Math 440 – Real Analysis II

Homework 6

Amandeep Gill

March 27, 2015

Problem 36 Let X be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, where $||\cdot||$ is defined for all $\vec{x} \in X$ as $||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$. Then for all $\vec{x}, \vec{y} \in X$, $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

Proof: For all $\vec{x}, \vec{y} \in X$, using the definition of $||\cdot||$ on X,

$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} + \vec{y} \rangle + \langle \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= ||\vec{x}||^2 + 2 \langle \vec{x}, \vec{y} \rangle + ||\vec{y}||^2 \end{aligned}$$

As this is a chain of strict equality, and since the inner product is a non-negative function, $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ only if $\langle \vec{x}, \vec{y} \rangle = 0$.

$\blacksquare$

Problem 37 Let X be a vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$, and define $||\cdot||$ as $||\vec{x}|| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$. $||\cdot||$ is a norm on X.

(a) $||\vec{x}|| \leqslant 0$ for all $\vec{x} \in X$

Proof: This follows from the fact that $0 \leqslant \langle \vec{x}, \vec{x} \rangle$ for all $\vec{x} \in X$, and that the square root function has no negative outputs on the non-negative reals.

$\blacksquare$

(b) $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}$

Proof: This follows from the fact that $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$, and that the square root function has only one zero, which is at 0.

$\blacksquare$

(c) $||c \cdot \vec{x}|| = |c| \cdot ||\vec{x}||$ for all $c \in \mathbb{R}$ and all $\vec{x} \in X$

Proof: Let $c \in \mathbb{R}$ and $\vec{x} \in X$, then $||c \cdot \vec{x}|| = \sqrt{\langle c\vec{x}, c\vec{x} \rangle} = \sqrt{c^2 \langle \vec{x}.\vec{x} \rangle}$. This is acceptable because it is a double application of property (d) of Orthogonal Functions twice. Since $\sqrt{c^2} = |c|$, $||c \cdot \vec{x}|| = |c| \cdot ||\vec{x}||$.

$\blacksquare$

Problem 37 continued

(d) $||\vec{x}+\vec{y}|| \leqslant ||\vec{x}|| + ||\vec{y}||$ for all $\vec{x}, \vec{y} \in X$

Proof: Let $\vec{x}, \vec{y} \in X$. By previous work, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + 2\langle \vec{x}, \vec{y}\rangle + ||\vec{y}||^2$. Because $\langle \vec{x}, \vec{y}\rangle \leqslant ||\vec{x}||\,||\vec{y}||$ by the Cauchy-Schwartz Inequality, so this gives $||\vec{x}+\vec{y}||^2 \leqslant ||\vec{x}||^2 + 2\,||\vec{x}||\,||\vec{y}|| + ||\vec{y}||^2 = (||\vec{x}|| + ||\vec{y}||)^2$. Thus, for all $\vec{x}, \vec{y} \in X$, $||\vec{x}+\vec{y}|| \leqslant ||\vec{x}|| + ||\vec{y}||$.

$\blacksquare$

Problem 38 Let $f(x) = \sin(\pi x)$, $\phi_1(x) = 1$, and $\phi_2(x) = x$. Find c_1 and c_2 such that $S_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ gives the best approximation of f on $[-1, 1]$.

Solution: Need to choose c_1 and c_2 such that $f \sim c_1\phi_1 + c_2\phi_2$

$$c_1 = \frac{\langle f, \phi_1\rangle}{||\phi_1||} = \int_0^1 \sin(\pi x)dx = \frac{\pi}{2}$$

$$c_2 = \frac{\langle f, \phi_2\rangle}{||\phi_2||} = \int_0^1 x\sin(\pi x)dx = \frac{1}{\pi}$$

Problem 39

(a)
$$\langle \phi_0, \phi_1\rangle = \tfrac{1}{2}x^2 - a_1x = \tfrac{1}{2} - a_1$$
$$\langle \phi_0, \phi_2\rangle = \tfrac{1}{3}x^3 - \tfrac{a_2}{2}x^2 - a_3x = \tfrac{1}{3} - \tfrac{a_2}{2} - a_3$$
$$\langle \phi_1, \phi_2\rangle = \tfrac{1}{4}x^4 - \tfrac{(1+a_1a_4)}{3}x^3 - \tfrac{(a_1a_2+a_3)}{2}x^2 + a_1a_3x$$
$$\langle \phi_1, \phi_2\rangle = \tfrac{a_1a_2}{2} + a_1a_3 - \tfrac{a_1+a_2}{3} - \tfrac{a_3}{2} + \tfrac{1}{4}$$
$$a_1 = \tfrac{1}{2}, a_2 = 1, a_3 = -\tfrac{1}{6}$$
$$\phi_0 = 1, \phi_1 = x - \tfrac{1}{2}, \phi_2 = x^2 - x + \tfrac{1}{6}$$

(b)
$$c_0 = \frac{\langle f, \phi_0\rangle}{||\phi_0||} = \int_0^1 \sin(\pi x)dx = \frac{\pi}{2}$$

$$c_2 = \frac{\langle f, \phi_2\rangle}{||\phi_2||} = \frac{\int_0^1 (x+\frac{1}{2})\sin(\pi x)dx}{\sqrt{\int_0^1 (x+\frac{1}{2})^2}} = \frac{1}{\pi}\sqrt[4]{\frac{3}{13}}$$

$$c_3 = \frac{\langle f, \phi_3\rangle}{||\phi_3||} = \frac{\int_0^1 (x^2-x+\frac{1}{6})\sin(\pi x)dx}{\sqrt{\int_0^1 (x^2-x+\frac{1}{6})^2}} = 2\sqrt{5}\left(\frac{\pi^2-12}{\pi^3}\right)$$

Problem 40 If f and f_n for $n \in \mathbb{N}$ are Riemann integrable functions on $[a, b]$, and $\{f_n\}$ converges uniformly to f on $[a, b]$ then $\{f_n\}$ converges in the mean to f on $[a, b]$.

Proof: Let f and f_n for $n \in \mathbb{N}$ be Riemann integrable functions on $[a, b]$, where $\{f_n\}$ converges uniformly to f on $[a, b]$, then there exists a number $n_\circ$ such that for all $\epsilon > 0$ and $n \in \mathbb{N}$, if $n > n_\circ$ then $|f - f_n| < \frac{\sqrt{\epsilon}}{\sqrt{b-a}}$. From this, we get $\int\limits_a^b (f - f_n)^2 dx < \int\limits_a^b (\frac{\sqrt{\epsilon}}{\sqrt{b-a}})^2 dx = \frac{\epsilon x}{b-a}\Big|_a^b = \epsilon$. Therefore, $\lim\limits_{n\to\infty} \int\limits_a^b (f - f_n)^2 dx = 0$ and $\{f_n\}$ converges in the mean to f on $[a, b]$.

■

Problem 41 For $n \in \mathbb{N}$, let f_n be the function

$$f_n = \begin{cases} \sqrt{n} & \text{if} \quad 0 < x < \frac{1}{n} \\ 0 & \text{if} \quad \quad\; x \geqslant \frac{1}{n} \end{cases}$$

defined on $[0,1]$. The sequence of functions $\{f_n\}$ converges pointwise to $f(x) = 0$, but does not converge in the mean.

Proof: Let $x \in [0,1]$ and $\epsilon > 0$ be given. If $x = 0$, then $f_n(x) = 0$ and $|f_n(x) - 0| = 0 < \epsilon$ for all $n \in \mathbb{N}$. Otherwise, using 340 Facts there exists an $n_\circ \in \mathbb{N}$ such that $\frac{1}{n_\circ} < x$. Therefore, for all $n > n_\circ$, $f_n(x) = 0$, so $|f_n(x) - 0| = 0 < \epsilon$, and $\{f_n\}$ is thus pointwise convergent to 0.

However, for all $n \in \mathbb{N}$, $\int\limits_0^1 (0 - f_n(x))^2 \, dx = \frac{1}{n} \cdot n = 1$, and

$\lim\limits_{n\to\infty} \int\limits_0^1 (f_n(x))^2 \, dx = 1$. Thus $\{f_n\}$ does not converge in the mean to 0.

■