Chapter-17
Application of Derivatives

Date:..............................
Page:..............................

Geometrical Meaning of Derivative

The derivative of $y = f(x)$ with respect to 'x' denoted by '$\frac{dy}{dx}$' or $f'(x)$ gives slope of Tangent to the curve.

Increasing function.

A function $y = f(x)$ is said to be increasing in the interval (a, b) if for every $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

In the case of increasing, the slope of Tangent at that point of the ~~curd~~ curve is positive. i.e $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$ or $f'(x) > 0$

Decreasing function

A function $y = f(x)$ is said to be decreasing in the interval (a, b) if for every $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

In the case of decreasing, the slope of Tangent at that point of the curve is negative.

i.e. $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$ or $f'(x) < 0$

Stationary Point

A point of the graph of the function $y = f(x)$ where Tangent is parallel to x-axis is known as stationary point.

Exercise- 17.1

1. i) Examine whether the function $f(x) = 15x^2 - 14x + 1$ is increasing or decreasing at $x = 2/5$ and $x = 5/2$.

Solution:

$f(x) = 15x^2 - 14x + 1$

Then, $\frac{dy}{dx} = 30x - 14$

At $x = \frac{2}{5}$, $\frac{dy}{dx} = 30 \cdot \frac{2}{5} - 14$

$= 12 - 14 = -2 < 0$

$\therefore f(x)$ is decreasing at $x = 2/5$

Also,

At $x = \frac{5}{2}$, $\frac{dy}{dx} = 30 \cdot \frac{5}{2} - 14$

$= 75 - 14$

$= 61 > 0$

$\therefore f(x)$ is increasing at $x = 5/2$

ii. Show that the function $f(x) = 2x^3 - 24x + 15$ is increasing at $x = 3$ and decreasing at $x = 3/2$.

Solution:

$f(x) = 2x^3 - 24x + 15$

Now, $\frac{dy}{dx} = 6x^2 - 24$

At $x = 3$, $\frac{dy}{dx} = 6 \cdot 3^2 - 24$

$= 30 > 0$

Since, $f'(x) > 0$, So it is increasing at $x = 3$

Also,

$x = 3/2$, $\frac{dy}{dx} = 6\left(\frac{3}{2}\right)^2 - 24 = -10.5 < 0$

Hence, $\frac{dy}{dx} < 0$, So ~~it is~~ $f(x)$ is decreasing at $x = 3/2$

2. i. Test whether the function $f(x) = 2x^2 - 4x + 3$ is increasing or decreasing on the interval $(1, 4]$

Solution:

$f(x) = 2x^2 - 4x + 3$

Now, $\frac{dy}{dx} = 4x - 4$

We know, $f(x)$ is increasing on $\frac{dy}{dx} > 0$

i.e. $4x - 4 > 0$

or, $x > 1$

i.e $(1, \infty)$

Also,

$f(x)$ is decreasing on $\frac{dy}{dx} < 0$

i.e. $4x - 4 < 0$

i.e. $x < 1$

i.e $(-\infty, 1)$

Hence, $f(x)$ is increasing on the interval $[1, 4]$

ii. Examine whether the function $f(x) = 16x - \frac{4}{3}x^3$ is decreasing or increasing on the interval $(-\infty, -2)$

Solution:

$f(x) = 16x - \frac{4}{3}x^3$

Now, $\frac{dy}{dx} = 16 - 4x^2$

We know, $f(x)$ is increasing on $\frac{dy}{dx} > 0$

i.e $16 - 4x^2 > 0$

i.e $4(4 - x^2) > 0$

i.e $(-\infty, 2)$

Also,

$f(x)$ is decreasing on $\frac{dy}{dx} < 0$

i.e. $16 - 4x^2 < 0$

i.e. $2 < x$

i.e $(2, \infty)$

$\cancel{\frac{dy}{dx}\ 4(2+x)(2-x) > 0}$

or, $\frac{dy}{dx} = 4(2+x)(2-x) < \cancel{0 \text{ for}} < \forall\ x \in (-\infty, -2)$

Hence, $f(x)$ is increasing for all $x \in (-\infty, -2)$

iii. Show that the function $f(x) = -x^3 + 6x^2 - 13x + 20$ is decreasing for all $x \in R$.

Solution:

$f(x) = -x^3 + 6x^2 - 13x + 20$

Now, $f'(x) = -3x^2 + 12x - 13$

Here,

$f'(x) = -3x^2 + 12x - 13 < 0$ for all $x \in R$

Hence, $f'(x)$ is decreasing ~~at~~ for all $x \in R$.

iv. Show that the function $f(x) = 4x - \frac{9}{x} + 6$ is increasing for all $x \in R$ except at $x = 0$.

Solution.

$f(x) = 4x - \frac{9}{x} + 6$

Now, $f'(x) = 4 + \frac{9}{x^2}$

Here,

$f'(x) = 4 + \frac{9}{x^2}$ is positive for all $x \in R$ except $x = 0$.

So, $f(x)$ is increasing for all ~~$x \in R$~~ $x \in R$ except $x = 0$.

Local Maxima

A function $y = f(x)$ is said to have the local maximum value or local maxima at $x = a$, if $f(a) > f(a \pm h)$, for sufficiently small positive value 'h'.

The local maxima is known as relative maxima of the function.

Local Minima

A function $y = f(x)$ is said to have local minimum value at $x = a$ if $f(a) < f(a \pm h)$, for sufficiently small positive value 'h'.

Absolute Maxima

A function $y = f(x)$ is said to have absolute maximum value or absolute maxima at $x = a$, if $f(a)$ is greatest of all its values for all 'x' belonging to the domain of the function.

In other word, $f(a)$ is absolute maximum value of $f(x)$ if $f(a) \geq f(x)$ for all ~~$x \in R$~~ $x \in D(f)$

Absolute Minima

A function $y = f(x)$ is said to have absolute minimum value or absolute minima at $x = a$ if $f(a)$ is smallest of all its value for all x belonging to the domain of the function.

In other word; $f(a)$ is absolute minimum value of $f(x)$ if $f(a) \leq f(x)$ for all $x \in D(f)$.

Procedure to find the Absolute Maxima and Minima

Let $y = f(x)$ be the given function defined in $[a, b]$

Now,

i. Find $\frac{dy}{dx}$ or $f'(x)$ i.e first derivative.

ii. Take $f'(x) = 0$, To solve for x, to get stationary point. Suppose value of x are 'c' and 'd'.

iii. Find the value of $f(x)$ at $x = a, b, c, d$.

The least value gives the absolute minimum value and greatest value gives the absolute maximum value.

Procedure to find Local Maxima and Local Minima

Let $y = f(x)$ be a function

i. Find $\frac{dy}{dx}$ or $f'(x)$ and $f''(x)$

ii. Making $f'(x) = 0$, to solve for x, to get stationary point.

Let $x = a$ be one of stationary point.

iii. Find $f''(a)$.

If $f''(a) < 0$ then $f(x)$ has maximum value at $x = a$, Maximum value is $f(a)$.

If $f''(a) > 0$ then $f(x)$ has minimum value at $x = a$, Minimum value is $f(a)$

If $f''(a) = 0$ then $f'''(a) \neq 0$ then $f(x)$ has neither maximum nor minimum value.

11/10

Date:

Page:

3. Find the intervals in which the following functions are increasing or decreasing.

i. $f(x) = 3x^2 - 6x + 5$

Solution:

$f'(x) = 6x - 6$

Let $f'(x) = 0$ then,

$6x - 6 = 0$

$\therefore x = 1$

Intervals	$6x - 1$
$-\infty, 1$	$-$
$1, \infty$	$+$

Hence, the function is increasing for $(1, \infty)$ and decreasing for $(-\infty, 1)$

ii. $f(x) = x^4 - \frac{1}{3}x^3$

Solution:

$f'(x) = 4x^3 - x^2$

Let $f'(x) = 0$

or, $4x^3 - x^2 = 0$

or, $x^2(4x - 1) = 0$

$\therefore x = 0$ or $1/4$

Intervals	x^2	$(4x-1)$	$x^2(4x-1)$
$-\infty, 1/4$	$+$	$-$	$-$
$1/4, \infty$	$+$	$+$	$+$

Hence,

The function is increasing for $(1/4, \infty)$ and decreasing for $(-\infty, 1/4)$

iii. $f(x) = 5x^3 - 135x + 22$

Solution:

$f'(x) = 15x^2 - 135$

Let $f'(x) = 0$ then

or, $15x^2 - 135 = 0$

or, $x^2 - 9 = 0$

$\therefore x = \pm 3$

Intervals	$(x+3)$	$(x-3)$	$(x+3)(x-3)$
$-\infty, -3$	-	-	+
$-3, 3$	+	-	-
$3, \infty$	+	+	+

Hence,

The function is increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 3)$.

iv. $f(x) = 6 + 12x + 3x^2 - 2x^3$

Solution:

$f'(x) = 12 + 6x - 6x^2$

Let $f'(x) = 0$ then

or, $12 + 6x - 6x^2 = 0$

or, $x^2 - x - 2 = 0$

or, $(x-2)(x+1) = 0$

$\therefore x = 2$ or -1

Intervals	$(x-2)$	$(x+1)$	$(x-2)(x+1)$
$-\infty, -1$	-	-	+
$-1, 2$	-	+	-
$2, \infty$	+	+	+

Hence,

The function is decreasing on $(-1, 2)$ and increasing on $(-\infty, -1) \cup (2, \infty)$.

Date:

Page:

$f(x) = x^3 - 12x$ defined on $[-3, 5]$

Solution:

$f'(x) = 3x^2 - 12$

Let $f'(x) = 0$

Then,

$$3x^2 - 12 = 0$$

or, $x^2 - 4 = 0$

or, $(x+2)(x-2) = 0$

$\therefore x = -2$ or 2

Intervals	$(x+2)$	$(x-2)$	$(x+2)(x-2)$
$[-3, -2)$	-	-	+
$(-2, 2)$	+	-	-
$(2, 5]$	+	+	+

Hence,

The function is increasing on $[-3, 2) \cup (2, 5]$ and decreasing on $(-2, 2)$

4. Find the absolute maximum and absolute minimum values of the following function on the given intervals:

i. $f(x) = 3x^2 - 6x + 4$ on $[-1, 2]$

Solution:

$f(x) = 3x^2 - 6x + 4$

Now, $f'(x) = 6x - 6$

Taking $f'(x) = 0$

or, $6x - 6 = 0$

$\therefore x = 1$

At $x = 1$, $f(x) = 3 \cdot 1^2 - 6 \cdot 1 + 4 = 1$

At $x = -1$, $f(x) = 3 \cdot (-1)^2 - 6(-1) + 4 = 13$

At $x = 2$, $f(x) = 3 \cdot 2^2 - 6 \cdot 2 + 4 = 4$

Hence,

Ab. Maximum value $= 13$

Absolute minimum value $= 1$

ii. $f(x) = 2x^3 - 9x^2$ on $[-2, 4]$

Solution:

$f(x) = 2x^3 - 9x^2$

Now, $f'(x) = 6x^2 - 18x$

Taking $f'(x) = 0$

i.e. $6x^2 - 18x = 0$

or, $6x(x-3) = 0$

$\therefore x = 0$ or 3

At $x = 0$, $f(x) = 0$

At $x = 3$, $f(x) = 2 \cdot 3^3 - 9 \cdot 3^2 = -27$

At $x = -2$, $f(x) = 2(-2)^3 - 9(-2)^2 = -52$

At $x = 4$, $f(x) = 2(4)^3 - 9(4)^2 = -16$

Hence,

Absolute maximum $= 0$, Absolute minimum $= -52$

iii. $f(x) = x^3 - 6x^2 + 9x$ on $[0, 5]$

Solution:

$f(x) = x^3 - 6x^2 + 9x$

Now, $f'(x) = 3x^2 - 12x + 9$

Taking $f'(x) = 0$

i.e. $3x^2 - 12x + 9 = 0$

or, $3(x^2 - 4x + 3) = 0$

or, $(x-3)(x-1) = 0$

$\therefore x = 3, 1$

At $x = 3$, $f(x) = 3^3 - 6.3^2 + 9.3 = 0$

At $x = 1$, $f(x) = 1^3 - 6.1^2 + 9.1 = 4$

At $x = 0$, $f(x) = 0^3 - 6.0^2 + 9.0 = 0$

At $x = 5$, $f(x) = 5^3 - 6.5^2 + 9.5 = 20$

Hence,

Absolute maximum value $= 20$

Absolute minimum value $= 0$

iv. $f(x) = 2x^3 - 15x^2 + 36x + 10$ on $[1, 4]$

Solution:

$f(x) = 2x^3 - 15x^2 + 36x + 10$

Now, $f'(x) = 6x^2 - 30x + 36$

Taking $f'(x) = 0$

i.e. $6x^2 - 30x + 36 = 0$

or, $x^2 - 5x + 6 = 0$

or, $(x-1)(x-5) = 0$

$\therefore x = 1, 5$

When, $x = 1$, $f(x) = 2.1^3 - 15.1^2 + 36.1 + 10 = 33$

At $x = 4$, $f(x) = 2.4^3 - 15.4^2 + 36.4 + 10 = 42$

Hence,

Absolute Maximum value $= 42$

Absolute ~~Max~~ Minimum Value $= 33$

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5. Find the Local maxima and minima and points of inflection.

i. $f(x) = 3x^2 - 6x + 3$

Solution:

$f(x) = 3x^2 - 6x + 3$

Then, $f'(x) = 6x - 6$ and $f''(x) = 6$

Let $f'(x) = 0$

i.e. $6x - 6 = 0$

$\therefore x = 1$

Put $x = 1$ in $f''(x)$

$f''(x) = 6 > 0$

$\therefore f(x)$ is minimum at $x = 1$

Minimum value $= f(1) = 3 \cdot 1^2 - 6 \cdot 1 + 3 = 0$

Let $f''(x) = 0$

or, $6 = 0$

$\therefore$ There is no point of inflection.

ii. $f(x) = x^3 - 12x + 8$

Solution:

$f'(x) = 3x^2 - 12$ and $f''(x) = 6x$

Let $f'(x) = 0$

i.e $3x^2 - 12 = 0$

$\therefore x = \pm 2$

Put $x = 2$ in $f''(x)$

$f''(2) = 6 \cdot 2 = 12 > 0$

$\therefore f(x)$ is ~~max~~ minimum at $x = 2$

Minimum value $= f(2) = 2^3 - 12 \cdot 2 + 8 = -8$

Let $f''(x) = 0$

or, $6x = 0$

$\therefore$ The point of inflection is at $x = 0$

Again,

Put $x = -2$ in $f''(x)$

$f''(-2) = 6 \times (-2) = -12 < 0$

$\therefore f(x)$ is maximum at $x = -2$

Maximum value $= f(-2) = (-2)^3 - 12(-2) + 8$

$= 24$

iii. $f(x) = x^3 - 6x^2 + 3$

solution:

$f'(x) = 3x^2 - 12x$ and $f''(x) = 6x - 12$

Let $f'(x) = 0$

i.e. $3x^2 - 12x = 0$

or, $x^2 - 4x = 0$

or, $x(x-4) = 0$

$\therefore x = 0, 4$

Let $f''(x) = 0$

or, $6x - 12 = 0$

or, $x - 2 = 0$

$\therefore x = 2$

$\therefore$ The point of inflection is at $x = 2$.

Put $x = 0$ in $f''(x)$

$f''(0) = 6.0 - 12 = -12 < 0$

$\therefore f(x)$ is maximum at $x = 0$

Maximum value $= f(0) = 0^3 - 6.0^2 + 3 = 3$

Put $x = 4$ in $f''(x)$

$f''(4) = 6.4 - 12 = 12 > 0$

$\therefore f(x)$ is minimum at $x = 4$

Minimum value $= f(x) = 4^3 - 6.4^2 + 3 = -29$

iv. $f(x) = 2x^3 - 15x^2 + 36x + 5$

Solution:

$f'(x) = 6x^2 - 30x + 36$ and $f''(x) = 12x - 30$

Let $f'(x) = 0$

Then, $\cancel{6x^2 - 15x = 0}$

$\cancel{\text{or, } 3x(2x-5) = 0}$

$6x^2 - 30x + 36 = 0$

or, $x^2 - 5x + 6 = 0$

or, $(x-2)(x-3) = 0$

$\therefore x = 2, 3$

Let $f''(x) = 0$

or, $12x - 30 = 0$

or, $2x - 5 = 0$

$\therefore x = \frac{5}{2}$

$\therefore$ The point of inflection is at $x = 5/2$

Put $x = 2$ in $f''(x)$

Then,

$f''(2) = 12 \cdot 2 - 30 = -6 > 0$

$\therefore f(x)$ has maximum value at $x = 2$

Maximum value $= f(2) = 2.2^3 - 15.2^2 + 36.2 + 5$
$= 33$

Put $x = 3$ in $f''(x)$

Then,

$f''(3) = 12 \times 3 - 30 = 6 > 0$

$\therefore f(x)$ has minimum value at $x = 3$

Minimum value $= f(3) = 2.3^3 - 15.3^2 + 36.3 + 5$
$= 32$

V. $f(x) = 2x^3 - 9x^2 - 24x + 3$

Solution:

$f'(x) = 6x^2 - 18x - 24$ and $f''(x) = 12x - 18$

Let $f'(x) = 0$

Then,

$6x^2 - 18x - 24 = 0$

or, $x^2 - 3x - 4 = 0$

or, $(x-4)(x+1) = 0$

$\therefore x = 4, -1$

Let $f''(x) = 0$

or, $12x - 18 = 0$

or, $6x - 9 = 0$

$\therefore x = 3/2$

$\therefore$ The inflection point is at $x = \frac{3}{2}$

Put $x = 4$ in $f''(x)$

Then,

$f''(4) = 12.4 - 18 = 30 > 0$

$\therefore f(x)$ has minimum value at $x = 4$

Minimum value $= f(4) = 2.4^3 - 9.4^2 - 24.4 + 3$
$= -109$

Again,

Put $x = -1$ in $f''(x)$

Then, $f''(-1) = 12(-1)^3 - 18$
$= -30 < 0$

$\therefore f(x)$ has maximum value at $x = (-1)$

$\therefore$ Maximum value $= f(-1) = 2 \times (-1)^3 - 9(-1)^2 - 24(-1) + 3$
$= 16$

vi. $f(x) = 4x^3 - 15x^2 + 12x + 7$

Solution:

$f'(x) = 12x^2 - 30x + 12$ and $f''(x) = 24x - 30$

Let $f'(x) = 0$

Then,

$12x^2 - 30x + 12 = 0$

or, $2x^2 - 5x + 2 = 0$

or, $(x-2)(2x-1) = 0$

$\therefore x = 2, 1/2$

Let $f''(x)\,\cancel{30}' = 0$

or, $24x - 30 = 0$

or, $\cancel{6x}\ 12x - 15 = 0$

$\therefore x = \cancel{3}\ \frac{5}{4}$

$\therefore$ The inflection point is at $x = 5/4$

Put $x = 2$ in $f''(x)$

Then,

$f''(2) = 24 \cdot 2 - 30 = 18 > 0$

$\therefore f(x)$ has minimum value at $x = 2$

Minimum value $= f(2) = 4 \cdot 2^3 - 15 \cdot 2^2 + 12 \cdot 2 + 7$

$= \cancel{109}\ 3$

Also,

Put $x = 1/2$

Then, $f''(x) = 24 \cdot \frac{1}{2} - 30 = -18 < 0$

$\therefore f(x)$ has maximum value at $x = 1/2$

Maximum value $= 4\left(\frac{1}{2}\right)^3 - 15\left(\frac{1}{2}\right)^2 + 12 \times \frac{1}{2} + 7$

$= \frac{39}{4}$

vii. $f(x) = 4x^3 - 6x^2 - 9x + 1$ on the interval $(-1, 2)$

Solution:

$f'(x) = 12x^2 - 12x - 9$ and $f''(x) = 24x - 12$

Let $f'(x) = 0$

Then,

$12x^2 - 12x - 9 = 0$

Let $f''(x) = 0$

or, $24x - 12 = 0$

$\therefore$ The inflection point is at $x = 1/2$

or, $4x^2 - 4x - 3 = 0$

or, $(2x-3)(2x+1) = 0$

$\therefore x = \frac{3}{2}, \frac{-1}{2}$

Put $x = 3/2$ in $f''(x)$

$f''(3/2) = 24 \times \frac{3}{2} - 12 = 24 > 0$

$\therefore f(x)$ has minimum value at $x = 3/2$

Minimum value $= 4\left(\frac{3}{2}\right)^3 - 6\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) + 1$

$= \frac{-25}{2}$

Also,

Put $x = -1/2$ in $f''(x)$

Then, $f''(-1/2) = 24 \times \left(\frac{-1}{2}\right) - 12 = -24 < 0$

$\therefore f(x)$ has maximum value at $x = -1/2$

Maximum value $= f(-1/2) = 4\left(\frac{-1}{2}\right)^3 - 6\left(\frac{-1}{2}\right)^2 - 9\left(\frac{-1}{2}\right) + 1$

$= \frac{7}{2}$

viii. $f(x) = x + \frac{100}{x} - 5$

Solution:

$f'(x) = 1 - \frac{100}{x^2}$ and $f''(x) = \frac{200}{x^3}$

Let $f'(x) = 0$

Then,

$1 - \frac{100}{x^2} = 0$

Let $f''(x) = 0$

or, $\frac{200}{x^3} = 0$

$\therefore$ There is no point of inflection.

or $x^2 = 100$

$\therefore x = \pm 10$

Put $x = 10$ at $f''(x)$

$$f''(10) = \frac{200}{10^3} = \frac{1}{5} > 0$$

$\therefore f(x)$ has minimum value at $x = 10$

Minimum value $= f(10) = x + \frac{100}{x^2}$

$$= 10 + \frac{100}{100}$$

$$= 10$$

Also, put $x = -10$ at $f''(x)$

Then, $f''(-10) = \frac{200}{(-10)^3}$

$$= -\frac{1}{5} < 0$$

$\therefore f(x)$ has maximum value at $x = -10$

Maximum value $= f(-10) = -10 - \frac{100}{-10}$

$$= -25$$

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6. Show that the following functions have neither maximum nor minimum value.

i. $f(x) = x^3 - 6x^2 + 24x + 4$

Solution:

$$f'(x) = 3x^2 - 12x + 24$$
$$= 3(x^2 - 4x + 8)$$
$$= 3(x^2 - 4x + 4 + 4)$$
$$= 3\{(x-2)^2 + 2^2\} \neq 0 \text{ for all real values of } x$$

So, $f(x)$ has neither maximum nor minimum value.

ii. $f(x) = x^3 - 6x^2 + 12x - 3$

Solution:

$f'(x) = 3x^2 - 12x + 12$ and $f''(x) = 6x - 12$

Put $f'(x) = 0$

Then,

$$3x^2 - 12x + 12 = 0$$

or, $x^2 - 4x + 4 = 0$

or, $(x-2)^2 = 0$

$\therefore x = 2$

Put $x = 2$ in $f''(x)$

Then, $f''(2) = 6 \cdot 2 - 12 = 0$

Now,

$f'''(x) = 6 \neq 0$

$\therefore f(x)$ has neither maximum nor minimum value at $x = 2$.

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7. Determine where the graph is concave upwards and where it is concave downwards of the following functions.

i. $f(x) = x^4 - 2x^3 + 5$

Solution:

$f'(x) = 4x^3 - \cancel{3x^2}\ 6x^2$

$f''(x) = 12x^2 - \cancel{6x}\ 12x$

$= 12x(x-1)$

Taking $f''(x) = 0$

or, $12x(x-1) = 0$

or, $x(x-1) = 0$

$\therefore x = 1, 0$

Point of inflection are $x = 0$ and $x = 1$

	x	$x-1$	$x(x-1)$
$(-\infty, 0)$	$-$	$-$	$+$
$(0, 1)$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$

The given function $f(x)$ is concave upward on the interval $(-\infty, 0)$ and $(1, \infty)$

And the function is concave downward on the interval $(0, 1)$

ii. $f(x) = x^4 - 8x^3 + 18x^2 - 24$

Solution:

$f'(x) = 4x^3 - 24x^2 + 36x$

$f''(x) = 12x^2 - 48x + 36$

Taking $f''(x) = 0$

or, $12x^2 - 48x + 36 = 0$

or, $x^2 - 4x + 3 = 0$

or, $(x-3)(x-1) = 0$

$\therefore x = 3, 1$

Point of inflection are $x = 3$ and $x = 1$

Now,

	$(x-3)$	$(x-1)$	$(x-3)(x-3)$
$(-\infty, 1)$	$-$	$-$	$+$
$(1, 3)$	$-$	$+$	$-$
$(3, \infty)$	$+$	$+$	$+$

The given function $f(x)$ is concave upward on the interval $(-\infty, 1)$ and $(3, \infty)$

And the function is concave downward on the interval $(1, 3)$

iii. $y = 3x^5 \cancel{+0} + 10x^3 + 15x$

Solution:

$f'(x) = 15x^4 + 30x^2 + 15$

$f''(x) = 60x^3 + 60x$

Taking $f''(x) = 0$

or, $60x^3 + 60x = 0$

or, $x(x^2+1) = 0$

$\therefore x = 0$ OR $x = \sqrt{-1}$ (Rejected)

$\therefore$ The point of inflection is at $x = 0$, Now

	x	x^2+1	$x(x^2+1)$
$-\infty, 0$	$-$	$+$	$-$
$0, \infty$	$+$	$+$	$+$

$\therefore$ The given function is concave down in $(-\infty, 0)$ and upward in $(0, \infty)$

iv. $f(x) = x^3 - 9x^2$ defined on $[-2, 5]$

Solution:

$f'(x) = 3x^2 - 18x$ and $f''(x) = 6x - 18$

Let $f''(x) = 0$ then

$6x - 18 = 0$

$\therefore x = 3$

Intervals	$6x-18$
$[-2, 3)$	$-$
$(3, 5]$	$+$

Hence,

The given function is concave upward on $(3, 5]$ and concave downward on $[-2, 3)$.

Concavity

The graph of the function $y = f(x)$ defined in an interval (a,b) is Concave upward (or Convex downward) if each point (except the point of contact) on the curve lies above any Tangent to it in that interval.

If each point on the curve lies below any Tangent to the curve in the interval then the graph of the function is said to be concave downward.

The graph of function $y = f(x)$ will be concave upward or concave downward according as $\frac{d^2y}{dx^2} = f''(x) > 0$ or

$$\frac{d^2y}{dx^2} = f''(x) < 0$$

Procedure to find Concave upward and downward

Let $y = f(x)$ be the function then,

i. Find second derivative i.e. $f''(x)$

ii. Find the interval within which $f''(x) > 0$ then conclude the graph of $y = f(x)$ will be concave upward in the interval.

iii. Again, find the interval within which $f''(x) < 0$ and conclude that the graph of function $y = f(x)$ will be concave downward in the interval.

Point of inflection

The point which divides the graph of the function from the shape of concave upward and concave downward where $\frac{d^2y}{dx} = f''(x) = 0$.

8. A man who has 144 metres of fencing material wishes to enclose a rectangular garden. find the maximum area he can enclose.

Solution:

Let x and y be the ~~area and~~ length and area of a rectangular garden. Then,

$$\text{Length } (L) = x \text{ cm}$$

$$\text{Perimeter } (P) = 144 \text{ cm}$$

$$\text{or, } 2(L+b) = 144$$

$$\therefore b = 72 - x$$

we know,

$$\text{Area } (A) = L \times b$$
$$= x(72-x)$$
$$= 72x - x^2$$

Now, $f'(x) = 72 - 2x$ and $f''(x) = -2$

Let $f'(x) = 0$

or, $72 - 2x = 0$

$\therefore x = 36$

put $x = 36$ in $f''(x)$

$\therefore f''(x) = -2 < 0$

$\therefore$ The area of rectangle will be maximum when length is 36 cm.

$\therefore$ Maximum area $= y = 72(36) - (36)^2$

$= 1296$ unit sq.

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9. Show that the rectangle of largest possible area, for a given perimeter, is a square.

Solution:

Let x be the length, y be the area and P be the perimeter of a rectangle. Then,

$$\text{Length } (L) = x \text{ cm}$$

$$\text{Perimeter } (P) = P \text{ cm}$$

or, $2(L+b) = P$

or, $2(x+b) = P$

$\therefore b = \frac{P}{2} - x$

We know,

$\text{Area}(A) = L \times b$

or, $y = x \cdot \left(\frac{P}{2} - x\right)$

or, $y = \frac{Px}{2} - x^2$

Also, $f'(x) = \frac{P}{2} - 2x$ and $f''(x) = -2$

Put $f'(x) = 0$ then

$\frac{P}{2} - 2x = 0$

$\therefore x = P/4$

Put $x = P/4$ in $f''(x)$ then

$f''(P/4) = -2 < 0$

$\therefore$ Area of rectangle is maximum ~~at~~ when length is $\frac{P}{4}$.

Also,

$b = \frac{P}{2} - x$

$= \frac{P}{2} - \frac{P}{4}$

$= \frac{P}{4}$

Here, Length and breath are same. So the given rectangle is a square.

10. A window is in the form of a rectangle surrounded by a semi-circle. If the ~~perim~~ total perimeter is 9 meters, find the radius of the semi-circle for the greatest window area.

Solution:

Let $2x$ be the length and y be the breadth of the rectangle of window.

We know, Total perimeter = Perimeter of rectangle + peri. of semi-circle

$$\text{or, } 9 = 2x + 2y + \pi x$$

$$\text{or, } \frac{9 - 2x - \pi x}{2} = y$$

Again,

Area of Window (A) = Area of rect. + Semi-circle

$$= (L \times b) + \frac{\pi r^2}{2}$$

$$= 2x \cdot y + \frac{\pi x^2}{2}$$

Putting the value of y in $2xy + \frac{\pi x^2}{2}$

$$= 2x\left(\frac{9 - 2x - \pi x}{2}\right) + \frac{\pi x^2}{2}$$

$$= 9x - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$= 9x - 2x^2 - \frac{1}{2}\pi x^2$$

Now, $f'(x) = 9 - 4x - \pi x$ and $f''(x) = -4 - \pi$

Put $f'(x) = 0$

$$\text{or } 9 - 4x - \pi x = 0$$

$$\text{or } 9 = x(4 + \pi)$$

$$\therefore x = \frac{9}{4 + \pi}$$

Putting the value of x in $f''(x)$

$= -(4+\pi) < 0$

$\therefore$ The area has maximum value when the radius of semi-circle is $\frac{9}{4+\pi}$

11 A closed cylindrical can is to be made so that its volume is 52 cm^3. Find its dimensions if the surface is to be a minimum

Solution:

Let, radius of cylindrical can $(r) = x$ cm

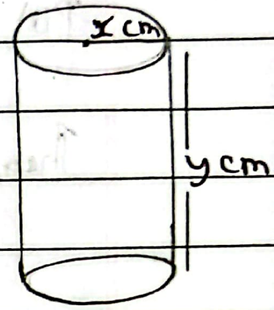
Height of cylindrical can $(h) = y$ cm

Volume $(V) = 52\ cm^3$

or, $\pi r^2 h = 52$

or, $\pi x^2 y = 52$

$\therefore y = \frac{52}{\pi x^2}$



And,

Total surface Area (TSA) $= 2\pi r^2 + 2\pi r h$

$= 2\pi x^2 + 2\pi x y$

$= 2\pi x^2 + 2\pi x\left(\frac{52}{\pi x^2}\right)$

$= 2\pi x\left(x + \frac{52}{\pi x^2}\right)$

$= 2\pi x\,\frac{(\pi x^3 + 52)}{\pi x^2}$

$= \frac{2\pi x^3}{x} + \frac{52 \times 2}{x}$

$= 2\pi x^2 + \frac{104}{x}$

$= 2\pi x^2 + 104 x^{-1}$

Now, $f'(x) = 4\pi x - 104x^{-2}$

$f''(x) = 4\pi + 208x^{-3}$

Put $f'(x) = 0$

or $4\pi x - 104x^{-2} = 0$

or, $4\pi x = 104x^{-2}$

or, $\pi x \times x^2 = 26$

or, $\pi x^3 = 26$

$\therefore x = \left(\frac{26}{\pi}\right)^{1/3}$

Put $x = \left(\frac{26}{\pi}\right)^{1/3}$ in $f''(x)$

Then, $4\pi + 208\left(\frac{26}{\pi}\right)$

$= 12\pi > 0$

$\therefore$ $f(x)$ has minimum dimension at $x = \left(\frac{26}{\pi}\right)^{1/3}$

And $y = \frac{52}{\pi x^2} = \frac{52x}{\pi\left(\frac{26}{\pi}\right)^{\frac{2}{3}} \cdot x}$ [$\because$ dividing by 'x']

$= \frac{52}{26} = 2x$

$= 2\left(\frac{26}{\pi}\right)^{1/3}$

$\therefore$ Height of can $= 2\left(\frac{26}{\pi}\right)^{1/3}$.