{\infty}$ whose limit is strictly greater than 1 then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges

Proof: Let $\{b_n\}$ have a subsequence that converges to some $b > 1$. Then $\limsup\limits_{n\to\infty}\{b_n\} \geqslant b$ since for all $n_\circ \in \mathbb{N}$ and $\epsilon > 0$ there is an $n > n_\circ$ such that $|b_n - b| < \epsilon$. Thus $\{b_n\}$ either diverges or converges to a number greater than 1, and the series diverges by the Root Test.

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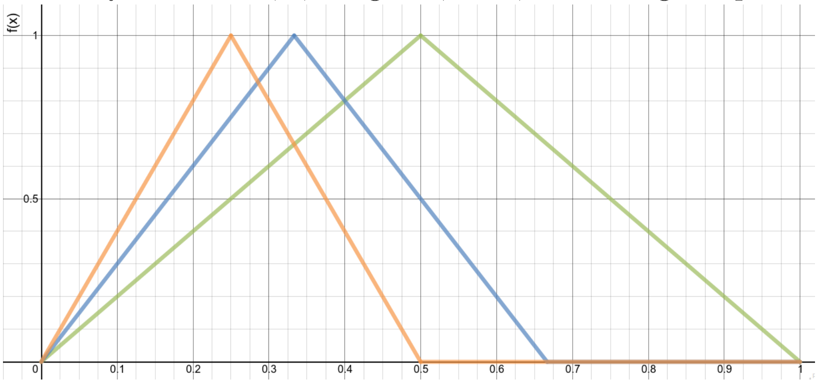
Problem 4 For each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} < x \leqslant \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leqslant 1 \end{cases}$$

(a) Plot of f_n for $n = 2, 3, 4$ – green, blue, and orange respectively:



(b) $\{f_n\}$ converges pointwise but not uniformly to some function f.

Proof: Let $x \in [0,1]$ be given. If $x = 0$ then $\lim_{n\to\infty} f_n = \lim_{n\to\infty} nx = 0$. Otherwise, if $0 < x \leqslant 1$ then there exists some $n_\circ \in \mathbb{N}$ such that $\frac{2}{n_\circ} < x$ and so for all $n > n_\circ$, $f_n(x) = 0$. Therefore $\{f_n\}$ converges pointwise to $f(x) = 0$. However letting $\epsilon = \frac{1}{2}$, $n > N$, and $x = \frac{1}{2n}$ for some $N \in \mathbb{N}$ gives $|f_n(x) - f(x)| = |nx - 0| = \left|\frac{n}{2n}\right| = \frac{1}{2} = \epsilon$, showing that $\{f_n\}$ is not uniformly convergent.

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(c) f_n is continuous for all n on [0,1], and the same holds for f.

(d) Parts (b) and (c) do not violate Corollary 8.3.2 because it does not address functions that do not converge uniformly.

Problem 5

(a) $\frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ when $x \in (-1, 1)$

Proof: Let $s_n = \sum\limits_{k=0}^{n} x^k$, then $s_n = 1 + x + x^2 + \cdots + x^n = \frac{1}{1-x} - \frac{x^n}{1-x}$. Since $|x| < 1$, $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1}{1-x} - \frac{x^n}{1-x} = \frac{1}{1-x}$.

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(b) Evaluate $f(x) = \int_0^{1/2} \frac{1}{1+x^6} dx$ as a power series.

Proof: By part (a), $\frac{1}{1+x^6} = \sum\limits_{k=0}^{\infty} (x^6)^k$ since $x^6 \in (-1, 1)$ for all $x \in \left[0, \frac{1}{2}\right]$. Substituting the series in the integral gives $\int_0^{1/2} \sum\limits_{k=0}^{\infty} (x^6)^k dx = \sum\limits_{k=0}^{\infty} \int_0^{1/2} (x^6)^k dx$ because the Power Series is uniformly convergent on its domain. This gives $f(x) = \sum\limits_{k=0}^{\infty} \frac{(x^6)^{k+1}}{k+1}$.

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Problem 6 Using $f(x) = \sum\limits_{k=1}^{\infty} f_k(x) = \sum\limits_{k=1}^{\infty} \frac{1}{1+k^2x}$ prove:

(a) That $f(x)$ converges uniformly on $x \in [a, \infty)$ for all $a > 0$

Proof: Since $a \leqslant x$ for all $x \in [a, \infty)$, $\frac{1}{1+k^2x} \leqslant \frac{1}{1+k^2a} < \frac{1}{k^2a}$ for all $k \in \mathbb{N}$. Since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2a}$ is a convergent P-series, $f(x)$ converges uniformly on $[a, \infty)$ by the Weierstrass M-test.

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(b) That $f(x)$ does not converge uniformly on $x \in (0, \infty)$

Proof: Let $\epsilon = \frac{1}{2}$, $k_\circ, m, n \in \mathbb{N}$ such that $k_\circ < m < n$, and $x = \frac{1}{n^2}$. Then by the Cauchy Criterion, $|f_n(x) - f_m(x)| = \left| \sum\limits_{k=1}^{n} \frac{1}{1+k^2x} - \sum\limits_{k=1}^{m} \frac{1}{1+k^2x} \right| = \sum\limits_{k=m+1}^{n} \frac{1}{1+k^2x} \geqslant \sum\limits_{k=m+1}^{n} \frac{1}{1+n^2x} = \sum\limits_{k=m+1}^{n} \frac{1}{2} \geqslant \frac{1}{2} = \epsilon$, and the series is therefore not uniformly convergent on $(0, \infty)$.

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Problem 7 Let $\{a_n\} = \left\{\frac{1}{2n-1}\right\}_{n\in\mathbb{N}} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ and $\{b_n\} = \left\{\frac{1}{2n}\right\}_{n\in\mathbb{N}} = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$

(a) The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = a_1 - b_1 + a_2 - b_2 + \cdots$ converges.

Proof: The series converges by the Alternating Series Test, and since it is the same term-by-term sequence of numbers as the series $\sum_{n=1}^{\infty}(a_n - b_n)$, it follows that $a_1 - b_1 + a_2 - b_2 + \cdots$ converges as well.

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(b) For each integer $j \geqslant 1$ let $n_j = 2^j - 1$, $a_{1+n_{j-1}} + a_{2+n_{j-1}} + \cdots + a_{n_j} \geqslant \frac{1}{4}$

Proof: By assumption, $a_{1+n_{j-1}} = \frac{1}{1+2(2^{j-1}-1)-1} = \frac{1}{2^j-1}$ and $a_{n_j} = \frac{1}{2(2^j-1)-1} = \frac{1}{2^{j+1}-3}$. The distance between the denominators is equal to $(2^{j+1}-3) - (2^j-1)+1 = 2(2^{j-1})$, giving 2^{j-1} terms in the sum s_j. Since each term in $s_{j_i} \geqslant \frac{1}{2^{j+1}-3} \geqslant \frac{1}{2^{j+1}}$, then $s_j \geqslant \frac{2^{j-1}}{2^{j+1}} = \frac{1}{4}$.

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(c) Let the rearrangement of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ be given as $\sum_{k=1}^{\infty}(s_k - b_k)$

Proof: Let $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty}(s_k - b_k)$, then the first four terms are:
$c_1 = a_1 - b_1$
$c_2 = a_2 + a_3 - b_2$
$c_3 = a_4 + a_5 + a_6 + a_7 - b_3$
$c_4 = a_8 + a_9 + a_10 + a_11 + a_12 + a_13 + a_14 + a_15 - b_4$

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(d) The rearrangement $\sum_{k=1}^{\infty} c_k$ diverges

Proof: $\lim_{k\to\infty} c_k = \lim_{k\to\infty} s_k - b_k$. Since $\lim_{k\to\infty} b_k = 0$, and $s_k \geqslant \frac{1}{4}$ by previous work, so $\lim_{k\to\infty} c_k \geqslant \frac{1}{4} \neq 0$. Thus the series fails the n^{th}-Term Test and is therefore divergent.

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