Chapter 13

Applications of Derivatives

Exercise 13.1

1. **Using L Hospital's rule, evaluate the following:**

a) $\lim\limits_{x\to 0} \dfrac{x^3 - 8}{x^2 - 4}$

Sol^n: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to 2} \frac{3x^2}{2x} = \lim_{x \to 2} \frac{3x^2}{2x} = \frac{3 \times 2}{2} = 3.$$

b) $\lim\limits_{x\to 1} \dfrac{x^4 - 3x^3 + 2}{2x^3 - 5x^2 + 3x}$

Sol^n: $\lim\limits_{x \to 1} \dfrac{x^4 - 3x^3 + 2}{2x^3 - 5x^2 + 3x}\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{4x^3 - 9x^2}{6x^2 - 10x + 3} = \frac{4 - 9}{6 - 10 + 3} = \frac{-5}{-1} = 5.$$

c) $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{x^2}$

Sol^n: $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{x^2}\left(\text{form } \dfrac{0}{0}\right) = \lim\limits_{x \to 0} \dfrac{e^x - 1}{2x}\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

d) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$

Sol^n: $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}(-1) + 2\sin x}{2\sin x \cos x}\left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}(-1) + 2\cos x}{2\{\sin(-\sin x) + \cos x.\ \cos x\}}$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{2(\cos^2 x - \sin^2 x)}$$

$$= \frac{e^0 + e^0 + 2\cos 0}{2(\cos^2 0 - \sin^2 0)} = \frac{1 + 1 + 2}{2(1 - 0)} = 2.$$

e) $\lim\limits_{x\to 1} \dfrac{1 - 2x + x^2}{1 + \log x - x}$

Sol^n: $\lim\limits_{x\to 1} \dfrac{1-2x + x^2}{1 + \log x - x} \quad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 1} \frac{-2 + 2x}{\frac{1}{x} - 1} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{2}{\frac{-1}{x^2}} = -2.$$

f) $\lim\limits_{x\to 0} \dfrac{\tan ax}{\tan bx}$

Sol^n: $\lim\limits_{x\to 0} \dfrac{\tan ax}{\tan bx} \quad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{a \sec^2 ax}{b \sec^2 bx} = \frac{a}{b}.$$

g) $\lim\limits_{x\to 0} \dfrac{x^2 - \sin^2 x}{x \sin x}$

Sol^n: $\lim\limits_{x\to 0} \dfrac{x^2 - \sin^2 x}{x \sin x} \quad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{2x - 2\sin x \cos x}{x\cos x + \sin x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2 - 2\{\sin x\,(-\sin x) + \cos x \cos x\}}{x\,(-\sin x) + \cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{2 - 2\,(\cos^2 x - \sin^2 x)}{-x \sin x + \cos x + \sin x}$$

$$= \frac{2 - 2\,(1 - 0)}{0 + 1 + 0} = \frac{0}{1} = 0.$$

h) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$

Sol^n: $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x} \quad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\sec x \,.\, \sec x \tan x}{-(-\sin x)}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \,.\, \tan x}{\sin x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \, \frac{\sin x}{\cos x}}{\sin x}$$

$$= \lim_{x\to 0} 2\sec^3 x = 2.$$

i) $\lim\limits_{x\to 0} \dfrac{x - \sin x \cos x}{x^3}$

Sol^n: $\lim\limits_{x\to 0} \dfrac{x - \sin x \cos x}{x^3}$ $\quad \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{1 - \sin x\,(-\sin x) + \cos x \cos x}{3x^2}$

$= \lim\limits_{x\to 0} \dfrac{1 - (\cos^2 x - \sin^2 x)}{3x^2}$ $\quad \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{1 + \sin^2 x - \cos^2 x}{3x^2}$ $\quad \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{2\sin x \cos x - 2\cos x(-\sin x)}{6x}$ $\quad = \lim\limits_{x\to 0} \dfrac{2\sin x \cos x + 2\sin x \cos x}{6x}$

$= \lim\limits_{x\to 0} \dfrac{2\sin 2x}{6x}$ $\quad = \lim\limits_{x\to 0} \dfrac{\sin 2x}{3x} \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{2\cos 2x}{3} = \dfrac{2}{3}.$

j) $\lim\limits_{x\to 0} \dfrac{(e^x - 1)\tan x}{x^2}$

Sol^n: $\lim\limits_{x\to 0} \dfrac{(e^x - 1)\tan x}{x^2}$ $\quad \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x \sec^2 x + \tan x\, e^x - \sec^2 x}{2x}$ $\quad \left(\text{form } \dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{e^x\, 2\sec x \sec x \tan x + \sec^2 x e^x + e^x \sec^2 x + e^x \tan x - 2\sec x \sec x \tan x}{2x} = \dfrac{0 + 1 + 1 + 0 - 0}{2} = 1.$

2. Find the limiting values of the following (Use L' Hospital's rule)

a) $\lim\limits_{x\to\infty} \dfrac{2x^2 + 3x}{1 + 5x^2}$

Sol^n: $\lim\limits_{x\to\infty} \dfrac{2x^2 + 3x}{1 + 5x^2}$ $\quad \left(\text{form } \dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{4x + 3}{10x}$ $\quad \left(\text{form } \dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{4}{10} = \dfrac{2}{5}.$

b) $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5}{2x^3 + 4x + 3}$

Sol^n: $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5}{2x^3 + 4x + 3}$ $\quad \left(\text{form } \dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{6x}{6x^2 + 4}$ $\quad \left(\text{form } \dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{6}{12x} = \lim\limits_{x\to\infty} \dfrac{1}{x}\dfrac{6}{12} = 0.$

c) $\displaystyle \lim_{x\to\frac{\pi}{2}} \frac{\sec 7x}{\sec 5x}$

Sol^n: $\displaystyle \lim_{x\to\frac{\pi}{2}} \frac{\sec 7x}{\sec 5x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos 5x}{\cos 7x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\sin 5x \,.\, 5}{-\sin 7x.\, 7}$$

$$= \frac{\sin\left(5\frac{\pi}{2}\right).\, 5}{\sin\left(7\frac{\pi}{2}\right).7} = \frac{5}{-7} = \frac{-5}{7}.$$

d) $\displaystyle \lim_{x\to 0} \frac{\log \sin x}{\cot x}$

Sol^n: $\displaystyle \lim_{x\to 0} \frac{\log \sin x}{\cot x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} = \lim_{x\to 0} \frac{-\cos x}{\operatorname{cosec} x}$$

$$= \lim_{x\to 0} (\sin x \cos x) = \sin 0 \times \cos 0 = 0.$$

e) $\displaystyle \lim_{x\to 0} \frac{\log x}{\log \cot x}$

Sol^n: $\displaystyle \lim_{x\to 0} \frac{\log x}{\log \operatorname{cost} x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{-\cot x}{\operatorname{cosec}^2 x}\right)$$

$$= \lim_{x\to 0} -\frac{\cos x}{\sin x}\frac{\sin^2 x}{x}$$

$$= -\lim_{x\to 0} \frac{\sin 2x}{2x} \left(\text{form } \frac{0}{0}\right)$$

$$= -\lim_{x\to 0} \frac{2\cos 2x}{2}$$

$$= \cos 2.0 = 1$$

f) $\lim_{x\to\infty} \frac{x^4}{e^x}$

Sol^n: $\lim_{x\to\infty} \frac{x^4}{e^x}$ $\left(\text{form } \frac{\infty}{\infty}\right)$

Applying L - Hospital rule until $\frac{\infty}{\infty}$ is vanished

$= \lim_{x\to\infty} \frac{4x^3}{e^x} \left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{12x^2}{e^x} \left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{24x}{e^x} \left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{24}{e^x} = 24 \times \frac{1}{e^\infty} = 24 \times 0 = 0.$

g) $\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}$

Sol^n: $\lim_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}$ $\left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{5\sec^2 5x}{\sec^2 x}$ $\left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{2\cos^2 x}{\cos^2 5x}$ $\left(\text{form } \frac{0}{0}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{-10 \cos x \sin x}{-2\cos 5x \sin 5x.\ 5}$ $\left(\text{form } \frac{0}{0}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\sin 10x}$ $\left(\text{form } \frac{0}{0}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{2\cos 2x}{10 \cos 10x}$

$= \frac{2(-1)}{10(-1)} = \frac{1}{5}.$

Hint and solution of MCQ's

1. Since $\infty + \infty = \infty$, is not the indeterminate form.
2. From L Hospital's rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)}$$

3. From L Hospital's rule

$\lim_{x\to c} \frac{f'}{g'}$ exists if both (a) and (b) satisfies

4. Using L Hospital's rule

$$\lim_{x \to 0} \frac{x^2}{4\sin x}\left(\text{from } \frac{0}{0}\right) = \lim_{x \to 0} \frac{2x}{4\cos x} = \frac{2.0}{4\cos 0} = 0$$

5. $$\lim_{x \to 0} \frac{\tan x}{x}\left(\text{from } \frac{0}{0}\right) = \lim_{x \to 0} \sec^2 x = \sec^2 0 = 1$$

6. $$\lim_{x \to 0} \frac{e^{3x} - 1}{2x}\left(\text{from } \frac{0}{0}\right) = \lim_{x \to 0} \frac{3.\ e^{3x}}{2} = \frac{3.\ e^0}{2} = \frac{3}{2}$$

7. $$\lim_{x \to 0} \frac{x - \sin x}{\sin^{-2} x}\left(\text{from } \frac{0}{0}\right) = \lim_{x \to 0} \frac{1 - \cos x}{2\sin x \cos x}\left(\text{from } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sin x}{2\cos 2x} = \frac{\sin 0}{2.\cos 0} = 0$$

8. $$\lim_{x \to 0} x^2 e^{-x} \text{ (form } 0 \times 1) = 0 \times e^o = 0$$

9. $$\lim_{x \to 0} \frac{xe^x - \log(1+x)}{\sin^2 x}\left(\text{from } \frac{0}{0}\right) = \lim_{x \to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2\sin x \cos x}$$

$$= \lim_{x \to 0} \frac{e^x(x+1) - \frac{1}{1+x}}{\sin 2x}\left(\text{from } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x.1 + (x+1)e^x + \frac{1}{(1+x)^2}}{2\cos 2x}\left(\text{from } \frac{0}{0}\right)$$

$$= \frac{e^0 + (0+1)e^0 + \frac{1}{(1+0)^2}}{2\cos 0} = \frac{1+1+1}{2} = \frac{3}{2}$$

10. $$\lim_{x \to \frac{1}{2}} \frac{\tan \pi x}{\sec^2 \pi x}\left(\text{form } \frac{\infty}{\infty}\right) = \lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{\cot \pi x}\left(\text{from } \frac{0}{0}\right)$$

$$= \lim_{x \to \frac{1}{2}} \cos \pi x.\ \sin \pi x$$

$$= \cos\pi.\frac{1}{2} \times \sin\pi.\frac{1}{2}$$

$$= 0 \times 1 = 0$$

11. $$\lim_{x \to 0} \frac{a\sin x - \pi\tan x}{x^2}\left(\text{from } \frac{0}{0}\right)$$

$= \lim_{x \to 0} \frac{a\cos x - \pi\sec^2 x}{2x}$ will take the $\left(\text{from } \frac{0}{0}\right)$, and hence limit is finite only if $a = \pi$

12. $$\lim_{x \to \theta} \frac{x\sin\theta - \theta\sin x}{\theta - x}\left(\text{from } \frac{0}{0}\right)$$

Using L' Hospital rule

$$= \lim_{x \to \theta} \frac{\sin\theta - \theta\cos x}{-1} \qquad [\boldsymbol{\theta} \textbf{\textit{ is constant}}]$$

$$= \frac{\sin\theta - \theta\cos\theta}{-1} = \theta\cos\theta - \sin\theta$$

13. $\lim_{x \to \infty} \frac{x^2}{e^{x/2}} \left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x \to \infty} \frac{2x}{\frac{1}{2} \cdot e^{x/2}}$

$= \lim_{x \to \infty} \frac{4x}{e^{x/2}} \left(\text{form } \frac{\infty}{\infty}\right)$

$= \lim_{x \to \infty} \frac{4}{\frac{1}{2} e^{x/2}} \quad = \frac{4}{\frac{1}{2} \cdot e^{\infty}} = \frac{4}{\infty} = 0$

14. $\lim_{x \to 0} \frac{1 - \cos kx}{x^2} = 8 \qquad \Rightarrow \lim_{x \to 0} \frac{k^2 \cos kx}{2} = 8$

$\Rightarrow \lim_{x \to 0} \frac{k \sin kx}{2x} = 8 \qquad \Rightarrow \frac{k^2 . \cos 0}{2} = 8$

$\Rightarrow k^2 = 16$

$\therefore k = \pm 4$

Exercise 13.2

1. Compute Δy, dy and Δy - dy when $y = \frac{x^2}{2} + 3x$, x = 2 and dx = 0.5

Sol^n: We have, $y = \frac{x^2}{2} + 3x$, x = 2 and dx = 0.5

$\therefore \quad dy = f'(x)dx = \left(\frac{2x}{2} + 3\right) dx = (x + 3)dx = (2 + 3) \times 0.5 = 2.5.$

$\therefore \quad \Delta y = f(x + \Delta x) - f(x)$

$= \left[\frac{(x + \Delta x)^2}{2} + 3(x + \Delta x)\right] - \left[\frac{x^2}{2} + 3x\right]$

$= \frac{1}{2} [(x + \Delta x)^2 - x^2) + 3\Delta x$

$= \frac{1}{2} [(2 + 0.5)^2 - 2^2] + 3 \times 0.5$

$= \frac{1}{2} [6.25 - 4] + 1.5 = \frac{2.25}{2} + 1.5 = 2.625.$

$\therefore \quad \Delta y - dy = 2.625 - 2.5 = 0.125.$

2. Find an approximate change in the volume of a cube of side x m, caused by increasing the sides by 1%. What is the percentage increment in the volume ?

Sol^n: Suppose that v be the volume of the cube $\therefore$ $v = x^3$.......(i)

$dx = 1\% \text{ of } x = x \times \frac{1}{100} = \frac{x}{100} = 0.01x = \Delta x$

$\therefore$ Increment in volume $= (x + 0.01x)^3 - x^3 = (1.01x)^3 - x^3 = 1.030301x^3$

$\therefore$ Percentage increment in volume $= \frac{0.03x^3}{x^3} \times 100 = 3\%.$