Theorem 10.7.5 If $\{f_n\}$ is a sequence of measurable, non-negative functions on a measurable set A, and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ almost everywhere, then

$$\int\limits_A f d\lambda \leqslant \liminf_{n\to\infty} \int\limits_A f_n d\lambda$$

Proof: Let $A_1 \cup A_2$ be a partition of A where A_1, A_2 are disjoint, measurable sets such that $A_1 = \{x : \lim\limits_{n\to\infty} f_n(x) \to f(x),\ x \in A\}$ and $\lambda(A_2) = 0$. By Theorem 10.7.4, $\int_A f_n d\lambda = \int_{A_1} f_n d\lambda + \int_{A_2} f_n d\lambda = \int_{A_1} f_n d\lambda + 0$, and similarly for f.

For all $k \in \mathbb{N}$, let $\psi_k(x) = \inf\{f_n(x) : n > k\}$, then $\psi_k(x) \leqslant \psi_{k+1}(x)$ and, because $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in A_1$, $\lim\limits_{k\to\infty} \psi_k(x) = f(x)$ for all $x \in A_1$. Thus $\int_{A_1} \psi_k d\lambda \leqslant \int_{A_1} f_n d\lambda$ for all $n > k$, and therefore $\int_{A_1} \psi_k d\lambda \leqslant \inf \int_{A_1} f_n d\lambda$ for all $n > k$. As $f(x) = \lim\limits_{k\to\infty} \psi_k(x)$ on A_1, $\int_{A_1} f d\lambda = \lim\limits_{k\to\infty} \int_{A_1} \psi_k d\lambda \leqslant \liminf\limits_{n\to\infty} \int_{A_1} f_n d\lambda$. Hence $\int_A f d\lambda \leqslant \liminf\limits_{n\to\infty} \int_A f_n d\lambda$

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