

Math 440 – Real Analysis II
Final Exam

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Problem 1 Determine which of the following series are divergent, conditionally convergent, or absolutely convergent.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k^2-1}$ converges conditionally.

Proof: Let $b_k = \frac{k}{2k^2-1}$. Since $b_{k+1} = \frac{k+1}{2(k+1)^2-1} = \frac{k+1}{2k^2+4k+1} < b_k$ for all $k \in \mathbb{K}$ and $\lim_{k \rightarrow \infty} b_k = 0$, by the Alternating Series Test $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k^2-1}$ converges conditionally. ■

(b) $\sum_{k=0}^{\infty} \frac{(-1)^{a_k}}{k^2+1}$ where $a_k = \lfloor \frac{k}{2} \rfloor$ converges absolutely.

Proof: Let $b_k = \left| \frac{(-1)^{a_k}}{k^2+1} \right| = \frac{1}{k^2+1}$. For all $k \in \mathbb{N}$, $b_k < \frac{1}{k^2}$, therefore since $\frac{1}{k^2}$ is a convergent P-series $\sum_{k=1}^{\infty} b_k$ converges. Thus $\sum_{k=0}^{\infty} \frac{(-1)^{a_k}}{k^2+1}$ converges absolutely. ■

Problem 2 Let $\{q_k\}_{k \in \mathbb{N}}$ be an enumeration of the rational numbers on $[0, 1]$, and for each $k \in \mathbb{N}$ define $C_k = \{q_1, q_2, \dots, q_k\}$. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \sum_{k=1}^{\infty} 2^{-k} \chi_{C_k}$.

(a) The series defining f converges uniformly on $[0, 1]$.

Proof: Let f be defined piecewise as

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \\ 2^{1-i} & \text{if } x = q_i \text{ for some } q_i \in \{q_k\}_{k \in \mathbb{N}} \end{cases}$$

Let $\epsilon > 0$ be given, $f_n(x) = \sum_{k=1}^n 2^{-k} \chi_{C_k}$, and let $m, n_o \in \mathbb{N}$ such that $m > n_o$ and $2^{-n_o} < \epsilon$. If $x \in [0, 1] \setminus \mathbb{Q}$, then $f_m(x) = 0$ and $|f_m(x) - f(x)| = 0 < \epsilon$. Otherwise, $x = q_i$ for some $i \in \mathbb{N}$.

case $i > m$:

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$$f_m(x) = 0 \text{ and } |f_m(x) - f(x)| = 2^{1-i} < 2^{-n_o} < \epsilon.$$

case $i \leq m$:

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$$f_m(x) = 2^{1-i} - 2^{-m} \text{ and } |f_m(x) - f(x)| = 2^{-m} < 2^{-n_0} < \epsilon.$$

Thus f_n converges to f uniformly on $[0, 1]$.

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(b) Calculate $f(q_1)$, $f(q_2)$, $f(q_3)$, and $f(\sqrt{2}/2)$

Solution:

$$f(q_1) = 1$$

$$f(q_2) = \frac{1}{2}$$

$$f(q_3) = \frac{1}{4}$$

$$f(\sqrt{2}/2) = 0$$

(c) f is Riemann integrable on $[0, 1]$ and $\int_0^1 f dx = 0$

Proof: For each $n \in \mathbb{N}$ let $\mathcal{P}_n = \{x_0, x_1, \dots, x_{2^n}\}$ partition $[0, 1]$ where $x_i = \frac{i}{2^n}$.

For each n , $\mathcal{U}(\mathcal{P}_n, f) \leq \sum_{k=1}^{2^n} 2^{1-k} \frac{1}{2^n} = 2^{1-n} - 2^{1-n-2^n}$ as the sup of an interval in the partition must be $f(q_1)$ or less, which means that the sup of one of the remaining intervals must be $f(q_2)$ or less, and so on up to $f(q_{2^n})$ in the final interval. Thus by Lemma 6.1.3

$$\inf \mathcal{U}(\mathcal{P}, f) \leq \lim_{n \rightarrow \infty} \mathcal{U}(\mathcal{P}_n, f) \leq \lim_{n \rightarrow \infty} 2^{1-n} - 2^{1-n-2^n} = 0$$

f is non-negative, so by Theorem 6.1.4,

$$0 \leq \sup \mathcal{L}(\mathcal{P}, f) \leq \inf \mathcal{U}(\mathcal{P}, f) \leq 0$$

Therefore $\int_0^1 f dx = 0$ by Definition 6.1.5.

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(c) f is Lebesgue integrable on $[0, 1]$ and $\int_0^1 f d\lambda = 0$

Proof: By Corollary 10.6.8 if the Riemann integral exists, then the Lebesgue integral exists and is equal to the Riemann integral. Thus $\int_0^1 f d\lambda = 0$.

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Problem 3 Let A be a measurable subset of \mathbb{R} , and let $\mathcal{L}(A)$ denote the set of all Lebesgue integrable functions on A . Given $f, g \in \mathcal{L}(A)$ and $c \in \mathbb{R}$, define $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for all $x \in \mathbb{R}$

(a) $\mathcal{L}(A)$ forms a vector space over \mathbb{R} .

Proof: Let $f, g \in \mathcal{L}(A)$ and $c \in \mathbb{R}$, then $\int_A (f + g) d\lambda = \int_A f d\lambda + \int_A g d\lambda$ by Theorem 10.7.8(a), as both f and g are Lebesgue integrable functions by assumption, so $(f + g)(x)$ is Lebesgue integrable and $\mathcal{L}(A)$ is closed under function addition. Similarly by Theorem 10.7.8(a), $\int_A cf d\lambda = c \int_A f d\lambda$, hence $(cf)(x)$ is Lebesgue integrable and $\mathcal{L}(A)$ is closed under scalar multiplication. Because the eight vector space axioms hold trivially, $\mathcal{L}(A)$ is a vector space over \mathbb{R} .

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(c) If $\|\cdot\| : \mathcal{L}(A) \rightarrow \mathbb{R}$ is defined as $\|f\| = \int_A |f| d\lambda$, then $\|\cdot\|$ is not a norm on $\mathcal{L}(A)$

Proof: Let $A = [0, 1]$ and $f(x) = \chi_E$ where $E = [0, 1] \setminus \mathbb{Q}$. Then $\|f\| = \int_A |f| d\lambda = \int_A \chi_E d\lambda$. By Theorem 10.6.10, $\int_A \chi_E d\lambda = \int_E 1 d\lambda + \int_{E^c} 0 d\lambda$. Let $x_n \in E$ such that $\bigcup_{n \in \mathbb{N}} \{x_n\} = E$, and let $\epsilon > 0$ be given. Then for each $I_n = (x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}})$, $E \subset \bigcup_{n \in \mathbb{N}} I_n$ and $\sum_{n \in \mathbb{N}} \lambda(I_n) = \epsilon$. Thus, by Theorem 6.1.11, $\lambda(E) = 0$ and $\int_E 1 d\lambda + \int_{E^c} 0 d\lambda = 1\lambda(E) + 0\lambda(E^c) = 0$. Since $f \neq 0$ for all $x \in [0, 1]$, $\|\cdot\|$ does not satisfy the property that $\|f\| = 0$ if and only if $f = 0$, and therefore $\|\cdot\|$ is not a norm on $\mathcal{L}(A)$.

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Problem (4) Let $a \in \mathbb{R}$ such that $0 < a < \frac{1}{2}$, and let $C_0 = [0, 1]$ be the first step in the generalized Cantor set with C_n comprised of 2^n disjoint intervals of length a^n such that $C_n \subset C_{n-1}$, and define

$$C_a = C_0 \cap C_1 \cap C_2 \cap \dots$$

(a) If $a \in \mathbb{R}$ with $0 < a < \frac{1}{2}$ then the set C_a is measurable with measure 0.

Proof: For each $n \in \mathbb{N}$, C_n is the union of 2^n disjoint intervals of length a^n , so by Theorem 10.4.5(b) $\lambda(C_n) = 2^n a^n = (2a)^n$. Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $c_n = \lambda(C_n) = (2a)^n$. Because the number of intervals is countable for all n , Theorem 10.4.5(b) holds for all n , and since $a < \frac{1}{2}$, $2a < 1$ and c_n is a monotonically decreasing sequence bounded by $[0, 1]$ and thus $\lim_{n \rightarrow \infty} c_n = 0$. As $C_a \subset C_n$, for each open cover $U \supset C_n$ so $C_a \subset U$ and so $\lambda^*(C_a) \leq \lambda^*(C_n)$ by Definition 10.3.1. By Theorem 10.3.4(a) $\lambda(C_n) = \lambda^*(C_n)$, hence $\lambda^*(C_a) \leq c_n$ for all n . Therefore $0 \leq \lambda_*(C_a) \leq \lambda^*(C_a) \leq \lim_{n \rightarrow \infty} c_n = 0$, and by Definition 10.3.4(a) C_a is measurable with $\lambda(C_a) = 0$. ■

(b) $\frac{\log 2}{\log(1/a)}$ is an upper bound on the Hausdorff dimension of C_a .

Proof: Let A_n be defined as a ball in \mathbb{R} such that $|A_n| = a^n$ is a δ_n cover for each $n \in \mathbb{N}$. The number of A_n balls needed to cover C_a is then $N = 2^n$. By Definition D.2.4, $\mathcal{H}_{\delta_n}^s(C_a) \leq \sum_{k=1}^N |A_n|^s = 2^n a^{sn}$. Therefore, as $\mathcal{H}^s(C_a) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(C_a) \leq \lim_{n \rightarrow \infty} 2^n a^{sn}$. Thus $s = \frac{\log 2}{\log(1/a)}$ is an upper bound on $\dim_H(C_a)$. ■

Problem 5 The Hausdorff dimension of any countable subset of \mathbb{R}^d is zero.

Proof: Let $\epsilon > 0$ be given, and let $X = \{x_1, x_2, \dots, x_k\}$ for some $k \in \mathbb{N}$ be a subset of \mathbb{R}^d with a δ_ϵ -cover defined as $\bigcup_{n=1}^k C_n$ where $x_n \in C_n$ and $|C_n| = \frac{\epsilon}{2^n}$. By Definition D.2.4, $\mathcal{H}_{\delta_\epsilon}^s \leq \sum_{n=1}^k \left(\frac{\epsilon}{2^n}\right)^s = \left(\epsilon - \frac{\epsilon}{2^k}\right)^s$. Thus $\lim_{\delta_\epsilon \rightarrow 0^+} \mathcal{H}_{\delta_\epsilon}^s \leq 0^s = 0$ for all $s > 0$ and therefore as k is arbitrary, $\dim_H(X) = 0$ for all $k \in \mathbb{N}$. ■

Problem 6 Let $\{E_k\}_{k \in \mathbb{N}}$ be a sequence of subsets of \mathbb{R} . If there exists a sequence $\{U_k\}_{k \in \mathbb{N}}$ of pairwise disjoint open sets such that $E_k \subset U_k$ for all $k \in \mathbb{N}$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda^*(E_k)$$

Proof: Let $\{U_k\}_{k \in \mathbb{N}}$ be a sequence of disjoint open sets such that $E_k \subset U_k$ for all k , and let U be the union of all U_k and E be the union of all E_k . By Theorem 10.4.4(a), $\lambda^*(E) \leq \sum_{k=1}^{\infty} \lambda^*(E_k)$.

case $\lambda^*(E) = \infty$:

From the above, $\infty \leq \sum_{k=1}^{\infty} \lambda^*(E_k)$ and so $\sum_{k=1}^{\infty} \lambda^*(E_k) = \lambda^*(E)$.

case $\lambda^*(E) < \infty$:

Because $U_i \cap U_j = \emptyset$ for all $i, j \in \mathbb{N}$ where $i \neq j$, E_i and E_j are similarly disjoint. Thus, given $\epsilon > 0$, by Definition 10.3.1 there exists an open set A where $E \subset A$, $m(A) < \infty$, and for each $k \in \mathbb{N}$, let $V_k \subset A$ be an open set such that $E_k \subset V_k \subset U_k$ and $\lambda^*(E_k) = m(V_k) - \frac{\epsilon}{2^k}$. Given that V_k is an open set $m(V_k) = \lambda(V_k)$ by Theorem 10.3.2(Ex2), and so from Theorem 10.4.5(b) and the fact that V_k are pairwise disjoint, $m(V) = \sum_{k=1}^{\infty} m(V_k)$ where V is the union of V_k for all k . V is a subset of A , so Theorem 10.2.4 gives $m(V) \leq m(A)$. Hence

$$\sum_{k=1}^{\infty} \lambda^*(E_k) = \sum_{k=1}^{\infty} \left(m(V_k) - \frac{\epsilon}{2^k} \right) \leq m(V) < \infty$$

Therefore $\sum_{k=1}^{\infty} \lambda^*(E_k) = m(V) - \epsilon = m(V)$, since ϵ is arbitrary. Similarly from Definition 10.3.1, for each $\epsilon > 0$ there exists an open set B such that $E \subset B \subset V$ and $\lambda^*(E) = m(B) - \epsilon$, and again since ϵ is arbitrary, $\lambda^*(E) = m(B)$. Thus

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda^*(E_k) + \lambda^*(E) &= m(V) + m(B) \\ \lambda^*(E) + \lambda^*(E) &\leq m(V) + m(B) \\ \lambda^*(E) + \lambda^*(E) &\leq \sum_{k=1}^{\infty} \lambda^*(E_k) + \lambda^*(E) \\ \lambda^*(E) &\leq \sum_{k=1}^{\infty} \lambda^*(E_k) \end{aligned}$$

Therefore $\lambda^*(E) = \sum_{k=1}^{\infty} \lambda^*(E_k)$.

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Problem 7 Let γ be given such that $0 < \gamma < 1$, and let the following be the generalized construction of the Cantor set with $C_0 = [0, 1]$ and C_{k+1} generated by removing the center open interval of length $\gamma/3^{k+1}$ from each of the 2^k interval of C_k . Define C_γ as

$$C_\gamma = C_0 \cap C_1 \cap C_2 \cap \cdots$$

(a) C_γ has positive measure with no intervals.

Proof: An alternate definition for C_γ can be defined as the set $[0, 1] \setminus A_\gamma$, where $A_\gamma = \bigcup_{n=1}^{\infty} A_n$ such that A_i, A_j are pairwise disjoint when $i \neq j$ and each A_n is the union of 2^{n-1} disjoint open intervals of length $\gamma/3^n$. For each n , $m(A_n) = \sum_{i=1}^{2^{n-1}} \frac{\gamma}{3^n}$ using Definition 10.2.2, and so from Theorem 10.3.2(Ex2) $\lambda(A_n) = \gamma \left(\frac{2^{n-1}}{3^n} \right)$. Because A_γ is a countable union of disjoint open sets, $\lambda(A_\gamma) = \sum_{n=1}^{\infty} \lambda(A_n) = \gamma$ by Theorem 10.4.5. Since C_γ is complementary to A_γ on $[0, 1]$, C_γ has measure $\lambda(C_\gamma) = 1 - \lambda(A_\gamma)$ by Corollary 10.4.3 and Theorem 10.3.7. Thus $\lambda(C_\gamma) = 1 - \gamma$. However, given $\epsilon > 0$ and $x \in C_\gamma$, let $I = [x - \epsilon, x + \epsilon]$. For some $n \in \mathbb{N}$, the length of each closed interval in C_n is $l = \frac{1-\gamma}{3^n} < \epsilon$. As $x \in C_n$, x is also in some I_n of length l with a gap in between I_n and the adjacent intervals, therefore $I_n \subset I$ and I contains all points that lie on the same interval as x , and as ϵ is arbitrary, $I = \{x\}$ and there are no intervals in C_γ . ■

(b) If $f : [0, 1] \rightarrow \mathbb{R}$ as $f(x) = \chi_{C_\gamma}(x)$, then f is not Riemann integrable.

Proof: Because there are no intervals on C_γ , each $x \in C_\gamma$ is a jump discontinuity of f . Hence the measure of the set of discontinuities of f has measure $1 - \gamma > 0$, and f is not Riemann integrable by Theorem 6.1.13. ■

(c) As defined in (b), f is Lebesgue integrable.

Proof: f is a simple, non-negative function defined on a measurable set, so $\int_{[0,1]} f d\lambda = \int_{C_\gamma} 1 d\lambda + \int_{A_\gamma} 0 d\lambda = (1 - \gamma)$ by Theorem 10.6.10(b). ■