

Math 440 – Real Analysis II

Exam 1

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Problem 1 Given the series $\sum_{k=1}^{\infty} p^{-k} k^p$, for what values of p does it:

(a) diverges when $-1 < p \leq 1$

Proof: If $p = 0$ then $p^{-k} k^p = \frac{1}{0}$, which is unbounded and therefore divergent.
 If $p = 1$ then $p^{-k} k^p = k$, which is unbounded and therefore divergent.
 If $0 < p < 1$ then $p = 1/r$ for some $r > 1$ and $p^{-k} k^p = r^k \sqrt[r]{k}$, which is unbounded and therefore divergent.
 If $-1 < p < 0$ then $p = -1/r$ for some $r > 1$ and $p^{-k} k^p = (-1)^k \frac{r^k}{\sqrt[r]{k}}$.
 Since r^k grows without bounds, and for all r , there exists some $k_o \in \mathbb{N}$ such that $r^{k_o} > \sqrt[r]{k_o}$, $(-1)^k \frac{r^k}{\sqrt[r]{k}}$ is unbounded and therefore divergent.

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(b) converge conditionally when $p = -1$

Proof: If $p = -1$ then $p^{-k} k^p = \frac{(-1)^k}{k}$, which is the oscillating harmonic series, and is conditionally convergent.

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(c) converge absolutely when $|p| > 1$

Proof: If $p < -1$ then $p^{-k} k^p = \frac{(-1)^k}{p^k k^p}$. Since $\left| \frac{(-1)^k}{p^k k^p} \right| = \frac{1}{p^k k^p} \leq \frac{1}{k^p}$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{p^k k^p}$ converges absolutely by the Comparison Test against a P-series where $P > 1$.

If $p > 1$ then using the Ratio Test leads to $\lim_{k \rightarrow \infty} \frac{p^{-k-1} (k+1)^p}{p^k k^p} = \frac{1}{p} < 1$, meaning the series converges. Because $|p^k k^p| = p^k k^p$, the series is absolutely convergent.

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Problem 2 If $\{a_n\}$ is a sequence with $a_n > 0$ for all n then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Proof: Let $\{a_n\}$ be a sequence with $a_n > 0$ for all n

(\implies) Assume $\sum_{n=1}^{\infty} a_n$ converges. Since $a_n > 0$, $a_n + 1 > 1$ which means that $\frac{a_n}{1+a_n} < a_n$ for all n . Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is a convergent series.

(\impliedby) Assume $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. This implies that $\lim_{n \rightarrow \infty} \frac{a_n}{1+a_n} = 0$, so $\lim_{n \rightarrow \infty} 1 - \frac{1}{1+a_n} = 0$, which means $\lim_{n \rightarrow \infty} a_n = 0$. $\{a_n\}$ is therefore a bounded sequence so there exists $m \in \mathbb{R}^+$ such that $m = \max\{a_n\}$. Then for all n , $\frac{a_n}{1+m} \leq \frac{a_n}{1+a_n}$ and by the Comparison Test $\sum_{n=1}^{\infty} \frac{a_n}{1+m}$ converges, and therefore so does $\sum_{n=1}^{\infty} a_n$. ■

Problem 3 Let $\{a_n\}$ be a sequence $a_n \neq 0$ for all n .

(a) Disprove: if $\left| \frac{a_{n+1}}{a_n} \right| < 1$ for all n then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Let $a_n = \frac{1}{n}$. For all n , $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n}{n+1} \right| < 1$, however $\sum_{n=1}^{\infty} a_n$ is the harmonic series and therefore does not converge. ■

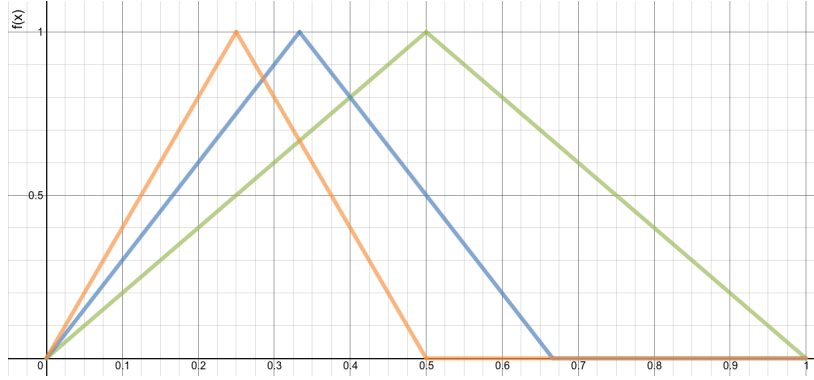
(b) If there exists a convergent subsequence of $\{b_n\} = \left\{ \sqrt[n]{|a_n|} \right\}_{n=1}^{\infty}$ whose limit is strictly greater than 1 then the series $\sum_{n=1}^{\infty} a_n$ diverges

Proof: Let $\{b_n\}$ have a subsequence that converges to some $b > 1$. Then $\limsup_{n \rightarrow \infty} \{b_n\} \geq b$ since for all $n_o \in \mathbb{N}$ and $\epsilon > 0$ there is an $n > n_o$ such that $|b_n - b| < \epsilon$. Thus $\{b_n\}$ either diverges or converges to a number greater than 1, and the series diverges by the Root Test. ■

Problem 4 For each $n \in \mathbb{N}$ let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leq 1 \end{cases}$$

(a) Plot of f_n for $n = 2, 3, 4$ – green, blue, and orange respectively:



(b) $\{f_n\}$ converges pointwise but not uniformly to some function f .

Proof: Let $x \in [0, 1]$ be given. If $x = 0$ then $\lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} nx = 0$. Otherwise, if $0 < x \leq 1$ then there exists some $n_o \in \mathbb{N}$ such that $\frac{2}{n_o} < x$ and so for all $n > n_o$, $f_n(x) = 0$. Therefore $\{f_n\}$ converges pointwise to $f(x) = 0$. However letting $\epsilon = \frac{1}{2}$, $n > N$, and $x = \frac{1}{2n}$ for some $N \in \mathbb{N}$ gives $|f_n(x) - f(x)| = |nx - 0| = \left|\frac{n}{2n}\right| = \frac{1}{2} = \epsilon$, showing that $\{f_n\}$ is not uniformly convergent.

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(c) f_n is continuous for all n on $[0, 1]$, and the same holds for f .

(d) Parts (b) and (c) do not violate Corollary 8.3.2 because it does not address functions that do not converge uniformly.

Problem 5

(a) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ when $x \in (-1, 1)$

Proof: Let $s_n = \sum_{k=0}^n x^k$, then $s_n = 1 + x + x^2 + \cdots + x^n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$. Since $|x| < 1$, $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = \frac{1}{1-x}$.

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(b) Evaluate $f(x) = \int_0^{1/2} \frac{1}{1+x^6} dx$ as a power series.

Proof: By part (a), $\frac{1}{1+x^6} = \sum_{k=0}^{\infty} (x^6)^k$ since $x^6 \in (-1, 1)$ for all $x \in [0, \frac{1}{2}]$. Substituting the series in the integral gives $\int_0^{1/2} \sum_{k=0}^{\infty} (x^6)^k dx = \sum_{k=0}^{\infty} \int_0^{1/2} (x^6)^k dx$ because the Power Series is uniformly convergent on its domain. This gives $f(x) = \sum_{k=0}^{\infty} \frac{(x^6)^{k+1}}{k+1}$.

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Problem 6 Using $f(x) = \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{1}{1+k^2x}$ prove:

(a) That $f(x)$ converges uniformly on $x \in [a, \infty)$ for all $a > 0$

Proof: Since $a \leq x$ for all $x \in [a, \infty)$, $\frac{1}{1+k^2x} \leq \frac{1}{1+k^2a}$ for all $k \in \mathbb{N}$. Thus, given any $\epsilon > 0$ let k_o be given such that $\frac{1}{1+k_o^2a} < \frac{\epsilon}{2}$

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