

# Math 440 – Real Analysis II

## Homework 1

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**Problem 1** Find the limit inferior and limit superior of each of the following sequences  $\{s_n\}$ .

(a)  $s_n = 2 - \frac{1}{n}$

$$\overline{\lim}\{s_n\} = \underline{\lim}\{s_n\} = \lim_{n \rightarrow \infty} s_n = 2$$

(b)  $s_n = n \bmod 4$

For all  $k \in \mathbb{N}$  the sequence  $\{s_k, s_{k+1}, s_{k+2}, \dots\}$  contains only the values  $\{0, 1, 2, 3\}$ . Thus  $\forall k, 0 \leq s_k \leq 3$ , so

$$\overline{\lim}\{s_n\} = 3$$

And

$$\underline{\lim}\{s_n\} = 0$$

(c)  $s_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} + \cos \pi n & \text{if } n \text{ is odd} \end{cases}$

For all  $n \in \mathbb{N}$ ,  $\frac{1}{n} + \cos \pi n \leq n$ , so the  $\sup\{s_k, s_{k+1}, s_{k+2}, \dots\}$  is dominated by the even terms of  $s_n$ . Therefore

$$\overline{\lim}\{s_n\} = \lim_{n \rightarrow \infty} 2n = \infty$$

By the same argument, the  $\inf\{s_k, s_{k+1}, s_{k+2}, \dots\}$  is dominated by the odd terms of  $s_n$ , which is split between two cases:  $s_k = \frac{1}{k} + 1$  and  $s_k = \frac{1}{k} - 1$ . Thus

$$\underline{\lim}\{s_n\} = \lim_{n \rightarrow \infty} \frac{1}{4n+3} - 1 = -1$$

(d)  $s_n = f(n)$ , where  $f: \mathbb{N} \rightarrow \mathbb{Q} \cap (0, 1)$  is a bijection

Let  $\epsilon > 0$  and  $\overline{S} = \{s_n: 1 - s_n < \epsilon\}$

By Math340 results we know that there exists an infinite number of rationals between any two distinct reals (in this case 1 and  $1 - \epsilon$ ). Therefore  $\overline{S}$  is a countably infinite set, and  $\overline{S} \setminus \{s_1, s_2, \dots, s_{k-1}\}$  yields a similarly infinite set. From this,

$$\overline{\lim}\{s_n\} = \sup \overline{S} = 1$$

Similarly, with  $\underline{S} = \{s_n: s_n < \epsilon\}$  we have

$$\underline{\lim}\{s_n\} = \inf \underline{S} = 0$$

**Problem 2** Give an example of a sequence  $\{s_n\}$  and a real number  $s$  such that  $s_n < s$  for all  $n$ , but  $\overline{\lim} s_n \geq s$

Let  $s_n = -\frac{1}{n}$  and  $s = 1$ , then  $\forall n, s_n < s$  and  $\overline{\lim} s_n = s$

**Problem 3** Let  $\{a_n\}$  and  $\{b_n\}$  be bounded sequences of nonnegative real numbers.

$$(a) \quad \overline{\lim}(a_n b_n) \leq (\overline{\lim} a_n)(\overline{\lim} b_n)$$

Proof: Let  $s_n = a_n b_n$

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**Problem 4** The series  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$  diverges

Proof: Let  $s_n = \sum_{k=1}^n \frac{1}{\sqrt{k}}$

Then the partial sum  $s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$

Since  $\frac{1}{\sqrt{k}}$  is monotone decreasing, we have

$$s_n \geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

So for all  $n \geq 1$ , the partial sum  $s_n \geq \sqrt{n}$

Therefore,  $\lim\{s_n\} \geq \lim\{\sqrt{n}\} = \infty$

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