Math 440 – Real Analysis II

Homework 7

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Problem 42 If $\{I_n\}_n$ is a finite or countable collection of disjoint open intervals with $\bigcup_n I_n \subset (a, b)$ then $\sum\limits_n m(I_n) \leqslant m((a, b))$

Proof: Let $\{I_n\}$ be a countable set of disjoint open intervals such that $\bigcup_n I_n \subset (a, b)$, and each interval $I_i = (a_i, b_i)$ where $i \in \mathbb{N}$ and $i \leqslant n$.

Because each interval is disjoint and each a_i and b_i is bounded below by a and above by b, the intervals can be arranged in such a way that $a \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant b$. By definition of measure $m(I_i) = b_i - a_i$, so $\sum\limits_n m(I_n) = (b_1 - a_1) + (b_2 - a_2) + \cdots$. From this, we can see that $(b_1 - a_1) + (b_2 - a_2) = (b_2 - a_1) - (a_2 - b_1) \leqslant (b_2 - a_1)$. Similarly, $(b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3) \leqslant (b_2 - a_1) + (b_3 - a_3) \leqslant (b_3 - a_1)$. Continuing the pattern $(b_1 - a_1) + \cdots + (b_i - a_i) \leqslant (b_i - a_1)$, and since $a \leqslant a_1 \leqslant b_i \leqslant b$, for all i, $(b_i - a_1) \leqslant (b - a) = m((b - a))$. Therefore $\sum\limits_n m(I_n) \leqslant m((a, b))$.

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Problem 43 The measure of the Cantor Set in $[0, 1]$ is 0.

Proof: Let P denote the Cantor Set. Defined as a sequence of steps where intervals are removed from the set $[0, 1]$, then let

$P_1 = (\frac{1}{3}, \frac{2}{3})$
$P_2 = \big((\frac{1}{9}, \frac{2}{9}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{7}{9}, \frac{8}{9})\big)$
$P_3 = \big((\frac{1}{27}, \frac{2}{27}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{13}{27}, \frac{14}{27}) \cup (\frac{7}{9}, \frac{8}{9}) \cup (\frac{25}{27}, \frac{26}{27})\big)$
$P_4 = \ldots$
Such that $P = [0, 1] \backslash \left(\lim\limits_{n\to\infty} P_n\right)$

Since P_n is a finite union of disjoint open sets,

$m(P_n) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$
$m(P_n) = \sum\limits_{k=0}^{n} \frac{1}{3}\left(\frac{2}{3}\right)^k$

Therefore $\lim\limits_{n\to\infty} m(P_n) = \sum\limits_{k=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$

Finally, by Definition 10.2.10, $m(P) = m([0, 1]) - \lim\limits_{n\to\infty} m(P_n) = 0$.

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Problem 44 If E is a finite subset of $\mathbb{R}$ then $m(E) = 0$.

Proof: Let $E = \{x_1, x_2, \ldots, x_n\}$ be a finite, ordered set of n points in $\mathbb{R}$ such that for some $a, b \in \mathbb{R}$, $a < x_1 < x_2 < \cdots < x_n < b$. Then $(a,b)\backslash E = (a, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup \cdots \cup (x_{n-1}, x_n) \cup (x_n, b)$. The measure of $(a,b) = b-a$ by the definition of measure, and since $(a,b)\backslash E$ is a disjoint union of open sets, $m\left((a,b)\backslash E\right) = (x_1 - a) + (x_2 - x_1) + \cdots + (b - x_n)$. Given that this is a telescoping sum, $m\left((a,b)\backslash E\right) = b - a$. Because finite sets are compact, $m(E) = m((a,b)) - m((a,b)\backslash E) = 0$.

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Problem 45 Calculate the following measures

(a) $m(\{1\} \cup [2,5]) = 3$
Since $\{1\}$ and $[2,5]$ are disjoint, the measure of the union is the $m(\{1\})+ m([2,5])$, which is $0 + 3$ by p.44 and Definition 10.2.1 respectively.

(b) $m([1,2) \cup (3,4]) = 2$
Since intervals of the form $[a,b)$ can be written as $\{a\} \cup (a,b)$, by previous work then $m([a,b)) = b - a$

(c) where P is the Cantor Set, $m([0,1]\backslash P) = 1$
Since $P^c = [0,1]\backslash P$ is a union of open sets, P^c is an open set. Therefore, by Theorem 10.2.15 $m(P \cap [0,1]) + m(P^c \cap [0,1]) = 1$, and because $m(P \cap [0,1]) = m(P)$ and $P^c \cap [0,1] = P^c$, $m(P^c) = 1$

Problem 46 If K_1 and K_2 are disjoint compact subsets of $\mathbb{R}$, then $m(K_1 \cup K_2) = m(K_1) + m(K_2)$

Proof: Let K_1, K_2 be disjoint, compact subsets of $\mathbb{R}$. Since the sets are compact in $\mathbb{R}$, they are both bounded and there exists $U = (a, b)$ such that $K_1 \cup K_2 \subset U$. Then by Definition 10.2.10, $m(K_1 \cup K_2) = m(U) - m(U \backslash (K_1 \cup K_2))$. However because, $U \backslash (K_1 \cup K_2) = (U \backslash K_1) \cap (U \backslash K_2)$, and since $(U \backslash K_1)$ and $(U \backslash K_2)$ open sets $m(U \backslash K_1) + m(U \backslash K_2) = m(U) + m((U \backslash K_1) \cap (U \backslash K_2)) = m(U) + m(U \backslash (K_1 \cup K_2))$ by Theorem 10.2.9. Using the fact that $m(K_1) = m(U) - m(U \backslash K_1)$ and $m(K_2) = m(U) - m(U \backslash K_2)$, we have that

$$\begin{aligned} m(K_1 \cup K_2) = & \quad m(U) - m(U \backslash (K_1 \cup K_2)) \\ = & \quad 2m(U) - (m(U \backslash K_1) + m(U \backslash K_2)) \\ = & \quad 2m(U) - (2m(U) + m(K_1) + m(K_1)) \\ = & \quad m(K_1) + m(K_1) \end{aligned}$$

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Problem 47 Let $E = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$, calculate $\lambda^*(E)$ and $\lambda_*(E)$

Proof: Let $m \in \mathbb{N}$ and $K_m = \{\frac{1}{n} \ : \ n < m\}$, since K_m is a finite, compact subset of E, and because $m(K_m) = 0$ is true for all $m \in \mathbb{N}$, then $\lambda_*(E) = 0$ as $\lambda_*(E) = \max\{m(K_m)\}$ by definition.

Let $\epsilon > 0$ be given and $U = \bigcup\limits_{n=1}^{\infty} \left(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon\right)$. As proved previously, there exists an $m \in \mathbb{N}$ such that for all $n > m$, $\left|\frac{1}{n} - 0\right| < \epsilon$, which means that $\bigcup\limits_{n=m}^{\infty} \left(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon\right) \subset (-\epsilon, 2\epsilon)$. Therefore, by Theorem 10.2.6, $m(U) \leqslant m\epsilon + 3\epsilon$. As $\lambda^*(E) \leqslant m(U)$ and because ϵ is arbitrary, $\lambda^*(E) = 0$.

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