Math 440 – Real Analysis II

Homework 9

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Problem 54 Let A be a measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be measurable. For each $n \in \mathbb{N}$, the function f_n is measurable where

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leqslant n \\ n & \text{if} \quad |f(x)| > n \end{cases}$$

Proof: Let $n \in \mathbb{N}$, $s \in \mathbb{R}$, and $E = \{x : f_n(x) > s, \ \forall x \in A\}$.

case $s < -n$:
Then $E = A$ since $f_n(x) > s$ for all $x \in A$. So E is measurable.

case $-n \leqslant s \leqslant n$:
If $f_n(x) > s$ then $f(x) > s$, hence $E = \{x : f(x) > s, \ \forall x \in A\}$ and is measurable by Theorem 10.5.3 for $f(x)$.

case $s > n$:
$f_n(x) \leqslant n$ for all $x \in A$, therefore $E = \emptyset$ and is measurable.

Thus $f_n(x)$ is measurable by Definition 10.5.1.

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Problem 55 Let f be a non-negative, bounded, measurable function on $[a, b]$. If E and F are measurable subsets of $[a, b]$ with $E \subset F$, then $\int\limits_E f d\lambda \leqslant \int\limits_F f d\lambda$.

Proof: Let $G = F \backslash E$, and let $\mathcal{P}_E = \{E_1, E_2, \ldots, E_n\}$, $\mathcal{P}_G = \{G_1, G_2, \ldots, G_m\}$ be partitions of E and G. Thus $\mathcal{P}_F = \mathcal{P}_E \cup \mathcal{P}_G$ is a partition of F, and by Definition 10.6.1

$$\begin{aligned} \mathcal{L}_L(\mathcal{P}_F, f) = & \ m_1\lambda(E_1) + \cdots m_n\lambda(E_n) + m_1\lambda(G_1) + \cdots m_m\lambda(G_m) \\ = & \sum_{k=1}^{n} m_k\lambda(E_k) + \sum_{i=1}^{m} m_i\lambda(G_i) \\ = & \ \mathcal{L}_L(\mathcal{P}_E, f) + \mathcal{L}_L(\mathcal{P}_G, f) \end{aligned}$$

Because f is bounded and non-negative, $\mathcal{L}_L(\mathcal{P}_E, f)$ and $\mathcal{L}_L(\mathcal{P}_G, f)$ are likewise both bounded and non-negative, $\mathcal{L}_L(\mathcal{P}_E, f) \leqslant \mathcal{L}_L(\mathcal{P}_F, f)$. Since any partition of F can be refined into partitions as shown above, this holds true for all arbitrary partitions of F. From Lemma 10.6.5 $\sup\limits_{\mathcal{P}_E} \mathcal{L}_L(\mathcal{P}_E, f) \leqslant \sup\limits_{\mathcal{P}_F} \mathcal{L}_L(\mathcal{P}_F, f)$, so $\int\limits_E f d\lambda \leqslant \int\limits_F f d\lambda$ by Definition 10.6.3.

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Problem 56 Let f be a bounded, measurable function on $[a, b]$. For each $c > 0$, $\lambda(\{x \in [a, b] : |f(x)| > c\}) \leqslant \frac{1}{c} \int\limits_{[a,b]} |f| \, d\lambda$

Proof: Let $E = \{x \in [a, b] : |f(x)| > c\}$ and $G = \{x \in [a, b] : |f(x)| \leqslant c\}$. Then E, G are measurable sets by Theorem 10.5.3 where $E \cup G = [a, b]$ and $E \cap G = \emptyset$, so $\mathcal{P} = \{E, G\}$ is a partition of $[a, b]$ such that $0 \leqslant |f(x)|$ for all $x \in G$ and $c \leqslant |f(x)|$ for all $x \in E$. Therefore, from Definition 10.6.1, $0\lambda(G) + c\lambda(E) \leqslant \mathcal{L}_L(\mathcal{P}, |f|) \leqslant \sup\limits_{\mathcal{Q}}(\mathcal{Q}, |f|) = \int\limits_{[a,b]} |f| \, d\lambda$. Hence $\lambda(E) \leqslant \frac{1}{c} \int\limits_{[a,b]} |f| \, d\lambda$

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Problem 57 Let f be a non-negative bounded measurable function on $[a, b]$.

(a) If $\int\limits_{[a,b]} f d\lambda = 0$ then $f = 0$ almost everywhere on $[a, b]$.

Proof: Let $\epsilon > 0$ and $E = \{x \in [a, b] : f(x) > \epsilon\}$. Then $\lambda(E) \leqslant \frac{1}{\epsilon} \int\limits_{[a,b]} |f| \, d\lambda$, by Problem 56. By assumption, $\int\limits_{[a,b]} |f| \, d\lambda = 0$, so $\lambda(E) \leqslant \frac{1}{\epsilon} \cdot 0 = 0$

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(b) If $A \subset [a, b]$ is measurable and $\int\limits_{A} f d\lambda = 0$ then $f = 0$ almost everywhere on A.

Proof: Let $\chi_A : [a, b] \to \mathbb{R}$ be defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Then by Definition 10.6.3, $\int\limits_{A} f d\lambda = \int\limits_{[a,b]} f\chi_A d\lambda$. Because $f\chi_A$ is a bounded non-negative measurable function defined on $[a, b]$, (a) gives $f\chi_A = 0$ almost everywhere on $[a, b]$, and as $f\chi_A$ can only be nonzero, and is equal to f, on A this implies that $f = 0$ a.e. on A.

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