c/w

Chapter-4
Sequence and series

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1. For any two numbers 'a' and 'b'. 'AM', GM and HM' between them are given by;

a. $A.M = \frac{a+b}{2}$

b. $G.M = \sqrt{ab}$

c. $H.M = \frac{2ab}{a+b}$

Proof :

a. Let A be the AM between a and b then a, A, b are in Arithmetic Sequence.

So, $A - a = b - A$

or, $A + A = a + b$

or, $2A = a + b$

or $A = \frac{a+b}{2}$

$\therefore AM = \frac{a+b}{2}$

b. Let G be the G.M between a and b then a, G, b are in geometric sequence.

or, $\frac{G}{a} = \frac{b}{G}$

or, $G^2 = ab$

or, $G = \sqrt{ab}$

$\therefore GM = \sqrt{ab}$

c. Let H be harmonic mean between a and b then a, H, b are in Harmonic Sequence. Then

$\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.S

$$\text{or, } \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\text{or, } \frac{1}{H} + \frac{1}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\text{or, } \frac{2}{H} = \frac{a+b}{ab}$$

$$\text{or, } \frac{2ab}{a+b} = H$$

$$\therefore H.M = \frac{2ab}{a+b}$$

2. The AM, GM and HM between any two unequal and positive number satisfy following condition:

a. $GM^2 = A.M \times H.M$

b. $AM > GM > H.M$

Proof:

Let a and b are two unequal and positive number then

$$A.M = \frac{a+b}{2}, \quad G.M = \sqrt{ab}, \quad H.M = \frac{2ab}{a+b}$$

a. $$AM \times HM = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$= ab$$

$$= (\sqrt{ab})^2$$

$$= GM^2$$

$$\therefore GM^2 = AM \times H.M$$

Proved.

b. $AM - GM = \frac{a+b}{2} - \sqrt{ab}$

$= \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{(\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2}{2}$

$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} > 0$

i.e. $AM - GM > 0$

$\rightarrow AM > GM$ —— (1)

Again,

We know that

$AM \times HM = GM^2$

or, $AM \times HM = GM \times GM$

or, $\frac{GM}{HM} = \frac{AM}{GM}$

$\rightarrow GM > HM$ —— (11)

Combining two relations, we get

$AM > GM > HM$

Proved

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Exercise - 4.1

1. Let A.M., G.M. and H.M. be the arithmetic mean, geometric mean and harmonic mean between two numbers.

a. If A.M. = 36 and H.M = 16, Find their G.M.

Solution:

We know,

$$GM^2 = AM \times HM$$

$$\text{or, } GM^2 = 36 \times 16$$

$$\text{or, } GM = \sqrt{576}$$

$$\therefore G.M = 24$$

b. If A.M. = 8 and G.M = 6, find their H.M.

Solution:

We know,

$$GM^2 = A.M \times H.M$$

$$\text{or, } 6^2 = 8 \times H.M$$

$$\therefore H.M = \frac{9}{2}$$

2.a. If H be the harmonic mean between a and b, prove that $\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$

Solution:

L.H.S

$$\frac{1}{H-a} + \frac{1}{H-b}$$

$$= \frac{1}{\frac{2ab}{a+b} - a} + \frac{1}{\frac{2ab}{a+b} - b}$$

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$$= \frac{1}{\frac{2ab - a^2 - ab}{a+b}} + \frac{1}{\frac{2ab - ab - b^2}{a+b}}$$

$$= \frac{a+b}{a(2b-a-b)} + \frac{a+b}{b(2a-a-b)}$$

$$= \frac{a+b}{a} \times \frac{1}{(b-a)} + \frac{a+b}{b} \times \frac{1}{(a-b)}$$

$$= \frac{a+b}{(a-b)} \left[-\frac{1}{a} + \frac{1}{b} \right]$$

$$= \frac{(a+b)}{(a-b)} \left[\frac{1}{b} - \frac{1}{a} \right]$$

$$= \frac{(a+b)}{(a-b)} \frac{(a-b)}{ab}$$

$$= \frac{a+b}{ab}$$

$$= \frac{1}{a} + \frac{1}{b}$$

Proved//

HW

b. If G is the geometric mean between a and b, show that

$$\frac{1}{G^2-a^2} + \frac{1}{G^2-b^2} = \frac{1}{G^2}$$

Solution:

L.H.S

$$\frac{1}{G^2-a^2} + \frac{1}{G^2-b^2}$$

$$= \frac{1}{(\sqrt{ab})^2-a^2} + \frac{1}{(\sqrt{ab})^2-b^2}$$

$$= \frac{1}{ab-a^2} + \frac{1}{ab-b^2}$$

$$= \frac{1}{a(b-a)} + \frac{1}{b(a-b)}$$

$$= \frac{-b+a}{ab(a-b)}$$

$$= \frac{1}{(\sqrt{ab})^2}$$

$$= \frac{1}{Gm^2}$$

Proved

c. If one G.M. 'G' and two 'A.M's' p and q are inserted between two numbers, prove that $G^2 = (2p-q)(2q-p)$

Solution:

Let the two numbers be 'a' and 'b' then,

$G = \sqrt{ab}$

or, $G^2 = ab$

Also, p and q are inserted between two numbers that is 'a' and 'b'.

$\therefore$ a, p, q, b are in A.P

Now, $p - a = q - p$

$\therefore a = 2p - q$ —— (i)

Also,

$q - p = b - q$

or, $b = 2q - p$ —— (ii)

Combining (i) and (ii)

$ab = (2p-q)(2q-p)$

or, $G^2 = (2p-q)(2q-p)$ proved//

d. If A be the A.M. and H, the H.M. between two quantities a and b, show that $\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$

Solution:

We know, $A = \frac{a+b}{2}$

$H = \frac{2ab}{a+b}$

Now, substituting the value in $\frac{a-A}{a-H} \times \frac{b-A}{b-H}$, we get

$$= \frac{a - \frac{a+b}{2}}{a - \frac{2ab}{a+b}} \times \frac{b - \frac{a+b}{2}}{b - \frac{2ab}{a+b}}$$

$$= \frac{(a-b)}{2} \times \frac{(a+b)}{2a(a-b)} \times \frac{(b-a)(a+b)}{2b(b-a)}$$

$$= \frac{(a+b)}{2a} \times \frac{(a+b)}{2b}$$

$$= \frac{(a+b)}{2} \times \frac{1}{\frac{2ab}{a+b}}$$

$$= \frac{A}{H}$$

Proved

e. If a, b, c are in A.P. Prove that $b^2 > ac$.

Solution:

$$b = \frac{a+c}{2}$$

or, $b^2 = \left(\frac{a+c}{2}\right)^2$

or, $b^2 = \frac{(a+c)^2}{4}$

Now, $b^2 - ac = \left(\frac{a+c}{2}\right)^2 - ac$

$$= \frac{(a+c)^2 - 4ab}{4}$$

$$= \frac{(a-c)^2}{4} > 0 \quad [\because (a-b)^2 = (a+b)^2 - 4ab]$$

3.a. If a, b, c are in A.P...., prove that b+c, c+a, a+b are in A.P. and bc, ca, ab are in H.P.

Solution:

Given, a, b, c are in A.P

Then, $b = \frac{a+c}{2}$

or $2b = a+c$

Now, we have to prove: b+c, c+a, a+b are in A.P

i.e. $c+a = \frac{b+c+a+b}{2}$

$$R.H.S = \frac{b+c+a+c}{2}$$

$$= \frac{2b+a+c}{2}$$

$$= \frac{a+c+a+c}{2}$$

$$= \frac{2(c+a)}{2}$$

$= c+a$ L.H.S proved

Again,

We have to prove; bc, ca, ab are in H.P

i.e. $ca = \frac{2bc.ab}{bc+ab}$ $\left[\because H.M = \frac{2ab}{a+b}\right]$

R.H.S

$$\frac{2bc.ab}{bc+ab}$$

$$= \frac{2ab^2c}{b(c+a)}$$

$$= \frac{2abc}{a+c}$$

$$= \frac{2a \cdot \frac{a+c}{2} \cdot c}{a+c} \quad \left[\because b = \frac{a+c}{2}\right]$$

$$= \frac{ac(a+c)}{(a+c)}$$

$= ac$ proved

b. If $\frac{1}{2}(x+y)$, y and $\frac{1}{2}(y+z)$ be in H.P. show that x, y, z are in G.P.

Solution:

Given, $\frac{(x+y)}{2}, y, \frac{y+z}{2}$ are in H.P

i.e. $$y = \frac{2\frac{(x+y)}{2} \cdot \frac{(y+z)}{2}}{\frac{(x+y)}{2} + \frac{(y+z)}{2}} \quad \left[\because H.M = \frac{2ab}{a+b}\right]$$

$$y = \frac{(x+y)(y+z)}{(x+y)(y+z)} \quad \text{--- (1)}$$

To prove:

x, y, z are in ~~G.M~~ G.P

i.e. $y = \sqrt{xz}$

~~LHS~~ Solving equation (1)

$y = (x+y)(y+z)$

or $y = \frac{xy + xz + y^2 + yz}{x+2y+z}$

or $xy + 2y^2 + yz = xy + xz + y^2 + yz$

or, $xy - xy + 2y^2 - y^2 + yz - yz = xz$

or, $y^2 = xz$

$\therefore y = \sqrt{xz}$ proved

c. If a, b, c are in H.P. Prove that $a(b+c), b(c+a), c(a+b)$ are in A.P.

Solution:

Given; a, b, c are in H.P

i.e. $b = \frac{2ac}{a+c} \longrightarrow b(a+c) = 2ac$

To prove:

$a(b+c), b(c+a), c(a+b)$ are in A.P

i.e. $b(c+a) = \frac{a(b+c) + c(a+b)}{2}$

R.H.S

$\frac{a(b+c) + c(a+b)}{2}$

$= \frac{ab + ac + ac + bc}{2}$

$= \frac{ab + bc + 2ac}{2}$

$= \frac{ab + bc + b(a+c)}{2}$

$= \frac{ab + bc + ab + bc}{2}$

$= \frac{2ab + 2bc}{2}$

$= \frac{2b(a+c)}{2}$

$= b(c+a)$

~~RH~~ LHS proved.

4.a. If x be the A.M between y, and z, y the G.M between z and x, prove that z will be the H.M between x and y.

Solution:

Given;

x be the A.M between y and z

i.e. $x = \frac{y+z}{2}$ —— (I)

y be the G.M between z and x

i.e $y = \sqrt{xz}$ —— (II)

To prove:

z be the H.M between x and y

i.e. $z = \frac{2xy}{x+y}$

From equation (I)

$$x = \frac{y+z}{2}$$

or, $x \cdot y = \left(\frac{y+z}{2}\right) y$

or, $xy = \frac{y^2+zy}{2}$

or, $2xy = zx + zy$ $\quad [\because y^2 = xz$, from (II)$]$

or, $2xy = z(x+y)$

or, $\frac{2xy}{(x+y)} = z$

$\therefore z = \frac{2xy}{(x+y)}$ proved.

b. If a, b, c are in A.P..., b, c, a are in H.P. Prove that c, a, b are in G.P.

Solution:

Given; a, b, c are in A.P

i.e. $b = \frac{a+c}{2}$ —— (i)

Also, b, c, a are in H.P

i.e. $c = \frac{2ab}{b+a}$ —— (ii)

To prove: c, a, b are in G.P

i.e. $a = \sqrt{bc}$

Now,

from equation (i) and (ii)

$c = \frac{2b \cdot a}{a+b}$

$c = \frac{(a+c)a}{a+b}$ $[\because 2b = a+c, \text{ from (i)}]$

or $ac + bc = a^2 + ac$

or $a^2 = bc$

$\therefore a = \sqrt{bc}$

proved.

c. If $a^x = b^y = c^z$ and a, b, c are in GP. Prove that x, y, z are in H.P

Solution:

Given;

~~let~~ $a^x = b^y = c^z = r$ (let)

Now,

$a^x = r^1$ $\quad$ $b^y = r^1$ $\quad$ $c^z = r^1$

$\therefore a = r^{1/x}$ $\quad$ $\therefore b = r^{1/y}$ $\quad$ $\therefore c = r^{1/z}$

Also, a, b, c are in G.P

i.e. $b = \cancel{a+c}\ \sqrt{ac}$

or $b^2 = ac$ — (i)

To prove:

x, y, z are in H.P

i.e. $y = \dfrac{2xz}{x+z}$

Now, from equation (i),

$b^2 = ac$

or $(r^{1/y})^2 = r^{1/x} \cdot r^{1/z}$

or, $r^{\frac{2}{y}} = r^{1/x + 1/z}$

or $\dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z}$

or $\dfrac{2}{y} = \dfrac{x+z}{xz}$

or $\dfrac{2xz}{y} = x+z$

$\therefore y = \dfrac{2xz}{x+z}$ proved

5.a. If x^2, b^2, y^2 are in A.P.., a, x, b and b, y, c both are in G.P., prove that a, b, c are in A.P.

Solution:

Given,

x^2, b^2, y^2 are in A.P

i.e. $b^2 = \dfrac{x^2+y^2}{2}$ — (i)

Also,

a, x, b are in G.P

i.e. $x = \sqrt{ab}$ — (ii)

b, y, c are in G.P

i.e. $y = \sqrt{bc}$ —— (iii)

To prove:

a, b, c are in A.P

i.e. $b = \frac{a+c}{2}$

from equation (i), (ii) and (iii).

$$b^2 = \frac{x^2+y^2}{2}$$

$$\text{or, } 2b^2 = ab + bc$$

$$\text{or, } b^2 = \frac{b(a+c)}{2}$$

$$\therefore b = \frac{a+c}{2}$$ proved

b. If a, b, c be in A.P., b, c, d in G.P and c, d, e in H.P.

Prove that a, c, e are in G.P.

Solution:

Given;

a, b, c are in A.P

i.e. $b = \frac{a+c}{2}$ —— (i)

b, c, d are in G.P

i.e. $c = \sqrt{bd}$ —— (ii)

~~To prove:~~

c, d, e in H.P

i.e. $d = \frac{2ce}{c+e}$ —— (iii)

To prove: a, c, e in G.P

i.e. $c = \sqrt{ae}$

Putting the value of b and d in equation (ii)

$$c = \sqrt{bd}$$

or, $c = \sqrt{\frac{a+c}{2} \times \frac{2ce}{c+e}}$

or, $c^2 = \frac{(a+c)ce}{c+e}$

or, $c^2 = \frac{ace + c^2e}{(c+e)}$

or, $\frac{c^2}{c} = \frac{ce(a+c)}{c(c+e)}$ [$\because$ dividing by 'c' both side]

or, $c = \frac{e(a+c)}{c+e}$

or, $c^2 + ce = ae + ce$

or, $c^2 = ae$

$\therefore c = \sqrt{ae}$ proved

6a. Find two numbers whose arithmetic mean is 25 and geometric mean is 20.

Solution:

Let two numbers be a and b

By question:

$A.M = 25$

or, $\frac{a+b}{2} = 25$

or, $a+b = 50$ —— (i)

Also,

$G.M = 20$

$\sqrt{ab} = 20$

or, $ab = 400$ —— (ii)

We know that

$(a-b)^2 = (a+b)^2 - 4ab$

or, $(a-b)^2 = 2500 - 1600$

or $(a-b)^2 = 900$

or $(a-b) = 30$ ——— (iii)

Solving equation (i) and (iii)

$a+b = 50$

$a-b = 30$

$2a = 80$

$\therefore a = 40$

putting the value of a in eq (i)

$40 + b = 50$

$\therefore b = 10$

b. If the G.M. and H.M between two positive numbers a and b are such that $G.M = \frac{5}{4} H.M$. Prove that $a:b = 4:1$.

Solution:

Let two numbers be a and b

By question:

$G.M = \frac{5}{4} H.M$

or $\sqrt{ab} = \frac{5}{4} \cdot \frac{2ab}{(a+b)}$ ——— (i)

To prove: $a:b = 4:1$

ie. $\frac{a}{b} = \frac{4}{1}$

$\therefore a = 4b$

Solving equation (i)

$\sqrt{ab} = \frac{5}{4} \cdot \frac{2ab}{(a+b)}$

Squaring both sides;

$$ab = \frac{100a^2b^2}{16(a+b)^2}$$

or, $$\frac{16}{100} = \frac{a^2b^2}{ab(a+b)^2}$$

or, $$\frac{16}{100} = \frac{ab}{a^2+2ab+b^2}$$

or, $16a^2 + 32ab + 16b^2 = 100ab$

or, $16a^2 + 16b^2 = 68ab$

or, $4(a^2+b^2) = 17ab$

or, $4a^2 - 17ab + 4b^2 = 0$

or, $4a^2 - 16ab - ab + 4b^2 = 0$

or, $(a-4b)(4a-b) = 0$

Either | OR
$a - 4b = 0$ | $4a - b = 0$
$\therefore a = 4b$ | $\therefore b = 4a$

So, the ratio is 4:1

7.a. If a, b and c are ~~in~~ the pth, qth, rth term of an A.P.., prove that $a(q-r) + b(r-p) + c(p-q) = 0$

Solution:

Given;

$t_p = a$

or, $t_1 + (p-1)d = a$ — (i) $[\because t_n = a + (n-1)d]$

Also,

$t_q = b$

or, $t_1 + (q-1)d = b$ — (ii)

Also,

$t_r = c$

or, $t_1 + (r-1)d = c$ — (iii)

To prove:

$a(q-r) + b(r-p) + c(p-q) = 0$

LHS:

$a(q-r) + b(r-p) + c(p-q)$

$= (t_1 + pd - d)(q-r) + (t_1 + qd - d)(r-p) + (t_1 + rd - d)(p-q)$

$= t_1q - t_1r + pdq - qdr - dq + dr + t_1r + qdr - rd - t_1p - pdq + pd$
$+ t_1p + pdr - dp - t_1q - rdq + dq$

$= 0$

Proved//

b. If a, b and c are the p^{th}, q^{th} and r^{th} term of a G.P.
Prove that. $a^{q-r} b^{r-p} c^{p-q} = 1$

Solution:

Let A be the 1^{st} term and R be the ~~common difference~~ common ratio.

Given,

$t_p = a$

or, $AR^{p-1} = a$ —— (I)

Also,

$t_q = b$

or, $AR^{q-1} = b$ —— (II)

Also,

$t_r = c$

or, $AR^{r-1} = c$ —— (III)

To prove:

$a^{q-r} b^{r-p} c^{p-q} = 1$

Substituting the value of a, b, c in L.H.S

$= (AR^{p-1})^{q-r} (AR^{q-1})^{r-p} (AR^{r-1})^{p-q}$

$= A^{(q-r)} \cdot R^{(p-1)(q-r)} \times A^{(r-p)} \cdot R^{(q-1)(r-p)} \times A^{(p-q)} \cdot R^{(r-1)(p-q)}$

$= A^0 R^0$

$= 1$ proved//