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Chapter-2
Perodic Motion

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Perodic Motion

In this motion a particle moves about the fix point in which the particle goes back and forth (up and down) about that point such that restoring force is directly proportional to a displacement from mean position and is directed towards mean position. [SHM - simple harmonic motion. Eg: Simp. Pendulum]

Mathematically,

$$F \propto x$$

$$\text{or, } \boxed{F = -kx}$$ Here, F = restoring force, x = displacement

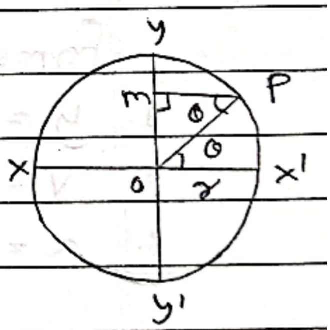
$$\text{or, } ma = -kx$$

$$\text{or, } a = -\frac{kx}{m}$$

$$\therefore a \propto x$$

Show that uniform circular motion executes (moves) SHM.

Let a particle 'P' moves in a circular path of radius 'r' with uniform speed 'V'. Here 'm' be the foot of the particle. When 'P' moves in the given circle, m shows to and fro motion along YY' axis.



In Δ PMO,

$$\sin\theta = \frac{OM}{OP} = \frac{y}{r}$$

$$\text{or, } y = r\sin\theta$$

$$\text{or, } y = r\sin\omega t$$

Velocity Equation:

$$V = \frac{dy}{dt} = \frac{d(r\sin\omega t)}{dt} = r\frac{d\sin\omega t}{dt} = r\omega\cos\omega t$$

$$\text{or, } V = r\omega\cos\omega t = r\omega\sqrt{1-\sin^2\omega t}$$

$$= r\omega\sqrt{1-\frac{y^2}{r^2}}$$

$$= r\omega \sqrt{\frac{r^2-y^2}{r^2}}$$

or, $V = r\omega \cos\omega t$
$= \omega \sqrt{r^2-y^2}$

Acceleration Equation

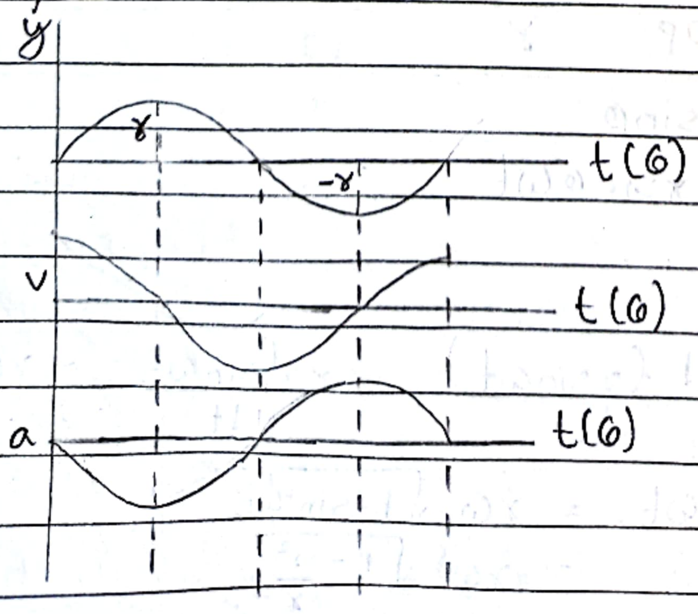
$$a = \frac{dV}{dt}$$
$$= \frac{d}{dt}(r\omega \cos\omega t)$$
$$= r\omega(-\sin\omega t\ \omega)$$
$$= -r\omega^2 \sin\omega t$$
$$= -\omega^2 r\sin\omega t$$
$$\therefore a = -\omega^2 y$$

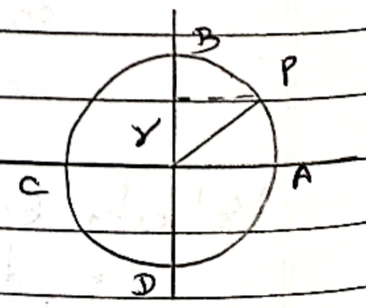
$\therefore$ A particle moving uniformly in a circular path shows SHM.

Formulas

1. $y = r\sin\theta = r\sin\omega t$, displacement equation
2. $V = r\omega\cos\omega t$
3. $a = -\omega^2 y = -\omega^2 r\sin\omega t$

Graph:





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Questions:

1. In displacement Equation, what is the value of y at B position.

Solution:

At B, $\theta = 90^\circ$, $y = r\sin\theta$

$= r\sin 90^\circ$

$= r$

$\therefore y = r$ (maximum displacement / amplitude)

2. In velocity equation, what is the value of 'v' at C position.

Solution:

At C, $\theta = 180^\circ$

Then, $v = r\omega\cos\theta$

$= r\omega\cos 180$

$= -r\omega$

3. In acceleration equation, what is the value of a at position A.

Solution:

At A, $\theta = 0^\circ$

Then, $a = -\omega^2 r\sin\theta$

or $a = -\omega^2 r\sin 0$

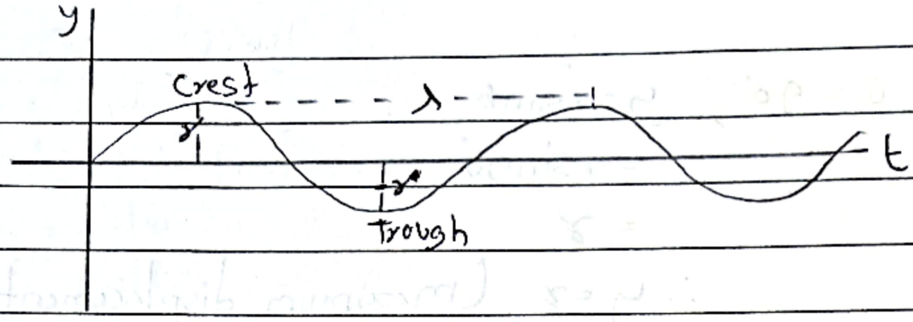
$\therefore a = 0$

4. What is the phase difference between y and v.

Ans: Displacement of the particle lack behind by 90°

Displacement Equation

$y = r\sin\theta = r\sin\omega t$



Time period (T) : Time required to complete one complete wave is called time period.

We have,

$$a = \omega^2 y$$

$$\text{or } \omega^2 = \frac{a}{y}$$

$$\text{or } \omega = \sqrt{\frac{a}{y}}$$

$$\text{or } \frac{\theta}{t} = \sqrt{\frac{a}{y}}$$

For time Period (T),

$$\text{or } \frac{2\pi}{T} = \sqrt{\frac{a}{y}}$$

$$\text{or, } \frac{T}{2\pi} = \sqrt{\frac{y}{a}}$$

$$\therefore T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

Note:

$\because \theta = \omega t$

or, $\omega = \frac{\theta}{t}$

for $t = T$,

$\theta = 2\pi$

Frequency (f)

Number of wave generated in 1 second is called frequency of a wave.

i.e. $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$

Unit of frequency = Hertz (Hz)

frequency of Kantipur FM = 96.1 Mega Hertz

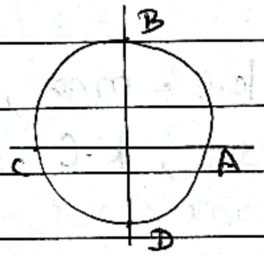
Time period of Kantipur FM $= \frac{1}{T}$

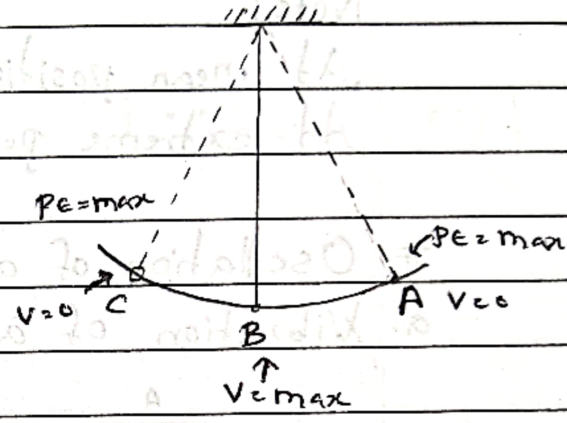
$$\text{or, } 96.1 \times 10^{6} = \frac{1}{T}$$

$$\cancel{\therefore T = 1.040 \times 10^{-5} \text{ sec.}}$$

$$\therefore T = 1.040 \times 10^{-8} \text{ seconds.}$$

Energy of a particle executing SHM





$$\text{Energy} = E_{kinetic} + E_{potential}$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} ky^2$$

We know,

$$V = r\omega \cos\omega t$$

$$= r\omega \sqrt{1 - \sin\omega t}$$

$$= r\omega \sqrt{1 - \frac{y^2}{r^2}}$$

$$= r\omega \frac{\sqrt{r^2 - y^2}}{r}$$

$$= \omega \sqrt{r^2 - y^2}$$

Also,

(spring constant) $k = m\omega^2$

$$E = \frac{1}{2} m\omega^2 (r^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

Here, $E_k = \frac{1}{2} m\omega^2 (r^2 - y^2)$

$$E_p = \frac{1}{2} m\omega^2 y^2$$

$E_{total} = E_k + E_p$

$= \frac{1}{2}mv^2 + \frac{1}{2}ky$

$= \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 y^2$

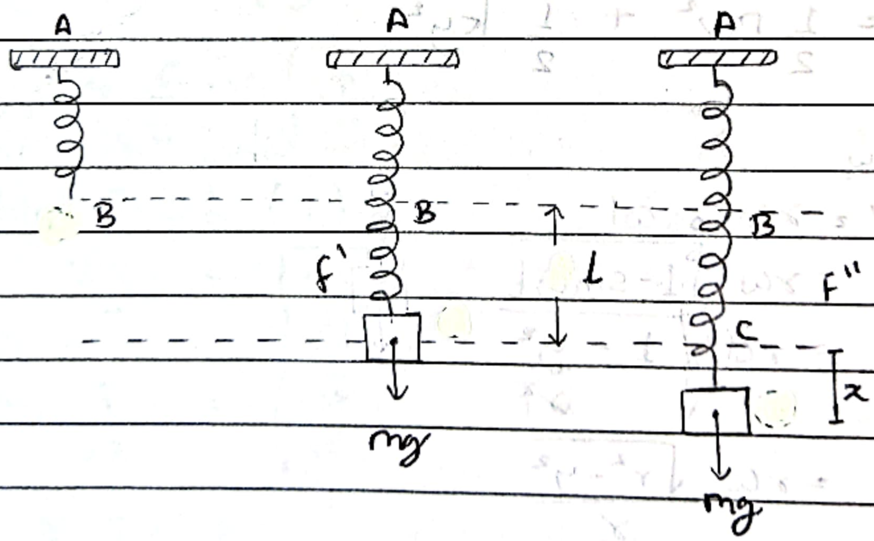
$= \frac{1}{2}m\omega^2(r^2 - y^2) + \frac{1}{2}m\omega^2 y^2$

$\therefore E_{total} = \frac{1}{2}m\omega^2 r^2$

Note:
At mean position, $y = 0$, K.E = max, PE = 0
At extreme position, $y = \pm r$, K.E = 0, PE = max

Oscillation of a loaded Spring

a. Vibration of a particle in vertical spring



Vertical mass spring system

When a load m is attached, the spring extends and Let 'l' be the elongation produced.
When the mass is at rest;
Restoring force (F') = mg

or, $\cancel{-ky}$ $-kl = mg$

$\therefore mg = -kl$ —(1)

Where, k is spring constant

When,

The load is pulled down through a small distance x, then the restoring force (f') is ;

$$F'' = -k(l+x)$$

Now,

When mass moves upward ;

$$F'' - mg = F$$

or, $F'' - mg = ma$

or, $-k(l+x) - mg = ma$

or, $-k(l+x) + kl = ma$

or, $-kl - kx + kl = ma$ [$\because$ from eq (i)]

or, $-kx = ma$

or, $a = \frac{-kx}{m}$ —— (ii)

$\therefore a \propto x$

Where $\frac{-k}{m}$ is a constant. Hence, the motion of a loaded spring is SHM.

~~We have,~~

$\cancel{a = \frac{kx}{m}}$

or, $\cancel{-\omega^2 y = \frac{-kx}{m}}$

For S.H.M, we have

$$a = -\omega^2 x \quad \text{—— (iii)}$$

Comparing (ii) and (iii), we get;

$$-\omega^2 = \frac{-k}{m}$$

or, $\omega = \sqrt{\frac{k}{m}}$

or, $\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

or, $T = 2\pi \sqrt{\frac{m}{k}}$

$\therefore T = 2\pi \sqrt{\frac{m}{k}}$

Also,

Frequency $(f) = \frac{1}{T} = \frac{1}{2\pi}\sqrt{k/m}$

b. Vibration of a particle in Horizontal Spring

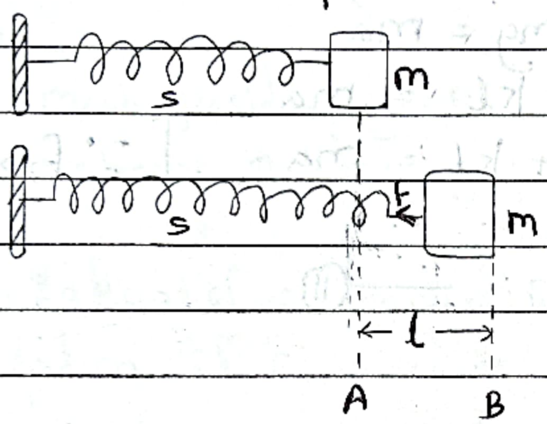


fig: mass oscillating horizontally on light sketched spring

Suppose a horizontally loaded spring as shown in the figure above.

Now, restoring force $(f) = -kl$

or, $ma = -kl$

or $a = -\frac{k}{m} l$ —— ①

$\therefore a \propto l$

$\therefore$ It shows simple harmonic motion.

For SHM;

$a = -\omega^2 l$ —— ⑪

Comparing ① and ⑪ we get

$\omega^2 = \frac{k}{m}$

or, $\omega = \sqrt{k/m}$

or, $\frac{2\pi}{T} = \sqrt{m/k}$

$\therefore T = 2\pi\sqrt{m/k}$

Show that a simple pendulum is simple harmonic. Obtain ~~the~~ an equation for its time period and frequency.

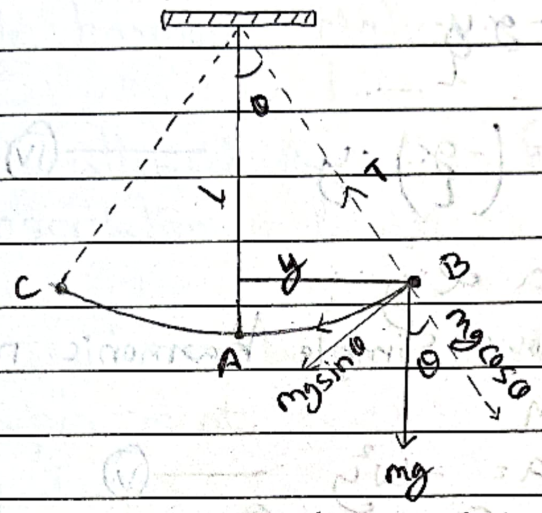


Fig: simple Pendulum

Let 'm' be the mass of the bob, l be the length of simple pendulum and θ be the angle made when it is displaced to 'y' distance from mean position 'A'.

At point B:

Force 'mg' acts downward and tension 'T' along the string. The component $mg\cos\theta$ balance tension and $mg\sin\theta$ acts as restoring force.

Then, $F = -mg\sin\theta$

or, $ma = -mg\sin\theta$

$\therefore a = -g\sin\theta$ —— (i)

For small value of θ, $\sin\theta \approx \theta$

from eq (i)

$\therefore a = -g\theta$ —— (ii)

From the figure, we can write;

$$\theta = \frac{\text{arc } AB}{\text{radius}}$$

or, $\theta = \frac{y}{l}$ $\quad$ [$\because$ for small θ = $\overset{\frown}{AB} \approx y$]

$\therefore$ $\theta = \frac{y}{l}$ ——— (III)

From (II) and (III).

$$a = -g \cdot \frac{y}{l}$$

or, $a = \left(-\frac{g}{l}\right) \cdot y$ ——— (IV)

$\therefore$ $a \propto y$

$\therefore$ It shows simple harmonic motion.

For SHM

$$a = -\omega^2 y \quad \text{——— (V)}$$

Comparing (IV) and (V)

$$\omega^2 = \frac{g}{l}$$

or, $\omega = \sqrt{\frac{g}{l}}$

or, $\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$

$\therefore$ $T = 2\pi \sqrt{\frac{L}{g}}$

Also,

Frequency $= \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$

Length of Second Pendulum

$T = 2$ sec, time period of second pendulum

We know,

$$T = 2\pi\sqrt{L/g}$$

$$\text{or, } 2 = 2 \times 3.14 \sqrt{\frac{L}{9.8}}$$

$$\therefore L = 0.992 \text{ cm}$$

Angular Simple Harmonic Motion

When a body is allowed to rotate freely about a given axis then the oscillation is known as angular oscillation

Let,

I = moment of Inertia of balance wheel

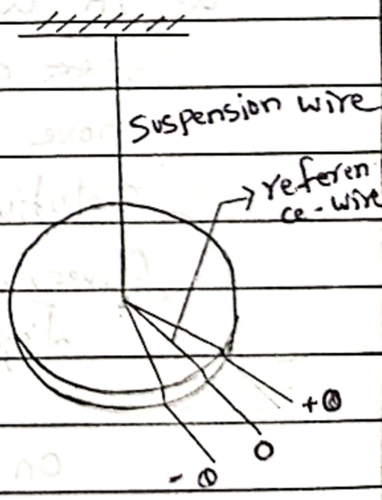
θ = Angular displacement

τ = restoring Torque

Now,

$$\tau \propto \theta$$

$$\therefore \tau = -k\theta \quad \text{—— (i)}$$



Where, k is the proportionality constant called torsion constant of the spring coil.

Again we know,

$$\tau = I\alpha \quad \text{—— (ii)}$$

From (i) and (ii)

$$I\alpha = -k\theta$$

$$\therefore \alpha = \frac{-k}{I}\theta \quad \text{—— (iii)}$$

Here, $\alpha = \left(\frac{-k}{I}\right)\theta$

i.e. $\alpha \propto \theta$, which indicates SHM.

For a body in SHM,

$$a = -\omega^2 x$$

Hence, the angular motion of balance wheel is ~~SH~~ in SHM and it is said to be in angular SHM.

Now,

$$\omega^2 = \frac{k}{I}$$

$$\text{or, } \omega^2 = 2\pi f = \sqrt{\frac{k}{I}}$$

$$\text{or, } f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$$

Now,

$$T = 2\pi\sqrt{I/k}$$

Numericals

a. In what time after its motion began will a particle ~~osca~~ oscilating according to the equation $y = 7\sin 0.5\pi t$ move from the mean position to maximum displacement.

Solution:

Given,

$$\text{displacement } (y) = 7\sin 0.5\pi t$$

$$\text{or, } r\sin\omega t = 7\sin 0.5\pi t$$

On comparing these two, we get;

$$r = 7 \text{ mm}$$

$$= 7\times10^{-3}\text{m}$$

$$\omega = 0.5 \text{ rad/sec}$$

$$\theta = \frac{\pi}{2}$$

According to question;

In Extreme position, $y = r$, $t = ?$

So, $x = 7 \sin 0.5\pi t$

or, $7 = 7 \sin 0.5\pi t$

or, $1 = \sin 0.5\pi t$

or, $\sin 90 = \sin 0.5\pi t$

or, $\frac{\pi}{2} = 0.5\pi t$

$\therefore t = \frac{1}{2 \times 0.5} = 1$ sec.

b. A body of mass 200g is executing simple harmonic motion with amplitude of 20 mm. The maximum force which acts upon it is 0.8N. Calculate its maximum velocity and its period of ~~oscilla~~ oscillation.

Solution:

Mass of body (m) = 200g = 200×10^{-3} kg

Amplitude (A) = 20mm = 20×10^{-3} m

Maximum force (f) = 0.8N

Maximum Velocity (V_{max}) = ?

Period of oscillation (T) = ?

We know, that

$$F_{max} = \frac{MV_{max}^2}{A}$$

$$\text{or, } 0.8 = \frac{200 \times 10^{-3} \times V_{max}^2}{20 \times 10^{-3}}$$

$$\therefore V_{max} = 0.28 \text{ m/s}$$

Again,

We know, $\omega A = V_{max}$

$$\text{or, } V_{max} = \frac{2\pi}{T} \cdot A$$

$$\text{or, } 0.28 = \frac{2 \cdot \pi \times 20 \times 10^{-3}}{T}$$

$\therefore T =$ ~~0.45~~ 0.45 sec.

c. The displacement 'y' of a mass vibrating with SHM is given by $y = 20\sin 10\pi t$, where y is in millimeter and time in second. What is;

i. Amplitude ii. The time period iii. Velocity at $t=0$

Solution:

Given, Displacement $(y) = 20\sin 10\pi t$ —— ①

Amplitude $(A) = ?$

Time period $(t) = ?$

Velocity at $t=0$ $(v) = ?$

We know,

$y = r\sin\omega t$ —— ⑪

On Comparing ① and ⑪, we get;

i. $r = 20\ mm = 20\times10^{-3}\ m$ ⟶ amplitude.

$\omega = 10\pi$

ii. Now,

$$\omega = \frac{2\pi}{T}$$

$$\text{or, } 10\pi = \frac{2\pi}{T}$$

$\therefore T = 0.2\ sec$

iii. Again,

Velocity at $t=0$;

We know, $V = r\omega\cos\omega t$

or $V = 20\times10^{-3} \times 10\pi \times \cos 10\pi\cdot0$

or $V = 20\times10^{-3} \times 10\pi \times \cos 0$

$\therefore V = 0.628\ m/s$

d. A simple pendulum has a period of 4.2 sec. ~~What~~ When the pendulum is shortened by 1 m, the period is 3.7 second. From these measurements, calculate the acceleration of free fall and the original length of the pendulum.

Solution:

Time Period $(T_1) = 4.2$ sec

Length shortened $= 1$ m

Time period $(T_2) = 3.7$ sec

Length $(l) = ?$

Acceleration of free fall $(g) = ?$

We have,

$$T_1 = 2\pi\sqrt{\frac{L}{g}} \quad \text{--- (i)}$$

and
$$T_2 = 2\pi\sqrt{\frac{L-1}{g}} \quad \text{--- (ii)}$$

from (i) and (ii)

$$\frac{T_2}{T_1} = \sqrt{\frac{L-1}{L}}$$

$$\text{or, } \left(\frac{T_2}{T_1}\right)^2 = \frac{L-1}{L}$$

$$\text{or, } \left(\frac{3.7}{4.2}\right)^2 = 1 - \frac{1}{L}$$

$$\text{or, } \frac{1}{L} = 1 - 0.776$$

$$\text{or, } \frac{1}{L} = 0.224$$

$$\therefore L = 4.5 \text{ m}$$

Again,

$$T_1 = 2\pi\sqrt{\frac{L}{g}}$$

$$\text{or, } 4.2 = 2\pi\sqrt{\frac{4.5}{g}}$$

$$\text{or, } g = \frac{4\pi^2 \times 4.5}{(4.2)^2}$$

$$\therefore g = 10 \text{ ms}^{-2}$$

e. A body oscillates in SHM with an amplitudes of 10 cm and with a frequency of 10 Hz. Compute

i. the maximum value of the velocity and acceleration.

ii. the acceleration and the velocity when the displacement is 5 cm.

Solution:

Here, $r = 10$ cm

Frequency (f) = 10 Hz

Max Velocity (V_{max}) = ?

Max. Acceleration (A_{max}) = ?

We know,

i. $V = \omega\sqrt{r^2 - y^2}$

or, $V_{max} = 2\pi f \sqrt{r^2 - y^2}$

The velocity is max. when $y = 0$

i.e. $V_{max} = 2 \times \pi \times 10 \sqrt{10^2 - 0}$

$\therefore V_{max} = 628.3$ cm/sec.

Again,

$a = -\omega^2 y$

or, $a = -2\pi f \cdot y$

or, $a = -(2\pi \times 10)^2 \times r$ $\quad$ [$\because$ Acc. is max at $y = r$]

or, $a = -(2\pi \times 10)^2 \times 10$

$\therefore a = 39478.4$ cm/sec

$= 394.78$ m/sec

ii. When $y = 5$ cm

$V = \omega\sqrt{r^2 - y^2}$

$= 20\pi\sqrt{10^2 - 5^2}$

$= 5.4$ m/s

Also,

$a = \cdot\omega^2 y$

$= (20\pi)^2 \cdot 5$

$= 197.39$ m/s^2