H/W

Chapter-10
Pair of straight line

Date: 2079/12/01
Page: 249

General Equation of Second Degree

The equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is known as general equation of second degree ~~which is represented by~~ which represents the pair of straight lines.

Homogeneous Equation

An equation ~~which~~ with two variables x and y in which each of the term has same degree, which is in the form $ax^2 + 2hxy + by^2 = 0$ which represent the pair of st. lines passing through the origin.

Angle between pair of st. line represented by $ax^2 + 2hxy + by^2 = 0$

If θ be the angle between two lines represented by $ax^2 + 2hxy + by^2 = 0$ then,

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

If two lines ~~rep~~ are perpendicular then $a + b = 0$

If two lines are coincidence, then $h^2 - ab = 0$

Note:

If $h^2 - ab > 0$, two lines are real and distinct.

If $h^2 - ab = 0$, two lines are coincident and real.

If $h^2 - ab < 0$, two lines are imaginary.

Bisectors of the angle between pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by;

$$h(x^2 - y^2) = (a-b)xy$$

Example:

$2x^2 + 5xy + 12y^2 = 0 \longrightarrow ax^2 + 2hxy + by^2 = 0$

Equation of bisector is given by:

$h(x^2-y^2) = (a-b)xy$

or, $\frac{5}{2}(x^2-y^2) = \cancel{(a-b)}\ (\cancel{8}\ 2-12)xy$

or, $5(x^2-y^2) = -20xy$

or, $x^2-y^2 = -4xy$

or, $x^2-y^2+4xy = 0$

or, $x^2+4xy-y^2 = 0$ is the req. equation.

Exercise - 10.1

1. Find a single equation representing the line pair.

a. $x+y=0$, $x+2y=0$

solution:

Equation given are;

$x+y=0$ — (i)

$x+2y=0$ — (ii)

Combining eq. (i) and (ii)

$(x+y)(x+2y) = 0$

or, $x^2+2xy+xy+2y^2=0$

or, $x^2+3xy+2y^2=0$ is the req. equations.

b. $ax-by=0$, $bx+ay=0$

solution:

Given, line pair are;

$ax-by=0$ — (i)

$bx+ay=0$ — (ii)

Combining equation (i) and (ii), we get

$(ax-by)(bx+ay)=0$

or, $abx^2+a^2xy-b^2xy-aby^2=0$

or, $abx^2+(a^2-b^2)xy-aby^2=0$ is the req. equations.

c. $x + y + 2 = 0$, $x + 2y + 1 = 0$

Solution:

Given equation of lines are;

$x + y + 2 = 0$ — (i)

$x + 2y + 1 = 0$ — (ii)

combining eq. (i) and (ii)

$(x + y + 2)(x + 2y + 1) = 0$

or, $x^2 + 2xy + x + xy + 2y^2 + y + 2x + 4y + 2 = 0$

or, $x^2 + 2y^2 + 3xy + 3x + 5y + 2 = 0$ is the required equation.

2. Determine the lines represented by each of the following equations.

a. $x^2 - 2xy = 0$

Solution:

$x^2 - 2xy = 0$

or, $x(x - 2y) = 0$

~~or, $x - 2y = 0$~~

Either OR

$x = 0$ $x - 2y = 0$

Hence,

$x = 0$ and $x - 2y = 0$ are the req. equations.

b. $x^2 - 5xy + 4y^2 = 0$

Solution:

$x^2 - 4xy - xy + 4y^2 = 0$

or, $x(x - 4y) - y(x - 4y) = 0$

or, $(x - 4y)(x - y) = 0$

Hence,

$x - 4y = 0$ and $x - y = 0$ are the req. equations.

H.W

c. $xy - 3x + 2y - 6 = 0$

Solution:

$xy - 3x + 2y - 6 = 0$

or, $x(y-3) + 2(y-3) = 0$

or, $(y-3)(x+2) = 0$

Hence

$y - 3 = 0$ and $x + 2 = 0$ are the required equations.

d. $x^2 + 2xy + y^2 - 2x - 2y - 15 = 0$

Solution:

$x^2 + 2xy - \cancel{2y}\ 2x + y^2 - 2y - 15 = 0$

or, $x^2 + (2y - 2)x + y^2 - 2y - 15 = 0$

Now,

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 1$, $b = 2y - 2$, $c = y^2 - 2y - 15$

We know,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or, $$x = \frac{-2y + 2 \pm \sqrt{(2y-2)^2 - 4 \cdot 1 (y^2 - 2y - 15)}}{2 \cdot 1}$$

or, $2x = \cancel{2y}\ \cancel{2y}\ 2 - y \pm \sqrt{4y^2 - 8y + 4 - 4y^2 + 8y + 60}$

or, $2x = 2 - y \pm \sqrt{64}$

or, $2x = 2 - y \pm 8$

Now,

Taking (+ve) sign

$2x = 2 - 2y + 8$

or, $2x + 2y - 10 = 0$

or, $x + y - 5 = 0$

Taking (-ve) sign

$2x = 2 - 2y - 8$

or, $2x + 2y + 6 = 0$

or, $x + y + 3 = 0$

Hence,

The required equations are $x + y - 5 = 0$ and $x + y + 3 = 0$.

Condition that general equation of second degree may represent the pair of lines.

If general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent the pair of straight line then,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

The point of intersection of pair of straight line given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is;

$$\left(\sqrt{\frac{f^2 - bc}{h^2 - ab}} , \sqrt{\frac{g^2 - ca}{h^2 - ab}} \right)$$

3. Find the angle between the following pair of lines.

a. $x^2 + 9xy + 14y^2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1$, $h = \frac{9}{2}$, $b = 14$

Let 'θ' be the angle between the ~~pair~~ lines then,

$$\tan\theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

or, $\tan\theta = \pm \dfrac{2\sqrt{(4.5)^2 - 14}}{1+14}$

or, $\tan\theta = \pm \dfrac{2\sqrt{20.25-14}}{15}$

or, $\tan\theta = \pm \dfrac{2\times 5/2}{15}$

or, $\tan\theta = \pm \dfrac{1}{3}$

$\therefore \theta = \tan^{-1}(\pm 1/3)$

b. $x^2 - 2xy\cot\theta - y^2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$ then,

$a = 1$ $\quad h = -\cot\theta$ $\quad b = -1$

Let 'θ' be the angle between the lines then,

$\tan\theta = \pm \dfrac{2\sqrt{h^2 - ab}}{a+b}$

or, $\tan\theta = \pm \dfrac{2\sqrt{(-\cot\theta)^2 + 1}}{1-1}$

or, $\tan\theta = \pm \dfrac{2\sqrt{(-\cot\theta)^2 + 1}}{0}$

or, $\tan\theta = \infty$

or, $\tan\theta = \tan 90^\circ$

$\therefore \theta = 90^\circ$

c. $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 1$, $h = -\dfrac{5}{2}$, $b = 4$

Let 'θ' be the angle between the lines then,

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

or, $$\tan\theta = \frac{\pm 2\sqrt{25/4 - 4}}{1+4}$$

or, $$\tan\theta = \pm 2 \cdot \frac{3}{2\times 5}$$

or, $$\tan\theta = \pm \frac{6}{10}$$

or, $$\tan\theta = \pm \frac{3}{5}$$

$$\therefore \theta = \tan^{-1}(\pm 3/5)$$

4. Find the equations of the bisectors of the angles between the pair of lines.

a. $3x^2 - 15xy + 2y^2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 3$, $h = \frac{-15}{2}$ $b = 2$

Equation of bisectors between the lines are:

$$h(x^2 - y^2) = (a-b)xy$$

or, $$\frac{-15}{2}(x^2 - y^2) = (3-2)xy$$

or, $$-15x^2 + 15y^2 = 2xy$$

or, $15x^2 - 15y^2 + 2xy = 0$ is the req. equation of the bisector.

b. $x^2 + 2xy \operatorname{cosec}\theta + y^2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$ we get,

$a = 1$, $h = \operatorname{cosec}\theta$, $b = 1$

Equation of bisector between the lines is

$h(x^2 - y^2) = (a-b)xy$

or, $\operatorname{cosec}\theta (x^2 - y^2) = (1-1)xy$

or, $x^2 - y^2 = 0$

$\therefore x^2 - y^2 = 0$ is the req. equation of the bisector.

c. $ax^2 + 2hxy + by^2 + k(x^2 + y^2) = 0$

Solution:

Equation of bisectors of the angle between the pair of lines is ~~giv~~

$h(x^2 - y^2) = \cancel{(a-b)xy}\ (a-b)xy$

Hence

The required equation is $h(x^2 - y^2) = (a-b)xy$.

H/W

Date:

Page:

5. Prove that each of the following equations represents a pair of lines.

a. $2x^2 + 7xy + 3y^2 - 4x - 7y + 2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get;

$a = 2$ $h = \frac{7}{2}$ $b = 3$, $g = -2$, $f = -\frac{7}{2}$, $c = 2$

Now,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & 7/2 & -2 \\ 7/2 & 3 & -7/2 \\ -2 & -7/2 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 7/2 & -2 \\ 7/2 & 3 & -7/2 \\ 2 & 7/2 & -2 \end{vmatrix}$$

$$= -1 \times 0$$

$$= 0$$

Hence, the general equation of second degree represents pair of straight line.

b. $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 6$ $h = -\frac{1}{2}$ $b = -12$, $g = -4$, $f = \frac{29}{2}$, $c = -14$

Now, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & -1/2 & -4 \\ -1/2 & -12 & 29/2 \\ -4 & 29/2 & -14 \end{vmatrix}$

$= 6\left(168 - \frac{841}{4}\right) + \frac{1}{2}\left(\frac{14}{2} + \frac{116}{2}\right)$

$- 4\left(\frac{-29}{4} - 48\right)$

$= \frac{-1014 + \cancel{190}\ 130 + 884}{4}$

$= 0/4 = 0$

6. Find the value of k so that ~~it may~~ the following equation may represent a line pair.

a. $x^2 + kxy + 2y^2 + 3x + 5y + 2 = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get;

$a = 1$, $h = \frac{k}{2}$ $b = 2$, $g = \frac{3}{2}$, $f = \frac{5}{2}$, $c = 2$

Since, the line represents the pair of lines, then

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $1.2.2 + 2.\frac{5}{2}.\frac{3}{2}.\frac{k}{2} - 1.\left(\frac{5}{2}\right)^2 - 2.\left(\frac{3}{2}\right)^2 - 2.\left(\frac{k}{2}\right)^2 = 0$

or, $4 + 7.5k - 6.25 - 4.5 - \frac{k^2}{2} = 0$

or, $7.5k - 6.75 - \frac{k^2}{2} = 0$

or, $15k - 13.5 - k^2 = 0$

or, ~~$k^2 - 15k - 13.5 = 0$~~

or, $4 + \frac{15k}{2} - \frac{25}{4} - \frac{9}{2} - \frac{k^2}{2} = 0$

or, $2k^2 - \cancel{9k - 6}\ 15k + 27 = 0$

or, $2k^2 - 9k - 6k + 27 = 0$

or $(2k - 9)(k - 3) = 0$

Either OR

$k = \frac{9}{2}$ $k = 3$

b. $2x^2 + 7xy + 3y^2 - 4x - 7y + k = 0$

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get;

$a = 2$, $h = \frac{7}{2}$, $b = 3$, $g = -2$, $f = -\frac{7}{2}$, $c = k$

If the given lines represent the pair of lines then,

$abc + \cancel{2g}\ 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $2 \cdot 3 \cdot k + 2\left(-\frac{7}{2}\right)(-2) \cdot \frac{7}{2} - 2\left(-\frac{7}{2}\right)^2 - 3(-2)^2 - k\left(\frac{7}{2}\right)^2 = 0$

or, $6k + 49 - \frac{49}{2} - 12 - \frac{49k}{4} = 0$

or, $\frac{-25k}{4} = \frac{-25}{2}$

$\therefore k = 2$

HW

Page:

7.a. Find the equation of the two lines represented by $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$. Prove that the two lines are parallel. Also, find the distance between them.

Solution:

Given equation is;

$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$

or, $\cancel{(x+3y)^2 + 4(x+3y) - 5 = 0}$

or, $x^2 + 6xy + 4x + 9y^2 + 12y - 5 = 0$

or, $x^2 + (6y+4)x + (9y^2 + 12y - 5) = 0$

Now,

Comparing the equation with $ax^2 + bx + c = 0$, we get

$a = 1$, $b = (6y+4)$ $c = (9y^2 + 12y - 5)$

We know,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or, $$x = \frac{-(6y+4) \pm \sqrt{(6y+4)^2 - 4.1(9y^2 + 12y - 5)}}{2 \times 1}$$

or, $2x = -(6y+4) \pm \sqrt{36y^2 + 48y + 16 - 36y^2 - 48y + 20}$

or, $2x = -(6y+4) \pm \sqrt{36}$

or, $2x = -(6y+4) \pm 6$

Taking +ve sign

$2x = -6y - 4 + 6$

or $2x + 6y - 2 = 0$

$\therefore x + 3y - 1 = 0$ —— (I)

Taking -ve sign

$2x = -6y - 4 - 6$

or $2x + 6y - 10 = 0$

$\therefore x + 3y + 5 = 0$ —— (II)

Slope of eq. (i), $m_1 = \frac{-1}{3}$

slope of eq (ii), $m_2 = -\frac{1}{3}$

Since, $m_1 = m_2$. They are parallel to each other.

Again,

By putting $y=0$ in equation (1), we get $x = 1$. So, the point that satisfy the equation is $(1,0)$

Also,

The length of perpendicular drawn from $(1,0)$ to the line (ii) i.e $x+3y-5=0$ is given by;

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \right|$$

$$= \left| \frac{1.1 + 3.0 + 5}{\sqrt{1+9}} \right|$$

$$= \frac{6}{\sqrt{10}}$$

$$= \frac{6\sqrt{10}}{10} = \frac{3\sqrt{10}}{5}$$

Hence, the distance between the parallel lines are $\frac{3\sqrt{10}}{5}$.

b. Find the equations of the two lines represented by the equation $2x^2 + 3xy + y^2 + 5x + 2y - 3 = 0$. Find their point of intersection and also the angle between them.

Solution:

Given equation of the line is;

$2x^2 + 3xy + y^2 + 5x + 2y - 3 = 0$

or, $2x^2 + 3xy + 5x + y^2 + 2y - 3 = 0$

or, $2x^2 + (3y+5)x + (y^2 + 2y - 3) = 0$

Now,

Comparing the equation with $ax^2 + bx + c = 0$, we get

$a = 2 \quad b = (3y+5) \quad c = y^2 + 2y - 3$

We know,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = -(3y+5) \pm \sqrt{(3y+5)^2 - 4.2.(y^2+2y-3)}$$

$$\text{or, } x = \frac{-(3y+5) \pm \sqrt{9y^2 + 30y + 25 - 8y^2 - 16y + 24}}{2 \times 2}$$

$$\text{or, } 4x = \frac{-(3y+5) \pm \sqrt{y^2 + 14y + 49}}{1}$$

$$\text{or, } 4x = -(3y+5) \pm (y+7)$$

Here,

Taking +ve sign

$4x = -3y - 5 + y + 7$

or, $4x + 2y - 2 = 0$

or, $2x + y - 1 = 0$ —— (i)

Taking -ve sign

$4x = -3y - 5 - y - 7$

or, $4x + 4y + 12 = 0$

or, $x + y + 3 = 0$ —— (ii)

Solving equation (i) and (ii)

$$2x + y = 1$$
$$\begin{array}{r} 2x + 2y = -6 \\ - \quad (-) \quad (+) \end{array}$$
$$\overline{\quad -y = 7 \quad}$$
$$\therefore y = -7$$

put $y = -7$ in $2x + y = 1$, we get $x = 4$

Hence, the point of intersection is $(4, -7)$

Also,

Slope of eq. (i), $m_1 = -2$

Slope of eq. (ii), $m_2 = -1$

Let 'θ' be the angle between the lines,

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 . m_2}\right)$$

or $$\tan\theta = \pm\left(\frac{-2+1}{1+(-2 \times -1)}\right)$$

or $$\tan\theta = \pm\left(\frac{-1}{3}\right)$$

$$\therefore \theta = \tan^{-1}\left(\pm 1/3\right)$$

8. If the lines pair $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y' = 0$ have the same bisectors, prove that $h(a' - b') = h'(a - b)$.

Solution.

The equation of bisector of $ax^2 + 2hxy + by^2 = 0$ is

$$h(x^2 - y^2) = (a - b)xy \quad \text{— (i)}$$

Also,

The equation of bisector of $a'x^2 + 2h'xy + b'y' = 0$ is

$$h'(x^2 - y^2) = (a' - b')xy \quad \text{— (ii)}$$

By question, the bisector are same, so,

$$\frac{h(x^2-y^2)}{h'(x^2-y^2)} = \frac{(a-b)xy}{(a'-b')xy}$$

or, $$\frac{h}{h'} = \frac{(a-b)}{(a'-b')}$$

or, $h(a'-b') = h'(a-b)$ proved

9.a. Find the equation of the straight lines through the origin and at right angles to the lines $x^2 - 5xy + 4y^2 = 0$.

Solution:

Given equation of st. lines is

$x^2 - 5xy + 4y^2 = 0$

or, $x^2 - \cancel{5xy}\ 4xy - xy + 4y^2 = 0$

or, $x(x-4y) - y(x-4y) = 0$

or, $(x-4y)(x-y) = 0$

Now,

Separate equation are:

$x - 4y = 0$ —— (I)

$x - y = 0$ —— (II)

Equation of lines perpendicular to eq (i) and (ii) and passing through the origin is given by;

$4x + y = 0$ —— (III)

$x + y = 0$ —— (IV)

Combining (III) and (IV) we get

$(4x+y)(x+y) = 0$

or, $4x^2 + 4xy + xy + y^2 = 0$

or, $4x^2 + 5xy + y^2 = 0$ is the req. equation.

b. Find the equation to the straight lines passing through (1,1) and parallel to the lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.

Solution:

Given, equation is

$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$

or, $x^2 - 5xy + x + 4y^2 + 2y - 2 = 0$

or, $x^2 + (1 - 5y)x + (4y^2 + 2y - 2) = 0$

Now,

Comparing the equation with $ax^2 + bx + c = 0$, we get

$a = 1$, $b = (1 - 5y)$ $c = 4y^2 + 2y - 2$

We know,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or, $$x = \frac{(5y-1) \pm \sqrt{(1-5y)^2 - 4.1(4y^2+2y-2)}}{2\times1}$$

or, $2x = (5y-1) \pm \sqrt{1 - 10y + 25y^2 - 16y^2 - 8y + 8}$

or, $2x = (5y-1) \pm \sqrt{9y^2 - 18y + 9}$

or, $2x = (5y-1) \pm \sqrt{(3y-3)^2}$

or, $2x = (5y-1) \pm (3y-3)$

Now,

Taking +ve sign

$2x = 5y - 1 + 3y - 3$

or, $2x - 8y + 4 = 0$

or, $x - 4y + 2 = 0$ —— (I)

Taking -ve sign

$2x = 5y - 1 - 3y + 3 = 0$

or, $2x - 2y - 2 = 0$

or, $x - y - 1 = 0$ —— (II)

Combining equation (i) and (ii)

$(x-4y+2)(x-y-1)=0$

or, $x^2-xy-x-4xy+4y^2+4y+2x-2y-2=0$

or, $x^2+x+4y^2+2y-\cancel{5y}\ 5xy-2=0$

or, x^2

c. Find the equation of the straight lines perpendicular to the lines given by $x^2+xy-6y^2+7x+31y-18=0$ and passing through the origin.

Solution:

Given, equation is;

$x^2+xy-6y^2+7x+31y-18=0$

or, $x^2+xy+7x-6y^2+31y-18=0$

or, $x^2+(y+7)x+(31y-6y^2-18)=0$

Now,

Comparing the equation with $ax^2+bx+c=0$, we get

$a=1$, $b=(y+7)$, $c=31y-6y^2-18$

We know,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{or, } x=\frac{-(y+7)\pm\sqrt{(y+7)^2-4\cdot1\cdot(31y-6y^2-18}}{2\cdot1}$$

or, $2x=-y+7\pm\sqrt{y^2+14y+49-124y+24y^2+72}$

or, $2x=7-y\pm\sqrt{25y^2-110y+121}$

or, $2x=7-y\pm\sqrt{(5y-11)^2}$

or, $2x=7-y\pm 5y-11$

Now,

Taking +ve sign

$2x=-7-y+5y-11$

or, $2x-4y+18=0$

or, $x - 2y + 9 = 0$ —— (i)

Also,

Taking -ve sign

$2x = -7 - y - 5y + 11 = 0$

or $2x + 6y - 4 = 0$

or, $x + 3y - 2 = 0$ —— (ii)

Now,

Equation of lines perpendicular to eq. (i) & (ii) and passing through the origin is; ~~given by;~~

$\cancel{y+2}\ 2x + y = 0$ and $3x - y = 0$

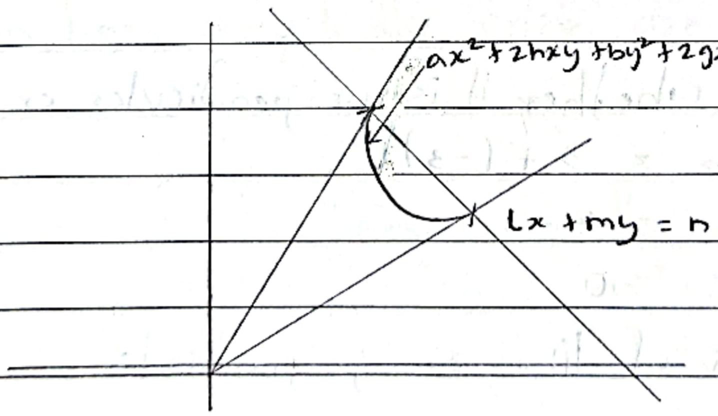
Combining them we get,

$(2x + y)(3x - y) = 0$

or $6x^2 - \cancel{2y}\ 2xy + 3xy - y^2 = 0$

or, $6x^2 + xy - y^2 = 0$ is the required equation of line.

Line joining the origin to intersection of a line and curve



The equation of the line joining the origin to the point of intersection of the line $lx + my \cancel{+c} = n$ and the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0$$

10.a Show that the line joining the origin to the point of intersection of the line $x+y=1$ with the curve $4x^2+4y^2+4x-2y-5=0$ are right angle to each other.

Solution:

Given equation of curve,

$4x^2+4y^2+4x-2y-5=0$

Equation of the line

$x+y=1$

Equation of pair of st. line joining the origin and intersection of curve is given by,

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2=0$$

$$\text{or, } 4x^2+4y^2+2(2x-y)\left(\frac{x+y}{1}\right)-5\left(\frac{x+y}{1}\right)^2=0$$

$$\text{or, } 4x^2+4y^2+(4x-2y)(x+y)-5(x^2+2xy+y^2)=0$$

$$\text{or, } 4x^2+4y^2+4x^2+4xy-2xy-2y^2-5x^2-10xy-5y^2=0$$

$$\text{or, } 3x^2-8xy-3y^2=0$$

Now,

To check whether it is perpendicular or not,

$$a+b=3+(-3)$$
$$=3-3$$
$$=0$$

$\therefore$ The pair of lines are perpendicular.

b. Show that the lines joining the origin to the points common to $x^2+hxy-y^2+gx+fy=0$ and $fx-gy=\lambda$ are ~~right~~ at right angles for all values of $\lambda\neq0$.

Solution:

Given equation of curve is

$x^2+hxy-y^2+gx+fy=0$

Equation of the line is
$fx - \cancel{gx}\, gy = \lambda$
Equation of st. line joining the origin and intersection of curve is given by:

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0$$

or, $x^2 + hxy - y^2 + gx\left(\frac{fx - gy}{\lambda}\right) + fy\left(\frac{fx - gy}{\lambda}\right) = 0$

or, $\lambda x^2 + \lambda hxy - \lambda y^2 + fgx^2 - g^2xy + f^2xy - fgy^2 = 0$
or, $\lambda x^2 + fgx^2 + \lambda hxy - g^2xy + f^2xy \cancel{+ fgx^2} - \lambda y^2 - fgy^2 = 0$
or, $(\lambda + fg)x^2 + (\lambda h - g^2 + f^2)xy + (-\lambda - fg)y^2 = 0$
Now,
Comparing the equation with $ax^2 + 2hxy + by^2 = 0$, we get
$a = (\lambda + fg)$, $b = (-\lambda - fg)$
Here,
$a + b = \lambda + fg - \lambda - fg$
$= 0$
So the given lines through the origin are at right angle to each other.

c. Find the equation of the lines joining the origin to the points of intersection of $x + 2y = 3$ and $4x^2 + 16xy - 12y^2 - 8x + 12y - 3 = 0$. Also, find the angle between the pair of two lines.
Solution:
Equation of the curve is
$\cancel{x + 2y}\; 4x^2 + 16xy - 12y^2 - 8x + 12y - 3 = 0$
Equation of line is
$x + 2y = 3$
Equation of the st. line joining the origin and the intersection of the curve is given by;

$$ax^2 + 2hxy + by^2 + 2(gx+fy)\left(\frac{lx+my}{n}\right)^2 + c\left(\frac{lx+my}{n}\right)^2 = 0$$

So,

$$4x^2 + 16xy - 12y^2 - 2(4x-6y)\left(\frac{x+2y}{3}\right) - 3\left(\frac{x+2y}{3}\right)^2 = 0$$

or, $4x^2 + 16xy - 12y^2 - 2\frac{(4x^2 + 8xy - 6xy - 12y^2)}{3} - (x^2 + 4xy + 4y^2) = 0$

or, $12x^2 + 48xy - 36y^2 - 8x^2 - 16xy + 12xy + 24y^2 - x^2 - 4xy - 4y^2 = 0$

or, $3x^2 + 40xy - 16y^2 = 0$ —— (1)

Now,

Comparing the equation (1) with $ax^2 + 2hxy + by^2 = 0$ we get

$a = 3 \quad h = 20 \quad b = -16$

Also,

Let 'θ' be the angle between the lines.

We know,

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

or, $\tan\theta = \pm\frac{2\sqrt{400 + 48}}{-13}$

or, $\tan\theta = \pm\frac{2\sqrt{448}}{13}$

or, $\tan\theta = \pm\frac{16\sqrt{7}}{13}$

$$\therefore \theta = \tan^{-1}\left(\pm\frac{16\sqrt{7}}{3}\right)$$

Hence,

the required equation is $3x^2 + 40xy - 16y^2 = 0$ and the angle between the lines is $\tan^{-1}\left(\pm\frac{16\sqrt{7}}{3}\right)$.

d. Find the equation to the pair of straight lines joining the origin to the intersection of the straight line $y = mx + c$ and the curve $x^2 + y^2 = a^2$, prove that they are at right angle if $2c^2 = a^2(1 + m^2)$.

Solution:

Given, equation of the curve is

$$x^2 + y^2 - a^2 = 0$$

Equation of the line is

$$y = mx + c$$

Equation of line joining the origin to the intersection of the curve is;

$$x^2 + y^2 - a^2\left(\frac{y - mx}{c}\right)^2 = 0$$

$$\text{or, } x^2 + y^2 - \frac{a^2(y^2 - 2mxy + m^2x^2)}{c^2} = 0$$

$$\text{or, } c^2x^2 + c^2y^2 - a^2y^2 + \cancel{2}mxy - m^2x^2 = 0$$

$$\text{or, } c^2x^2 + 2mxy + c^2y^2 - a^2y^2 - m^2x^2 = 0$$

$$\text{or, } c^2x^2 - m^2x^2 + 2mxy + (c^2 - a^2)y^2 = 0$$

$$\text{or, } (c^2 - m^2)x^2 + 2mxy + (c^2 - a^2)y^2 = 0$$

$$\text{or, } c^2x^2 + c^2y^2 - a^2y^2 + 2a^2mxy - a^2m^2x^2 = 0$$

$$\text{or, } c^2x^2 - a^2m^2x^2 + 2a^2mxy + c^2y^2 - a^2y^2 = 0$$

$$\text{or, } (c^2 - a^2m^2)x^2 + 2a^2mxy + (c^2 - a^2)y^2 = 0$$

Now,

Comparing the equation with $ax^2 + 2hxy + by^2 = 0$ we get

$a = (c^2 - a^2m^2)$ ~~$h = a^2m$~~ $b = c^2 - a^2$

here,

The two lines are perpendicular to each other.

i.e. $a + b = 0$

$$\text{or, } c^2 - a^2m^2 + c^2 - a^2 = 0$$

$$\text{or, } 2c^2 = a^2 + a^2m^2$$

or, $2c^2 = a^2(1+m^2)$

Proved

11. Prove that the two straight lines $(x^2+y^2)\sin^2\alpha = (x\cos\theta - y\sin\theta)^2$ include an angle 2α.

Solution:

Given equation of st. line is

$(x^2+y^2)\sin^2\alpha = (x\cos\theta - y\sin\theta)^2$

or, $x^2\sin^2\alpha + y^2\sin^2\alpha = x^2\cos^2\theta - 2x\cos\theta . y\sin\theta + y^2\sin^2\theta$

or, $x^2\sin^2\alpha - x^2\cos^2\theta + 2x\cos\theta . y\sin\theta + y^2\sin^2\alpha - y^2\sin^2\theta = 0$

or, $(\sin^2\alpha - \cos^2\theta)x^2 + 2x\cos\theta . y\sin\theta + (\sin^2\alpha - \sin^2\theta)y^2 = 0$

Now,

Comparing the equation with $ax^2 + 2hxy + by^2 = 0$, we get

$a = \sin^2\alpha - \cos^2\theta$, $h = \sin\theta . \cos\theta$ & $b = \sin^2\alpha - \sin^2\theta$

Also,

Let ϕ be the angle between the lines then,

$$\tan\phi = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{(\sin\theta . \cos\theta)^2 - (\sin^2\alpha - \cos^2\theta)(\sin^2\alpha - \sin^2\theta)}}{\sin^2\alpha - \cos^2\theta + \sin^2\alpha - \sin^2\theta}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{\sin^2\theta . \cos^2\theta - \sin^4\alpha + \sin^2\alpha . \sin^2\theta + \sin^2\alpha . \cos^2\theta - \sin^2\theta . \cos^2\theta}}{2\sin^2\alpha - 1}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{\sin^2\alpha . \cos^2\theta + \sin^2\alpha . \sin^2\theta - \sin^4\alpha}}{\cancel{\cos 2\theta} - \cos 2\alpha}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{\sin^2\alpha(\sin^2\theta + \cos^2\theta) - \sin^4\alpha}}{\cos 2\alpha}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{\sin^2\alpha - \sin^4\alpha}}{\cos 2\alpha}$$

or, $$\tan\phi = \frac{\pm 2\sqrt{\sin^2\alpha(1 - \sin^2\alpha)}}{\cos 2\alpha}$$

Date: ....
Page: ....

or, $\tan\phi = \pm 2\dfrac{\sqrt{\sin^2\alpha \cdot \cos^2\alpha}}{\cos 2\alpha}$

or, $\tan\phi = \pm \dfrac{2\sin\alpha \cdot \cos\alpha}{\cos 2\alpha}$

or, $\tan\phi = \pm \dfrac{\sin 2\alpha}{\cos 2\alpha}$

or, $\tan\phi = \pm \tan 2\alpha$

Hence,
The angle between the lines include 2α.

12. Find the condition that one of the lines $ax^2 + 2hxy + by^2 = 0$ may coincide with one of the lines $a_1x^2 + 2h_1xy + by^2 = 0$.

Solution: