# Chapter-2
## Binomial Theorem

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### Exercise - 2.1

Expand each of the following by the binomial ~~theory and~~ theorem and simplify:

1.a. $(a+b)^7$

Solution:

$(a+b)^7 = c(7,0).a^7b^0 + c(7,1)a^6b^1 + c(7,2)a^5b^2 + c(7,3)a^4b^3 + c(7,4)a^3b^4 + c(7,5)a^2b^5 + c(7,6)a^1b^6 + c(7,7)b^7$

$= 1.a^7 + 7a^6b + \frac{6.7}{2.1}a^5b^2 + \frac{7.6.5}{3.2.1}a^4b^3 + \frac{7.6.5}{3.2.1}a^3b^4 + \frac{7.6}{2.1}a^2b^5 + 7ab^6 + b^7$

$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

b. $(2x-3y)^4$

Solution:

$(2x-3y)^4$

$= c(4,0)(2x)^4 - c(4,1)(2x)^3(3y) + c(4,2)(2x)^2(3y)^2 - c(4,3)(2x)^1(3x)^3 - c(4,4)(3x)^4$

$= \frac{4!}{0!4!} \times 16x^4 - \frac{4!}{3!1!} \times 8x^3.3y + \frac{4!}{2!2!}4x^2(9y^2) - \frac{4!}{1!3!} \times 2x\,27y^4 - \frac{4!}{0!4!} \times 81y^4$

$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

c. $(2x+y^2)^5$

Solution:

$c(5,0)(2x)^5 + c(5,1)(2x)^4(y^2) + c(5,2)(2x)^3y^4 + c(5,3)(2x)^2(y^2)^3 + c(5,4)(2x)^1(y^2)^4 + c(5,5)\cancel{(2}(y^2)^5$

$= 32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + \cancel{10}\, y^{10}$

d. $\left(\frac{x}{2} + \frac{2}{y}\right)^5$

Solution:

$= C(5,0)\left(\frac{x}{2}\right)^5 + C(5,1)\left(\frac{x}{2}\right)^4\left(\frac{2}{y}\right) + C(5,2)\left(\frac{x}{2}\right)^3\left(\frac{2}{y}\right)^2$

$+ C(5,3)\left(\frac{x}{2}\right)^2\left(\frac{2}{y}\right)^3 + C(5,4)\left(\frac{x}{2}\right)^1\left(\frac{2}{y}\right)^4 +$

$C(5,5)\left(\frac{x}{2}\right)^0\left(\frac{2}{y}\right)^5$

$= \frac{x^5}{32} + \frac{5}{8}\frac{x^4}{y} + \frac{5x^3}{y^2} + 20\frac{x^2}{y^3} + 40\frac{x}{y^4} + \frac{32}{y^5}$

e. $\left(x^2 - \frac{1}{x}\right)^5$

Solution:

$= C(5,0)(x^2)^5 + C(5,1)(x^2)^4\left(\frac{1}{x}\right) + C(5,2)(x^2)^3\left(\frac{1}{x}\right)^2$

$+ C(5,3)(x^2)^2\left(\frac{1}{x}\right)^3 + C(5,4)(x^2)^4\left(\frac{1}{x}\right)^2 + C(5,5)$

$x^0\left(\frac{1}{x}\right)^5$

$= x^{10} - 5x^7 + 10x^4 - 10x^4 - 10x + \frac{5}{x^2} - \frac{1}{x^5}$

2. Find the stated terms in the following cases.

a. Seventh term of $(2x+y)^{12}$

Solution:

$7^{th}$ term $= t_7 = t_{6+1} = C(12,6)(2x)^{12-6} \cdot y^6$

$= C(12, 6)\,(2x)^6 y^6$

b. fourth term of $\left(2x^2 + \frac{1}{x}\right)^8$

Solution:

4th term $= t_{3+1} = C(8,3)\,(2x^2)^{8-3} \cdot \left(\frac{1}{x}\right)^3$

$= \cancel{C(8,3)}$

$= 1792x^7$

c. fifth term of $\left(x - \frac{2}{x}\right)^7$

Solution:

5th term $= t_5 = t_{4+1} = C(7,4)\,(x)^{7-4}\left(\frac{2}{x}\right)^4$

$= \frac{560}{x}$

3. Find the general terms of the following:

a. $\left(x^2 + \frac{1}{x}\right)^6$

Solution:

Let $t_{r+1}$ be the general ~~formula~~ term ~~of~~ in the expansion of $\left(x^2 + \frac{1}{x}\right)^6$

Then, $t_{r+1} = C(6, r)\,(x^2)^{6-r} \cdot \left(\frac{1}{x}\right)^r$

$= C(6, r)\, x^{6x-3r}$

b. $\left(\frac{a}{b} + \frac{b}{a}\right)^{2n+1}$

Solution:

Let $t_{r+1}$ be the general term in the expansion $\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1}$

Then,

$$t_{r+1} = C(2n+1, r)\left(\frac{a}{b}\right)^{2n+1-r}\left(\frac{b}{a}\right)^{r}$$

$$= C(2n+1, r)\left(\frac{a}{b}\right)^{2n-2r+1}$$

4. Find the coefficients of;

a. $x^5$ in the expansion of $\left(x+\frac{1}{2x}\right)^7$

Solution:

$$t_n = t_{r+1} = C(7, r)(x)^{7-r}\cdot\left(\frac{1}{2x}\right)^{r}$$

$$= \frac{C(7, r)(x^{7-2r})}{2^r}$$

from question:

$7-2r=5$

$\therefore r = +1$

Hence, Coeff. of $x^5 = C(7,1)$

$$= \frac{7}{2}$$

b. $x^2$ in the expansion of $\left(x^3+\frac{a}{x}\right)^{10}$

Solution:

$$t_n = t_{r+1} = C(10, r)(x^3)^{10-r}\left(\frac{a}{x}\right)^{r}$$

$$= C(10, r)\, x^{30-4r}\cdot a^r$$

from question $30-4r=2$

$\therefore r = 7$

Hence, Coefficient $x^2$ $= C(10,7)\, a^7$

$= 120 \cdot a^7$

c. $x^6$ in the expansion of $\left(3x^2 - \frac{1}{3x}\right)^9$

Solution:

$t_n = t_{r+1} = C(9,r)\,(3x^2)^{9-r} \cdot \left(\frac{1}{3x}\right)^r (-1)^r$

$= C(9,r) \cdot 3^{9-r} \cdot x^{18-r} \cdot \frac{1}{3^r \cdot x^r}$

$= C(9,r)\; 3^{9-\cancel{2}r} \cdot x^{18-2r} \cdot (-1)^r.$

$= C(9,r) \cdot 3^{9-2r} \cdot x^{18-3r}$

By question:

$18 - 3r = 6$

$\therefore r = 4$

Hence, Coefficient of $x^6$ $= C(9,4) \cdot 3^{9-2\times4}\,(-1)^4$

$= 378$

5. Find the term independent (free) of 'x' in the expansion of:

a. $\left(x^2 + \frac{1}{x}\right)^{12}$

Solution:

Let $t_{r+1}$ be the term independent of $x$.

Then, $t_{r+1} = C(12,r)\,(x^2)^{12-r}\left(\frac{1}{x}\right)^r$

$= C(12,r)\, x^{24-3r} \cdot 1^r$

For the term to be independent of 'x'

$24 - 3r = 0$

$\therefore r = 8$

$\therefore t_{8+1} = t_9$ is the term which is free from $x$.

So, $t_{8+1} = C(12,8)\,(x^2)^{12-8} \cdot \left(\frac{1}{x}\right)^8$

$= C(12, 8)$

$= 495$

b. $\left(2x + \frac{1}{3x^2}\right)^9$

Solution:

Let $t_{r+1}$ be the term which is free from 'x'.

$t_{r+1} = C(9, r)\,(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$

$= C(9,r) \cdot 2^{9-r} \cdot x^{9-r} \cdot \frac{1^r}{3^r \cdot x^{2r}}$

$= C(9,r) \cdot 2^{9-r} \cdot x^{9-3r} \cdot \frac{1}{3^r}$

For the term to be independent from 'x'

$9 - 3r = 0$

$\therefore r = 3$

So, $t_4$ is independent of 'x'.

Then,

$t_{3+1} = C(9,3)\,(2x)^{9-3}\left(\frac{1}{3x^2}\right)^3$

$= \frac{1792}{9}$

c. $\left(x + \frac{1}{x}\right)^{2n}$

Solution:

Let $t_{r+1}$ be the term independent of $x$ then,

$t_{r+1} = C(2n, r)\, x^{2n-r} \cdot \left(\frac{1}{x}\right)^r$

$= C(2n, r)\, x^{2n-2r} \cdot 1^r$

For the term to be independent of 'x'.

$\cancel{2n - 2r = 0}$ $2n - 2r = 0$

$\therefore r = n$

So, $t_{n+1}$ is the independent term from 'x'.

$t_{n+1} = C(2n, n)\, x^{2n-n} \cdot \frac{1}{x^n}$

$= \frac{2n!}{n!\,n!}$

d. $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

Solution:

Let $t_{r+1}$ be the term independent of $x$ then,

$t_{r+1} = C(9, r)\left(\frac{3x^2}{2}\right)^{9-r}(-1)^r \cdot \left(\frac{1}{3x}\right)^r$

$= C(9, r)\left(\frac{3}{2}\right)^{9-r} \cdot x^{18-2r} \cdot \frac{1}{3^r} \cdot \frac{1}{x^r}$

$= C(9, r)\left(\frac{3}{2}\right)^{9-r} \cdot x^{18-3r} \cdot \frac{1}{3^r}$

For the term to be independent of 'x'

$18 - 3r = 0$

$\therefore r = 6$

Hence, $t_7$ or 7th term is independent of $x$.

Then, $t_{6+1} = C(9, 6)\left(\frac{3x^2}{2}\right)^{9-6}\left(\frac{1}{3x}\right)^6$

$= C(9, 6)\left(\frac{3}{2}\right)^3 \cdot \left(\frac{1}{3}\right)^6$

$= 84 \times \frac{27}{8} \times \frac{1}{729}$

$= \frac{7}{18}$

6.a. Write down the fourth term in the expression of $\left(Px + \frac{1}{x}\right)^n$. If this term is independent of $x$. find the value of $n$. With this value of $n$, calculate the value of P given that the fourth term is $\frac{5}{2}$.

Solution:

fourth term $= t_{3+1} = c(n,0)\,(Px)^{n-3}\,(1/x)^3$

$= c(n,0)\; p^{n-3}.\, x^{n-3}.\, \frac{1}{x^3}$ —— ①

$= c(n,0)\; p^{n-3}.\, x^{n-6}$

Since, the term is independent of $x$, we can write;

$n - 6 = 0$

$\therefore n = 6$

We know, By ~~Question~~ Question:

$\frac{5}{2} = c(n,0)\; p^{n-3}.\, x^{n-3}.\, \frac{1}{x^3}$

$\frac{5}{2} = c(6,0)\; p^{n-3}$

$= c(6,0)\; p^3$

or, $\frac{5}{2} = \frac{6!}{3!3!}\; p^3$

or $p^3 = \frac{5 \times 3! \times 3!}{2 \times 6!}$

$\therefore P = \frac{1}{2}$

b. If the coefficient of $x^{-1}$ in the expansion of $\left(x + \frac{k}{x^2}\right)^5$ is 90. find the value of k.

Solution:

Let $t_{r+1} = c(5,r)\, x^{5-r}\left(\frac{k}{x^2}\right)^r$

$= c(5,r)\, x^{5-r}.\, k^r.\, x^{-2r}$

We know,

$$x^{5-3r} = x^{-1}$$

or $5 - 3r = -1$

$\therefore r = 2$

Hence, third term $(t_3)$ has the coefficient $x^{-1}$.

Again, By Question:

$$C(5,r)\, k^r = 90$$

or $k^2 = \dfrac{90}{C(5,2)}$

$\therefore k = \pm 3$

7. Find the middle term or terms in the expansion of:

a. $\left(x + \dfrac{1}{x}\right)^{18}$

Solution.

Here, n is the even no. So there is only one mid-term.

$$\left(x + \frac{1}{x}\right)^{18} = t_{\frac{n}{2}+1}$$

$$= t_{\frac{18}{2}+1}$$

$$= t_{9+1}$$

Now,

$$t_{9+1} = C(18,9)\, x^{18-9} \left(\frac{1}{x}\right)^9$$

$$= C(18,9)\, x^9 \cdot \frac{1}{x^9}$$

$$= \frac{18!}{9!\,9!}$$

b. $\left(2x + \frac{1}{x}\right)^{17}$

Solution:

Here, n is odd. So there will be 2 mid-terms:

$$\left(2x + \frac{1}{x}\right)^{17} = t_{\frac{n+1}{2}}, \; t_{\frac{n+1}{2}+1}$$

$$= t_{\frac{17+1}{2}}, \; t_{\frac{17+1}{2}+1}$$

$$= t_9, \; t_{10}$$

Now,

$$t_9 = t_{8+1} = C(17, 8)\,(2x)^{17-8} \cdot \left(\frac{1}{x}\right)^8$$

$$= \frac{17!}{9!\,8!} \cdot 2^9 x$$

Also,

$$t_{9+1} = C(17, 9)\,(2x)^{17-9} \cdot \left(\frac{1}{x}\right)^9$$

$$= \frac{17!}{8!\,9!} \times \frac{2^8}{x}$$

c. $\left(\frac{x}{a} - \frac{a}{x}\right)^{2n+1}$

Solution:

There will be two terms since 2n+1 is odd number.

$$\left(\frac{x}{a} - \frac{a}{x}\right)^{2n+1} = t_{\frac{n+1}{2}}, \; t_{\frac{n+1}{2}+1}$$

$$= t_{\frac{2n+2}{2}}, \; t_{\frac{2n+2}{2}+1}$$

$$= t_{(n+1)}, \; t_{(n+2)}$$

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$$t_{n+1} = t_{(n+1-1)+1} = t_{(n+1)} = C(2n+1, n)\left(\frac{x}{a}\right)^{2n+1-n} \cdot \left(\frac{a}{x}\right)^{n} \cdot (-1)^n$$

$$= \frac{(2n+1)!}{n!\,(n+1)!} \; \frac{x^{n+1}}{a^{n+1}} \cdot \frac{a^n}{x^n} \cdot (-1)^n$$

$$= (-1)^n \cdot \frac{(2n+1)!}{n!\,(n+1)!} \cdot \frac{x}{a}$$

Again,

$$(n+2)^{th} \text{ term} = t_{(n+1)+1} = C(2n+1, n+1) \cdot \left(\frac{x}{a}\right)^{2n+1-(n+1)} \cdot (-1)^{n+1} \left(\frac{a}{x}\right)^{n+1}$$

$$= \frac{(2n+1)!}{(n+1)!\,n!} \; \frac{x^n}{a^n} \cdot (-1)^{n+1} \cdot \frac{a^{n+1}}{x^{n+1}}$$

$$= \frac{(2n+1)!}{(n+1)!\,n!} \cdot \frac{x^n}{a^n} \cdot (-1)^{n+1} \cdot \frac{a}{x}$$

d. $\left(ax - \frac{1}{ax}\right)^{2n}$

Solution:

$$\left(ax - \frac{1}{ax}\right)^{2n} = \cancel{t_{\frac{n+1}{2}}} \; t_{\frac{n}{2}+1}$$

$$= t_{n+1}$$

Now,

$$t_{n+1} = t_{(n+1-1)+1} = C(2n, n)\,(ax)^{2n-n} \cdot (-1)^n \cdot \left(\frac{1}{ax}\right)^n$$

$$= \frac{2n!}{n!\,n!} \cdot a^n x^n \cdot \frac{1}{a^n x^n} \cdot (-1)^n$$

$$= \frac{2n!}{n!\,n!} \times (-1)^n$$

8. Show that the middle term in the expansion of

a. $(1+x)^{2n}$ is $\frac{1.3.5.\ldots(2n-1)}{n!} 2^n x^n$

Solution:

$(1+x)^{2n} = t_{\frac{2n}{2}+1}$

$= t_{n+1}$

Now, $t_{n+1} = C(2n, n)\, 1^{2n-n} . x^n$

$= \frac{2n!}{n!n!} . x^n$

$= \frac{2n (2n-1). (2n-2). \ldots\ldots 4 3.2.1\, x^n}{n!n!}$

$= \frac{1.2.3.4.\ldots(2n-1).2n.(2n-2)\, x^n}{n!n!}$

$= \frac{\{1.3.5.\ldots(2n-1)\}\{2.4.6\ldots(2n-2)2n.x^n\}}{n!n!}$

$= \frac{\{1.3.5\ldots(2n-1)\}\, 2^n(1.2.3.\ldots n)x^n}{n!n!}$

$= \frac{1.3.5\ldots(2n-1)\, x^n . 2^n}{n!}$

b. $\left(x - \frac{1}{x}\right)^{2n}$ is $\frac{1.3.5\ldots\ldots(2n-1)(-2)^n}{n!}$

Solution:

$\left(x - \frac{1}{x}\right)^{2n} = t_{\frac{2n}{2}+1}$

$= t_{n+1}$

Now, $t_{n+1} = C(2n, n).(x)^{2n-n} . (-1)^n \left(\frac{1}{x}\right)^n$

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$= \frac{2n!}{n!n!} x^n \cdot (-1)^n \cdot \frac{1}{x^n}$

$= \frac{1.2.3. \ldots\ldots (2n-2)(2n-1)\, 2n. (-1)^n}{n!n!}$

$= \frac{\{1.3.5 \ldots\ldots (2n-1)\} \{2, 4, 6 \ldots (2n-1)\, 2n\, (-1)^n\}}{n!n!}$

$= \frac{\{1, 3, 5 \ldots (2n-1)\} 2^n \{1, 2, 3 \ldots 2n)\, (-1)^n\}}{n!\, n!}$

$= \frac{\{1, 3, 5 \ldots 2n-1\} 2^n \{\cancel{1, 2, 3, \ldots n}\} (-1)^n\}}{n!\cancel{n!}}$

$= \frac{\{1, 3, 5 \ldots 2n-1\} \cdot 2^n (-1^n)}{n!}$

$= \frac{1, 3, 5 \ldots\ldots (2n-1) \cdot 2^n}{n!}$

Proved

In the expansion of $(1+x)^{21}$, the coefficient of $(2r+1)^{th}$ term is equal to the coefficient of $(3r+2)^{th}$ term. find $r$.

Solution:

~~$(2r+1)^{th}$ term $= t_{2r+1} = C(n, 2r)\, C(21, 2r)\, (2r)$~~

Here, $n = 21$, $a = 1$, $x = x$

Then,

$(2r+1)^{th}$ term $= t_{2r+1} = C(21, 2r) \cdot (1)^{n-r} \cdot x^{2r}$

b. In the expansion of $(1+x)^{2n+1}$, the coefficient of $x^r$ and $x^{r+1}$ are equal. Find $r$.

Solution:

Let $t_{r+1} = C(2n+1, r) \cdot x^r$ ——— (i)

Also,

$t_{(r+1)+1} = C(2n+1, r+1) \cdot x^{r+1}$ ——— (ii)

By question: (Coefficient are equal)

$C(2n+1, r) = C(2n+1, r+1)$

or $r + r + 1 = 2n + 1$

$\therefore n = r$

Note:
$C(n,r) = C(n,r')$
Then, $r + r' = n$

10. Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms of $(1+x)^{2n-1}$.

Solution:

For the middle term of $(1+x)^{2n}$

$= t_{\frac{n}{2}+1}$

$= t_{\frac{2n}{2}+1}$

$= t_{n+1}$

For the middle term of $(1+x)^{2n-1}$

$= t_{\frac{n+1}{2}}, t_{\frac{n+1}{2}+1}$

$= t_{\frac{2n-1+1}{2}}, t_{\frac{2n-1+1}{2}+1}$

$= t_{\frac{2n}{2}}, t_{\frac{2n+2}{2}}$

$= t_n, t_{n+1}$

~~Sum of Coefficient of $t_{n+1}$ $(1+x)^{2n}$ =~~

Now,

$$t_{n+1} = C(2n, n) \cdot x^n$$

$$= \frac{2n!}{n!n!} \times x^n \quad \text{——— (i)}$$

Also,

$$t_n = t_{(n-1)+1} = C(2n-1, n-1)\, x^{n-1}$$

$$= \frac{(2n-1)!}{n!(n-1)!\cancel{(n-1)!}} \times x^{n-1} \quad \text{——— (ii)}$$

Also,

$$t_{n+1} = C(2n-1, n)\, x^n$$

$$= \frac{(2n-1)!}{\cancel{(n+1)!}(n-1)!\,n!} \times x^n \quad \text{——— (iii)}$$

By Question: Coefficient of $(1+x)^{2n}$ = ~~Coeff~~ Sum of Coeff. of $(1+x)^{2n-1}$

Then, $\frac{2n!}{n!n!} = \frac{(2n-1)!}{(n-1)!n!} + \frac{(2n-1)!}{n!(n-1)!\cancel{(n-1)}}$

or $\frac{2n!}{n!n!} = \frac{2(2n-1)!}{n!(n-1)!}$

$$= \frac{2(2n-1)!}{n!(n-1)!} \times \frac{2n}{2n}$$

$$= \frac{2 \times 2n(2n-1)!}{2n \times n!(n-1)!}$$

$$= \frac{2 \times 2n!}{2n! \times n(n-1)!} \qquad \left\{ \because n(n-1)! = n! \right\}$$

$$= \frac{2 \times (2n)!}{2(n)!\, n!}$$

$$= \frac{2n!}{n!n!}$$

LHS = RHS proved.