Chapter 17

# Linear Programming

## Exercise 17.1

1. **Reformulate the following LP problem into standard form.**

   a) **Maximize P = 20x + 30y s.t. $3x + y \le 15$, $x + 3y \le 12$, $x, y \ge 0$**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$$3x + y + r.1 + s.0 + p.0 = 15$$
$$x + 3y + r.0 + s.1 + p.0 = 12$$
$$-20x - 30y + r.0 + s.0 + p.1 = 0$$

   b) **Maximize F = 7x + 5y s.t. $4x + 3y \le 48$, $2x + y \le 20$, $x, y \ge 0$**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$$4x + 3y + r.1 + s.0 + p.0 = 48$$
$$2x + y + r.0 + s.1 + p.0 = 12$$
$$-7x - 5y + r.0 + s.0 + p.1 = 0$$

2. **Using simplex method, find the optimal solutions of the following LP problems.**

   a) **Maximize Z = 2x + y s.t. $x + 2y \le 10$, $x + y \le 6$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$$x + 2y + r.1 + s.0 + z.0 = 10$$
$$x + y + r.0 + s.1 + z.0 = 6$$
$$-2x - y + r.0 + s.0 + z.1 = 0$$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 1 | 2 | 1 | 0 | 0 | 10 | 10/1 = 10 |
| s | $\boxed{1}$ | 1 | 0 | 1 | 0 | 6 | 6/1 = 6 |
| | – 2 | – 1 | 0 | 0 | 1 | 0 | 6 < 10 |

Here, the last row contain negative entry. So, the solution Z = 0 is not optimal and – 2 is the most negative entry. So, the first column is pivot column and first row is pivot row. So, 1 is pivot element.

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 2R_2$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 0 | 1 | 1 | -1 | 0 | 4 | |
| s | 1 | 1 | 0 | 1 | 0 | 6 | |
| | 0 | 1 | 0 | 2 | 1 | 12 | |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. Z = 12 when x = 6, y = 0 such that

$Z = 2x + y \Rightarrow 12 = 2 \times 6 + 0 \qquad \Rightarrow 12 = 12$(true)

**b) Maximize Z = x + 3y s.t. x + y ≤ 5, 3x + y ≤ 15,**

**Sol$^n$:** Let r and s be non-negative slack variable, then the standard form of given LPP is

$$x + y + r.1 + s.0 + z.0 = 5$$
$$3x + y + r.0 + s.1 + z.0 = 15$$
$$-x - 3y + r.0 + s.0 + z.1 = 0$$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 1 | $\boxed{1}$ | 1 | 0 | 0 | 5 | 5/1 = 5 |
| s | 3 | 1 | 0 | 1 | 0 | 15 | 15/1 = 15 |
| | – 1 | – 3 | 0 | 0 | 1 | 0 | 6 < 10 |

Here, the last row contain negative entry. So, Z = 0 is not the optimal solution and – 3 is the most negative entry. So, the first column is pivot column and first row is pivot row. So, 1 is pivot element.

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 3R_1$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | 0 | 0 | 5 | ….. |
| s | 2 | 0 | -1 | 1 | 0 | 10 | ….. |
| | 2 | 0 | 3 | 0 | 1 | 15 | |

Since all entries in the last row are non-negative, so the solution is optimal.

∴ max. Z = 15 when x = 0, y = 5 such that

$Z = x + 3y$ $\Rightarrow 15 = 0 + 3 \times 5$ $\Rightarrow 15 = 15$(true)

**c) Maximize F = 5x + 12y s.t. 3x + y ≤ 12, x + 2y ≤ 12,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$$3x + y + r.1 + s.0 + F.0 = 12$$
$$x + 2y + r.0 + s.1 + F.0 = 12$$
$$-5x - 12y + r.0 + s.0 + F.1 = 0$$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 3 | 1 | 1 | 0 | 0 | 12 | 12/1 = 12 |
| s | 1 | $\boxed{2}$ | 0 | 1 | 0 | 12 | 12/2 = 6 |
| | – 5 | – 12 | 0 | 0 | 1 | 0 | 6 < 12 |

Here, the last row contain negative entry. So, F = 0 is not the optimal solution. In the last row – 12 is the most negative entry. So, the second column is pivot column and first row is pivot row. So, 2 is pivot element.

Applying $R_2 \to \frac{R_2}{2}$, we get

| B.V. | x | y | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 3 | 1 | 1 | 0 | 0 | 12 | …. |
| s | 1/2 | 1 | 0 | 1/2 | 0 | 6 | …. |
| | – 5 | – 12 | 0 | 0 | 1 | 0 | |

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 12R_2$, we get

| B.V. | x | y | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 5/2 | 0 | 1 | -1/2 | 0 | 6 | 12 |
| s | 1/2 | 1 | 0 | 1/2 | 0 | 6 | 6 |
| | 1 | 0 | 0 | 6 | 1 | 72 | |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. F = 72 when x = 0, y = 6 such that

F = 5x + 12y $\Rightarrow 72 = 0 + 12\times6 \Rightarrow 72 = 72$ (true)

**d) Max. $Z = 4x - 6y$ s.t. $2x - 3y \le 8$, $x + y \le 24$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$2x - 3y + r.1 + s.0 + Z.0 = 8$

$x + y + r.0 + s.1 + Z.0 = 24$

$-4x - 6y + r.0 + s.0 + Z.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | $\boxed{2}$ | -3 | 1 | 0 | 0 | 8 | 8/2 = 4 |
| s | 1 | 1 | 0 | 1 | 0 | 24 | 24/1 = 24 |
| | – 4 | 6 | 0 | 0 | 1 | 0 | 4 < 24 |

Here, the last row contain negative entry. So, Z = 0 is not the optimal solution and – 4 is the most negative entry. So, the first column is pivot column and first row is pivot row hence 2 is pivot element.

Applying $R_1 \to \frac{R_1}{2}$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | -3/2 | 1/2 | 0 | 0 | 4 | ….. |
| s | 1 | 1 | 0 | 1 | 0 | 24 | ….. |
| | – 4 | 6 | 0 | 0 | 1 | 0 | |

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 4R_1$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | -3/2 | 1/2 | 0 | 0 | 8 | ….. |
| s | 0 | 5/2 | -1/2 | 1 | 0 | 20 | ….. |
| | 0 | 0 | 2 | 0 | 1 | 16 | |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. Z = 16 when x = 4, y = 0 such that

Z = 4x - 6y $\Rightarrow 16 = 4\times4 - 0 \Rightarrow 16 = 16$ (true)

**e) Maximize $U = 25x + 45y$ s.t. $x + 3y \le 21$, $2x + 3y \le 24$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$x + 3y + r.1 + s.0 + U.0 = 21$

$2x + 3y + r.0 + s.1 + U.0 = 24$

$-25x - 45y + r.0 + s.0 + U.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | U | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 1 | $\boxed{3}$ | 1 | 0 | 0 | 21 | 21/3 = 7 |
| s | 2 | 3 | 0 | 1 | 0 | 24 | 24/3 = 8 |
| | – 25 | –45 | 0 | 0 | 1 | 0 | 7 < 8 |

Here, the last row contain negative entry. So, U = 0 is not the optimal solution. Since – 4 is the most negative entry, the second column is pivot column and first row is pivot row hence 3 is pivot element.

Applying $R_1 \to \frac{R_1}{3}$, we get

| B.V. | x | y | r | s | U | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 1/3 | 1 | 1/3 | 0 | 0 | 7 | …….. |
| s | 2 | 3 | 0 | 1 | 0 | 24 | ……. |
| | – 25 | –45 | 0 | 0 | 1 | 0 | …… |

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 + 45R_1$, we get

| B.V. | x | y | r | s | U | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 1/3 | 1 | 1/3 | 0 | 0 | 7 | 7/1/3 = 21 |
| s | $\boxed{1}$ | 0 | –1 | 1 | 0 | 3 | 3/1 = 3 |
| | – 10 | 0 | 15 | 0 | 1 | 315 | 3 < 21 |

Here, the last row still contains negative value – 10. So, the first column is pivot column and second row is pivot row hence 1 is pivot element.

Applying $R_1 \to R_1 - 1/3R_2$ and $R_3 \to R_3 + 10R_2$, we get

| B.V. | x | y | r | s | U | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 0 | 1 | 2/3 | –1/3 | 0 | 6 | ………. |
| x | 1 | 0 | -1 | 1 | 0 | 3 | ………. |
| | 0 | 0 | 5 | 10 | 1 | 345 | ……. |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. U = 345 when x = 3, y = 6 such that

U = 25x + 45y $\Rightarrow$ 345 = 25×3 +45×6

$\Rightarrow$ 345 = 345 (true)

**f) Maximize P = 8x + 10y s.t. x + 2y ≤ 30, 2x + 2y ≤ 40,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$x + 2y + r.1 + s.0 + p.0 = 30$

$2x + 2y + r.0 + s.1 + p.0 = 40$

$-8x - 10y + r.0 + s.0 + p.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | P | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 1 | $\boxed{2}$ | 1 | 0 | 0 | 30 | 30/2 = 15 |
| s | 2 | 2 | 0 | 1 | 0 | 40 | 40/2 = 20 |
| | – 8 | –10 | 0 | 0 | 1 | 0 | 15 < 20 |

Here, the last row contain negative entry. So, P = 0 is not the optimal solution. Since – 10 is the most negative entry, the second column is pivot column and first row is pivot row hence 2 is pivot element.

Applying $R_1 \to \frac{R_1}{2}$, we get

| B.V. | x | y | r | s | P | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 1/2 | 1 | 1/2 | 0 | 0 | 15 | ……… |
| s | 2 | 2 | 0 | 1 | 0 | 40 | ……… |
| | – 8 | –10 | 0 | 0 | 1 | 0 | …… |

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 10R_1$, we get

| B.V. | x | y | r | s | P | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 1/2 | 1 | 1/2 | 0 | 0 | 15 | 15/1/2 = 30 |
| s | $\boxed{1}$ | 0 | –1 | 1 | 0 | 10 | 10/1 = 10 |
| | – 3 | 0 | 5 | 0 | 1 | 150 | 10 < 30 |

Here, the last row still contains negative value – 3. So, the first column is pivot column and second row is pivot row hence 1 is pivot element.

Applying $R_1 \to R_1 - 1/2R_2$ and $R_3 \to R_3 + 3R_2$, we get

| B.V. | x | y | r | s | P | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| y | 0 | 1 | 1 | –1/2 | 0 | 10 | ……… |
| s | 1 | 0 | -1 | 1 | 0 | 10 | ……… |
| | 0 | 0 | 2 | 3 | 1 | 180 | ……. |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. P = 180 when x = 10, y = 10 such that

P = 8x + 10y $\Rightarrow$ 180 = 8×10 +10×10 $\Rightarrow$ 180 = 180 (true)

**g) Maximize $F = 5x_1 + 3x_2$ s.t. $2x_1 + x_2 \le 40$, $x_1 + 2x_2 \le 50$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$2x_1 + x_2 + r.1 + s.0 + F.0 = 40$

$x_1 + 2x_2 + r.0 + s.1 + F.0 = 50$

$-5x_1 - 3x_2 + r.0 + s.0 + F.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | $x_1$ | $x_2$ | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| R | $\boxed{2}$ | 1 | 1 | 0 | 0 | 40 | 40/2 = 20 |
| S | 1 | 2 | 0 | 1 | 0 | 50 | 50/1 = 50 |
| | – 5 | –3 | 0 | 0 | 1 | 0 | 20 < 50 |

Here, the last row contain negative entry. So, F = 0 is not the optimal solution. Since – 5 is the most negative entry in the last row, the first column is pivot column and first row is pivot row hence 2 is pivot element.

Applying $R_1 \rightarrow \frac{R_1}{2}$ , we get

| B.V. | $x_1$ | $x_2$ | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1/2 | 1/2 | 0 | 0 | 20 | ……… |
| S | 1 | 2 | 0 | 1 | 0 | 50 | ……… |
| | – 5 | –3 | 0 | 0 | 1 | 0 | …… |

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 5R_1$, we get

| B.V. | $x_1$ | $x_2$ | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1/2 | 1/2 | 0 | 0 | 20 | 20/1/2 = 40 |
| S | 0 | 3/2 | -1/2 | 1 | 0 | 30 | 30/3/2 = 20 |
| | 0 | -1/2 | 5/2 | 0 | 1 | 100 | 20 < 40 |

Here, the last row still contains negative value – 1/2. So, the second column is pivot column and second row is pivot row hence 3/2 is pivot element.

Applying $R_2 \rightarrow R_2 \times 2/3$, we get

| B.V. | $x_1$ | $x_2$ | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1/2 | 1/2 | 0 | 0 | 20 | ……… . |
| $x_2$ | 0 | 1 | -1/3 | 2/3 | 0 | 20 | ……… . |
| | 0 | -1/2 | 5/2 | 0 | 1 | 100 | …… . |

Applying $R_1 \rightarrow R_1 - 1/2\times R_1$ and $R_3 \rightarrow R_3 + 1/2\times R_2$, we get

| B.V. | $x_1$ | $x_2$ | r | s | F | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 2/3 | -1/3 | 0 | 10 | ……… . |
| $x_2$ | 0 | 1 | -1/3 | 2/3 | 0 | 20 | ……… . |
| | 0 | 0 | 7/3 | 1/3 | 1 | 110 | …… . |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. F = 110 when $x_1 = 10$, $x_2 = 20$ such that

$F = 5x_1+3x_2 \quad \Rightarrow 110 = 5\times10 + 3\times20 \quad \Rightarrow 110 = 110$ (true)

**h) Maximize C = 7x + 5y s.t. 4x + 3y ≤ 48, 2x + y ≤ 20,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$4x + 3y + r.1 + s.0 + C.0 = 48$

$2x + y + r.0 + s.1 + C.0 = 20$

$-7x - 5y + r.0 + s.0 + C.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | C | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 4 | 3 | 1 | 0 | 0 | 48 | 48/4 = 12 |
| s | $\boxed{2}$ | 1 | 0 | 1 | 0 | 20 | 20/2 = 10 |
| | – 7 | –5 | 0 | 0 | 1 | 0 | 10 < 12 |

Here, the last row contain negative entry. So, C = 0 is not the optimal solution. Since – 7 is the most negative entry in the last row, the first column is pivot column and second row is pivot row hence 2 is pivot element.

Applying $R_2 \rightarrow \frac{R_2}{2}$, we get

| B.V. | x | y | r | s | C | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 4 | 3 | 1 | 0 | 0 | 48 | ……… |
| y | 1 | 1/2 | 0 | 1/2 | 0 | 10 | ……… |
| | – 7 | –5 | 0 | 0 | 1 | 0 | …… |

Applying $R_1 \to R_1 - 4R_2$ and $R_3 \to R_3 + 7R_2$, we get

| B.V. | x | y | r | s | C | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 0 | $\boxed{1}$ | 1 | -2 | 0 | 8 | 8/1 = 8 |
| y | 1 | 1/2 | 0 | 1/2 | 0 | 10 | 10/1/2 = 20 |
| | 0 | -3/2 | 0 | 7/2 | 1 | 70 | 8 < 20 |

Here, the last row still contains negative value-3/2. So, the second column is pivot column and first row is the pivot row hence 1 is pivot element.

Applying $R_2 \to R_2 - 1/2R_1$ and $R_3 \to R_3 + 3/2R_1$, we get

| B.V. | x | y | r | s | C | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 1 | –2 | 0 | 8 | ………. |
| y | 1 | 0 | -1/2 | 3/2 | 0 | 6 | ………. |
| | 0 | 0 | 3/2 | 1/2 | 1 | 82 | ……. |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. C = 82 when x = 6, y = 8 such that

$C = 7x+5y \quad \Rightarrow 82 = 7\times6 + 5\times8 \quad \Rightarrow 82 = 82$ (true)

**i) Maximize $Z = 5x + 5y$ s.t. $2x + y \le 20$, $2x + 3y \le 24$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$2x + y + r.1 + s.0 + z.0 = 20$

$2x + 3y + r.0 + s.1 + z.0 = 24$

$-5x - 5y + r.0 + s.0 + z.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | $\boxed{2}$ | 1 | 1 | 0 | 0 | 20 | 20/2 = 10 |
| s | 2 | 3 | 0 | 1 | 0 | 24 | 24/2 = 12 |
| | – 5 | – 5 | 0 | 0 | 1 | 0 | 10 < 12 |

Here, the last row contain negative entry. So, Z = 0 is not the optimal solution. Since, – 5 is the most negative entry, the first column is pivot column and first row is pivot row hence 2 is pivot element.

Applying $R_1 \to \frac{R_1}{2}$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 1/2 | 1/2 | 0 | 0 | 10 | ……… |
| s | 2 | 3 | 0 | 1 | 0 | 24 | ……… |
| | – 5 | – 5 | 0 | 0 | 1 | 0 | …… |

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 5R_1$, we get

| B.V. | x | y | r | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 1/2 | 1/2 | 0 | 0 | 10 | 10/1/2 = 20 |
| s | 0 | $\boxed{2}$ | -1/2 | 1 | 0 | 4 | 4/2 = 2 |
| | 0 | -5/2 | 5/2 | 0 | 1 | 50 | 2 < 20 |

Here, the last row still contains negative value – 5/2. So, the second column is pivot column and second row is the pivot row hence 2 is pivot element.

Applying $R_2 \rightarrow \frac{R_2}{2}$, we get

| B.V. | x | y | R | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 1/2 | ½ | 0 | 0 | 10 | ……… |
| y | 0 | 1 | -1/4 | 1/2 | 0 | 2 | ……… |
| | 0 | -5/2 | 5/2 | 0 | 1 | 50 | …… |

Again, applying $R_1 \rightarrow R_1 - 1/2R_2$ and $R_3 \rightarrow R_3 + 5/2R_2$, we get

| B.V. | x | y | R | s | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 0 | -3/4 | -1/4 | 0 | 9 | ………. |
| y | 0 | 1 | -1/2 | 1/2 | 0 | 2 | ………. |
| | 0 | 0 | 5/4 | 5/4 | 1 | 55 | ……. |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ Max. Z = 55 when x = 9, y = 2 such that

$Z = 5x + 5y \Rightarrow 55 = 5\times 9 + 5\times 2$ $\Rightarrow 55 = 55$ (true)

**j) Max $Z = 5x_1 + 7x_2$ subject to $2x_1 + 3x_2 \leq 13$, $3x_1 + 2x_2 \leq 12$, $x_1, x_2 \geq 0$**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$2x_1 + 3x_2 + r + 0.s + 0.z = 13$

$3x_1 + 2x_2 + 0.r + 0.s + 0.z = 12$

$-5x_1 - 7x_2 + 0.r + 0.s + 1.z = 0$

The initial simplex tableau is given below:

| B.V. | $x_1$ | $x_2$ | r | S | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 2 | 3 | 1 | 0 | 0 | 13 | 13/3 = 4.33 |
| s | 3 | 2 | 0 | 1 | 0 | 12 | 12/2 = 6 |
| | -5 | -7 | 0 | 0 | 1 | 0 | 4.33 < 6 |

Here, the last row contain negative entry. So, Z = 0 is not the optimal solution. Since, – 7 is the most negative entry, the second column is pivot column and first row is pivot row hence 3 is pivot element.

Applying $R_2 \rightarrow \frac{R_2}{3}$, we get

| B.V. | $x_1$ | $x_2$ | r | S | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_2$ | 2/3 | 1 | 1/3 | 0 | 0 | 13/3 | ……… |
| s | 3 | 2 | 0 | 1 | 0 | 12 | ……… |
| | -5 | -7 | 0 | 0 | 1 | 0 | ……… |

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + 7R_1$, we get

| B.V. | $x_1$ | $x_2$ | r | S | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_2$ | $\frac{2}{3}$ | 1 | $\frac{1}{3}$ | 0 | 0 | $\frac{13}{3}$ | $\frac{13/3}{2/3} = 6.5$ |
| s | $\frac{5}{3}$ | 0 | $\frac{2}{3}$ | 1 | 0 | $\frac{10}{3}$ | $\frac{10/3}{5/3} = 2$ |
| | $-\frac{1}{3}$ | 0 | $\frac{7}{3}$ | 0 | 1 | $\frac{91}{3}$ | 2 < 6.5 |

Here, the last row still contains negative value – 1/3. So, the first column is pivot column and second row is the pivot row hence5/3 is pivot element.

Applying $R_1 \rightarrow \frac{3}{5} \times R_1$, we get

| B.V. | $x_1$ | $x_2$ | r | S | Z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_2$ | $\frac{2}{3}$ | 1 | $\frac{1}{3}$ | 0 | 0 | $\frac{13}{3}$ | ……. |
| $x_1$ | 1 | 0 | $-\frac{2}{5}$ | $\frac{3}{5}$ | 0 | 2 | ……… |
| | $\frac{-1}{3}$ | 0 | $\frac{7}{3}$ | 0 | 1 | $\frac{91}{3}$ | ……… |

Applying $R_1 \rightarrow R_1 - \frac{2}{3} R_2$ and $R_3 \rightarrow R_3 + \frac{1}{3} R_2$, we get

| B.V. | $x_1$ | $x_2$ | r | s | Z | RHS |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 3/5 | - 2/5 | 0 | 3 |
| $x_1$ | 1 | 0 | - 2/5 | 3/5 | 0 | 2 |
| | 0 | 0 | 11/5 | 1/5 | 1 | 31 |

Since all entries in the last row are non – negative, the solution of the given LP problem is optimal.

$\therefore$ Max. Z = 31 when $x_1 = 2$ and $x_2 = 3$ such that

$Z = 5x_1 + 7x_2$ $\Rightarrow 31 = 5 \times 2 + 7 \times 3$ $\Rightarrow 31 = 31$ (true)

**k) Maximize g = 15x + 12y s.t. 2x + 3y ≤ 21, 3x + 2y ≤ 24,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$2x + 3y + r.1 + s.0 + g.0 = 20$

$2x + 3y + r.0 + s.1 + g.0 = 24$

$-15x - 12y + r.0 + s.0 + g.1 = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is,

| B.V. | x | y | r | s | g | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 2 | 3 | 1 | 0 | 0 | 21 | 21/2 = 10.5 |
| s | 3 | 2 | 0 | 1 | 0 | 24 | 24/3 = 8 |
| | – 15 | – 12 | 0 | 0 | 1 | 0 | 8 < 10.5 |

Here, the last row contain negative entry. So, g = 0 is not the optimal solution. Since – 15 is the most negative entry in the last row, the first column is pivot column and second row is pivot row hence 2 is pivot element.

Applying $R_2 \rightarrow \frac{R_2}{3}$, we get

| B.V. | x | y | r | s | g | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 2 | 3 | 1 | 0 | 0 | 21 | ……. |
| x | 1 | 2/3 | 0 | 1/3 | 0 | 8 | ……. |
| | – 15 | – 12 | 0 | 0 | 1 | 0 | …… |

Applying $R_1 \rightarrow R_1 - 2R_2$ and $R_3 \rightarrow R_3 + 15R_2$, we get

| B.V. | x | y | r | s | g | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 0 | 5/3 | 1 | -2/3 | 0 | 5 | 5/5/3 = 3 |
| x | 1 | 2/3 | 0 | 1/3 | 0 | 8 | 8/2/3 = 12 |
| | 0 | – 2 | 0 | 5 | 1 | 120 | 3 < 12 |

Here, the last row still contains negative value – 2. So, the second column is pivot column and first row is the pivot row hence 5/3 is pivot element.

Applying $R_1 \to \frac{3}{5} \times R_1$, we get

| **B.V.** | **X** | **y** | **R** | **s** | **g** | **RHS** | **Ratio** |
|---|---|---|---|---|---|---|---|
| **y** | 0 | 1 | 3/5 | -2/5 | 0 | 3 | ……… |
| **x** | 1 | 2/3 | 0 | 1/3 | 0 | 8 | ……… |
| | 0 | – 2 | 0 | 5 | 1 | 120 | …… |

Applying $R_2 \to R_2 - 2/3R_1$ and $R_3 \to R_3 + 2R_1$, we get

| **B.V.** | **g** | **y** | **R** | **s** | **g** | **RHS** | **Ratio** |
|---|---|---|---|---|---|---|---|
| **y** | 0 | 1 | 3/5 | -2/5 | 0 | 3 | ………. |
| **x** | 1 | 0 | 2/5 | 3/5 | 0 | 6 | ………. |
| | 0 | 0 | 6/5 | 21/5 | 1 | 126 | ……. |

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ Max. g = 126 when x = 6, y = 3 such that

$g = 15x + 12y \quad \Rightarrow 126 = 15 \times 6 + 12 \times 3 \quad \Rightarrow 126 = 126$ (true)

**l) Maximize $z = 10x_1 + 12x_2$ s.t. $3x_1 + x_2 \le 12$, $x_1 + 2x_2 \le 14$,**

**Sol$^n$:** Let r and s be non-negative slack variables, then the standard form of given LPP is

$$3x_1 + x_2 + 1.r + 0.s + 0.z = 12$$
$$x_1 + 2x_2 + 0.r + 0.s + 0.z = 14$$
$$-10x_1 - 12x_2 + 0.r + 0.s + 1.z = 0$$

The initial simplex tableau is given below:

| B.V. | $x_1$ | $x_2$ | R | s | z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 3 | 1 | 1 | 0 | 0 | 12 | 12/1 = 12 |
| s | 1 | 2 | 0 | 1 | 0 | 14 | 14/2 = 7 |
| | – 10 | – 12 | 0 | 0 | 1 | 0 | 7 < 12 |

Here, the last row contain negative entry. So, z = 0 is not the optimal solution. Since – 12 is the most negative entry in the last row, the second column is pivot column and second row is pivot row hence 2 is pivot element.

Applying $R_2 \to \frac{R_2}{2}$, we get

| B.V. | $x_1$ | $x_2$ | R | s | z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 3 | 1 | 1 | 0 | 0 | 12 | ……… |
| $x_2$ | 1/2 | 1 | 0 | 1/2 | 0 | 7 | ……… |
| | – 10 | – 12 | 0 | 0 | 1 | 0 | ……… |

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 12R_2$, we get

| B.V. | $x_1$ | $x_2$ | R | s | z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| r | 5/2 | 0 | 1 | –1/2 | 0 | 5 | $\frac{5}{5/2} = 2$ |
| $x_2$ | 1/2 | 1 | 0 | 1/2 | 0 | 7 | $\frac{7}{1/2} = 14$ |
| | – 4 | 0 | 0 | 6 | 1 | 84 | 2 < 14 |

Here, the last row still contains negative value – 4. So, the first column is pivot column and first row is the pivot row hence 5/2 is pivot element.

Applying $R_2 \to \frac{2}{5} \times R_2$, we get

| B.V. | $x_1$ | $x_2$ | R | S | z | RHS | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 2/5 | –1/5 | 0 | 2 | ……… |
| $x_2$ | 1/2 | 1 | 0 | 1/2 | 0 | 7 | ………… |
| | – 4 | 0 | 0 | 6 | 1 | 84 | ……… |

Now, perform the operation $R_2 \to R_2 - 1/2\ R_1$ and $R_3 \to R_3 + 4R_1$,

| B.V. | $x_1$ | $x_2$ | R | s | P | RHS |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 2/5 | −1/5 | 0 | 2 |
| $x_2$ | 0 | 1 | -1/5 | 3/5 | 0 | 6 |
| | 0 | 0 | 8/5 | 26/5 | 1 | 92 |

Since all entries in the last row are non-negative, so the solution is optimal.

∴ Max. $z = 92$ when $x_1 = 2$, $x_2 = 6$ such that

$z = 10x_1 + 12x_2 \quad \Rightarrow 92 = 10\times2 + 12\times6 \quad \Rightarrow 92 = 92$ (true)

**3. Find the dual problem corresponding to each of the following linear programming problems**

**a) Maximize $P = 8x + 36y$ s.t**
**$2x + 6y \le 18$**
**$x + 6y \le 12$**
**$x, y \ge 0$**

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 6 & 18 \\ 1 & 6 & 12 \\ \hline 8 & 36 & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{constraints} \\ \leftarrow \text{objective function} \end{array}$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 6 & 36 \\ \hline 18 & 12 & 0 \end{array}\right)$$

∴ The dual problem of given LPP is minimize $P^* = 18u + 12v$ s.t.
$2u + v \ge 8$
$6u + 6v \ge 36 \Rightarrow u + v \ge 6$
$u, v \ge 0$.

**b) Maximize $Z = 30x + 20y$ s.t.**
**$3x + y \le 15$**
**$x + 3y \le 12$**
**$x, y \ge 0$**

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 3 & 1 & 15 \\ 1 & 3 & 12 \\ \hline 30 & 20 & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{constraints} \\ \leftarrow \text{objective function} \end{array}$$

$$A^T = \left(\begin{array}{cc|c} 3 & 1 & 30 \\ 1 & 3 & 20 \\ \hline 15 & 12 & 0 \end{array}\right)$$

∴ The dual problem of given LPP is
Minimize $Z^* = 15u + 12v$ s.t
$3u + v \ge 30$
$u + 3v \ge 20$
$u, v \ge 0$

**c) Minimize W = 12x + 18y s.t.**

**$x + 2y \geq 8$**

**$4x + 4y \geq 24$**

**$x, y \geq 0$**

The augmented matrix of given LP problem is

$$A = \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 4 & 4 & 24 \\ \hline 12 & 18 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 4 & 12 \\ 2 & 4 & 18 \\ \hline 8 & 24 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $W^* = 8u + 24v$ s.t.

$u + 4v \leq 12$

$2u + 4v \leq 18$

$u, v \geq 0$

**d) Minimize P = 32x + 84y s.t.**

**$x + 3y \geq 50$**

**$2x + 4y \geq 80$**

**$x, y \geq 0$**

The augmented matrix of given LPP is

$$A = \left(\begin{array}{cc|c} 1 & 3 & 50 \\ 2 & 4 & 80 \\ \hline 32 & 84 & 0 \end{array}\right) \qquad A^T = \left(\begin{array}{cc|c} 1 & 2 & 32 \\ 3 & 4 & 84 \\ \hline 50 & 80 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $P^* = 50u + 80v$ s.t

$u + 2v \leq 32$

$3u + 4v \leq 84$

$u, v \geq 0$

**4. Find the optimal solution of the following LP problems**

**a) Minimize $Z = 20x + 12y$ s.t**

**$2x + 2y \geq 20$**

**$5x + y \geq 5$**

**$x, y \geq 0$**

The augmented matrix of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 2 & 20 \\ 5 & 1 & 5 \\ \hline 20 & 12 & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \leftarrow \\ \leftarrow \end{array}\right] \text{constraints} \\ \leftarrow \text{objective function} \end{array}$$

$$A^T = \left(\begin{array}{cc|c} 2 & 5 & 20 \\ 2 & 1 & 12 \\ \hline 20 & 5 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $Z^* = 20u + 5v$ s.t.

$2u + 5v \le 20$

$2u + v \le 12$

$u, v \ge 0$

Introducing the non-negative slack variables x and y, the LP problem is

$2u + 5v + x = 20$

$2u + v + y = 12$

$20u + 5v = Z^*$

i.e. $2u + 5v + x + 0.y + 0.Z^* = 20$

$2u + v + 0.x + y + 0.Z^* = 12$

$-20u - 5v + 0.x + 0.y + Z^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | x | y | Z* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 2 | 5 | 1 | 0 | 0 | 20 | $\frac{20}{2}$ |
| y | $\boxed{2}$ | 1 | 0 | 1 | 0 | 12 | $\frac{12}{2}$ |
| | –20 | –5 | 0 | 0 | 1 | 0 | |

The most negative entry is last row is – 20 ⇒ u-column is pivot column and the least ratio dividing R.H.S. by pivot column elements is $\frac{12}{2} = 6$ ⇒ row second is pivot row. Hence 2 is the pivot entry.

$R_2 : \frac{1}{2} R_2$

| Basic variables | u | v | x | y | Z* | R.H.S. |
|---|---|---|---|---|---|---|
| x | 2 | 5 | 1 | 0 | 0 | 20 |
| u | $\boxed{1}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 6 |
| | –20 | –5 | 0 | 0 | 1 | 0 |

$R_1 : R_1 - 2R_2$, $R_3 : R_3 + 20 R_2$

| Basic variables | u | v | x | y | z* | R.H.S. |
|---|---|---|---|---|---|---|
| x | 0 | 4 | 1 | –1 | 0 | 8 |
| u | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 6 |
| | 0 | 5 | 0 | 10 | 1 | 120 |

Since all entries in the last row are non-negative, the solution is optimal

∴ Max. $Z^* = 120$ at $u = 6$, $v = 0$ ⇒ Min. $Z = 120$ when $x = 0$, $y = 10$

**b) Minimize $Z = 24x + 6y$ s.t.**

**$3x + y \ge 40$**

**$4x + 2y \ge 25$**

**$x, y \ge 0$**

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 3 & 1 & 40 \\ 4 & 2 & 25 \\ \hline 24 & 6 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 3 & 4 & 24 \\ 1 & 2 & 6 \\ \hline 40 & 25 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $Z^* = 40u + 25v$ s.t.

$3u + 4v \leq 24$

$u + 2v \leq 6$

$u, v \geq 0$

Introducing the non-negative slack variables x and y, the LP problem is

$3u + 4v + x = 24$
$u + 2v + y = 6$
$40u + 25v = Z^*$

i.e. $3u + 4v + x + 0.y + 0Z^* = 24$
$u + 2v + 0.x + y + 0.Z^* = 6$
$-40u - 25v + 0.x + 0.y + Z^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | x | y | Z* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 3 | 4 | 1 | 0 | 0 | 24 | $24/3$ |
| y | 1 | 2 | 0 | 1 | 0 | 6 | $6/1$ |
| | –40 | –25 | 0 | 0 | 1 | 0 | |

The most negative entry is last row is – 40⇒ u-column is pivot column and the least ratio dividing R.H.S. by pivot column element is $\frac{6}{1}$ ⇒ row second is pivot row. Hence 1 is the pivot entry.

$R_1: R_1 - 3R_2$, $R_3: R_3 + 40\,R_2$

| Basic variables | u | v | x | y | Z* | R.H.S. |
|---|---|---|---|---|---|---|
| x | 0 | - 2 | 1 | –3 | 0 | 6 |
| u | 1 | 2 | 0 | 1 | 0 | 6 |
| | 0 | 55 | 0 | 40 | 1 | 240 |

Since all entries in the last row are non-negative, the solution is optimal

∴ Max. $Z^* = 240$ at $u = 6, v = 0$ ⇒ Min. $Z = 240$ when $x = 0, y = 40$

**c) Minimize** $\mathbf{P = 6x + 20y}$ **s.t.**

$\mathbf{2x + y \geq 6}$

$\mathbf{-3x + y \leq -9}$

$\mathbf{x, y \geq 0}$

The given problem can be rewrite as

Minimize $P = 6x + 20y$ s.t.

$2x + y \geq 6$

$3x - y \geq 9$

$x, y \geq 0$

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 1 & 6 \\ 3 & -1 & 9 \\ \hline 6 & 20 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & 20 \\ \hline 6 & 9 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $C^* = 3u + 4v$ s.t.

$u + 2v \le 8$

$2u + 2v \le 10$

$u, v \ge 0.$

Introducing the non-negative slack variables x and y, the LPP in standard form is

$u + 2v + x = 8$

$2u + 2v + y = 10$

$3u + 4v = C^*$

i.e. $u + 2v + x + 0.y + 0.C^* = 8$

$2u + 2v + 0.x + y + 0.C^* = 10$

$-3u - 4v + 0.x + 0.y + C^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | x | y | C* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 2 | 1 | 0 | 0 | 8 | $\frac{8}{2}$ |
| y | 2 | 2 | 0 | 1 | 0 | 10 | $\frac{10}{2}$ |
| | –3 | –4 | 0 | 0 | 1 | 0 | |

The most negative entry in last row is – 4 ⇒ v-column is pivot column and the least ratio dividing R.H.S. by pivot column element is $\frac{8}{2}$ ⇒ row first is pivot row. Hence 2 is the pivot entry.

$R_1: \frac{1}{2} R_1$

| Basic variables | u | v | x | y | C* | R.H.S. |
|---|---|---|---|---|---|---|
| v | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 4 |
| y | 2 | 2 | 0 | 1 | 0 | 10 |
| | – 3 | –4 | 0 | 0 | 1 | 0 |

$R_2: R_2 - 2R_1,$ $R_3 : R_3 + 4 R_1$

| Basic variables | u | v | x | y | C* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| v | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 4 | $\frac{4}{1/2} = 8$ |
| y | 1 | 0 | -1 | 1 | 0 | 2 | $\frac{2}{1} = 2$ |
| | -1 | 0 | 2 | 0 | 1 | 16 | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is – 1 ⇒ u-column is pivot column and the least ratio dividing R.H.S. by pivot column element is $\frac{2}{1}$ ⇒ row second is pivot row. Hence 1 is the pivot entry.

$R_1: R_1 - \frac{1}{2} R_2,$ $R_3 : R_3 + R_2$

| Basic variables | u | v | x | y | C* | R.H.S. |
|---|---|---|---|---|---|---|
| v | 0 | 1 | 1 | $-\frac{1}{2}$ | 0 | 3 |
| u | 1 | 0 | -1 | 1 | 0 | 2 |
| | 0 | 0 | 1 | 1 | 1 | 18 |

$\therefore$ Max.$C^* = 18$ at $u = 2, v = 3$ $\Rightarrow$ Min. $C = 18$ at $x = 1, y = 1$

**e) Minimize** $U = 3x_1 + x_2$ **s.t.**

$2x_1 + x_2 \geq 14$

$x_1 - x_2 \geq 4$

$x_1, x_2 \geq 0$

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 1 & 14 \\ 1 & -1 & 4 \\ \hline 3 & 1 & 0 \end{array}\right) \qquad A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -1 & 1 \\ \hline 14 & 4 & 0 \end{array}\right)$$

$\therefore$ The dual of given LPP is

Maximize $U^* = 14u + 4v$ s.t.

$2u + v \leq 3$

$u - v \leq 1$

$u, v \geq 0$

Introducing the non-negative slack variables $x_1$ and $x_2$, the LP problem is

$2u + v + x_1 = 3$

$u - v + x_2 = 1$

$14u + 4v = U^*$

i.e. $2u + v + x_1 + 0.x_2 + 0.U^* = 3$

$u - v + 0.x_1 + x_2 + 0.U^* = 1$

$-14u - 4v + 0.x_1 + 0.x_2 + U^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | $x_1$ | $x_2$ | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 2 | 1 | 1 | 0 | 0 | 3 | $\frac{3}{2}$ |
| $x_2$ | 1 | -1 | 0 | 1 | 0 | 1 | $\frac{1}{1}$ |
| | – 14 | - 4 | 0 | 0 | 1 | 0 | |

The most negative entry is last row is – 14 $\Rightarrow$ u-column is pivot column and least ratio dividing R.H.S. by pivot column element is $\frac{1}{1} \Rightarrow$ row second is pivot row. Hence 1 is the pivot entry.

$R_1: R_1 - 2R_2$, $\qquad R_3: R_3 + 14 R_2$

| Basic variables | u | v | $x_1$ | $x_2$ | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 0 | 3 | 1 | -2 | 0 | 1 | $\frac{1}{3}$ |
| u | 1 | - 1 | 0 | 1 | 0 | 1 | $\frac{1}{-1}$ |
| | 0 | -18 | 0 | 14 | 1 | 14 | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is – 18 ⇒ v-column is pivot column and only the positive ratio dividing R.H.S. by pivot column element is $\frac{1}{3}$ ⇒ row first is pivot row. Hence 3 is the pivot entry.

$R_1$: $\frac{1}{3}R_1$

| Basic variables | u | v | $x_1$ | $x_2$ | U* | R.H.S. |
|---|---|---|---|---|---|---|
| v | 0 | 1 | $\frac{1}{3}$ | $-\frac{2}{3}$ | 0 | $\frac{1}{3}$ |
| u | 1 | -1 | 0 | 1 | 0 | 1 |
| | 0 | -18 | 0 | 14 | 1 | 14 |

Operating $R_2$: $R_2 + R_1$ $R_3$: $R_3 + 18R_1$

| Basic variables | u | v | $x_1$ | $x_2$ | U* | R.H.S. |
|---|---|---|---|---|---|---|
| v | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | $\frac{1}{3}$ |
| u | 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\frac{4}{3}$ |
| | 0 | 0 | 6 | 2 | 1 | 20 |

∴ Max.U* = 20 at u = $\frac{4}{3}$, v = $\frac{1}{3}$ ⇒Min. U = 20 at x = 6, y = 2

**f) Minimize Z = 4x + 8y s.t.**

$$\mathbf{2x + y \geq 6}$$
$$\mathbf{x + 2y \geq 6}$$
$$\mathbf{x, y \geq 0}$$

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 6 \\ \hline 4 & 8 & 0 \end{array}\right) \qquad A^T = \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & 8 \\ \hline 6 & 6 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $Z^* = 6u + 6v$ s.t.

$$2u + v \leq 4$$
$$u + 2v \leq 8$$
$$u, v \geq 0$$

Introducing the non-negative slack variables x and y, the LP problem is

$$2u + v + x = 4$$
$$u + 2v + y = 8$$
$$6u + 6v = Z^*$$

i.e.
$$2u + v + x + 0.y + 0.Z^* = 4$$
$$u + 2v + 0.x + y + 0.Z^* = 8$$
$$-6u - 6v + 0.x + 0.y + Z^* = 0$$

The initial simplex table of the problem is given by

| Basic variables | u | v | x | y | Z* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 2 | 1 | 1 | 0 | 0 | 4 | $\frac{4}{2}$ |
| y | 1 | 2 | 0 | 1 | 0 | 8 | $\frac{8}{1}$ |
| | -6 | -6 | 0 | 0 | 1 | 0 | |

The negative entry is last row are both – 6. So, let us take u-column is pivot column and least ratio dividing R.H.S. by pivot column elements is $\frac{4}{2} \Rightarrow$ row first is pivot row. Hence 2 is the pivot entry.

$R_1$: $\frac{1}{2} R_1$,

| Basic variables | u | v | x | y | Z* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 2 |
| y | 1 | 2 | 0 | 1 | 0 | 8 |
| | -6 | -6 | 0 | 0 | 1 | 0 |

$R_2$: $R_2 - R1$, $R_3$: $R_3 + 6R_1$

| Basic variables | u | v | x | y | Z* | R.H.S. | |
|---|---|---|---|---|---|---|---|
| u | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 2 | $\frac{2}{1/2} = 4$ |
| y | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 1 | 0 | 6 | $\frac{6}{3/2} = 4$ |
| | 0 | -3 | 3 | 0 | 1 | 12 | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is – 3 ⇒ v-column is pivot column and the ratios dividing R.H.S. by pivot column element are both 4 ⇒ row second shall be now the pivot row. Hence $\frac{3}{2}$ is the pivot entry.

$R_2$: $\frac{2}{3} R_2$

| Basic variables | u | v | x | y | Z* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 2 |
| v | 0 | 1 | $-\frac{1}{3}$ | $\frac{2}{3}$ | 0 | 4 |
| | 0 | -3 | 3 | 0 | 1 | 12 |

$R_1$: $R_1 - \frac{1}{2} R_2$, $R_3$: $R_3 + 3R_2$

| Basic variables | u | v | x | y | Z* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | 0 | $\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | 0 |
| v | 0 | 1 | $-\frac{1}{3}$ | $\frac{2}{3}$ | 0 | 4 |
| | 0 | 0 | 2 | 2 | 1 | 24 |

∴ Max.C* = 24 at u = 0, v = 4 ⇒ Min. C = 24 at x = 2, y = 2

**g) Minimize U = 24x + 16y s.t.**

**$2x + y \geq 10$**

**$8x + 8y \geq 64$**

**$x \geq 0, y \geq 0$**

The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 1 & 10 \\ 8 & 8 & 64 \\ \hline 24 & 16 & 0 \end{array}\right) \qquad A^T = \left(\begin{array}{cc|c} 2 & 8 & 24 \\ 1 & 8 & 16 \\ \hline 10 & 64 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $U^* = 10u + 64v$ s.t.

$2u + 8v \le 24$

$u + 8v \le 16$

$u, v \ge 0.$

Introducing the non-negative slack variables x and y, the LP problem is

$2u + 8v + x = 24$

$u + 8v + y = 16$

$10u + 64v = Z^*$

i.e. $2u + 8v + x + 0.y + 0.U^* = 24$

$u + 8v + 0.x + y + 0.U^* = 16$

$-10u - 64v + 0.x + 0.y + U^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | x | y | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 2 | 8 | 1 | 0 | 0 | 24 | $\frac{24}{8} = 3$ |
| y | 1 | 8 | 0 | 1 | 0 | 16 | $\frac{16}{8} = 2$ |
| | -10 | -64 | 0 | 0 | 1 | 0 | |

The negative entry is last row is $-64 \Rightarrow$ v-column is pivot column and least ratio dividing R.H.S. by pivot column element is $\frac{16}{8} = 2 \Rightarrow$ row second is pivot row. Hence 8 is the pivot entry.

$R_2$: $\frac{1}{8}R_2$,

| Basic variables | u | v | x | y | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 2 | 8 | 1 | 0 | 0 | 24 | |
| v | $\frac{1}{8}$ | 1 | 0 | $\frac{1}{8}$ | 0 | 2 | |
| | -10 | -64 | 0 | 0 | 1 | 0 | |

$R_1$: $R_1 - 8R_2$, $R_3$: $R_3 + 64R_2$

| Basic variables | u | v | x | y | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | 0 | 1 | -1 | 0 | 8 | $\frac{8}{1} = 8$ |
| v | $\frac{1}{8}$ | 1 | 0 | $\frac{1}{8}$ | 0 | 2 | $\frac{2}{1/8} = 16$ |
| | -2 | 0 | 0 | 8 | 1 | 128 | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is $-2 \Rightarrow$ u-column is pivot column and the ratios dividing R.H.S. by pivot column element is $\frac{8}{1} = 8 \Rightarrow$ row first is the pivot row. Hence 1 is the pivot entry.

$R_2$: $R_2 - \frac{1}{8}R1$, $R_3$: $R_3 + 2R_1$

| Basic variables | u | v | x | y | U* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| u | 1 | 0 | 1 | -1 | 0 | 8 | |
| v | 0 | 1 | $-\frac{1}{8}$ | $\frac{1}{4}$ | 0 | 1 | |
| | 0 | 0 | 2 | 6 | 1 | 144 | |

∴ Max. U* = 144 at u = 0, v = 4 ⇒ Min. U = 144 at x = 2, y = 6

**h) Minimize** $F = 20x_1 + 48x_2$ **s.t.**

$x_1 + 3x_2 \geq 11$

$x_1 + 2x_2 \geq 9$

$x_1, x_2 \geq 0.$

The augmented matrix form of given system is

$$A = \left(\begin{array}{cc|c} 1 & 3 & 11 \\ 1 & 2 & 9 \\ \hline 20 & 48 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 1 & 20 \\ 3 & 2 & 48 \\ \hline 11 & 9 & 0 \end{array}\right)$$

∴ The dual of given LPP is

Maximize $F^* = 11u + 9v$ s.t.

$u + v \leq 20$

$3u + 2v \leq 48$

$u, v \geq 0$

Introducing the non-negative slack variables $x_1$ and $x_2$, the LP problem is

$u + v + x_1 = 20$

$3u + 2v + x_2 = 48$

$11u + 9v = F^*$

i.e. $u + v + x_1 + 0.x_2 + 0.F^* = 20$

$3u + 2v + 0.x_1 + x_2 + 0.F^* = 48$

$-11u - 9v + 0.x_1 + 0.x_2 + F^* = 0$

The initial simplex table of the problem is given by

| Basic variables | u | v | $x_1$ | $x_2$ | F* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 0 | 0 | 20 | $\frac{20}{1} = 20$ |
| $x_2$ | 3 | 2 | 0 | 1 | 0 | 48 | $\frac{48}{3} = 16$ |
| | -11 | -9 | 0 | 0 | 1 | 0 | |

The most negative entry in last row is – 11 ⇒ u-column is pivot column and least ratio dividing R.H.S. by pivot column element is $\frac{48}{3} = 16$ ⇒ row second is pivot row. Hence 3 is the pivot entry.

$R_2$: $\frac{1}{3}R_2$,

| Basic variables | u | v | $x_1$ | $x_2$ | F* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 0 | 0 | 20 | |
| u | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 | 16 | |
| | -11 | -9 | 0 | 0 | 1 | 0 | |

$R_1: R_1 - R2$, $R_3: R_3 + 11R_2$

| Basic variables | u | v | $x_1$ | $x_2$ | F* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| $x_1$ | 0 | $\frac{1}{3}$ | 1 | $-\frac{1}{3}$ | 0 | 4 | $\frac{4}{1/3} = 12$ |
| u | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 | 16 | $\frac{16}{2/3} = 24$ |
| | 0 | $-\frac{5}{3}$ | 0 | $\frac{11}{3}$ | 1 | 176 | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is $-\frac{5}{3} \Rightarrow$ v-column is pivot column and the ratios dividing R.H.S. by pivot column element is $\frac{4}{1/3} = 12 \Rightarrow$ row first is the pivot row. Hence $\frac{1}{3}$ is the pivot entry.

$R_1$: 3R1

| Basic variables | u | v | $x_1$ | $x_2$ | F* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| v | 0 | 1 | 3 | -1 | 0 | 12 | |
| u | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 | 16 | |
| | 0 | $-\frac{5}{3}$ | 0 | $\frac{11}{3}$ | 1 | 176 | |

$R_2: R_2 - \frac{2}{3}R1$, $R_3: R_3 + \frac{5}{3}R_1$

| Basic variables | u | v | $x_1$ | $x_2$ | F* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| v | 0 | 1 | 3 | -1 | 0 | 12 | |
| u | 1 | 0 | -2 | $\frac{5}{3}$ | 0 | 8 | |
| | 0 | 0 | 5 | 2 | 1 | 196 | |

$\therefore$ Max. F* = 196 at u = 8, v = 12 $\Rightarrow$ Min. C = 196 at $x_1 = 5$, $x_2 = 2$

**5. a) A small scale dealer deals in a rice and wheat. He has Rs. 1500 for investment. A bag of rice costs him Rs. 180 and a bag of wheat Rs. 120. He has a storage capacity of 10 bags only. He sells a bag of rice at a profit of Rs. 11 and a bag of wheat at a profit of Rs. 8. How many bags of each must he buy to make a maximum profit? Also find the maximum profit.**

**Sol$^n$**: Let x and y are the nuber of rice and wheat bags.

Here, the dealer sells a bag of rice at a profit of Rs. 11 and a bag of wheat at a profit of Rs. 8.

Therefore profit function is given by P = 11x + 8y

Also, A bag of rice costs him Rs. 180 and a bag of wheat Rs. 120. He has Rs. 1500 for investment. Therefore

$180x + 120y \le 1500 \quad \Rightarrow 3x + 2y \le 25$

Also, He has a storage capacity of 10 bags only. Therefore, $x + y \le 10$

$\therefore$ the corresponding LP problem is formulated as

Maximize P = 11x + 8y s.t.

$3x + 2y \le 25$

$x + y \le 10$

$x, y \ge 0$

Introducing the non-negative slack variables r and s, the LP problem is

$3x + 2y + r = 25$

$x + y + s = 10$

$11x + 8y = P$

i.e. $3x + 2y + r + 0.s + 0.P = 25$
$x + y + 0.r + s + 0.P = 10$
$-11x - 8y + 0.r + 0.s + P = 0$
The initial simplex table of the problem is given by

| Basic variables | x | y | r | s | P | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| r | 3 | 2 | 1 | 0 | 0 | 25 | $\frac{25}{3} = 8.33$ |
| s | 1 | 1 | 0 | 1 | 0 | 10 | $\frac{10}{1} = 10$ |
| | -11 | -8 | 0 | 0 | 1 | 0 | |

The most negative entry in last row is $-11 \Rightarrow$ x-column is pivot column and least ratio dividing R.H.S. by pivot column element is $\frac{25}{3} = 8.33 \Rightarrow$ row first is pivot row. Hence 3 is the pivot entry.

$R_1$: $\frac{1}{3} R_1$,

| Basic variables | x | y | s | r | P | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{25}{3}$ | |
| r | 1 | 1 | 0 | 1 | 0 | 10 | |
| | -11 | -8 | 0 | 0 | 1 | 0 | |

$R_2$: $R_2 - R1$, $R_3$: $R_3 + 11R_1$

| Basic variables | x | y | s | r | P | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{25}{3}$ | $\frac{25/3}{2/3} = \frac{25}{2}$ |
| r | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 1 | 0 | $\frac{5}{3}$ | $\frac{5/3}{1/3} = 5$ |
| | 0 | $-\frac{2}{3}$ | $\frac{11}{3}$ | 11 | 1 | $\frac{275}{3}$ | |

Since all entries in the last row are not non-negative, the solution is not optimal. Again, the most negative entry is last row is $-\frac{2}{3} \Rightarrow$ y-column is pivot column and the ratios dividing R.H.S. by pivot column element is $\frac{5/3}{1/3} = 5 \Rightarrow$ row second is the pivot row. Hence $\frac{1}{3}$ is the pivot entry.

$R_2$: 3R2

| Basic variables | x | y | r | s | P | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{25}{3}$ | |
| y | 0 | 1 | -1 | 3 | 0 | 5 | |
| | 0 | $-\frac{2}{3}$ | $\frac{11}{3}$ | 11 | 1 | $\frac{275}{3}$ | |

$R_1$: $R_1 - \frac{2}{3} R2$, $R_3$: $R_3 + \frac{2}{3} R_2$

| Basic variables | x | y | r | s | P | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 0 | 0 | 1 | -2 | 0 | 5 | |
| y | 0 | 1 | -1 | 3 | 0 | 5 | |
| | 0 | 0 | 3 | 13 | 1 | 95 | |

$\therefore$ Max. P = 95 at x = 5, y = 5

**b) A farmer can obtain two types of fertilizers. A bag of fertilizer M cost Rs. 50 and contains 6 kg of Nitrogen and 3kg of Potassium. A bag of fertilizer N costs Rs. 40 and contains 3 kg of Nitrogen and 3kg of Potassium. If a farmer wants to add at least 30 kg of Nitrogen and at least 18 kg of Potassium to each plot of land, how many bags of each types of fertilizer should be used to minimize the cost of fertilizing a plot of land? Also find the minimum cost.**

Let x and y are no. of bags of fertilizer M and N.

Since each bag of fertilizer M cost Rs. 50 and fertilizer N cost Rs. 40, the cost function C = 50x + 40y

Each bag of fertilizer M has 6 kg Nitrogen and fertilizer N has 3kg Nitrogen and at least 30 kg of Nitrogen is to add then we have

$6x + 3y \geq 30$

$\Rightarrow \quad 2x + y \geq 10$

Similarly, each bag of fertilizer M has 3 kg potassium and fertilizer N has 3 kg potassium and at least 18 kg of potassium is to add, then

$3x + 3y \geq 18$

$\Rightarrow \quad x + y \geq 6$

$\therefore$ The LPP for given problem is

Minimize $C = 50x + 40y$ s.t.

$2x + y \geq 10$

$x + y \geq 6$

$x, y \geq 0.$

Now, we solve the problem using simplex. The augmented matrix form of given LPP is

$$A = \left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 1 & 6 \\ \hline 50 & 40 & 0 \end{array}\right) \qquad A^T = \left(\begin{array}{cc|c} 2 & 1 & 50 \\ 1 & 1 & 40 \\ \hline 10 & 6 & 0 \end{array}\right)$$

$\therefore$ The dual of given LPP is

Maximize $C^* = 10u + 6v$ s.t.

$2u + v \leq 50$

$u + v \leq 40$

$u, v \geq 0$

Introducing non-negative slack variables x and y we have

$2u + v + x = 50$

$u + v + y = 40$

$-10u - 6v = C^*$

i.e. $\quad 2u + v + x + 0.y + 0.C^* = 50$

$u + v + 0.x + y + 0.C^* = 40$

$-10u - 6v + 0.x + 0.y + C^* = 0$

The initial simplex table to solve the problem is

| Basic variable | u | v | x | y | C* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| x | 2 | 1 | 1 | 0 | 0 | 50 | 50/2 = 25 |
| y | 1 | 1 | 0 | 1 | 0 | 40 | 40/1 = 40 |
| | −10 | − 6 | 0 | 0 | 1 | 0 | |

The most negative in last raw is − 10⇒u- column is pivot column. The least ratio dividing R.H.S. by pivot column element is 50/2= 25 ⇒row second is pivot element

$R_1$: $1/2R_1$

| Basic variable | u | v | x | y | C* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | 1/2 | 1/2 | 0 | 0 | 25 |
| y | 1 | 1 | 0 | 1 | 0 | 40 |
| | − 10 | -6 | 0 | 0 | 1 | 0 |

Operating, $R_2$: $R_2 - R_1$, $R_3$: $R_3 + 10R_2$

| Basic variable | u | v | x | y | C* | R.H.S. | Ratio |
|---|---|---|---|---|---|---|---|
| u | 1 | 1/2 | 1/2 | 0 | 0 | 25 | $\frac{25}{1/2}$ =50 |
| y | 0 | 1/2 | -1/2 | 1 | 0 | 15 | $\frac{15}{1/2}$ = 30 |
| | 0 | – 1 | 5 | 0 | 1 | 250 | |

Here the solution is not optimal since the cost raw consists non-negative entries $-1 \Rightarrow$v – column is pivot column. The least ratio dividing R.H.S by pivot column element is $\frac{15}{1/2}$ = 30, row second is pivot row.

Operating, $R_2$: $2R_2$

| Basic variable | u | v | x | y | C* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | 1/2 | 1/2 | 0 | 0 | 25 |
| v | 0 | 1 | –1 | 2 | 0 | 30 |
| | 0 | – 1 | 5 | 0 | 1 | 250 |

Operating $R_1$: $R_1 - \frac{1}{2}R_2$ and $R_3$: $R_3 + R_2$

| 0 | u | v | x | y | C* | R.H.S. |
|---|---|---|---|---|---|---|
| u | 1 | 0 | 1 | –1/2 | 0 | 10 |
| v | 0 | 1 | –1 | 2 | 0 | 30 |
| | 0 | 0 | 4 | 2 | 1 | 280 |

Here, the solution is optimal as we have the last raw consisting non-negative entries.

$\therefore$ Minimum of C* = 280 at u = 10, v = 30 $\Rightarrow$ Maximum of C = 280 at x = 4, y = 2

**Hint and Solution of MCQ's**

1. The different terms used in LPP are all of objective function, decision variable and constraints
2. To get the maximum profit with limited resourced, the method used in the LPP is a mathematical method.
3. The aim of LPP is to optimize the objective function.
4. The variable used to change the greater than inequality into equality is surplus variable.
   To change greater than inequality, $A(x) \geq b$ in to the equality of the form $A(x) + s = b$, the value s is called surplus variable for $s \leq 0$
5. The presentation of objective function $Z = 3x + 4y$ with non-negative slack variables r and s is
   $-3x - 4y + 0.r + 0.S + Z = 0$
6. The entering variable is the most negative value in pivot column.
7. The outgoing or departing variable is the variable in the row containing the least value of
   RHS entry÷ corresponding pivot column entry
8. The pivot element in the simplex tableau is any element common to pivot column and pivot row.
9. In the tableau most negative in last row = – 5
   $\Rightarrow$ Column 1st is entering column
   RHS ÷ Corresponding to pivot column entry are such that:
   $\frac{6}{3} < \frac{4}{1}$ $\Rightarrow$Row 1st is pivot raw
   $\therefore$ common element of 1st column 1st row is 3 which is the pivot element.
10. The optimal value of objective function is obtained when all entries in the last row of simplex tableau is non-negative.