Math 440 – Real Analysis II

Final Exam

Amandeep Gill

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Problem 1 Determine which of the following series are divergent, conditionally convergent, or absolutely convergent.

(a) $\sum\limits_{k=1}^{\infty} \frac{(-1)^k k}{2k^2-1}$ converges conditionally.

Proof: Let $b_k = \frac{k}{2k^2-1}$. Since $b_{k+1} = \frac{k+1}{2(k+1)^2-1} = \frac{k+1}{2k^2+4k+1} < b_k$ for all $k \in \mathbb{K}$ and $\lim\limits_{k\to\infty} b_k = 0$, by the Alternating Series Test $\sum\limits_{k=1}^{\infty} \frac{(-1)^k k}{2k^2-1}$ converges conditionally.

$\blacksquare$

(b) $\sum\limits_{k=0}^{\infty} \frac{(-1)^{a_k}}{k^2+1}$ where $a_k = \lfloor \frac{k}{2} \rfloor$ converges absolutely.

Proof: Let $b_k = \left| \frac{(-1)^{a_k}}{k^2+1} \right| = \frac{1}{k^2+1}$. For all $k \in \mathbb{N}$, $b_k < \frac{1}{k^2}$, therefore since $\frac{1}{k^2}$ is a convergent P-series $\sum\limits_{k=1}^{\infty} b_k$ converges. Thus $\sum\limits_{k=0}^{\infty} \frac{(-1)^{a_k}}{k^2+1}$ converges absolutely.

$\blacksquare$

Problem 2 Let $\{q_k\}_{k\in\mathbb{N}}$ be an enumeration of the rational numbers on $[0,1]$, and for each $k \in \mathbb{N}$ define $C_k = \{q_1, q_2, \ldots, q_k\}$. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sum\limits_{k=1}^{\infty} 2^{-k}\chi_{C_k}$.

(a) The series defining f converges uniformly on $[0,1]$.

Proof: Let f be defined piecewise as

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1]\backslash\mathbb{Q} \\ 2^{1-i} & \text{if } x = q_i \text{ for some } q_i \in \{q_k\}_{k\in\mathbb{N}} \end{cases}$$

Let $\epsilon > 0$ be given, $f_n(x) = \sum\limits_{k=1}^{n} 2^{-k}\chi_{C_k}$, and let $m, n_\circ \in \mathbb{N}$ such that $m > n_\circ$ and $2^{-n_\circ} < \epsilon$. If $x \in [0,1]\backslash\mathbb{Q}$, then $f_m(x) = 0$ and $|f_m(x) - f(x)| = 0 < \epsilon$. Otherwise, $x = q_i$ for some $i \in \mathbb{N}$.

case $i > m$:

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$f_m(x) = 0$ and $|f_m(x) - f(x)| = 2^{1-i} < 2^{-n_\circ} < \epsilon$.

case $i \leqslant m$:

$\vdots$

$f_m(x) = 2^{1-i} - 2^{-m}$ and $|f_m(x) - f(x)| = 2^{-m} < 2^{-n_\circ} < \epsilon$.

Thus f_n converges to f uniformly on $[0, 1]$.

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(b) Calculate $f(q_1)$, $f(q_2)$, $f(q_3)$, and $f(\sqrt{2}/2)$

Solution:

$$\begin{aligned} f(q_1) &= 1 \\ f(q_2) &= \tfrac{1}{2} \\ f(q_3) &= \tfrac{1}{4} \\ f(\sqrt{2}/2) &= 0 \end{aligned}$$

(c) f is Riemann integrable on $[0, 1]$ and $\int_0^1 f dx = 0$

Proof: For each $n \in \mathbb{N}$ let $\mathcal{P}_n = \{x_0, x_1, \ldots, x_{2^n}\}$ partition $[0, 1]$ where $x_i = \frac{i}{2^n}$. For each n, $\mathcal{U}(\mathcal{P}_n, f) \leqslant \sum_{k=1}^{2^n} 2^{1-k}\frac{1}{2^n} = 2^{1-n} - 2^{1-n-2^n}$ as the sup of an interval in the partition must be $f(q_1)$ or less, which means that the sup of one of the remaining intervals must be $f(q_2)$ or less, and so on up to $f(q_{2^n})$ in the final interval. Thus by Lemma 6.1.3

$$\inf \mathcal{U}(\mathcal{P}, f) \leqslant \lim_{n\to\infty} \mathcal{U}(\mathcal{P}_n, f) \leqslant \lim_{n\to\infty} 2^{1-n} - 2^{1-n-2^n} = 0$$

f is non-negative, so by Theorem 6.1.4,

$$0 \leqslant \sup \mathcal{L}(\mathcal{P}, f) \leqslant \inf \mathcal{U}(\mathcal{P}, f) \leqslant 0$$

Therefore $\int_0^1 f dx = 0$ by Defintion 6.1.5.

$\blacksquare$

(c) f is Lebesgue integrable on $[0, 1]$ and $\int_0^1 f d\lambda = 0$

Proof: By Corollary 10.6.8 if the Riemann integral exists, then the Lebesgue integral exists and is equal to the Riemann integral. Thus $\int_0^1 f d\lambda = 0$.

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Problem 3 Let A be a measurable subset of $\mathbb{R}$, and let $\mathcal{L}(A)$ denote the set of all Lebesgue integrable functions on A. Given $f, g \in \mathcal{L}(A)$ and $c \in \mathbb{R}$, define $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for all $x \in \mathbb{R}$

(a) $\mathcal{L}(A)$ forms a vector space over $\mathbb{R}$.

Proof: Let $f, g \in \mathcal{L}(A)$ and $c \in \mathbb{R}$, then $\int\limits_A (f+g)d\lambda = \int\limits_A f d\lambda + \int\limits_A f d\lambda$ by Theorem 10.7.8(a), as both f and g are Lebesgue integrable functions by assumption, so $(f+g)(x)$ is Lebesgue integrable and $\mathcal{L}(A)$ is closed under function addition. Similarly by Theorem 10.7.8(a), $\int\limits_A cf d\lambda = c\int\limits_A f d\lambda$, hence $(cf)(x)$ is Lebesgue integrable and $\mathcal{L}(A)$ is closed under scalar multiplication. Because the eight vector space axioms hold trivially, $\mathcal{L}(A)$ is a vector space over $\mathbb{R}$.

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(c) If $||\cdot|| : \mathcal{L}(A) \to \mathbb{R}$ is defined as $||f|| = \int\limits_A |f|\, d\lambda$, then $||\cdot||$ is not a norm on $\mathcal{L}(A)$

Proof: Let $A = [0,1]$ and $f(x) = \chi_E$ where $E = [0,1]\backslash\mathbb{Q}$. Then $||f|| = \int\limits_A |f|\, d\lambda = \int\limits_A \chi_E d\lambda$. By Theorem 10.6.10, $\int\limits_A \chi_E d\lambda = \int\limits_E 1 d\lambda + \int\limits_{E^c} 0 d\lambda$. Let $x_n \in E$ such that $\bigcup\limits_{n\in\mathbb{N}} \{x_n\} = E$, and let $\epsilon > 0$ be given. Then for each $I_n = (x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}})$, $E \subset \bigcup\limits_{n\in\mathbb{N}} I_n$ and $\sum\limits_{n\in\mathbb{N}} \lambda(I_n) = \epsilon$. Thus, by Theorem 6.1.11, $\lambda(E) = 0$ and $\int\limits_E 1 d\lambda + \int\limits_{E^c} 0 d\lambda = 1\lambda(E) + 0\lambda(E^c) = 0$. Since $f \neq 0$ for all $x \in [0,1]$, $||\cdot||$ does not satisfy the property that $||f|| = 0$ if and only if $f = 0$, and therefore $||\cdot||$ is not a norm on $\mathcal{L}(A)$.

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Problem (4) Let $a \in \mathbb{R}$ such that $0 < a < \frac{1}{2}$, and let $C_0 = [0,1]$ be the first step in the generalized Cantor set with C_n comprised of 2^n disjoint intervals of length a^n such that $C_n \subset C_{n-1}$, and define

$$C_a = C_0 \cap C_1 \cap C_2 \cap \cdots$$

(a) If $a \in \mathbb{R}$ with $0 < a < \frac{1}{2}$ then the set C_a is measurable with measure 0.

Proof: For each $n \in \mathbb{N}$, C_n is the union of 2^n disjoint intervals of length a^n, so by Theorem 10.4.5(b) $\lambda(C_n) = 2^n a^n = (2a)^n$. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $c_n = \lambda(C_n) = (2a)^n$. Because the number of intervals is countable for all n, Theorem 10.4.5(b) holds for all n, and since $a < \frac{1}{2}$, $2a < 1$ and c_n is a monotonically decreasing sequence bounded by $[0,1]$ and thus $\lim\limits_{n\to\infty} c_n = 0$. As $C_a \subset C_n$, for each open cover $U \supset C_n$ so $C_a \subset U$ and so $\lambda^*(C_a) \leqslant \lambda^*(C_n)$ by Definition 10.3.1. By Theorem 10.3.4(a) $\lambda(C_n) = \lambda^*(C_n)$, hence $\lambda^*(C_a) \leqslant c_n$ for all n. Therefore $0 \leqslant \lambda_*(C_a) \leqslant \lambda^*(C_a) \leqslant \lim\limits_{n\to\infty} c_n = 0$, and by Definition 10.3.4(a) C_a is measurable with $\lambda(C_a) = 0$.

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(b) $\frac{\log 2}{\log(1/a)}$ is an upper bound on the Hausdorff dimension of C_a.

Proof: Let A_n be defined as a ball in $\mathbb{R}$ such that $|A_n| = a^n$ is a δ_n cover for each $n \in \mathbb{N}$. The number of A_n balls needed to cover C_a is then $N = 2^n$. By Definition D.2.4, $\mathcal{H}^s_{\delta_n}(C_a) \leqslant \sum\limits_{k=1}^{N} |A_n|^s = 2^n a^{sn}$. Therefore, as $\mathcal{H}^s(C_a) = \lim\limits_{\delta\to 0^+} \mathcal{H}^s_\delta(C_a) \leqslant \lim\limits_{n\to\infty} 2^n a^{sn}$. Thus $s = \frac{\log 2}{\log(1/a)}$ is an upper bound on $\dim_H(C_a)$.

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Problem 5 The Hausdorff dimension of any countable subset of $\mathbb{R}^d$ is zero.

Proof: Let $\epsilon > 0$ be given, and let $X = \{x_1, x_2, \ldots, x_k\}$ for some $k \in \mathbb{N}$ be a subset of $\mathbb{R}^d$ with a δ_ϵ-cover defined as $\bigcup\limits_{n=1}^{k} C_n$ where $x_n \in C_n$ and $|C_n| = \frac{\epsilon}{2^n}$. By Definition D.2.4, $\mathcal{H}^s_{\delta_\epsilon} \leqslant \sum\limits_{n=1}^{k} (\frac{\epsilon}{2^n})^s = \left(\epsilon - \frac{\epsilon}{2^k}\right)^s$. Thus $\lim\limits_{\delta_\epsilon\to 0^+} \mathcal{H}^s_{\delta_\epsilon} \leqslant 0^s = 0$ for all $s > 0$ and therefore as k is arbitrary, $\dim_H(X) = 0$ for all $k \in \mathbb{N}$.

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Problem 6 Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}$. If there exists a sequence $\{U_k\}_{k\in\mathbb{N}}$ of pairwise disjoint open sets such that $E_k \subset U_k$ for all $k \in \mathbb{N}$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda^*(E_k)$$

Proof: Let $\{U_k\}_{k\in\mathbb{N}}$ be a sequence of disjoint open sets such that $E_k \subset U_k$ for all k, and let U be the union of all U_k and E be the union of all E_K. By Theorem 10.4.4(a), $\lambda^*(E) \leqslant \sum_{k=1}^{\infty} \lambda^*(E_k)$.

case $\lambda^*(E) = \infty$:

From the above, $\infty \leqslant \sum_{k=1}^{\infty} \lambda^*(E_k)$ and so $\sum_{k=1}^{\infty} \lambda^*(E_k) = \lambda^*(E)$.

case $\lambda^*(E) < \infty$:

Because $U_i \cap U_j = \emptyset$ for all $i, j \in \mathbb{N}$ where $i \neq j$, E_i and E_j are similarly disjoint. Thus, given $\epsilon > 0$, by Definition 10.3.1 there exists an open set A where $E \subset A$, $m(A) < \infty$, and for each $k \in \mathbb{N}$, let $V_k \subset A$ be an open set such that $E_k \subset V_k \subset U_k$ and $\lambda^*(E_k) = m(V_k) - \frac{\epsilon}{2^k}$. Given that V_k is an open set $m(V_k) = \lambda(V_k)$ by Theorem 10.3.2(Ex2), and so from Theorem 10.4.5(b) and the fact that V_k are pairwise disjoint, $m(V) = \sum_{k=1}^{\infty} m(V_k)$ where V is the union of V_k for all k. V is a subset of A, so Theorem 10.2.4 gives $m(V) \leqslant m(A)$. Hence

$$\sum_{k=1}^{\infty} \lambda^*(E_k) = \sum_{k=1}^{\infty} \left(m(V_k) - \frac{\epsilon}{2^k} \right) \leqslant m(V) < \infty$$

Therefore $\sum_{k=1}^{\infty} \lambda^*(E_k) = m(V) - \epsilon = m(V)$, since ϵ is arbitrary. Similarly from Definition 10.3.1, for each $\epsilon > 0$ there exists an open set B such that $E \subset B \subset V$ and $\lambda^*(E) = m(B) - \epsilon$, and again since ϵ is arbitrary, $\lambda^*(E) = m(B)$. Thus

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda^*(E_k) + \lambda^*(E) &= m(V) + m(B) \\
\lambda^*(E) + \lambda^*(E) &\leqslant m(V) + m(B) \\
\lambda^*(E) + \lambda^*(E) &\leqslant \sum_{k=1}^{\infty} \lambda^*(E_k) + \lambda^*(E) \\
\lambda^*(E) &\leqslant \sum_{k=1}^{\infty} \lambda^*(E_k)
\end{aligned}$$

Therefore $\lambda^*(E) = \sum_{k=1}^{\infty} \lambda^*(E_k)$.

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