Graph Coloring Using State Space Search

Sandeep Dasgupta Tanmay Gangwani Mengjia Yan Novi Singh

University of Illinois, Urbana-Champaign

1 Introduction

As a subject with many applications, graph coloring is one of the most studied NP-hard problems. Due to its time complexity, graph coloring is an excellent candidate for implementation with a parallelized architecture, and as such has become our choice for our project. Of the many ways that graph coloring can be adapted for parallel programming there are two main approaches in literature: the iterative and state-space search methods. The iterative approach begins by dividing the vertices of the graph to be colored into different groups, each of which is assigned to a node in the cluster. In the context of Charm++, this would be akin to dividing the vertices among a set of chares. At this point each chare would independently color its assigned vertices, and once all vertices had been colored the chares would communicate with each other to see if each of their respective vertices had any neighbors on other chares that violate the constraints of the coloring problem. Once color conflicts have been assessed, the chares would seek to mediate these problems through another round of independent coloring among the chares, a process that would be repeated until the graph has been successfully colored or shown to be uncolorable with the given number of colors. An example implementation is present by Boman et al. [1].

In contrast, the state-space search method seeks to start with the initial graph and color it by creating child chares for each possible color assignment among the vertices. This process of coloration and child creation is repeated until a solution to the coloring problem is found or an exhaustive search has yielded no solution. By doing so, the program explores all different possible colorations of the graph for a given number of colors. A state-space search exploration could find only the first valid solution and terminate; find all possible solutions; or find the most *optimal* solution among all possibilities. Sun et al. [4] have developed a general framework for solving state-space search problems like graph coloring and N-Queens.

State-space search has an advantage over the iterative approach in that it does not require the large number of message exchanges between chares to mediate coloring conflicts. However, the state-space search does require the use of many more chares than the iterative approach and could incur higher memory overhead. Given the Charm runtime features likes load balancing, prioritized message execution and the general emphasis on over-decomposition, we implement the state-space approach for finding the **first** valid coloring of an input graph, with a given number of colors.

2 Program Structure

Our implementation of state-space search is outlined in the structure of our .ci file. At the start of the program a root node chare is created by the main chare. This chare is responsible for preconditioning the input graph for the state-space exploration and is also the chare where the final result is accumulated. Next, the root-node chare selects a graph vertex to color based on heuristics, and spawns further chares which form the nodes of the state-space tree. In effect, each chare receives a partially colored graph as input and is responsible for finding a valid coloring

.ci file structure

1: **Main Chare**
    Spawn root-node of state space search
2: **Root-node Chare**
    Graph-preprocessing algorithms (precoloring, vertex removal, subgraph detection)
    Spawn (child) node chares with value-ordering
    Wait for response from children
3: **Node Chare**
    Graph-preprocessing algorithms (vertex removal, subgraph detection)
    Depending on grain-size
        Spawn (child) node chares with value-ordering, *or*
        Call sequential coloring algorithm
    Wait for response from children (if any)
    Merge childrens status, propagate response to parent
4: **Group Chares**
    Collect statistics, terminate search of subtrees or whole process
5: **Message Categories**
    Bitvector prioritization for node chare seeds
    Expedited privilege for result signals

(if one exists) for the entire graph starting from *that* initial state. It does so by pruning the graph using vertex removal and subgraph detection, and then spawning child chares to further distribute the search. In certain cases (section 3.5), the chare calls a sequential coloring algorithm instead of creating child chares.

We use the Charm++ *group* feature for bookkeeping purposes and for kill-chasing (section 4). The spawning of new node chares is loosely ordered using bit-vector prioritization (section 3.4). We used expedited message privileges for result messages that flow from leaf nodes of the state-space all the way up to the root node. This prevents delays in the termination of the program after a valid first solution has been found to the graph coloring problem.

3 Heuristics for State-Space Search

Due to the NP-hard nature of the problem, the number of all possible states in the state-space search is exponential in size, and as such heuristics to reduce the number of states that are searched are necessary for any tractable computation to be possible. This section outlines the details. The ideas are borrowed from the work of Kale et al. [2].

3.1 Pre-Coloring

The first of these heuristics is a preprocessing technique, namely the pre-coloring of the graph. We find the 3-vertex cliques in the graph, and assign them the lowest available color ensuring that a vertex does not get any of the colors assigned to its neighbors. However, we found that this technique is best applied to sparse graphs only. As dense graphs tend to have many, many 3-vertex cliques, this preprocessing technique often results in an unoptimized sequential coloring of a large part of the input graph. This may result in un-colorability of the graph even though it may be colorable with the given number of colors by using an optimized approach.

3.2 Vertex Removal

By noting that a vertex with degree less than the number of available colors must always have a valid possible coloring, we can remove such a vertex from the graph (pushing it onto a local stack), color the rest of the graph, and re-include the removed vertex, assigning it the lowest available color. It is important to note that this removal can be performed recursively. That is, the removal of a vertex from the graph may lower the degree of its neighboring vertices to the point that this heuristic could then be applied to them.

3.3 Next Vertex Selection

An intuitive and simple heuristic guides our process for selecting the next vertex to be colored. By picking the vertex with the least available number of potential colors, we can move more efficiently through the state-space. Two related optimizations include **impossibility-testing** and **forced-move**. In the former, if after a vertex is colored, one of its neighbors has no available colors left, then the state for that coloring is not generated since it can never lead to a valid coloring of the entire graph. The latter dictates that after a vertex is colored, if the number of possible colors at any neighbor is reduced to 1, then the neighbor is colored in the same step. This process is repeated recursively since the coloring of a neighbor might lead to forced-move for the neighbor of the neighbor.

3.4 ValueOrdering & Bit-vector Prioritization

Value ordering provides a lexicographical ordering of state-space search nodes to prioritize some nodes over others. This prioritization is based on the fact that the set of configurations resulting from the possible coloring of a graph vertex should be explored in the order of decreasing likelihood of getting a valid coloring for the entire graph. The color that needs to be explored first must be the one that affects the neighbors the least. To calculate this ordering, we find for each coloring of the vertex, the sum of the number of remaining possible colors for all its neighbors. *Ranks* are assigned to the colors based on the calculated sum - highest rank assigned to the color which results in the maximum sum. Based on this ranking, bit-vector priorities are assigned to the newly spawned child chares.

With respect to the state-space tree, this results in the left sibling of a node chare having a higher priority than the right sibling, leading to a *broom-stick sweep*. Chare seeds are picked up by the Charm scheduler based on their priority. This adds some notion of determinism in the otherwise non-deterministic exploration. It also provides relatively consistent (monotonic) speedups as the number of available PEs is increased. This idea has been presented by Saletore et al. [3].

3.5 Grain-size Control

In Charm terminology, grain-size is the amount of work done by each chare. For the graph coloring problem, this corresponds to the number of vertices colored by each chare. Non-leaf chares in the state-space subtree perform minimal amount of work by coloring a single vertex only, and spawning more chares to work on its behalf. For our experiments, we define grain-size as a threshold G. If the number of uncolored vertices at a chare falls below G, then instead of creating new chares, the chare become a leaf node and colors the remaining subtree locally.

We use grain-size as a parameter to determine the number of chares created in the system. Picking close to optimal grain-size is of paramount importance; too low and a huge number of chares are generated, incurring parallelization overhead, and a threshold that is too high results in very few chares and an under-utilization of processing power.

The sequential algorithm run at each leaf chare is a stack based, worklist approach that employs vertex removal and value ordering. In some cases, the sequential algorithm could take an excessive amount of time to explore the entire subtree of nodes. The large entry method could hog the PE leading to undesirable effects. To mitigate this, the sequential coloring is made **pre-emptable**, which is to say that the coloring method returns after a set timeout value. A new call to the sequential coloring method is inserted in the scheduler queue at the same PE. On invocation, the method picks off from where it left at the time of preemption. By doing this, we prevent the blocking of expedited messages, such as success or failure messages, and don't unnecessarily delay the termination of the program.

An important note to the selection of grain size is that it had to be manually fine-tuned to find a good value as variations in graph structure and edge density would result in vastly different grain size optimums.

3.6 Detection of Independent Subgraphs

This heuristic is to detect independent subgraphs, an , $\mathcal{O}(V+E)$ operation, in the input graph and then to run the coloring algorithm on each subgraph separately. By splitting the graph into two or more disconnected pieces, we often obtain large speedups as the problem is exponential in nature and evaluating multiple smaller subgraphs results in extremely fewer possible states as compared to evaluating a single, larger graph.

This heuristic also leads to a changing of the structure of the state-space tree in that it transforms it into an AND-OR tree. In the single graph approach, either the left OR right subtree (or at least one of many) has to find a solution to the coloring problem for an overall solution to be found, and failure in one subtree does not necessitate the failure of the entire search. However, in the case of an AND-OR tree, when splitting the problem into multiple, independent subgraphs it becomes necessary for a solution to exist for all of the subgraphs (the left AND right subtrees have to yield solutions), and failure for any one subgraph implies failure for the entire search. Additionally, higher priority is given to that subtree of an AND node which has less uncolored vertices, with the hope that it would finish faster.

4 Kill Chasing

There are conditions under which a node chare should be prohibited from spawning new chares. The issue of delivering the prohibition signal before the actual spawning is referred to as kill-chasing. We employ group chares to prevent the spawning of new chares if a solution to the graph coloring problem has been determined in some part of the state-space search tree. Before creating any children, a chare must obtain *permission* from the group chare on its PE. When a solution is found in the system, all the group chares are notified immediately with expedited messages, and prevent any node chares from further creating progenies. Similarly, in the case of an AND node, if one of the subtree reports a failed coloring, the information is broad-casted to the group chares, and node chares in the sibling subtrees are prevented from creating further chares.

5 Setup

Table 1 shows the graphs along with their features which we used for our experimentations. Each of the graphs, G1-G6, were randomly generated by a python script. We varied the number of vertices, average edge density, and maximum number of allowed colors, and additionally generated both truly random graphs and graph that were composed of two or more independent subgraphs. All of the graphs in the table were colorable save for two. Graphs G5 and G6 have subgraph partitions.

All runs were done using 12 PEs on the Taub campus cluster. For each configuration, we report the average of 3 runs.

	Vertices	Edge Density	#Colors	Colorable
G1	300	12	9	Yes
G2	300	8	4	No
G3	1000	5	5	Yes
G4	500	7	6	Yes
G5	1000	6	6	Yes
G6	450	6	5	No

Table 1: Summary Of Graphs

6 Results

	Vertices	Grain Size	Timeout	Priority Bits	Value Ordering	Sub-graph Detection	Execution Time (s)	#Chares
G1	300	290	10	Enable	Enable	Enable	18	1220
				Disable			31	189105
G2	300	260	10	Enable			27	186329
		280					16	124601
		290					7	2681
		300					82	1
G3	1000	960	10	Enable	Enable	Enable	41	235
			30				214	585
G4	500	480	10	Enable	Enable	Enable	102	642
					Disable		133	669
G5	1000	960	5	Enable	Enable	Enable	0.01	4
						Disable	35	2610
G6	450	410	5	Enable	Enable	Enable	0.02	4
						Disable	5.77	49646

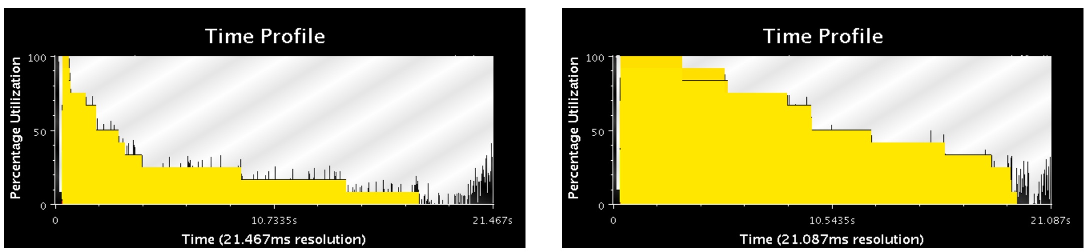
Table 2: Performance Data

The performance data is summarized in Table 2. The columns in the table refer to the configurable knobs in the program. For each of the graphs, we toggle some feature (heuristic), and measure the impact. We report the application running time in seconds, and the number of chares created in the system. Time is measured using *CkTimer* functionality, and the number of chares are calculated using groups.

For G1, enabling bit-vector prioritization improves performance by 1.7x. This is because bit-vector prioritization directs the state-space exploration to a path with a high likelihood of a solution. In G2, we evaluate the impact of changing the grain size for a graph with 300 vertices. A grain size of 300 is akin to sequential coloring, and is completed in 82s. As the grain size is varied from 260 to 290, the performance improves. This is because increasing grain size results in lesser number of chares being created, and hence lesser parallelism overheads. For G3, we varied the time for which the sequential coloring algorithm is allowed to run uninterrupted. We observed that changing the value from 30s to 10s results in a performance improvement of 5.2x. This is because a lower timeout value results in speedy delivery of the success (or failure) messages from the leaf chare to the root chare, resulting in prompt termination of the program. G4 shows the effect of using the value ordering heuristic. Ordering the configurations resulting from the coloring of a particular vertex leads to an improvement of 1.3x. Graphs G5 and G6 present the huge potential of subgraph detection. G5 and G6 are partitioned graphs. For the scenario with subgraph detection disabled, the application creates many chares which explore the state-space. This incurs some parallelization overhead. With subgraph detection enabled, the partitions (subgraphs) are colored sequentially since the number of uncolored vertices in each partition falls below the grain size. This results in the creation of just 4 chares - one root chare, and one each for the 3 partitions in G5 and G6.

7 Seed Balancer

To further improve the execution time and processor utilization in our program, we sought to employ a seed balancer. This balancer divides the creation of new chares across processors to even out the distribution of the workload. More specifically, we tried "work stealing" and "NeighborLB" as strategies for the seed balancer. In the case of the former, we found little difference in processor utilization compared to an execution without a seed balancer, but for the latter we were able to note a major difference. In Figure 1(b), generated using Projections, utilization remains higher for a longer period of time with NeighborLB enabled as compared to without a seed balancer, Figure 1(a). Figure 2 shows the distribution of the chares across the PEs with and without the seed balancer. Without the seed balancer, all the PEs get roughly the same number of chares. This ultimately leads to lower overall processor utilization since some leaf chares take more time to complete than others. A seed balancer can sense such un-evenness and migrate more seeds away from a processor where the leaf chares have a longer execution time. This results in some PEs getting more chares than others — PE9 gets more 1000 more chares than PE7 — but the overall processor utilization is improved, and application time reduced.



(a) (b)

Figure 1: Percentage Utilization on graph G2 with grainsize 260, timeout = 10s (a) Without Seed balancer (b) With "NeighborLB" Seed Balancer

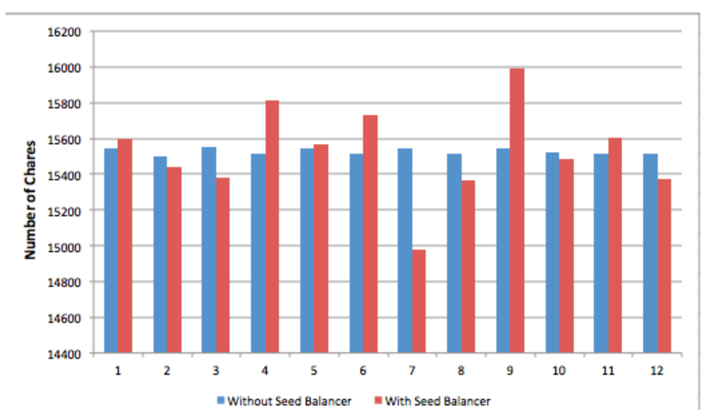


Figure 2: Distribution of seeds on each of the 12 PEs

8 Core Scaling

For an un-colorable graph with no subgraphs (such as G2), all the states need to be explored before a failure can be reported. Adding cores is useful is such a scenario since the state-space

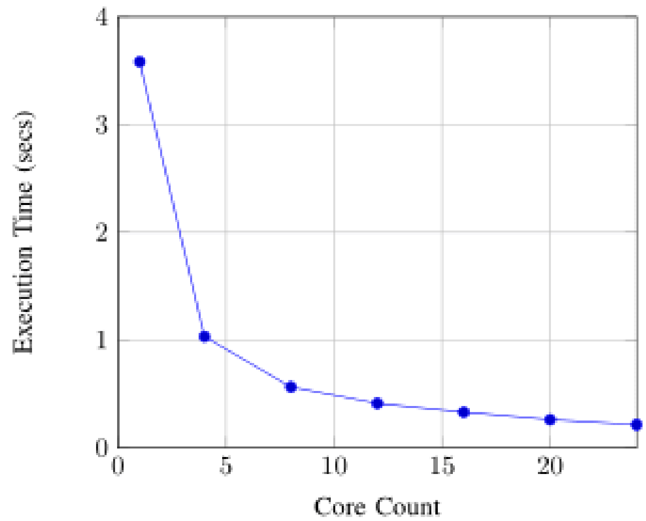


Figure 3: Graph G2, with 4 colors - vertices 300, edge-density 8

exploration work gets distributed. As shown in Figure 3, the execution time for G2 reduces with increasing core count. We observe diminishing returns after adding a certain number of PEs. This could be attributed to that fact with more PEs, the ancestry of a particular node in the state-space subtree could be spread out among different PEs, leading to more network overheads as result signals propagate from children to parent.

For colorable graphs, we could not establish a consistent trend with increasing number of PEs. We observe that the entire state-space is not explored for a colorable graph, unlike an un-colorable one. Adding more processors, hence, is not expected to yield the same benefits. Secondly, our heuristics are strong enough to guide the search towards a successful coloring. Since our implementation terminates after the first solution, work done by extra processors is not always required. Moreover, the extra processors could add useless work on the path of the processor which ultimately produces the first coloring, and hence may negatively impact the overall application execution time. We attribute the observed discrepancies to these effect. Bit-vector prioritization, in theory, helps to obtain monotonic speedups with additional processors. We note, however, that to work perfectly, such a solution would require a system with a **central** scheduling queue for the chare seeds.

9 Conclusion

Our project sought to employ as many heuristics as possible to mitigate the exponential nature of the graph coloring problem, and this desire combined with experimentation of other execution parameters has allowed us to create an effective program for graph coloring. Many of our difficulties lay in fine-tuning our heuristics to work with the large variation in graph structure that could occur, as well as working with the non-deterministic effects in the exploration of the state-space tree. In the future, this work could be expanded upon to more clearly express the impact of each heuristic on a graph, and to change the application of certain heuristics (for example, grain-size) automatically depending on the characteristics of the graph.

References

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