

# Math 440 – Real Analysis II

## Homework 5

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**Problem 29** Find the radius of convergence of each of the following powerseries

(a)  $\sum_{k=1}^{\infty} \frac{3^k}{k^3} x^k$

**Proof:** Since  $\frac{3^k}{k^3} x^k = \frac{(3x)^k}{k^3}$ , if  $|x| \leq 1/3$  then  $|3x| \leq 1$  and  $\frac{|3x|^k}{k^3} \leq \frac{1}{k^3}$  for all  $k$ . The series is thus uniformly convergent for  $x \in [-\frac{1}{3}, \frac{1}{3}]$  by the Weierstrass M-test. For  $x \notin [-\frac{1}{3}, \frac{1}{3}]$ ,  $|3x| > 1$  which leads to  $\lim_{k \rightarrow \infty} \frac{|3x|^k}{k^3}$  being unbounded and the series being divergent by the Limit Test. ■

(b)  $\sum_{k=0}^{\infty} \frac{1}{4^k} (x+1)^{2k}$

**Proof:** As with part (a), using the product rule gives  $\frac{1}{4^k} (x+1)^{2k} = \left(\frac{x+1}{2}\right)^{2k}$ . Because this is geometric, for the series to be convergent  $\left|\frac{x+1}{2}\right|$  must be strictly less than 1. Therefore, the series converges for  $x \in (-3, 1)$  and is otherwise divergent. ■

(c)  $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^k x^k$

**Proof:** By algebra,  $\left(1 - \frac{1}{k}\right)^k x^k = \left(x - \frac{x}{k}\right)^k$ . Since  $\lim_{k \rightarrow \infty} \left(x - \frac{x}{k}\right) = x$ , then  $\lim_{k \rightarrow \infty} \left(x - \frac{x}{k}\right)^k \neq 0$  if  $x \geq 1$  and the series is divergent. Given that  $\left|x - \frac{x}{k}\right|^k \leq |x|^k$  for all  $k \in \mathbb{N}$ , and  $\sum_{k=1}^{\infty} x^k < \infty$  for all  $x \in (-1, 1)$ , the series  $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^k x^k$  is uniformly convergent on  $x \in (-1, 1)$  by the Weierstrass M-test. ■

**Problem 30** Show that  $|\sqrt[3]{1+x} - (1 + \frac{1}{3}x - \frac{1}{9}x^2)| < \frac{5}{81}x^3$  for all  $x > 0$ , and approximate  $\sqrt[3]{1.2}$  and  $\sqrt[3]{2}$ .

**Proof:** Let  $f(x) = \sqrt[3]{1+x}$ . Then the  $n^{\text{th}}$ -order Taylor Polynomial at 0 is  $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$ . Thus,  $T_2(f, 0)(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$  with the remainder  $R_2(f, 0)(x) = \frac{f^{(3)}(\xi)}{3!}x^3 = \frac{5x^3}{81(1+\xi)^{\frac{8}{3}}} < \frac{5}{81}x^3$  for all  $\xi \in (0, \infty)$ . From  $f(x) = T_2(f, 0)(x) + R_2(f, 0)(x)$  we have that  $f - T_2 = R_2 < \frac{5}{81}x^3$ .

- (a)  $\sqrt[3]{1.2} \approx 1.24$  with an error less than 0.1067
- (b)  $\sqrt[3]{2} \approx 1.22$  with an error less than 0.4938

■

**Problem 31** Determine all values of  $p \in \mathbb{R}$  such that the given sequence is in  $l^2$

- (a)  $\{p^k\}_{k=1}^{\infty}$  for  $p \in (-1, 1)$

**Proof:**  $\{p^k\}_{k=1}^{\infty} \in l^2$  if and only if  $\sum_{k=1}^{\infty} p^{2k} < \infty$ . Since this is a geometric series, it converges when  $p^2 \in (-1, 1)$ , which is equivalent to  $p \in (-1, 1)$ .

■

- (a)  $\left\{\frac{k^p}{p^k}\right\}_{k=1}^{\infty}$  for  $p \in (-\infty, -1] \cup (1, \infty)$

**Proof:**  $\left\{\frac{k^p}{p^k}\right\}_{k=1}^{\infty} \in l^2$  if and only if  $\sum_{k=1}^{\infty} \left(\frac{k^p}{p^k}\right)^2 = \sum_{k=1}^{\infty} \frac{k^{2p}}{p^{2k}} < \infty$ . Using the Ratio Test yields  $\lim_{k \rightarrow \infty} \left(\frac{(k+1)^{2p}}{p^{2(k+1)}}\right) \left(\frac{p^{2k}}{k^{2p}}\right) = \frac{1}{p^2}$ , so the series converges when  $|p| > 1$ . Adding in the special case where  $p = -1$  and  $\sum_{k=1}^{\infty} \frac{k^{2p}}{p^{2k}} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ , the sequence is in  $l^2$  when  $p \in (-\infty, -1] \cup (1, \infty)$

■

**Problem 32** For each  $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , define  $\|\vec{x}\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ . Prove  $\|\cdot\|$  satisfies the a norm on  $\mathbb{R}^n$ .

(a)  $\|\vec{x}\| \geq 0$  for all  $\vec{x} \in \mathbb{R}^n$ .

**Proof:** For all  $i$ ,  $|x_i| \geq 0$  and by definition  $\|\vec{x}\| \geq |x_i|$ , so  $\|\vec{x}\| \geq 0$ . ■

(b)  $\|\vec{x}\| = 0$  if and only if  $\vec{x} = \vec{0}$ .

**Proof:**  $\|\vec{x}\| = 0 \Leftrightarrow$  for all  $i$ ,  $|x_i| \leq 0$ , so  $|x_i| = 0$  and  $\vec{x} = \vec{0}$ . ■

(c)  $\|c \cdot \vec{x}\| = |c| \cdot \|\vec{x}\|$  for all  $c \in \mathbb{R}$  and all  $\vec{x} \in \mathbb{R}^n$ .

**Proof:**  $\|c \cdot \vec{x}\| = \|(cx_1, cx_2, \dots, cx_n)\| = \max\{|cx_1|, |cx_2|, \dots, |cx_n|\} = |cx_i| = |c| \cdot |x_i| = |c| \cdot \|\vec{x}\|$  ■

(d)  $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$  for all  $\vec{x}, \vec{y} \in \mathbb{R}^n$

**Proof:**  $\|\vec{x} + \vec{y}\| = |x_i + y_i| \leq |x_i| + |y_i| \leq \|\vec{x}\| + \|\vec{y}\|$  since  $|x_i| \leq \|\vec{x}\|$  and  $|y_i| \leq \|\vec{y}\|$ . ■

**Problem 33** Let  $(X, \|\cdot\|)$  be a normed linear space.

(a) A sequence  $\{\vec{x}_n\}_{n=1}^\infty$  of vectors converges to  $\vec{x} \in X$  if for all  $\epsilon > 0$ , there exists  $n_o \in \mathbb{N}$  and  $m > n_o$  such that  $\|\vec{x}_m - \vec{x}\| < \epsilon$ .

A sequence  $\{\vec{x}_n\}_{n=1}^\infty$  of vectors is Cauchy if for all  $\epsilon > 0$ , there exists  $n_o \in \mathbb{N}$  and  $m, n > n_o$  such that  $\|\vec{x}_m - \vec{x}_n\| < \epsilon$ .

(b) If a sequence  $\{\vec{x}_n\}_{n=1}^\infty$  of vectors converges to  $\vec{x} \in X$ , then it is Cauchy.

**Proof:** Let  $\{\vec{x}_n\}_{n=1}^\infty$  sequence of vectors that converges to  $\vec{x} \in X$ . Then there exists  $\epsilon > 0$  and  $n_o \in \mathbb{N}$  such that for all  $n_1, n_2 > n_o$ ,  $\|\vec{x}_{n_1} - \vec{x}\| < \frac{\epsilon}{2}$  and  $\|\vec{x}_{n_2} - \vec{x}\| < \frac{\epsilon}{2}$ . This gives  $\|\vec{x}_{n_1} - \vec{x}\| + \|\vec{x}_{n_2} - \vec{x}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ . Since  $\|\vec{x}_{n_2} - \vec{x}\| = |-1| \cdot \|\vec{x} - \vec{x}_{n_2}\|$ , we have  $\epsilon > \|\vec{x}_{n_1} - \vec{x}\| + \|\vec{x} - \vec{x}_{n_2}\| \geq \|\vec{x}_{n_1} - \vec{x}_{n_2}\|$  ■

**Problem 34** For each  $n \in \mathbb{N}$ , let  $\vec{e}_n$  be the sequence in  $l^2$  defined such that

$$\vec{e}_n(k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

Show that the Bolzano-Weierstrass theorem fails in  $l^2$

**Proof:** Let bounded in  $l^2$  be defined as  $\|\vec{x}_n\|_2 \leq c$  for some  $c \geq 0$  and all  $n \in \mathbb{N}$ . By this definition of bounded, since  $\|\vec{e}_n\|_2 = 1$  for all  $n$  then  $\{\vec{e}_n\}$  is bounded and monotone.

Let  $\epsilon = \sqrt{2}$  and  $n_o \in \mathbb{N}$  with  $n_o < n < m$ , then  $\|\vec{e}_n - \vec{e}_m\|_2 = \sqrt{0 + 0 + \dots + 1 + \dots + 1 + \dots + 0} = \sqrt{2} = \epsilon$ . Thus the sequence  $\{\vec{e}_n\}$  is not Cauchy and therefore does not converge. Because it this sequence is both bounded and monotone, but not convergent, the Bolzano-Weierstrass Theorem does not hold true in  $l^2$ .

■