Math 440 – Real Analysis II

Homework 1

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January 29, 2015

Problem 1 Find the limit inferior and limit superior of each of the following sequences $\{s_n\}$.

(a) $s_n = 2 - \frac{1}{n}$

$$\overline{\lim}\{s_n\} = \underline{\lim}\{s_n\} = \lim_{n\to\infty} s_n = 2$$

(b) $s_n = n \bmod 4$

For all $k \in \mathbb{N}$ the sequence $\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ contains only the values $\{0, 1, 2, 3\}$. Thus $\forall k$, $0 \leqslant s_k \leqslant 3$, so

$$\overline{\lim}\{s_n\} = 3$$

And

$$\underline{\lim}\{s_n\} = 0$$

(c) $s_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} + \cos \pi n & \text{if } n \text{ is odd} \end{cases}$

For all $n \in \mathbf{N}$, $\frac{1}{n} + \cos \pi n \leq n$, so the $\sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ is dominated by the even terms of s_n. Therefore

$$\overline{\lim}\{s_n\} = \lim_{n\to\infty} 2n = \infty$$

By the same argument, the $\inf\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ is dominated by the odd terms of s_n, which is split between two cases: $s_k = \frac{1}{k} + 1$ and $s_k = \frac{1}{k} - 1$. Thus

$$\underline{\lim}\{s_n\} = \lim_{n\to\infty} \frac{1}{4n+3} - 1 = -1$$

(d) $s_n = f(n)$, where $f\colon \mathbb{N} \to \mathbb{Q} \cap (0, 1)$ is a bijection

Let $\epsilon > 0$ and $\overline{S} = \{s_n \colon 1 - s_n < \epsilon\}$
By Math340 results we know that there exists an infinite number of rationals between any two distinct reals (in this case 1 and $1-\epsilon$). Therefore $\overline{S}$ is a countably infinite set, and $\overline{S} \setminus \{s_1, s_2, \ldots, s_{k-1}\}$ yields a similarly infinite set. From this,

$$\overline{\lim}\{s_n\} = \sup \overline{S} = 1$$

Similarly, with $\underline{S} = \{s_n \colon s_n < \epsilon\}$ we have

$$\underline{\lim}\{s_n\} = \inf \underline{S} = 0$$