# Math 440 – Real Analysis II

Homework 6

Amandeep Gill

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**Problem 36** Let $X$ be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, where $||\cdot||$ is defined for all $\vec{x} \in X$ as $||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$. Then for all $\vec{x}, \vec{y} \in X$, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

**Proof:** For all $\vec{x}, \vec{y} \in X$, using the definition of $||\cdot||$ on $X$,

$$\begin{aligned} ||\vec{x}+\vec{y}||^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\ &= \langle \vec{x}, \vec{x}+\vec{y} \rangle + \langle \vec{y}, \vec{x}+\vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= ||\vec{x}||^2 + 2\langle \vec{x}, \vec{y} \rangle + ||\vec{y}||^2 \end{aligned}$$

As this is a chain of strict equality, and since the inner product is a non-negative function, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ only if $\langle \vec{x}, \vec{y} \rangle = 0$.

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**Problem 37** Let $X$ be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, and define $||\cdot||$ as $||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$. $||\cdot||$ is a norm on $X$.

**(a)** $||\vec{x}|| \leqslant 0$ for all $\vec{x} \in X$

**Proof:** This follows from the fact that $0 \leqslant \langle \vec{x}, \vec{x} \rangle$ for all $\vec{x} \in X$, and that the square root function has no negative outputs on the non-negative reals.

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**(b)** $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}$

**Proof:** This follows from the fact that $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$, and that the square root function has only one zero, which is at 0.

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**(c)** $||c \cdot \vec{x}|| = |c| \cdot ||\vec{x}||$ for all $c \in \mathbb{R}$ and all $\vec{x} \in X$

**Proof:** Let $c \in \mathbb{R}$ and $\vec{x} \in X$, then $||c \cdot \vec{x}|| = \sqrt{\langle c\vec{x}, c\vec{x} \rangle} = \sqrt{c^2 \langle \vec{x}.\vec{x} \rangle}$. This is acceptable because it is a double application of property (d) of Orthogonal Functions twice. Since $\sqrt{c^2} = |c|$, $||c \cdot \vec{x}|| = |c| \cdot ||\vec{x}||$.

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**Problem 37** continued

**(d)** $||\vec{x}+\vec{y}|| \leqslant ||\vec{x}|| + ||\vec{y}||$ for all $\vec{x}, \vec{y} \in X$

**Proof:** Let $\vec{x}, \vec{y} \in X$. By previous work, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + 2\langle \vec{x}, \vec{y}\rangle + ||\vec{y}||^2$. Because $\langle \vec{x}, \vec{y}\rangle \leqslant ||\vec{x}||\,||\vec{y}||$ by the Cauchy-Schwartz Inequality, so this gives $||\vec{x}+\vec{y}||^2 \leqslant ||\vec{x}||^2 + 2\,||\vec{x}||\,||\vec{y}|| + ||\vec{y}||^2 = (||\vec{x}|| + ||\vec{y}||)^2$. Thus, for all $\vec{x}, \vec{y} \in X$, $||\vec{x}+\vec{y}|| \leqslant ||\vec{x}|| + ||\vec{y}||$.

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**Problem 38** Let $f(x) = \sin(\pi x)$, $\phi_1(x) = 1$, and $\phi_2(x) = x$. Find $c_1$ and $c_2$ such that $S_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ gives the best approximation of $f$ on $[-1, 1]$.

**Solution:** Need to choose $c_1$ and $c_2$ such that

$\int\limits_{-1}^{1} \sin^2(\pi x)dx - {c_1}^2 \int\limits_{-1}^{1} {f_1}^2(x)dx - {c_2}^2 \int\limits_{-1}^{1} {f_2}^2(x)dx$ is minimized.

$\int\limits_{-1}^{1} {f_1}^2(x)dx = 2$ and $\int\limits_{-1}^{1} {f_2}^2(x)dx = 2$

Since $\int\limits_{-1}^{1} \sin^2(\pi x)dx = 1$ we need $2c_1 + 2c_2 = 1$ or $c_1 = \frac{1}{2} - c_2$, so choose $\underline{c_1 = 0 \text{ and } c_2 = \frac{1}{2}}$

**Problem 40** If $f$ and $f_n$ for $n \in \mathbb{N}$ are Riemann integrable functions on $[a, b]$, and $\{f_n\}$ converges uniformly to $f$ on $[a, b]$ then $\{f_n\}$ converges in the mean to $f$ on $[a, b]$.

**Proof:** Let $f$ and $f_n$ for $n \in \mathbb{N}$ be Riemann integrable functions on $[a, b]$, where $\{f_n\}$ converges uniformly to $f$ on $[a, b]$, then there exists a number $n_\circ$ such that for all $\epsilon > 0$ and $n \in \mathbb{N}$, if $n > n_\circ$ then $|f - f_n| < \frac{\sqrt{\epsilon}}{\sqrt{b-a}}$. From this, we get $\int\limits_{a}^{b} (f - f_n)^2 dx < \int\limits_{a}^{b} (\frac{\sqrt{\epsilon}}{\sqrt{b-a}})^2 dx = \frac{\epsilon x}{b-a}\Big|_a^b = \epsilon$. Therefore, $\lim\limits_{n\to\infty} \int\limits_{a}^{b} (f - f_n)^2 dx = 0$ and $\{f_n\}$ converges in the mean to $f$ on $[a, b]$.

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**Problem 41** For $n \in \mathbb{N}$, let $f_n$ be the function

$$f_n = \begin{cases} \sqrt{n} & \text{if } \quad 0 < x < \frac{1}{n} \\ 0 & \text{if } \quad \phantom{0 <{}} x \geqslant \frac{1}{n} \end{cases}$$

defined on $[0, 1]$. The sequence of functions $\{f_n\}$ converges pointwise to $f(x) = 0$, but does not converge in the mean.

**Proof:** Let $x \in [0, 1]$ and $\epsilon > 0$ be given. If $x = 0$, then $f_n(x) = 0$ and $|f_n(x) - 0| = 0 < \epsilon$ for all $n \in \mathbb{N}$. Otherwise, using 340 Facts there exists an $n_\circ \in \mathbb{N}$ such that $\frac{1}{n_\circ} < x$. Therefore, for all $n > n_\circ$, $f_n(x) = 0$, so $|f_n(x) - 0| = 0 < \epsilon$, and $\{f_n\}$ is thus pointwise convergent to 0.

However, for all $n \in \mathbb{N}$, $\int\limits_0^1 (0 - f_n(x))^2 \, dx = \frac{1}{n} \cdot n = 1$, and

$\lim\limits_{n\to\infty} \int\limits_0^1 (f_n(x))^2 \, dx = 1$. Thus $\{f_n\}$ does not converge in the mean to 0.

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