

Math 440 – Real Analysis II

Homework 5

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Problem 29 Find the radius of convergence of each of the following powerseries

$$(a) \sum_{k=1}^{\infty} \frac{3^k}{k^3} x^k$$

Proof: Since $\frac{3^k}{k^3} x^k = \frac{(3x)^k}{k^3}$, if $|x| \leq 1/3$ then $|3x| \leq 1$ and $\frac{|3x|^k}{k^3} \leq \frac{1}{k^3}$ for all k . The series is thus uniformly convergent for $x \in [-\frac{1}{3}, \frac{1}{3}]$ by the Weierstrass M-test. For $x \notin [-\frac{1}{3}, \frac{1}{3}]$, $|3x| > 1$ which leads to $\lim_{k \rightarrow \infty} \frac{|3x|^k}{k^3}$ being unbounded and the series being divergent by the Limit Test. ■

$$(b) \sum_{k=0}^{\infty} \frac{1}{4^k} (x+1)^{2k}$$

Proof: As with part (a), using the product rule gives $\frac{1}{4^k} (x+1)^{2k} = \left(\frac{x+1}{2}\right)^{2k}$. Because this is geometric, for the series to be convergent $\left|\frac{x+1}{2}\right|$ must be strictly less than 1. Therefore, the series converges for $x \in (-3, 1)$ and is otherwise divergent. ■

$$(c) \sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^k x^k$$

Proof: By algebra, $\left(1 - \frac{1}{k}\right)^k x^k = \left(x - \frac{x}{k}\right)^k$. Since $\lim_{k \rightarrow \infty} \left(x - \frac{x}{k}\right) = x$, then $\lim_{k \rightarrow \infty} \left(x - \frac{x}{k}\right)^k \neq 0$ if $x \geq 1$ and the series is divergent. Given that $\left|x - \frac{x}{k}\right|^k \leq |x|^k$ for all $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} x^k < \infty$ for all $x \in (-1, 1)$, the series $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^k x^k$ is uniformly convergent on $x \in (-1, 1)$ by the Weierstrass M-test. ■

Problem 30 Show that $|\sqrt[3]{1+x} - (1 + \frac{1}{3}x - \frac{1}{9}x^2)| < \frac{5}{81}x^3$ for all $x > 0$, and approximate $\sqrt[3]{1.2}$ and $\sqrt[3]{2}$.

Proof: Let $f(x) = \sqrt[3]{1+x}$. Then the n^{th} -order Taylor Polynomial at 0 is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$. Thus, $T_2(f, 0)(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$ with the remainder $R_2(f, 0)(x) = \frac{f^{(3)}(\xi)}{3!}x^3 = \frac{5x^3}{81(1+\xi)^{\frac{8}{3}}} < \frac{5}{81}x^3$ for all $\xi \in (0, \infty)$. From $f(x) = T_2(f, 0)(x) + R_2(f, 0)(x)$ we have that $f - T_2 = R_2 < \frac{5}{81}x^3$.

1. $\sqrt[3]{1.2} \approx 1.24$ with an error less than 0.1067
2. $\sqrt[3]{2} \approx 1.22$ with an error less than 0.4938

■

Problem 31 Determine all values of $p \in \mathbb{R}$ such that the given sequence is in l^2

- (a) $\{p^k\}_{k=1}^{\infty}$ for $p \in (-1, 1)$

Proof: $\{p^k\}_{k=1}^{\infty} \in l^2$ if and only if $\sum_{k=1}^{\infty} p^{2k} < \infty$. Since this is a geometric series, it converges when $p^2 \in (-1, 1)$, which is equivalent to $p \in (-1, 1)$.

■

- (a) $\left\{\frac{k^p}{p^k}\right\}_{k=1}^{\infty}$ for $p \in (-\infty, -1] \cup (1, \infty)$

Proof: $\left\{\frac{k^p}{p^k}\right\}_{k=1}^{\infty} \in l^2$ if and only if $\sum_{k=1}^{\infty} \left(\frac{k^p}{p^k}\right)^2 = \sum_{k=1}^{\infty} \frac{k^{2p}}{p^{2k}} < \infty$. Using the Ratio Test yields $\lim_{k \rightarrow \infty} \left(\frac{(k+1)^{2p}}{p^{2(k+1)}}\right) \left(\frac{p^{2k}}{k^{2p}}\right) = \frac{1}{p^2}$, so the series converges when $|p| > 1$. Adding in the special case where $p = -1$ and $\sum_{k=1}^{\infty} \frac{k^{2p}}{p^{2k}} = \sum_{k=1}^{\infty} \frac{1}{k^2}$, the sequence is in l^2 when $p \in (-\infty, -1] \cup (1, \infty)$

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Problem 32 For each $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, define $\|\vec{x}\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$. Prove $\|\cdot\|$ satisfies the a norm on \mathbb{R}^n .

(a) $\|\vec{x}\| \geq 0$ for all $\vec{x} \in \mathbb{R}^n$.

Proof: For all i , $|x_i| \geq 0$ and by definition $\|\vec{x}\| \geq |x_i|$, so $\|\vec{x}\| \geq 0$. ■

(b) $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.

Proof: $\|\vec{x}\| = 0 \Leftrightarrow$ for all i , $|x_i| \leq 0$, so $|x_i| = 0$ and $\vec{x} = \vec{0}$. ■

(c) $\|c \cdot \vec{x}\| = |c| \cdot \|\vec{x}\|$ for all $c \in \mathbb{R}$ and all $\vec{x} \in \mathbb{R}^n$.

Proof: $\|c \cdot \vec{x}\| = \|(cx_1, cx_2, \dots, cx_n)\| = \max\{|cx_1|, |cx_2|, \dots, |cx_n|\} = |cx_i| = |c| \cdot |x_i| = |c| \cdot \|\vec{x}\|$ ■

(d) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$

Proof: $\|\vec{x} + \vec{y}\| = |x_i + y_i| \leq |x_i| + |y_i| \leq \|\vec{x}\| + \|\vec{y}\|$ since $|x_i| \leq \|\vec{x}\|$ and $|y_i| \leq \|\vec{y}\|$. ■

Problem 33 Let $(X, \|\cdot\|)$ be a normed linear space.

(a) A sequence $\{\vec{x}_n\}_{n=1}^\infty$ of vectors converges to $\vec{x} \in X$ if for all $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ and $m > n_o$ such that $\|\vec{x}_m - \vec{x}\| < \epsilon$.

A sequence $\{\vec{x}_n\}_{n=1}^\infty$ of vectors is Cauchy if for all $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ and $m, n > n_o$ such that $\|\vec{x}_m - \vec{x}_n\| < \epsilon$.

(b) If a sequence $\{\vec{x}_n\}_{n=1}^\infty$ of vectors converges to $\vec{x} \in X$, then it is Cauchy.

Proof: Let $\{\vec{x}_n\}_{n=1}^\infty$ sequence of vectors that converges to $\vec{x} \in X$. Then there exists $\epsilon > 0$ and $n_o \in \mathbb{N}$ such that for all $n_1, n_2 > n_o$, $\|\vec{x}_{n_1} - \vec{x}\| < \frac{\epsilon}{2}$ and $\|\vec{x}_{n_2} - \vec{x}\| < \frac{\epsilon}{2}$. This gives $\|\vec{x}_{n_1} - \vec{x}\| + \|\vec{x}_{n_2} - \vec{x}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Since $\|\vec{x}_{n_2} - \vec{x}\| = |-1| \cdot \|\vec{x} - \vec{x}_{n_2}\|$, we have $\epsilon > \|\vec{x}_{n_1} - \vec{x}\| + \|\vec{x} - \vec{x}_{n_2}\| \geq \|\vec{x}_{n_1} - \vec{x}_{n_2}\|$ ■

Problem 34 For each $n \in \mathbb{N}$, let \vec{e}_n be the sequence in l^2 defined such that

$$\vec{e}_n(k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

Show that the Bolzano-Weierstrass theorem fails in l^2

Proof: Let bounded in l^2 be defined as $\|\vec{x}_n\|_2 \leq c$ for some $c \geq 0$ and all $n \in \mathbb{N}$. By this definition of bounded, since $\|\vec{e}_n\|_2 = 1$ for all n then $\{\vec{e}_n\}$ is bounded and monotone.

Let $\epsilon = \sqrt{2}$ and $n_o \in \mathbb{N}$ with $n_o < n < m$, then $\|\vec{e}_n - \vec{e}_m\|_2 = \sqrt{0 + 0 + \dots + 1 + \dots + 1 + \dots + 0} = \sqrt{2} = \epsilon$. Thus the sequence $\{\vec{e}_n\}$ is not Cauchy and therefore does not converge. Because it this sequence is both bounded and monotone, but not convergent, the Bolzano-Weierstrass Theorem does not hold true in l^2 .

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