Math 440 – Real Analysis II

Homework 10

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Problem 63 Let $a \in \mathbb{R}$ such that $0 < a < \frac{1}{2}$, and let $C_0 = [0, 1]$ be the first step in the generalized Cantor set with C_n comprised of 2^n disjoint intervals of length a^n such that $C_n \subset C_{n-1}$, and define

$$C_a = C_0 \cap C_1 \cap C_2 \cap \cdots$$

Assuming the box-counting dimension of C_a exists, find $\dim_B(C_a)$.

Solution: Let $n \in \mathbb{N}$ and $r_n = a^n$. The number of intervals of length r_n needed to cover C_a, $N_{r_n}(C_a) = 2^n$. Then

$$\begin{aligned} \dim_B(C_a) &= \lim_{r_n \to 0} \frac{\log N_{r_n}(C_a)}{-\log r_n} \\ &= \lim_{n \to \infty} \frac{\log 2^n}{-\log a^n} \\ &= \lim_{n \to \infty} \frac{n \log 2}{n \log \frac{1}{a}} \\ &= \frac{\log 2}{\log \frac{1}{a}} \end{aligned}$$

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Problem 64 The box-counting dimension of any finite subset of $\mathbb{R}$ is zero.

Proof: Let $E = \{x_1, x_2, \ldots, x_n\}$ be a subset of $\mathbb{R}$. For all $r \in \mathbb{R}$ such that $r > 0$, $1 \leqslant N_r(E) \leqslant n$. Hence $\frac{\log 1}{-\log r} \leqslant \frac{\log N_r(E)}{-\log r} \leqslant \frac{\log n}{-\log r}$. Since $\frac{\log 1}{-\log r} = 0$ and $\lim\limits_{r \to 0} \frac{\log n}{-\log r} = 0$, $0 \leqslant \lim\limits_{r \to 0} \frac{\log N_r(E)}{-\log r} \leqslant 0$. Thus $\dim_B(E) = 0$ for all finite subsets E of $\mathbb{R}$.

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Problem 65 If $E \subset F \subset \mathbb{R}^n$ and the box-counting dimensions exist for E and F, then $\dim_B(E) \leqslant \dim_B(F)$.

Proof: Let E, E^c be disjoint subsets of $\mathbb{R}^n$ such that $E \cup E^c = F \subset \mathbb{R}^n$, and let $n = N_r(E)$ with C_r the accompanying box cover of E. If $E^c \subset C_r$, then $F \subset C_r$ and $N_r(F) = N_r(E)$. Otherwise additional r-cubes must be unioned to C_r in order to cover E^c, and so $N_r(F) > N_r(E)$. Since r is arbitrary, $N_r(E) \leqslant N_r(F)$ and $\frac{\log N_r(E)}{-\log r} \leqslant \frac{\log N_r(F)}{-\log r}$ for all $r \in \mathbb{R}^+$. Therefore, because the box-counting dimensions for E and F exist by assumption, $\dim_B(E) \leqslant \dim_B(F)$.

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Problem 66 Find an upper bound on the Hausdorff dimension of the Menger sponge.

Solution: Let M be the set defined in the construction of the Menger sponge and M_n be defined as a ball in $\mathbb{R}^3$ such that $|M_n| = \frac{\sqrt{3}}{3^n}$ for each $n \geqslant 0$. Then the number of M_n balls needed to cover M is $N = 20^n$. By Definition D.2.4, $\mathcal{H}^s_\delta(M) = \inf\left\{\sum_{i\geqslant 1} |U_i|^s : \{U_i\}_{i\geqslant 1} \text{ is a } \delta\text{-cover of } M\right\}$. Therefore $\mathcal{H}^s_\delta(M) \leqslant \sum_{i=1}^{N} |M_n|^s = \sum_{i=1}^{20^n} \left(\frac{\sqrt{3}}{3^n}\right)^s = \frac{20^n}{3^{sn}}\sqrt{3}^s$ for all δ-covers of M, and thus $\mathcal{H}^s(M) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(M) \leqslant \lim_{n\to\infty} \frac{20^n}{3^{sn}}\sqrt{3}^s$. Thus $s = \frac{\log 20}{\log 3}$ is an upper bound on $\dim_H(M)$.

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Problem 67 For each $s > 0$, $\mathcal{H}^s$ is an outer measure on $\mathbb{R}^n$.

Proof: Let $\mathcal{H}^s : 2^{\mathbb{R}^n} \to [0, \infty]$ be defined as $\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E)$. By Ex4 on page 113 this limit is well defined for all subsets E in $\mathbb{R}^n$, and given that $\mathcal{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$ by part(a) of Theorem D.2.5,

(i) $\mathcal{H}^s(\emptyset) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(\emptyset) = \lim_{\delta\to 0^+} 0 = 0$, since $\mathcal{H}^s_\delta(\emptyset) = 0$ for all δ-covers.

(ii) Let $A \subset B \subset \mathbb{R}^n$, then for all δ-covers, $\mathcal{H}^s_\delta(A) \leqslant \mathcal{H}^s_\delta(B)$. Thus $\lim_{\delta\to 0^+} \mathcal{H}^s_\delta(A) \leqslant \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(B)$ and $\mathcal{H}^s(A) \leqslant \mathcal{H}^s(B)$.

(iii) Let $\{A_i\}_{i\geqslant 1}$ be a countable collection of subsets of $\mathbb{R}^n$, and let $A = \bigcup_{i\geqslant 1} A_i$. If $\sum_{i\geqslant 1} \mathcal{H}^s(A_i) = \infty$, then $\mathcal{H}^s(A) \leqslant \sum_{i\geqslant 1} \mathcal{H}^s(A_i)$ is true by definition. Assume then that $\sum_{i\geqslant 1} \mathcal{H}^s(A_i) < \infty$. for each A_i, $\mathcal{H}^s_\delta(A_i) \leqslant \mathcal{H}^s(A_i)$ as $\mathcal{H}^s_\delta$ is a monotone decreasing function on δ. Thus $\sum_{i\geqslant 1} \mathcal{H}^s_\delta(A_i) \leqslant \sum_{i\geqslant 1} \mathcal{H}^s(A_i)$, for all δ-covers. Additionally, $\mathcal{H}^s_\delta(A) \leqslant \sum_{i\geqslant 1} \mathcal{H}^s_\delta(A_i)$. Hence

$$\mathcal{H}^s(A) = \lim_{\delta\to^+} \mathcal{H}^s_\delta(A) \leqslant \lim_{\delta\to 0^+} \sum_{i\geqslant 1} \mathcal{H}^s_\delta(A_i) \leqslant \lim_{\delta\to 0^+} \sum_{i\geqslant 1} \mathcal{H}^s(A_i)$$

Thus $\mathcal{H}^s$ satisfies all properties of Definition D.2.3 and is therefore an outer measure on $\mathbb{R}^n$.

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