Math 440 – Real Analysis II

Exam 2

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Problem 1

(a) The sequence $\{\cos(nx)\}_{n=0}^{\infty}$ is orthogonal on $[-\pi, \pi]$.

Proof: Let $m, n \in \mathbb{N} \cup \{0\}$.

case $m \neq n$:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx &= \int_{-\pi}^{\pi} \tfrac{1}{2}(\cos(mx - nx) + \cos(mx + nx))dx \\ &= \tfrac{1}{2}\left(\tfrac{1}{m-n} sin((m-n)x) + \tfrac{1}{m+n} sin((m+n)x)\right)_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

case $m = n \neq 0$:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx &= \int_{-\pi}^{\pi} \cos^2(mx)dx \\ &= \left(\frac{sin(2mx)}{m} + \pi\right)_{-\pi}^{\pi} \\ &= \pi \neq 0 \end{aligned}$$

case $m = n = 0$:
Since $\cos(0x) = 1$, $\langle 1, 1\rangle = 2\pi \neq 0$

Therefore the sequence $\{\cos(nx)\}_{n=0}^{\infty}$ is orthogonal on $[-\pi, \pi]$.

$\blacksquare$

(b) Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ with respect to the sequence $\{\cos(nx)\}_{n=0}^{\infty}$.

Solution: $f(x) \sim \sum\limits_{n=0}^{\infty} c_n \cos(nx)$
$c_\circ = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{\pi}{2}$
$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi n^2}(\pi n \sin(\pi n) + \cos(\pi n) - 1)$

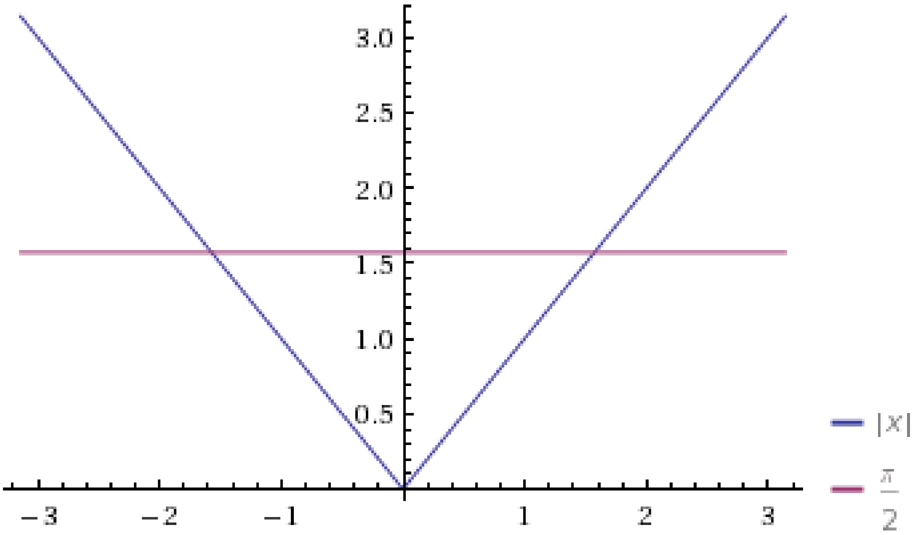
$$c_n = \begin{cases} 0 & \text{if } n = \text{even} \\ -\frac{4}{\pi n^2} & \text{if } n = \text{odd} \end{cases}$$

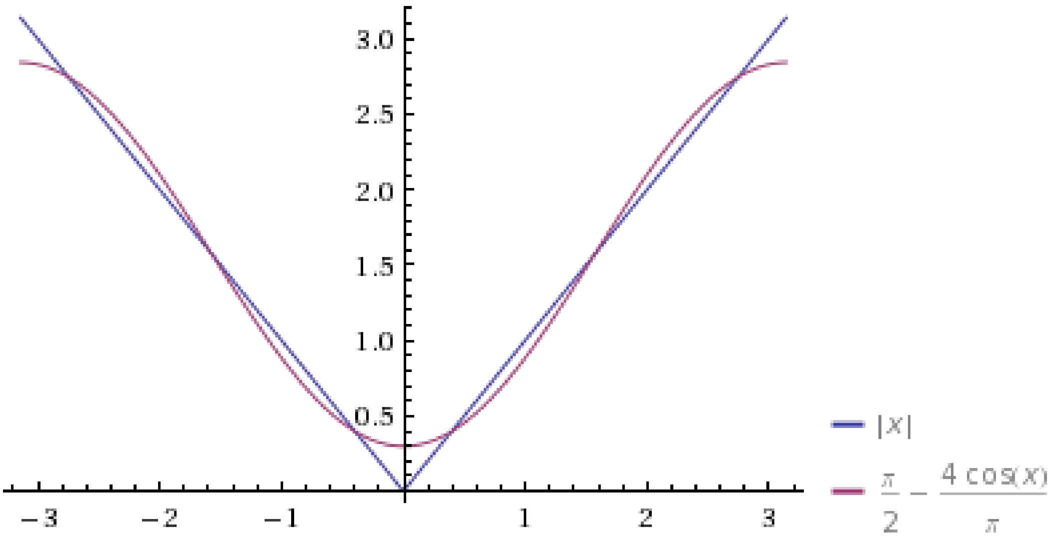
Therefore:

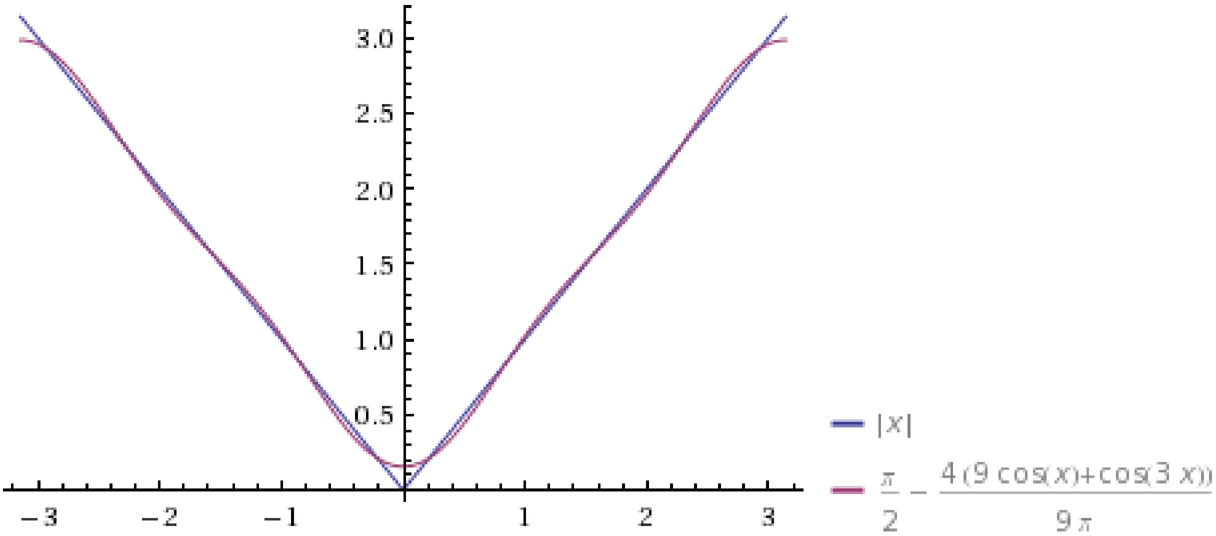
$$\begin{aligned} \sum_{n=0}^{\infty} c_n \cos(nx) &= \tfrac{\pi}{2} - \tfrac{4}{\pi}\cos(x) - \tfrac{4}{9\pi}\cos(3x) - \tfrac{4}{25\pi}\cos(5x) - \cdots \\ &= \tfrac{\pi}{2} - \tfrac{4}{\pi}\left[\cos(x) + \tfrac{1}{9}\cos(3x) + \tfrac{1}{25}\cos(5x) + \cdots\right] \\ &= \tfrac{\pi}{2} - \tfrac{4}{\pi}\sum_{n=0}^{\infty} \tfrac{1}{(2n+1)^2}\cos((2n+1)x) \end{aligned}$$

Problem 1 continued...

(c) Graph the first four n^{th} partial sums from (b) with $f(x)$.






3.0
2.5
2.0
1.5
1.0
0.5
−3
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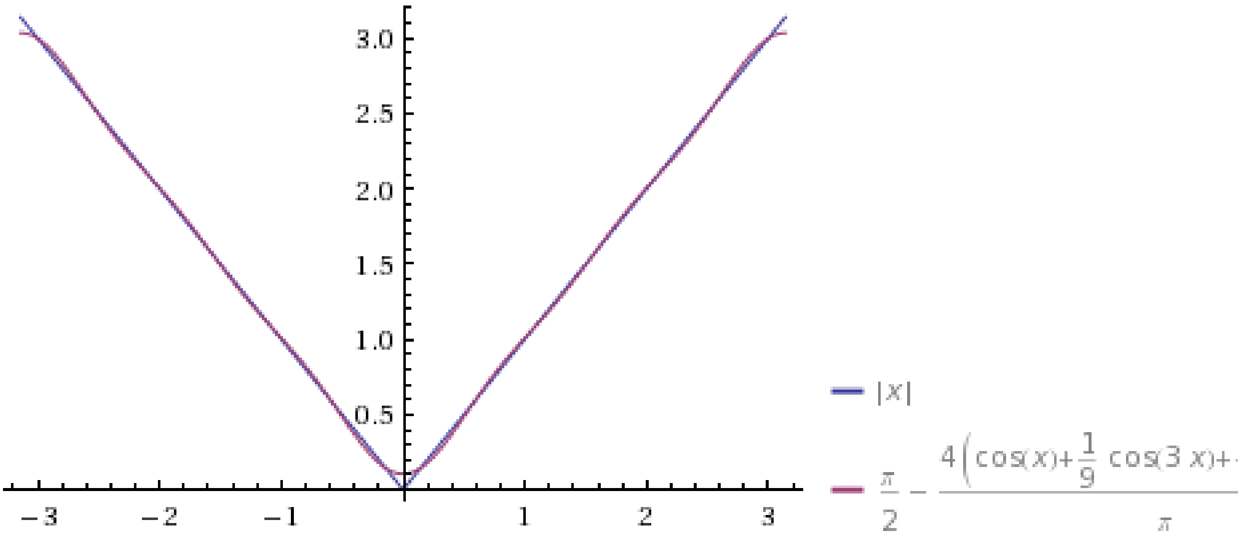
−1
1
2
3
$|x|$
$\frac{\pi}{2} - \frac{4(9\cos(x)+\cos(3x))}{9\pi}$


3.0
2.5
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1.0
0.5
−3
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1
2
3
$|x|$
$\frac{\pi}{2} - \frac{4\left(\cos(x)+\frac{1}{9}\cos(3x)+\right.}{\pi}$

Problem 2 Let $K \neq \emptyset$, and $\mathcal{F} = \{f : K \to \mathbb{R} : \exists M \text{ such that } |f(x)| \leqslant M, \forall x \in K\}$. For each $f \in \mathcal{F}$, define $||f|| = \sup_{x \in K} |f(x)|$.

(a) $\mathcal{F}$ is a vector space.

Proof: Let $f, g \in \mathcal{F}$ such that $|f(x)| \leqslant M_1$ and $|g(x)| \leqslant M_2$ for all $x \in K$. Then $|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant M_1 + M_2$ for al $x \in K$, thus $(f + g) \in \mathcal{F}$ and $\mathcal{F}$ is closed under vector addition.

Let $f \in \mathcal{F}$ such that $|f(x)| \leqslant M$ and $c \in \mathbb{R}$. From this, we have $|c \cdot f(x)| = |c| \cdot |f(x)| \leqslant |c| \cdot M$ for all $x \in K$, which means that $c \cdot f \in \mathcal{F}$ for all $c \in \mathbb{R}$ and $\mathcal{F}$ is closed under scalar multiplication.

Since the eight vector space axioms are assumed trivially, $\mathcal{F}$ is a vector space.

$\blacksquare$

(b) $(\mathcal{F}, ||\cdot||)$ is a normed vector space.

Proof: Let $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$

(a) $||f|| \geqslant 0$ since for all $x \in K$, $|f(x)| \geqslant 0$, and thus $\sup_{x \in K} |f(x)| \geqslant 0$

(b) $f(x) = 0$ for all $x \in K$ implies that $\sup_{x \in K} |0| = 0$, and $\sup_{x \in K} |f(x)| = 0$ means that $|f(x)| \leqslant 0$, and thus $f(x) = 0$. Therefore $||f|| = 0$ if and only if $f(x) = 0$ for all $x \in K$.

(c) $||c \cdot f|| = \sup_{x \in K} |c \cdot f(x)|$. Because $|c \cdot f(x)| = |c| \cdot |f(x)|$ is bounded, $\sup_{x \in K} |c \cdot f(x)| = |c| \cdot \sup_{x \in K} |f(x)| = |c| \cdot ||f||$.

(d) For all $x \in K$, $|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)| = ||f(x)|| + ||g(x)||$. Thus $||f + g|| = \sup_{x \in K} (f(x) + g(x)) \leqslant ||f|| + ||g||$.

Therefore $(\mathcal{F}, ||\cdot||)$ is a normed vector space.

$\blacksquare$

Problem 3 Every finite or countable infinite subset E of $\mathbb{R}$ is Lebesgue measurable with $\lambda(E) = 0$

Proof: Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, and $E = \{x_1, x_2, \ldots, x_n\}$ be ordered such that $a < x_i < x_{i+1} < b$ for all $i < n$. E is a finite, and therefore compact subset of (a, b), so $m(E) = m(a, b) - m\left((a, b) \backslash E\right)$. Since $(a, b) \backslash E = (a, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, b)$ is a union of disjoint open intervals, $m\left((a, b) \backslash E\right) = (x_1 - a) + (x_2 - x_1) + \cdots + (b - x_n) = b - a = m(a, b)$. Thus $m(E) = (a - b) - (a - b) = 0$, and because n was arbitrary $m(E) = 0$ for all countable subsets of $\mathbb{R}$.

$\blacksquare$

Problem 4 Define $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 2x^2 & \text{if} \quad x \in [0, 1] \backslash \mathbb{Q} \\ 1 & \text{if} \quad x \in [0, 1] \cap \mathbb{Q} \end{cases}$$

f is a measurable function

Proof: By the definition, f is a measurable function if and only if $E = \{x \in [0, 1] : f(x) \geqslant s\}$ is has a measure for all $s \in \mathbb{R}$.

case $s \leqslant 0$:
$E = [0, 1]$ and is therefore measurable.

case $s \geqslant 2$:
$E = \emptyset$ and has measure 0 by definition.

case $0 < s \leqslant 1$:
$E = \left(\left[0, \sqrt{\frac{s}{2}}\right] \cap \mathbb{Q}\right) \cup \left[\sqrt{\frac{s}{2}}, 1\right]$. Because $\left[0, \sqrt{\frac{s}{2}}\right] \cap \mathbb{Q}$ is a countable set, it is measurable by previous work. Thus E is a union of measurable sets and hence has measure by Theorem 10.4.1.

case $1 < s < 2$:
$E = \left[\sqrt{\frac{s}{2}}, 1\right] \backslash \mathbb{Q}$. By Corollary 10.4.3 E is measurable because the complimentary set $\left[\sqrt{\frac{s}{2}}, 1\right] \cap \mathbb{Q}$ has measure 0.

Thus f is a measurable function.

$\blacksquare$

Problem 5 Let $\{A_k\}$ be a countable collection of measurable sets. If $\lambda(A_i \cap A_j) = 0$ for all $i \neq j$, then $\lambda\left(\bigcup\limits_{k=1}^{\infty} A_k\right) = \sum\limits_{k=1}^{\infty} \lambda(A_k)$.

Proof: Let $n \in \mathbb{N}$ and $U_n = \lambda\left(\bigcup\limits_{k=1}^{n} A_k\right)$, then

$U_1 = \lambda(A_1)$

$U_2 = \lambda(A_1 \cup A_2)$. By Theorem 10.4.1, $\lambda(A_1) + \lambda(A_2) = \lambda(A_1 \cup A_2) + \lambda(A_1 \cap A_2) = \lambda(A_1 \cup A_2)$, since $\lambda(A_1 \cap A_2) = 0$ by assumption. So $U_2 = \lambda(A_1) + \lambda(A_2)$

$U_3 = \lambda(A_1 \cup A_2 \cup A_3)$. By substitution, $\lambda(A_1) + \lambda(A_2) + \lambda(A_3) = \lambda(A_1 \cup A_2) + \lambda(A_3) = \lambda(A_1 \cup A_2 \cup A_3) + \lambda((A_1 \cap A_3) \cup (A_2 \cap A_3))$. Since, definition of Lebesgue measure and the finite extension of Theorem 10.4.4(a), $\lambda((A_1 \cap A_3) \cup (A_2 \cap A_3)) \leqslant \lambda(A_1 \cap A_3) + \lambda(A_2 \cap A_3) = 0$. Thus, $U_3 = \lambda(A_1) + \lambda(A_2) + \lambda(A_3)$.

Assume $U_m = \sum\limits_{k=1}^{m} \lambda(A_k)$ for $1 < m < n$. Then,

$$\begin{aligned}
U_{m+1} &= U_m + \lambda(A_{m+1}) \\
&= \lambda(A_1 \cup A_2 \cup \cdots \cup A_m) + \lambda(A_{m+1}) \\
&= \lambda(A_1 \cup A_2 \cup \cdots \cup A_{m+1}) + \lambda((A_1 \cup A_2 \cup \cdots \cup A_m) \cap A_{m+1}) \\
&= \lambda(A_1 \cup A_2 \cup \cdots \cup A_{m+1}) \text{ by similar inequality as step 3}
\end{aligned}$$

Inductively then, $U_n = \sum\limits_{k=1}^{n} \lambda(A_k)$, and because n was arbitrary this holds true for all $n \in \mathbb{N}$, so $\lambda\left(\bigcup\limits_{k=1}^{\infty} A_k\right) = \sum\limits_{k=1}^{\infty} \lambda(A_k)$.

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Problem 6 Let $\epsilon > 0$ be given, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions defined on $[a,b]$ that converges point-wise to $f : [a,b] \to \mathbb{R}$. Define $E_k = \{x : |f_n(x) - f(x)| < \epsilon \text{ for all } n \geqslant k\}$.

(a) E_k is measurable for all $k \in N$

Proof: Let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of functions where $g_k(x) = \sup\limits_{n \geqslant k} |f_n(x) - f(x)|$. Since each f_n are real-valued functions, for each $x \in [a,b]$ the sequence $\{f_n(x)\}_{n=k}^{\infty}$ is a sequence of real values that converges to $f(x)$. Thus the sequence is bounded, and therefore $\{|f_n(x) - f(x)|\}$ has a supremum and by Theorem 10.5.9 $g_k(x)$ is a measurable function. Hence the set $\{x : g_k(x) \leqslant \epsilon\}$ is measurable by Theorem 10.5.3, and because $|f_n(x) - f(x)| \leqslant g_k(x)$ for all $n \geqslant k$, $\{x : |f_n(x) - f(x)| \leqslant \epsilon \text{ for all } n \geqslant k\}$ has a measure. This is equivalent to E_k being measurable.

$\blacksquare$

(b) $\lim\limits_{k\to\infty} \lambda(E_k^c)$ where $E_k^c = [a,b] \backslash E_k$.

Proof: If $|f_n(x) - f(x)| < \epsilon$ for all $n \geqslant k$, then $|f_n(x) - f(x)| < \epsilon$ for all $n \geqslant k+1$ as well, so $E_k \subset E_{k+1}$ and $E_k^c \supset E_{k+1}^c$. Thus $\cap_{k\in\mathbb{N}} E_k^c = [a,b] \backslash (\cup_{k\in\mathbb{N}} E_k)$. Given that $\{f_n\}_{n=1}^{\infty}$ is pointwise convergent for all $x \in [a,b]$, $\cup_{k\in\mathbb{N}} E_k = [a,b]$ and $\cap_{k\in\mathbb{N}} E_k^c = \emptyset$. Therefore $\lim\limits_{k\to\infty} \lambda(E_k^c) = \lambda(\emptyset) = 0$.

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