Chapter-7
Complex Number

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An ordered pair of real number is called Complex number.
Generally, we denote it by z.

Note:

i. If $z = (a,b)$ and $w = (c,d)$ be two complex number then

$z+w = (a+c,\ b+d)$

$z \cdot w = (ac-bd,\ bc+ad)$

Theorem

$i^2 = -1$

proof:

$i = (0,1)$

$= (0,-1)(0,1)$

$= (0-1,\ 0+0)$

$= -1$

Power of i

$i^2 = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = (i^2)^3 \cdot i = -i$

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Exercise - 7.1

1. Evaluate:

a. $(1,0)^2$

Solution:

$= 1^2$

$= 1$

b. $(1,0)^5$

Solution:

$= 1^5$

$= 1$

c. $(0,1)^5$

Solution:

$= (i)^5$

$= (i)^4 \cdot i$

$= i \cdot 1$

$= i$

d. $(0,1)^{11}$

Solution:

$= (i)^{11}$

$= (i)^4 \cdot (i)^4 \, (i)^2 \cdot i$

$= 1 \cdot 1 \cdot (-1) \cdot i$

$= -i$

2. Find the value of x and y in each of the following.

a. $(x,y) = (2,3) + (3,2)$

Solution:

$(x,y) = (2+3,\ 3+2)$

$= (5,5)$

$\therefore x = 5$ & $y = 5$

b. $(x,y) = (2,1) + (-2,-1)$

Solution:

$(x,y) = (2-2,\ 1-1)$

$= (0,0)$

$\therefore x = 0$ & $y = 0$

c. $(x,y) = (2,3) - (3,2)$

or, $(x,y) = (2-3,\ 3-2)$

$= (-1,1)$

$\therefore x = -1$ & $y = 1$

d. $(2,3) = (1,1) + (x,y)$

Solution:

$(2,3) = (1+x, 1+y)$

So,

$1+x = 2$ $\qquad$ $1+y = 3$

$\therefore x = 1$ $\qquad$ $\therefore y = 2$

e. $(x,y) = (1,1).(2,3)$

solution:

$(x,y) = (1\cdot 2 - 1\cdot 3, 1\cdot 2 + 3\cdot 1)$

$= (2-3, 2+3)$

$= (-1, 5)$

$\therefore x = -1$ & $y = 5$

f. $(x,y) = \dfrac{(1,1)}{(3,4)}$

Solution:

$(x,y) = \dfrac{1+i}{3+4i} \times \dfrac{3-4i}{3+4i}$

$= \dfrac{3-4i+3i-4i^2}{3^2-(4i)^2}$

$= \dfrac{3-4i+3i+4}{9+16}$

$= \dfrac{7-i}{25}$

$= \dfrac{7}{25}, \dfrac{-i}{25}$

$\therefore x = \dfrac{7}{25}$ & $y = \dfrac{-i}{25}$

3. Simplify:

a. $\sqrt{-9} + \sqrt{-25} - \sqrt{-36}$

Solution:

$= \sqrt{9i^2} + \sqrt{25i^2} - \sqrt{36i^2}$

$= 3i + 5i - 6i$

$= 2i$

b. $(3 - \sqrt{-4})(2 + \sqrt{-9})$

Solution:

$= (3 - 2i)(2 + 3i)$

$= 6 + 9i - 4i - 6i^2$

$= 6 + 5i + 6$

$= 12 + 5i$

c. $\sqrt{-16} \cdot \sqrt{-1}$

Solution:

$= 4i \cdot i$

$= 4i^2$

$= -4 \quad [\because i^2 = -1]$

d. $3i^2 + i^3 + 9i^4 - i^7$

Solution:

$= 3(-1) + (-i) + 9(1) - i^4 . (i)^2 . i.$

$= -3 - i + 9 - (1 \times -1i)$

$= -3 - i + 9 + i$

$= 6$

e. $\frac{1}{i} - \frac{1}{i^2} + \frac{1}{i^3} - \frac{1}{i^4}$

Solution:

$= \frac{1}{i} - \frac{1}{-1} + \frac{1}{-i} - \frac{1}{1}$

$= \frac{1}{i} + \frac{1}{1} - \frac{1}{i} - \frac{1}{1}$

$= 0$

f. $\frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4}$

Solution:

$= \frac{1}{i} + \frac{1}{-1} + \frac{1}{-i} + \frac{1}{1}$

$= \frac{1}{i} - 1 - \frac{1}{i} + 1$

$= 0$

4. Prove that:

a. $(1+i)^4 \cdot \left(1 + \frac{1}{i}\right)^4 = 16$

Solution:

$= \left[(1+i)\ (1-i)\right]^4$

$= \left[1^2 - i^2\right]^4$

$= [1+1]^4$

$= 2^4$

$= 16$

b. $(1-i^3)^6 \cdot \left(1 - \frac{1}{i^3}\right)^6 = 32$

Solution:

$= (1-i^3)^6 \quad (1+i^3)^6$

$= [(1-i^3)\ (1+i^3)]^6$

$= [1^2 - i^6]^6$

$= [1+1]^6$

$= 2^6$

$= 32 + 32$

5. Find the real numbers x and y if.

a. $x + iy = (2-3i) \cdot (3-2i)$

Solution:

$(x+iy) = (6 - 4i - 9i + 6i^2)$

$= (6 - 13i - 6)$

$= (0 - 13i)$

$= (0, -13i)$

$\therefore x = 0$ & $y = -13$

b. $(x-1)i + (y+1) = (1+i)(4-3i)$

Solution:

$(x-1)i + (y+1) = 4 - 3i + 4i - 3i^2$

or, $(y+1)\ (x-1)i = 4 + i + 3$

or, $(y+1)\ (x-1)i = 7 + i$

or, $(y+1)\ (x-1)i = (7, i)$

Comparing the corresponding elements:

$y + 1 = 7$ $\quad\quad$ $(x-1)i = i$

$\therefore y = 6$ $\quad\quad$ or, $x - 1 = 1$

$\therefore x = 2$

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6.a Show that $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i}$ is purely a real number.

Solution:

Given;

$$\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$$

$$= \frac{(3+2i)(2+5i) + (3-2i)(2-5i)}{2^2 - 25i^2}$$

$$= \frac{(6-10) + i(15+4) + (6-10) + i(-4-15)}{4+25}$$

$$= \frac{-4 + 19i - 4 - 19i}{29}$$

$$= \frac{-8}{29}$$ which is purely a real number.

b. Show that $\dfrac{(2+3i)^2 - (2-3i)^2}{(2+3i)^2 + (2-3i)^2}$ is a Complex number.

Solution:

Given;

$$\frac{(2+3i)^2 - (2-3i)^2}{(2+3i)^2 + (2-3i)^2}$$

$$= \frac{4 + 12i + 9i^2 - 4 + 12i - 9i^2}{4 + 12i + 9i^2 + 4 - 12i + 9i^2}$$

$$= \frac{24i}{8 - \cancel{12}\, 9 - 9}$$

$$= \frac{24i}{8-18}$$

$$= \frac{24i}{-10}$$

$$= \frac{-12i}{5}$$ which is a complex number.

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7. If $z_1 = 1 + i$ and $z_2 = 2 - 3i$, find $z_1 . z_2$, $\frac{z_1}{z_2}$ and $z_1^2 + z_2^2$. Also, verify that $z_1 z_2 = z_2 . z_1$

Solution:

Given;

$z_1 = 1 + i$

$z_2 = 2 - 3i$

Now,

$$z_1 . z_2 = (1+i)(2-3i)$$
$$= 2 - 3i + 2i - 3i^2$$
$$= 2 - i + 3$$
$$= 5 - i$$

Also,

$$\frac{z_1}{z_2} = \frac{1+i}{2-3i}$$
$$= \frac{(1+i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)}$$
$$= \frac{2 + 3i + 2i + 3i^2}{4 - 9i^2}$$
$$= \frac{2 + \cancel{8i}\ 5i - 3}{4+9}$$
$$= \cancel{\frac{2+8i}{13}} \quad \frac{-1+5i}{13}$$
$$= \frac{1}{13}(-1 + 5i)$$

Again,

$$(z_1)^2 + (z_2)^2 = (1+i)^2 + (2-3i)^2$$
$$= 1 + 2i + i^2 + 2^2 - 12i + 9i^2$$
$$= 1 + 2i - 1 + 4 - 12i - 9$$
$$= -5 - 10i$$

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To prove:

$z_1 . z_2 = z_2 . z_1$

We know;

$z_1 . z_2 = (5-i)$ [from above solution]

Also,

$z_2 . z_1 = (2-3i)(1+i)$

$= 2 + 2i - 3i - 3i^2$

$= 2 - i + 3$

$= 5 - i$

$\therefore$ L.H.S = RHS proved

Conjugate of Complex number

If $z = a + ib$ be a complex number then conjugate of z is defined and denoted by $\bar{z} = a - ib$

Properties of Conjugate

Let $z = a + ib$ and $W = c + id$ are two complex number.

Then,

(i) $Re(z) = \frac{1}{2}(z + \bar{z})$

(ii) $Im(z) = \frac{1}{2}(z - \bar{z})$

(iii) $\overline{z + w} = \bar{z} + \bar{w}$

(iv) $\overline{zw} = \bar{z} . \bar{w}$

(v) $\bar{\bar{z}} = z$

(vi) $(\bar{z})^2 = \overline{z^2}$

(vii) $z . \bar{z}$ = a real number.

Proofs:

(i) $Re(z) = \frac{1}{2}(z+\bar{z})$

Given;

$z = a+ib$, $w = c+id$

$\bar{z} = a-ib$, $\bar{w} = c-id$

Now,

$z+\bar{z} = a+ib+\cancel{c+}a-ib$

$= 2a$

or, $a = \frac{1}{2}(z+\bar{z})$

$\therefore Re(z) = \frac{1}{2}(z+\bar{z})$

(ii) $z-\bar{z} = a+ib-a+ib$

$= 2ib$

or, $ib = \frac{z-\bar{z}}{2}$

or, $ib = \frac{1}{2}(z-\bar{z})$

$\therefore Im(z) = \frac{1}{2}(z-\bar{z})$

(iii) $z+w = a+ib+c+id$

$= a+c+i(b+d)$

$\overline{z+w} = (a+c)-i(b+d)$

Also,

$\bar{z}+\bar{w} = a-ib+c-id$

$= a+c-i(b+d)$

$\therefore \overline{z+w} = \bar{z}+\bar{w}$

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(iv) $z \cdot w = (a+ib)(c+id)$
$= (ac-bd) + i(bc+ad)$

$\overline{z \cdot w} = (ac-bd) - i(bc+ad)$

Also,

$\bar{z} \cdot \bar{w} = (a-ib)(c-id)$
$= (ac-bd) - i(bc+ad)$

$\therefore \overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(v) $(\bar{z})^2 = (a-ib)^2$
$= a^2 - 2aib - b^2$
$= a^2 - b^2 - 2iab$

$z^2 = (a+ib)^2$
$= a^2 + 2aib + i^2b^2$
$= a^2 - b^2 + 2aib$

Also,

$\overline{(z^2)} = a^2 - b^2 - 2aib$

$\therefore (\bar{z})^2 = \overline{(z^2)}$

(vi) $\bar{z} = a - ib$

Also,

$\bar{\bar{z}} = a + ib$

Also,

$z = a + ib$

$\therefore \bar{\bar{z}} = z$

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(vii) $z.\bar{z} = (a+ib)(a-ib)$

$= a^2 - i^2 b^2$

$= a^2 + b^2$ which is a real number.

Absolute Value

Let $z = a+ib$ be a complex number then absolute value of z is defined and denoted by;

$$|z| = \sqrt{a^2+b^2}$$

Example:

$$z = 3+4i$$

$$|z| = \sqrt{3^2+4^2}$$

$$= \sqrt{9+16}$$

$$= \sqrt{25} = 5$$

Properties of Absolute Value.

Let $z = a+ib$ and $W = c+id$ are two complex number then;

i. $|z| = |\bar{z}|$

ii. $|z.w| = |z|.|w|$

iii. $|z| = 0$ iff $z=0$

iv. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

v. $z.\bar{z} = |z|^2$

vi. $Re(z) \leq |z|$ and $Im(z) \leq |z|$

Solution:

Proofs:

Given;

$z = a+ib$, $\bar{z} = a-ib$

$W = c+id$, $\bar{W} = c-id$

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i. $|z| = \sqrt{a^2+b^2}$

Also,

$$|\bar{z}| = \sqrt{a^2+(-b)^2}$$
$$= \sqrt{a^2+b^2}$$
$$\therefore |z| = |\bar{z}|$$

ii. $|z \cdot w| = |z| \cdot |w|$

Solution:

$$z \cdot w = (a+ib)(c+id)$$
$$= (ac-bd) + i(bc+ad)$$

$$|z \cdot w| = \sqrt{(ac-bd)^2 + (bc+ad)^2}$$
$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$
$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$
$$= \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}$$
$$= |z| \cdot |w|$$

iii. $|z| = 0$ iff $z = 0$

Solution:

Let $z = a+ib$ and $|z| = 0$

$$|z| = \sqrt{a^2+b^2}$$
$$\text{or, } 0 = \sqrt{a^2+b^2}$$
$$\therefore a = 0 \ \& \ b = 0$$

Again,

$$z = 0$$
$$= (0,0)$$
$$|z| = 0$$

$\therefore |z| = 0$ iff $z = 0$ proved//

iv. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

Solution:

$$\frac{z}{w} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2}, \frac{(bc-ad)i}{c^2+d^2}$$

Also

$$\left|\frac{z}{w}\right| = \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2}$$

$$= \sqrt{\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{|z|}{|w|}$$

v. $z \cdot \bar{z} = |z|^2$

Solution:

$$z \cdot \bar{z} = (a+ib)(a-ib)$$

$$= a^2+b^2$$

$$= \left(\sqrt{a^2+b^2}\right)^2$$

$$= (|z|)^2$$

$$= |z|^2$$

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vi) $Re(z) \leq |z|$ and $Im(z) \leq |z|$

Solution:

Let $z = a + ib$

Then, $Re(z) = a$, $Im(z) = b$

Also,

$$|z| = \sqrt{a^2 + b^2}$$

Since, $a^2 \leq a^2 + b^2$

$$\rightarrow a \leq \sqrt{a^2 + b^2}$$

$$\rightarrow Re(z) \leq |z|$$

Similarily,

$$b \leq \sqrt{a^2 + b^2}$$

$$\rightarrow b \leq |z|$$

$$\rightarrow Im(z) \leq |z|$$