Product of Vectors

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Exercise - 9.1

1.a. Find the scalar product of each of the following pair of vectors.

i. (3, 4) and (2, 1)

Solution:

Let $\vec{a} = (3, 4)$ and $\vec{b} = (2, 1)$

Then, $\vec{a} \cdot \vec{b} = (3, 4) \cdot (2, 1)$

$= (a_1 b_1 + a^2 b^2)$

$= (3 \times 2 + 4 \times 1)$

$= 10$

ii. (−3, 2, 1) and (1, 2, 3)

Solution:

Let $\vec{a} = (-3, 2, 1)$ and $\vec{b} = (1, 2, 3)$

Then $\vec{a} \cdot \vec{b} = (-3, 2, 1)(1, 2, 3)$

$= -3 + 4 + 3$

$= 4$

iii. $5\vec{i} + 2\vec{j} + 3\vec{k}$ and $-2\vec{i} - 3\vec{j} + 6\vec{k}$

Solution:

Let $\vec{a} = (5\vec{i} + 2\vec{j} + 3\vec{k})$ and $\vec{b} = (-2\vec{i} - 3\vec{j} + 6\vec{k})$

Then $\vec{a} \cdot \vec{b} = (5\vec{i} + 2\vec{j} + 3\vec{k})(-2\vec{i} - 3\vec{j} + 6\vec{k})$

$= \cancel{10\vec{i}}\ -10 - 6 + 18$

$= 2$

b. For what value of m is the following pair of vectors perpendicular (orthogonal)?

i. (3, −4, 7) and (2, 5, m)

Solution:

Let $\vec{a} = (3, -4, 7)$ and $\vec{b} = (2, 5, m)$

When they are perpendicular then,

$\vec{a} \cdot \vec{b} = 0$

or, $6 - 20 + 7m = 0$

$\therefore m = 2$

ii. $m\vec{i} - 2\vec{j} + \vec{k}$ and $m\vec{i} + m\vec{j} + \vec{k}$

Solution:

Let $\vec{a} = m\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = m\vec{i} + m\vec{j} + \vec{k}$

When, they are orthogonal then;

$\vec{a} \cdot \vec{b} = 0$

or, $m^2 - 2m + 1 = 0$

or, $(m - 1)^2 = 0$

$\therefore m = +1$

2. Find the cosine of the angle between the following pair of vectors.

i. $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

Solution:

Let $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

Now,

$\vec{a} \cdot \vec{b} = 6 - 2 + 8$

$= 12$

Also,

Magnitude of a $= \sqrt{9 + 1 + 4}$

$= \sqrt{14}$

Magnitude of b $= \sqrt{4 + 4 + 16}$

$= \sqrt{24}$

Then, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab}$

$= \dfrac{20}{\sqrt{14}\cdot\sqrt{24}}$

$\therefore \cos\theta = \dfrac{\sqrt{21}}{7}$

ii. $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ ~~ab~~ and $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$

Solution:

$\vec{a}\cdot\vec{b} = 1 - 6 + 6$

$= 1$

Also, Magnitude of a $= \sqrt{1+4+9}$

$= \sqrt{14}$

Magnitude of b $= \sqrt{1+9+4}$

$= \sqrt{14}$

Now,

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab}$

$= \dfrac{1}{\sqrt{14}\cdot\sqrt{14}}$

$\therefore \cos\theta = \dfrac{1}{14}$

3. If $\vec{i}, \vec{j}, \vec{k}$ are three mutually perpendicular unit vectors and $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} - \vec{k}$ are two vectors, obtain the values of a, b, $\vec{a}\cdot\vec{b}$ and $\cos\theta$. Also, find the projection of $\vec{a}$ on $\vec{b}$ and projection of $\vec{b}$ on $\vec{a}$.

Solution:

$\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} - 3\vec{j} - \vec{k}$

Now,

Magnitude of a $= \sqrt{1+4+1}$

$= \sqrt{6}$

Magnitude of b $= \sqrt{4+9+1}$

$= \sqrt{14}$

Also,

$\vec{a}\cdot\vec{b} = (\vec{i}-2\vec{j}+\vec{k})(2\vec{i}-3\vec{j}-\vec{k})$

$= 2+6-1$

$= 7$

Also,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$$

$$= \frac{7}{\sqrt{6}\cdot\sqrt{14}}$$

$$= \frac{7}{2\sqrt{21}}$$

Also,

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{7}{\sqrt{14}}$

Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{7}{\sqrt{6}}$

4a. Show that the line AB is perpendicular to CD if A, B, C, D are the points (2, 3, 4), (5, 4, -1), (3, 6, 2) and (1, 2, 0).

Solution:

We have A(2, 3, 4), B(5, 4, -1), C(3, 6, 2) and D(1, 2, 0)

Now,

$\vec{AB} = (5-2)\hat{i} + (4-3)\hat{j} + (-1-4)\hat{k}$

$= 3\hat{i} + \hat{j} - 5\hat{k}$

Also,

$\vec{CD} = (1-3)\hat{i} + (2-6)\hat{j} + (0-2)\hat{k}$

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$= -2\hat{i} - 4\hat{j} - 2\hat{k}$

Now,

$\vec{AB} \cdot \vec{CD} = (3\hat{i} + \hat{j} - 5\hat{k}) \cdot (-2\hat{i} - 4\hat{j} - 2\hat{k})$

$= 3(-2) + 1\cdot(-4) + (-5)(-2)$

$= -6 - 4 + 10$

$= 0$

b. If the coordinates of P, Q, R and S be $(1,2,2)$, $(2,4,0)$, $(-3,0,1)$ and $(-1,-2,2)$ respectively, show that the projection of PQ on RS is equal to the projection of RS on PQ. Also, show that the angle between PQ and RS is $\cos^{-1}(-4/9)$.

Solution:

We have, $P(1,2,2)$, $Q(2,4,0)$, $R(-3,0,1)$ and $S(-1,-2,2)$

$\vec{PQ} = 1\hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{RS} = 2\hat{i} - 2\hat{j} + \hat{k}$

Projection of $\vec{PQ}$ on $\vec{RS}$ is given by

$= \dfrac{\vec{PQ} \cdot \vec{RS}}{|\vec{RS}|}$

$= \dfrac{(\hat{i} + 2\hat{j} - 2\hat{k})(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{4+4+1}}$

$= \dfrac{2-4-2}{3}$

$= \dfrac{-4}{3}$

Again,

Projection of $\vec{RS}$ on $\vec{PQ}$ is given by:

$= \dfrac{\vec{PQ} \cdot \vec{RS}}{|\vec{PQ}|}$

$= \dfrac{2-4-2}{\sqrt{1+4+4}} = \dfrac{-4}{3}$

Again,

$$\cos\theta = \frac{\vec{PQ}\cdot\vec{RS}}{|\vec{PQ}||\vec{RS}|}$$

or $$\cos\theta = \frac{2-4-2}{\sqrt{1+4+4}\cdot\sqrt{4+4+1}}$$

or $$\cos\theta = \frac{-4}{9}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-4}{9}\right)$$

5.a. Find the angles of a triangle whose vertices are $7\vec{j}+10\vec{k}$, $-\vec{i}+6\vec{j}+6\vec{k}$ and $-4\vec{i}+9\vec{j}+6\vec{k}$ respectively

Solution:

Let the given vertices be $A(7\vec{j}+10\vec{k})$, $B(-\vec{i}+6\vec{j}+6\vec{k})$ and $C(-4\vec{i}+9\vec{j}+6\vec{k})$

$\vec{AB} = -\vec{i}-\vec{j}-4\vec{k}$

$\vec{BC} = -3\vec{i}+3\vec{j}+0\vec{k}$

$\vec{AC} = -4\vec{i}+2\vec{j}+4\vec{k}$

Here,

Angle between $\vec{AB}$ and $\vec{AC}$ is given by;

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\cdot|\vec{c}|}$$

or, $$\cos\theta = \frac{(-1,-1,-4)\,(-4,2,4)}{\sqrt{1+1+16}\cdot\sqrt{16+4+16}}$$

or $$\cos\theta = \frac{4-2+16}{\sqrt{18}\cdot\sqrt{36}}$$

or $$\cos\theta = 1/\sqrt{2}$$

$$\therefore \theta = 45^\circ$$

Also

Angle between $\vec{BC}$ and $\vec{BA}$ is given by;

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$$\cos B = \frac{(-3\vec{i} + 3\vec{j})(\vec{i} + \vec{j} + 4\vec{k})}{\sqrt{9+9}\ \sqrt{1+1+16}}$$

or, $$\cos B = \frac{(-3+3+0)}{\sqrt{18}.\sqrt{18}}$$

$$\therefore B = \cos^{-1}(0)$$
$$= 90^\circ$$

Again,

The angle between $\vec{CA}$ and $\vec{CB}$ is given by;

$$\cos C = \frac{(4\vec{i} - 2\vec{j} - 4\vec{k})(3\vec{i} - 3\vec{j} + 0\vec{k})}{\sqrt{16+4+16}\ .\ \sqrt{9+9}}$$

or, $$\cos C = \frac{12+6+0}{\sqrt{18}.6}$$

or, $$C = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\therefore C = 45^\circ$$

b. Show that the three points whose position vectors are $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4\vec{k}$ form the sides of a right angled triangle. Also, show that the remaining two angles of a triangle are $\cos^{-1}\sqrt{\frac{35}{41}}$ and $\cos^{-1}\sqrt{\frac{6}{41}}$ respectively

Solution:

Let the given vertices of triangle be $A(2\vec{i} - \vec{j} + \vec{k})$, $B(\vec{i} - 3\vec{j} - 5\vec{k})$ and $C(3\vec{i} - 4\vec{j} - 4\vec{k})$ then,

$\vec{AB} = (-\vec{i} - 2\vec{j} - 6\vec{k})$

$\vec{BC} = (2\vec{i} - \vec{j} + \vec{k})$

$\vec{CA} = (-\vec{i} + 3\vec{j} + 5\vec{k})$

Now, Angle between $\vec{AB}$ and $\vec{AC}$ is given by;

$$\cos A = \frac{(-\vec{i}, -2\vec{j} - 6\vec{k})(\vec{i} - 3\vec{j} - 5\vec{k})}{\sqrt{1+4+36}\ \sqrt{1+9+25}}$$

or, ~~cos A~~ $\cos A = \dfrac{-1+6+30}{\sqrt{41}\cdot\sqrt{35}}$

$$\therefore A = \cos^{-1}\left(\sqrt{\frac{35}{41}}\right)$$

Again,

Angle between $\vec{BC}$ and $\vec{BA}$ is given by;

$$\cos B = \frac{(2\vec{i} - \vec{j} + \vec{k})(\vec{i} + 2\vec{j} + 6\vec{k})}{\sqrt{4+1+1}\cdot\sqrt{1+4+36}}$$

$$\text{or, } \cos B = \frac{2-2+6}{\sqrt{6}\cdot\sqrt{41}}$$

$$\therefore B = \cos^{-1}\sqrt{\frac{6}{41}}$$

Also,

Angle between $\vec{CA}$ and $\vec{CB}$ is given by;

$$\cos C = \frac{(-\vec{i} + 3\vec{j} + 5\vec{k})(-2\vec{i} + \vec{j} - \vec{k})}{\sqrt{1+9+25}\cdot\sqrt{4+1+1}}$$

$$\text{or, } \cos C = \frac{2+3-5}{\sqrt{35}\cdot\sqrt{6}}$$

$$\text{or, } \cos C = 0$$

$$\therefore C = 90^\circ$$

Hence, the triangle ABC is a right angled triangle.

6. If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them, show that; $\frac{1}{2}|\vec{a} - \vec{b}| = \sin\frac{\theta}{2}$

Solution:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= 1\cdot 1\cdot\cos\theta$$
$$= \cos\theta$$

Now,

$$|\vec{a} - \vec{b}| = (\vec{a} - \vec{b})^2$$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$= 1 + 1 - 2\cos\theta$

$= 2(1 - \cos\theta)$

Now,

$|\vec{a} - \vec{b}|^2 = 2(1 - \cos\theta)$

or $|\vec{a} - b|^2 = 2.2\sin^2\frac{\theta}{2}$

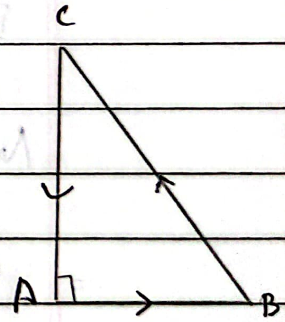
or $|\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$

$\therefore \frac{1}{2}|\vec{a} - \vec{b}| = \sin\frac{\theta}{2}$

Proved

7. In a right angled triangle ABC, right angles at A, show that $AB^2 + AC^2 = BC^2$.

Solution:



According to the triangle law of vector addition

$\vec{BC} = \vec{BA} + \vec{AC}$

or, $\vec{BC} = -\vec{AB} + \vec{AC}$

Squaring both side, we get;

$BC^2 = (\vec{AC} - \vec{AB})^2$

or $BC^2 = AC^2 - 2\vec{AC}\cdot\vec{AB} + AB^2$

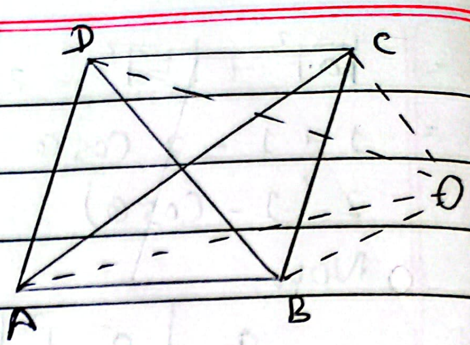
or $BC^2 = AC^2 - 2.0 + AB^2$ $\{\vec{AC}\cdot\vec{AB} = 0°\}$

or $BC^2 = AC^2 + AB^2$

$\therefore AB^2 + AC^2 = BC^2$ Proved

8. If A, B, C and D are any four points, show that $\vec{AB}\cdot\vec{CD} + \vec{BC}\cdot\vec{AD} + \vec{CA}\cdot\vec{BD} = 0$

Solution:



Let O be the origin, then

$\vec{AB} = \vec{AO} + \vec{OB}$

$= -\vec{OA} + \vec{OB}$ —— (i)

$\vec{BC} = \vec{BO} + \vec{OC}$

$= -\vec{OB} + \vec{OC}$ —— (ii)

$\vec{CA} = \vec{CO} + \vec{OA}$

$= -\vec{OC} + \vec{OA}$ —— (iii)

$\vec{CD} = \vec{CO} + \vec{OD}$

$= -\vec{OC} + \vec{OD}$ —— (iv)

$\vec{AD} = \vec{AO} + \vec{OD}$

$= -\vec{OA} + \vec{OD}$ —— (v)

$\vec{BD} = \vec{BO} + \vec{OD}$

$= -\vec{OB} + \vec{OD}$ —— (vi)

Now,

$\vec{AB} \cdot \vec{CD} + \vec{BC} \cdot \vec{AD} + \vec{CA} \cdot \vec{BD}$

$= (-\vec{OA} + \vec{OB})(-\vec{OC} + \vec{OD}) + (-\vec{OB} + \vec{OC})(\vec{OA} + \vec{OD}) + (-\vec{OC} + \vec{OA})(-\vec{OB} + \vec{OD})$

$= \vec{OA} \cdot \vec{OC} - \vec{OA} \cdot \vec{OD} - \vec{OC} \cdot \vec{OB} + \vec{OB} \cdot \vec{OD} + \vec{OC} \cdot \vec{OA} + \vec{OC} \cdot \vec{OD} - \cancel{\vec{OA}}$
$- \vec{OA} \cdot \vec{OB} - \vec{OB} \cdot \vec{OD} + \vec{OC} \cdot \vec{OB} - \vec{OC} \cdot \vec{OD} + \vec{OA} \cdot \vec{OD} - \vec{OA} \cdot \vec{OB}$

$= 0$

Proved.

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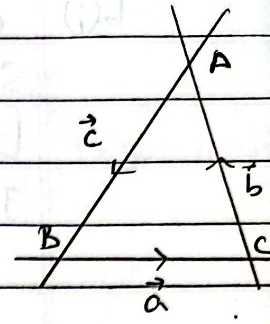
9. Prove vectorically that;

a(i) $b = c\cos A + a\cos C$



Solution:

Let $\vec{AB} = \vec{c}$, $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$

In any triangle,

$$\vec{AB} + \vec{BC} + \vec{CA} = 0$$

or, $\vec{CA} = -\vec{BC} - \vec{AB}$

or, $\vec{b} = -\vec{a} - \vec{c}$

Multiplying both sides by $\vec{b}$

$$\vec{b}.\vec{b} = -\vec{a}.\vec{b} - \vec{b}\vec{c}$$

or, $b^2 = -ab\cos(\pi - C) - bc\cos(\pi - A)$

or, $b^2 = +ab\cos C + bc\cos A$

$\therefore$ $b = a\cos C + c\cos A$

Proved

ii. $C = a\cos B + b\cos A$

Solution:

Let $\vec{AB} = \vec{c}$, $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$

In any triangle;

$$\vec{AB} + \vec{BC} + \vec{CA} = 0$$

or $\vec{AB} = -\vec{BC} - \vec{CA}$

or $\vec{c} = -\vec{a} - \vec{b}$

or $\vec{c}.\vec{c} = -\vec{a}.\vec{c} - \vec{b}.\vec{c}$

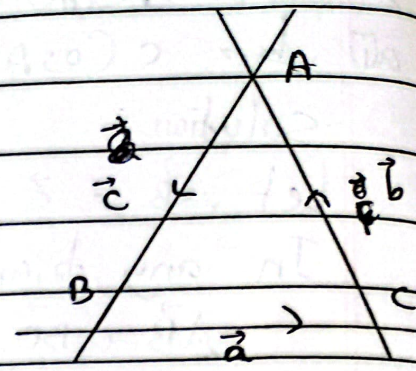
or $c^2 = -ac\cos(\pi - B) - bc\cos(\cos\pi - A)$

or $c^2 = +ac\cos B + bc\cos A$

$\therefore$ $c = a\cos B + b\cos A$

Proved

b(i) $a^2 = b^2 + c^2 - 2bc\cos A$



Solution:

Let $\vec{BC} = \vec{a}$, $\vec{AB} = \vec{c}$ and $\vec{CA} = \vec{b}$

In any triangle,

$\vec{AB} + \vec{BC} + \vec{CA} = 0$

or $\vec{BC} = -\vec{AB} - \vec{CA}$

or $\vec{a} = -\vec{c} - \vec{b}$

s.b.s

$a^2 = \{-(\vec{c} + \vec{b})\}^2$

or $a^2 = c^2 + 2\vec{b}\vec{c} + b^2$

or $a^2 = c^2 + 2bc\cos(\pi - A) + b^2$

or, $a^2 = c^2 - \cancel{\cos A}\ 2bc\cos A + b^2$

or $a^2 = b^2 + c^2 - 2bc\cos A$

Proved

ii. $b^2 = c^2 + a^2 - 2ca\cos B$

Solution:

Let $\vec{AB} = \vec{c}$, $\vec{BC} = \vec{a}$ and $\vec{CA} = \vec{b}$

In any triangles;

$\vec{AB} + \vec{BC} + \vec{CA} = 0$

or $\vec{CA} = -\vec{AB} - \vec{BC}$

or $\vec{b} = -\vec{c} - \vec{a}$

s.b.s

$b^2 = \{-(\vec{c} + \vec{a})\}^2$

or $b^2 = c^2 + 2\vec{c}\vec{a} + a^2$

or $b^2 = c^2 - 2ac\cos B + a^2$

or $b^2 = c^2 + a^2 - 2ca\cos B$

Proved

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c. $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

Solution:

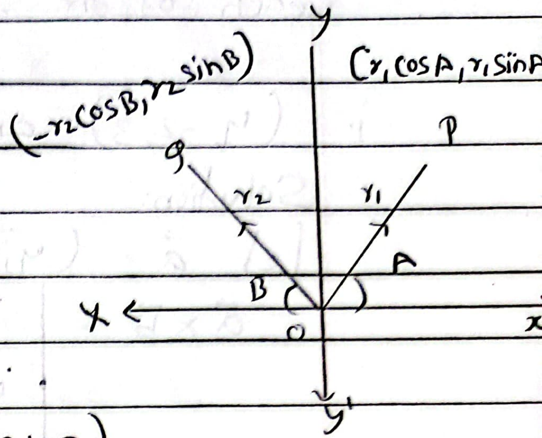
Let XOX' and ~~XO~~ YOY' be the axes.

$OP = r_1$ and $OQ = r_2$

Coordinates of P and Q are;

$\vec{OP} = (r_1 \cos A, r_1 \sin A)$

$\vec{OQ} = (-r_2 \cos B, r_2 \sin B)$



Now,

$\vec{OP} \cdot \vec{OQ} = (r_1 \cos A, r_1 \sin A)(-r_2 \cos B, r_2 \sin B)$

$= -r_1 . r_2 \cos A \cdot \cos B + r_1 \cdot r_2 \sin A \cdot \sin B$

$= -r_1 . r_2 (\cos A \cdot \cos B - \sin A \cdot \sin B)$ ——— (i)

Again,

$\vec{OP} \cdot \vec{OQ} = OP . OQ \cos\theta$

$= r_1 . r_2 \cos[\pi - (A+B)]$ ——— (ii)

from (i) and (ii)

$r_1 . r_2 \cos[\pi - (A+B)] = -r_1 r_2 [\cos A \cdot \cos B - \sin A \cdot \sin B]$

or, $-\cos(A+B) = -[\cos A \cdot \cos B - \sin A \cdot \sin B]$

$\therefore \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

Proved

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