Formulas for Pre-Olympiad Competition Math

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This is a compilation of various formulas that are useful for competition math. I've ordered them by topic (geometry, number theory, algebra, and counting/probability). It is designed to be a reference - not a study guide. The starred (*) formulas are ones you must know for competition math, as they are very useful and come up in nearly every competition. The others listed are good to know, fun to learn, and are used occasionally, but aren't necessary for scoring well. Thanks to all the AoPSers (especially mathwiz0803) who contributed through their time and suggestions! For more in depth explanations for each of these, visit AoPS Wiki or search for explanations on Youtube. I hope this helps!

1 Geometry

1.1 Area of a Triangle*

$$A = \frac{bh}{2} = rs = \frac{1}{2}ab\sin\theta = \frac{abc}{4R}$$

Where A is the area, b is the base, and h is the height. In the second equation, r is the inradius and s is the semiperimeter (which is half the perimeter). In the third equation, θ is the angle between two sides a, b of the triangle. In the final equation, a, b, c are the sides of the triangle with circumradius R.

1.2 Area of a Square (and kite/rhombus)*

$$A = bh = s^2 \text{ or alternatively } \frac{d_1 \cdot d_2}{2}$$

Where A is the area, b is the base, h is the height, s is the side length, and $d_{1,2}$ are the lengths of the diagonals. The prior equation only applies to squares. The latter formula applied to any quadrilateral with perpendicular diagonals (such as kites and rhombi).

1.3 Area of a Rectangle*

$$A = bh$$

Where A is the area, b is the base, and h is the height.

1.4 Area of a Trapezoid*

$$A = \frac{(b_1 + b_2)(h)}{2}$$

Where A is the area, b_1 and b_2 are bases, and h is the height.

1.5 Area of a Regular Hexagon*

$$A = \frac{3\sqrt{3}s^2}{2}$$

Where A is the area and s is the side length. Deriving this by breaking the hexagon into six equilateral triangles and then 12 right triangles is a useful exercise.

1.6 Area of a Regular Polygon*

$$A = \frac{ap}{2} \text{ or } \frac{ns^2}{4\tan\left(\frac{180}{n}\right)}$$

Where A is the area, a is the apothem, p is the perimeter, n is the number of sides, and s is the side length.

1.7 Volume/Surface Area of a Cone*

$$V = \frac{\pi r^2 h}{3},\ SA = \pi r^2 + \pi r l$$

Where V is the volume, SA is the surface area, r is the radius of the circular base, h is the height, and l is the slant height.

1.8 Volume/Surface Area of a Sphere*

$$V = \frac{4\pi r^3}{3},\ SA = 4\pi r^2$$

Where V is the volume, SA is the surface area, and r is the radius of the sphere (which is radius of the central cross section/the base of the semisphere).

1.9 Volume/Surface Area of a Cube*

$$V = s^3,\ SA = 6s^2$$

Where V is the volume, SA is the surface area, and s is the length of a side.

1.10 Volume/Surface Area of a Pyramid*

$$V = \frac{1}{3}bh,\ SA = 2sl + b$$

Where V is the volume, SA is the surface area, b is the area of the base, h is the height, l is the slant height, and s is the length of a side of the base. Note that a pyramid can have a base of any polygon, but if none is specified, assume a square base. A pyramid with a triangular base is known as a tetrahedron.

1.11 Volume/Surface Area of a Cylinder*

$$V = \pi r^2 h,\ SA = 2\pi r^2 + 2\pi rh$$

Where V is the volume, SA is the surface area, r is the radius of the circular base, and h is the height.

1.12 Volume/Surface Area of a Prism*

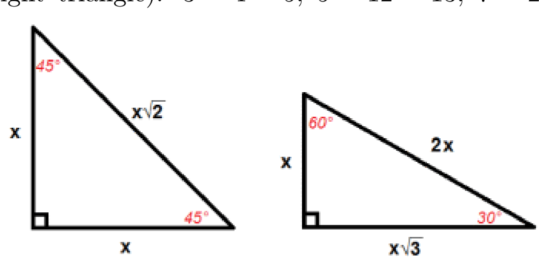
$$V = lwh,\ SA = 2(lw + lh + wh)$$

Where V is the volume, SA is the surface area, l is the length, w is the width, and h is the height.

1.13 Pythagorean Theorem and Right Triangles*

$$a^2 + b^2 = c^2$$

Where c is the hypotenuse of a right triangle with legs a and b. Note that there are some "special" right triangles. These include right triangles with angle measures $45° - 90° - 45°$ and $30° - 60° - 90°$. The prior type of right triangle has the property that if either leg (they are identical) has length x, the hypotenuse has length $x\sqrt{2}$. Similarly, the latter type of right triangle has the property that if the side opposite the $30°$ angle has length x, the side opposite the $60°$ angle has length $x\sqrt{3}$, and the side opposite the $90°$ angle has length $2x$. You should also memorize some common Pythagorean triples (if a triangle has these side lengths, or these side lengths multiplied by some factor, it is a right triangle): $3 - 4 - 5$, $5 - 12 - 13$, $7 - 24 - 25$, and $8 - 15 - 17$.



45° - 45° - 90° Triangle 30° - 60° - 90° Triangle

1.14 Distance Formula*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Where (x_1, y_1) and are points a coordinate plane and d is the distance between them. This is essentially the Pythagorean Theorem restated for points on a plane.

1.15 Heron's Formula*

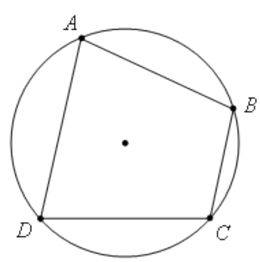
$$A = \sqrt{(s)(s-a)(s-b)(s-c)}$$

Where A is the area and s is the semiperimeter of the triangle with sides a, b, c.

1.16 Cyclic Quadrilaterals*

A quadrilateral is cyclic if and only if the quadrilateral can be inscribed in a circle. Here are some of the fundamental properties of cyclic quadrilaterals.

1. Opposite angles add to $180°$.
2. A convex quadrilateral is cyclic if and only if the four perpendicular bisectors to the sides are concurrent. This common point is the circumcenter.
3. In cyclic quadrilateral $ABCD$, $\angle ABD = \angle ACD$, $\angle BCA = \angle BDA$, $\angle BAC = \angle BDC$, $\angle CAD = \angle CBD$



1.17 Ptolemy's Theorem*

$$ab + cd = ef$$

Where $ABCD$ is a cyclic quadrilateral with side lengths a, b, c, d and diagonals e, f (with a opposite b and c opposite d).

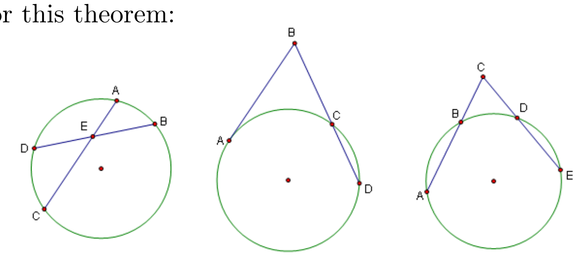
1.18 Brahmagupta's Formula

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Where K is the area and s is the semiperimeter of the quadrilateral with sides a, b, c, d. For this formula to work, the quadrilateral must be cyclic.

1.19 Power of a Point*

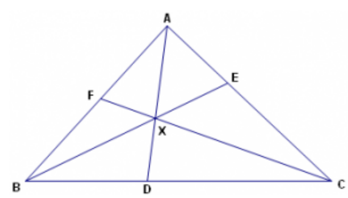
There are three cases for this theorem:



This theorem states that in the leftmost diagram (two intersecting internal chords), $\overline{AE} \cdot \overline{EC} = \overline{DE} \cdot \overline{EB}$, in the middle diagram (a tangent and a secant that meet at a point) $\overline{AB}^2 = \overline{BC} \cdot \overline{BD}$, and in the last diagram (secants that intersect outside the circle) $\overline{CB} \cdot \overline{CA} = \overline{CD} \cdot \overline{CE}$. A good exercise is to try and prove these with similar triangles.

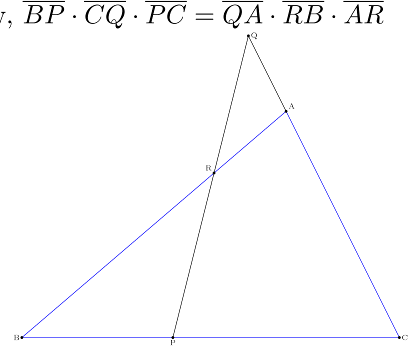
1.20 Ceva's Theorem

Ceva's theorem states that in with points D, E, F on sides $\overline{BC}, \overline{AC}, \overline{AB}$ respectively, are concurrent if, and only if, $\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1$



1.21 Menelaus' Theorem

This configuration shows up from time to time, though not often. The theorem states that given a configuration as shown below, $\overline{BP} \cdot \overline{CQ} \cdot \overline{PC} = \overline{QA} \cdot \overline{RB} \cdot \overline{AR}$

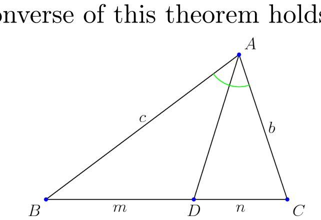


1.22 Arcs and Angles in a Circle*

An arc is a portion of the circle's circumference measured in degrees. The measure of an angle formed by the center of a circle and two point on the circumference is equal to the measure of the intercepted (subtended) arc. An angle formed by three points on the circumference of the circle is equal to $\frac{1}{2}$ the measure of the subtended arc.

1.23 Angle Bisector Theorem*

The Angle Bisector Theorem states that given $\triangle ABC$ and angle bisector $\overline{AD}$, where D is on side $\overline{BC}$, then $\frac{c}{m} = \frac{b}{n}$. Likewise, the converse of this theorem holds as well



1.24 Trigonometric Identities*

Note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$. Also $\sec\theta = \frac{1}{\cos\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$. Therefore, the identities for $\tan, \cot, \sec, \csc$ are easily derived from the identities for sin and cos.

Double Angle: $\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ Negative Angles: $\sin\cos -\theta = \sin\cos\theta$ $\cos\sin -\theta = \cos - \sin\theta = \cos\sin\theta$ Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1$ $\cot^2\theta + 1 = \csc^2\theta$ $\tan^2\theta + 1 = \sec^2\theta$ Addition/Subtraction Identities: $\sin a \pm B = \sin a \cos B \pm \sin B \cos a$ $\cos a \pm B = \cos a \cos B \pm \sin a \sin B$ Half Angle Identities: $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\theta}{2}}$ $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+cos\theta}{2}}$

1.25 Triangle Inequality*

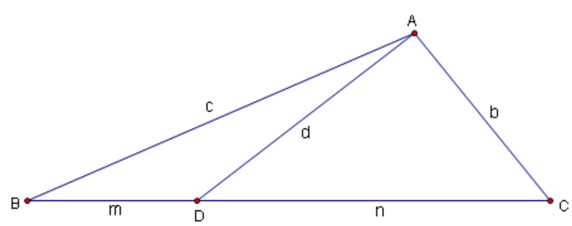
The Triangle Inequality says that in nondegenerate $\triangle ABC$: $\overline{AB} + \overline{BC} > \overline{AC}$, $\overline{BC} + \overline{AC} > \overline{AB}$, $\overline{AC} + \overline{AB} > \overline{BC}$

1.26 Pick's Theorem

$A = I + \frac{1}{2}B - 1$ Where A is the area, I is the number of lattice points in the interior, and B is the number of lattice points on the boundary of a figure in the coordinate plane.

1.27 Stewart's Theorem*

Take $\triangle ABC$ with sides of length a, b, c opposite vertices A, B, C respectively. If cevian $\overline{AD}$ is drawn so that $\overline{BD} = m, \overline{DC} = n, \overline{AD} = d$ we have that $man + dad = bmb + cnc$, which can be remembered using the mnemonic device, "A man and his dad put a bomb in the sink." This theorem isn't used very often in competition math, but it can trivialize otherwise difficult problems when it does come up.



1.28 (Extended) Law of Sines*

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ Where a, b, c are sides of a triangle, each opposite its respective angle A, B, C. R is the circumradius.

1.29 Law of Cosines*

For a triangle with sides a, b, c and opposite angles A, B, C respectively, the Law of Cosines states $c^2 = a^2 + b^2 - 2ab\cos C$

1.30 Shoelace Theorem*

Shoelace Theorem Suppose the polygon has vertices $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$ listed in clockwise order. Then the area of P is $\frac{1}{2}|(a_1b_2 + a_2b_3 + ... + a_nb_1) - (b_1a_2 + b_2a_3 + ... + b_na_1)|$ The Shoelace Theorem gets its name because if one lists the coordinates in a column and marks the pairs of coordinates to be multiplied, the resulting image looks like laced-up shoes.

$$(a_1, b_1)$$
$$(a_2, b_2)$$
$$...$$
$$(a_n, b_n)$$
$$(a_1, b_1)$$

2 Number Theory

2.1 Sum of an Arithmetic Series*

$S_n = \frac{n}{2}(a_1 + a_n)$ Where n is the number of terms, S_n is the sum, and a_1 is the first term.

2.2 Sum of the first n terms of a Geometric Series*

$S = \frac{a_1(1-r^n)}{1-r}$ Where S is the sum, a_1 is the first term, and r is the common ratio with $|r| < 1$.

2.3 Sum of an Infinite Geometric Series*

$S = \frac{a_1}{1-r}$ Where r is the common ratio, S is the sum, and a_1 is the first term. Note that $|r| < 1$ for this formula to work (otherwise the sum doesn't converge).

2.4 Sum of an Arithmetic Series*

The sum of the first odd numbers is simply n^2. The sum of the first n numbers $\frac{n(n+1)}{2}$. The sum of the first n even numbers is $n(n + 1)$.

2.5 Sum of the First n (Even/Odd/Both) Integers*

The sum of the first n odd positive integers is simply n^2. The sum of the first n even positive integers is $n(n + 1)$. The sum of the first n positive integers is given by $\frac{n(n+1)}{2}$.

2.6 Number/Sum of Divisors*

Let $p_1^x p_2^y ... p_m^n$ be the prime factorization of sum number a. The number of divisors a has is given by $(x + 1)(y + 1)...(n + 1)$. Similarly, the sum of the divisors of a is given by $(p_1^0 + p_1^1 ... p_1^x)(p_2^0 + p_2^1 ... p_2^y)...(p_m^0 + p_m^1 ... p_m^n)$.

2.7 Chinese Remainder Theorem

The Chinese Remainder Theorem (or CRT) allows you to solve a system of linear congruences. Let $m_1, m_2, ..., m_r$ be a collection of pairwise relatively prime integers. Then the system of simultaneous congruences $x \equiv a_1 (\mod m_1), x \equiv a_2 (\mod m_2), ..., x \equiv a_r (\mod m_r)$ has a unique solution modulo $M = m_1m_2, \Delta\Delta\Delta m_r$, for any given integers $a_1, a_2, ..., a_r$. This is easier to understand in an example and is very well explained in this video: https://www.youtube.com/watch?v=Y5RcMWiUyyE

2.8 Chicken McNugget Theorem

The Chicken McNugget Theorem states that for any two relatively prime positive integers, m, n, the greatest integer that cannot be written in the form $am + bn$ (the greatest number that can not be expressed as a sum of the two numbers) where a, b are positive integers is $mn - m - n$. It follows that there are $\frac{(m-1)(n-1)}{2}$ positive integers which cannot be expressed as the sum of some number of s and s.

2.9 Euler's Totient/Phi Function

$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_n})$ Where n is any positive integer and p_n are prime divisors of n. This gives the number of relatively prime positive integers less than or equal to some number n. This is often used in modular arithmetic problems since Euler's Theorem states $a^{\phi(n)} \equiv 1 \mod n$.

2.10 Wilson's Theorem

Wilson's Theorem is rather uncommon, but it is very powerful when you use it since it's a bijection. Also, note that Wilson's Theorem provides a primality test. However, there is no quick way to compute $p!$. It states that for any prime p, $(p-1)! \equiv -1 \mod p$

2.11 Trivial Inequality

Yes, this is a real thing. It states that $x^2 \geq 0$ for all real x (now you know why it is called "trivial"). You don't really need this for anything, but many other well known theorems and inequalities are based on this.

2.12 Fibonacci Numbers

Define a sequence F_n such that $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}$. The first few terms look like $0, 1, 1, 2, 3, 5, 8, \ldots$ The quotient of two consecutive terms approaches the golden ratio: $\frac{1+\sqrt{5}}{2}$

2.13 Pigeonhole Principle

If we distribute n balls into k boxes such that $n > k$ then at least one box must have multiple balls.

3 Algebra

3.1 Logarithm Rules*

Logarithmic to Exponential: $\log_a b = x \Rightarrow a^x = b$
Addition: $\log_a b + \log_a c = \log_a bc$
Subtraction: $\log_a b - \log_a c = \log_a \frac{b}{c}$
Exponent Reducing: $\log_a b^n = n \log_a b$
Change of Base: $\log_a b = \frac{\log_c b}{\log_c a}$
Reciprocals: $\log_a b = \frac{1}{\log_b a}$

3.2 Vieta's Formulas*

Vieta's formulas relate the coefficients of a polynomial to its roots. This set of formulas is one of the most useful in competition math. These state that the sum of the roots for a quadratic polynomial $ax^2 + bx + c$ is $\frac{-b}{a}$ and that the product of the roots is $\frac{c}{a}$. This extends to higher degree polynomials as well. For example, if the cubic polynomial $ax^3 + bx^2 + cx + d$ has roots r, s, t, $r + s + t = \frac{-b}{a}$, $rst = \frac{-d}{a}$, and $rs + rt + st = \frac{c}{a}$. Notice the signs alternate (with b being negative and alternating thereafter). This takes some practice to get used to and I recommend you visit the AoPS page on Vieta's Formulas for further details.

3.3 Common Factorizations and SFFT*

$a^2 - b^2 = (a+b)(a-b)$
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$
$ab+a+b+1 = (a+1)(b+1)$
$ab-a-b+1 = (a-1)(b-1)$
SFFT, or Simon's Favorite Factoring, refers to the strategy of adding some number to both sides of a polynomial so that it will factor and then breaking the side that is an integer into its prime factors to find posible solutions.

3.4 Quadratic Formula and Discriminant*

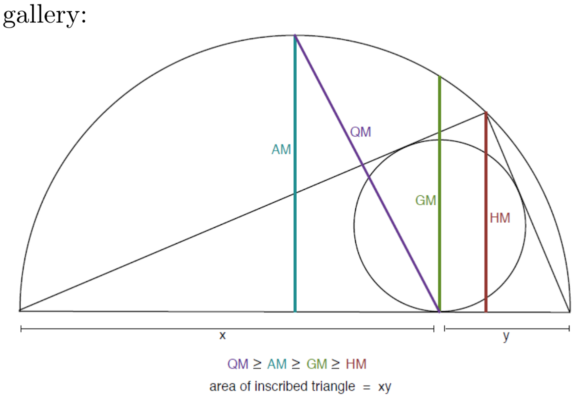
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $d = b^2 - 4ac$

Where a, b, c are the coefficients of the x^2, x^1, x^0 terms respectively. If the discriminant is , the quadratic will have one real solution. If the discriminant is negative, the quadratic will have no real solutions. If the discriminant is positive, the quadratic will have two real solutions.

3.5 RMS-AM-GM-HM*

$\sqrt{\frac{x_1^2+\ldots+x_n^n}{n}} \geq \frac{x_1+\ldots x_n}{n} \geq \sqrt[n]{x_1\ldots x_n} \geq \frac{n}{\frac{1}{x_1}+\ldots+\frac{1}{x_n}}$

This essentially states that the root-mean square is greater than or equal to the arithmetic mean is greater than or equal to the geometric mean is greater than or equal to the harmonic mean. The most important part of this inequality is AM-GM, which states that the arithmetic mean is greater than or equal to the geometric mean. Equality holds when all the s are the same. It is often used for maximization and minimization problems. Here is an interesting "proof" for this from AoPS Proofs Without Words gallery:



3.6 Location of Roots (DeMoivre's Theorem*

Let $\cos(\theta) + i\sin\theta = \text{cis}(\theta)$. DeMoivre's Theorem allows complex numbers in polar form - that is $r \cdot cis(\theta)$ - to be raised to raised to a power. It states that for a rational x and integral n, $\text{cis}(x)^n = \text{cis}(nx)$. This can be used to find the n^{th} root of a number/polynomial. See here for more information: https://www.ck12.org/book/CK-12-Trigonometry—Second-Edition/section/6.7/

4 Counting and Probability

4.1 Permutations*

${}_nP_k = \frac{n!}{(n-k)!}$ Where n is the total number of objects from which you are choosing k objects (order does matter).

4.2 Combinations*

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Where n is the total number of objects from which you are choosing k objects (order doesn't matter).

4.3 Pascals Identity

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ Where n is the total number of objects from which you are choosing k objects. Note that this can be easily observed from Pascal's Triangle. This identity can be extended to sums of three or more binomial coefficients

4.4 Binomial Theorem and Sum of Row in Pascal's Triangle*

The binomial theorem states that $(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. This is easier to understand by doing practice expansions. The coefficients of the powers of a and b are given in Pascals Triangle. The sum of the n^{th} row in the triangle is 2^{n-1} (as shown below).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

4.5 Fundamental Theorem of Counting*

If one event has n possible outcomes and another event has m possible outcomes, the total number of events if both events occur is $m \times n$. We can recursively prove this for multiple events.

4.6 Burnside's Lemma

Burnside's Lemma is a way to count objects if you need to account for rotations and reflections. It has appeared on Olympiads and even AMCs (it can be used for a variety of problems, though it may not always be the fastest method). The video below provides a great explanation and example for this lemma and I recommend you watch it. https://www.youtube.com/watch?v=pheASxiQk2I

4.7 Stars and Bars*

The number of ways to distribute n indistinguishable items into k distinguishable boxes, where each box must receive at least one item, is $\binom{n-1}{k-1}$Alternatively, the number of ways to distribute indistinguishable balls in distinguishable boxes, where some box(es) may remain empty, is $\binom{n+k-1}{k-1}$.

4.8 Expected Value*

If the expected value is $E(X)$, $E(X) = \sum_i P(X_i)V(X_i)$,where $P(X_i)$ is the probability of event X_i occurring, and $V(X_i)$ is the value of outcome X_i. If $X_1, X_2, ...X_k$ are several events (independent or not), then $E(X_1 + X_2 + ... + X_k) = E(X_1) + E(X_2) + ... + E(X_k)$. Basically, you can sum expected values (weighted averages). For clarity, here is an example: You roll two dice. What is the expected sum? We can compute the expected value of a dice roll. Since all outcomes are equally likely, the weighted average for one roll of a dice is simply the average, or 3.5. Since we can sum expected values, the expected value of the sum is $3.5 + 3.5 = \boxed{7}$

Resources/Practice Recommendations

-AoPS Volumes 1+2 for AMC and AIME preparation
-EGMO by Evan Chen for AIME to Olympiad geometry
-For AMC/AIME/USA(J)MO practice, visit the AoPS past competitions pages
-AoPS community/forums, videos, and AoPS Wiki for reference (**make an AoPS account**!!!)
-This may seem redundant, but the **best way to improve is to practice problems from the competitions you want to improve for**
-Handouts by Evan Chen (for very high level AIME to Olympiad): http://web.evanchen.cc/olympiad.html
-Shameless self advertising:) https://artofproblemsolving.com/community/c476370