c/w

Chapter-2
Functions

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Function
A function from a set A to a set B is a relation or rule that associates each element of A with a unique element of B.

Domain
Let $f: A \rightarrow B$, is a function then the set of values of ~~x whic~~ $x \in A$ for which function is define is said to be the domain and denoted by 'D(f)'. Thus,

$$D(f) = \{x : x \in A \text{ for which function is defined}\}$$

Range
If a function $f: A \rightarrow B$, the set of values of $y \in B$ corresponding to each $x \in A$ which runs over the domain of the function is known as range of f.

Types of function

a. One to one or Injective functions.

A function f from A to B i.e $f: A \rightarrow B$ is said to be one-to-one or injective if distinct element (Pre image) in A have distinct image of B.

In symbol, for all $x, y \in A$

$$x \neq y \rightarrow f(x) \neq f(y)$$

OR

$$f(x) = f(y) \rightarrow x = y$$

b. Onto function

A function f from a set A to set B i.e. $f: A \rightarrow B$ is said to be onto or ~~sub~~ surjective, if every emelemt of B is an image of at least one element of A.

Exercise - 2.1

1. Let $X = \{a, b, c\}$ and $Y = \{p, r, s\}$. Determine which of the following relations from X to Y are functions. Give reasons for your answers.

a. $R_1 = \{(a,p), (a,r), (b,r), (c,s)\}$

Solution:

No, it is not a function as first two ordered pair have the same domain.

b. $R_2 = \{(a,p), (b,r)\}$

Solution.

No, it is not a function because, $c \in X$ corresponds with no element of Y.

c. $R_3 = \{(a,s), (b,s), (c,s)\}$

Solution:

Yes, it is a function because every element of X associate with a unique element of Y.

Domain; $D(f) = \{a, b, c\}$

$R(f) = \{s\}$

d. $R_4 = \{(a,p), (b,p), (c,s)\}$

Solution:

Yes, it is a function because every element of X associate with a unique element of Y.

$D(f) = \{a, b, c\}$

$R(f) = \{p, s\}$

e. $R_5 = \{(a,p), (b,r), (c,s)\}$

Solution:

Yes, it is a function because every element of X are associate with unique element of Y.

$D(f) = \{a, b, c\}$
$R(f) = \{p, r, s\}$

2. If $f: A \to B$ where A and $B \subset R$, is defined by $f(x) = 1 - x$, find the images of $1, \frac{3}{2}, -1, 2 \in A$.

Solution:
We have, $f(x) = 1 - x$
Now,
$f(1) = 1 - 1 = 0$
$f(3/2) = 1 - \frac{3}{2} = -\frac{1}{2}$
$f(-1) = 1 - (-1) = 2$
$f(2) = 1 - 2 = -1$

3. Let i) $f(x) = x + 2$ ii) $f(x) = 2|x| + 3x$ in the interval $-1 \leq x \leq 2$. Find,

Solution:
i) ~~$f(x) = x+2$~~
When, $f(x) = x + 2$
a. $f(-1) = -1 + 2 = 1$
b. $f(0) = 0 + 2 = 2$
c. $f(1) = 1 + 2 = 3$
d. $f(2) = 2 + 2 = 4$
e. $f(-2)$ = function is not define as $-2 \notin$ domain definition
f. $f(-3)$ = function not define as -3 doesn't belong to domain definition.

ii. $f(x) = 2|x| + 3x$
Solution:
a. $f(-1) = 2|-1| + 3(-1)$
$= 2 - 3$
$= -1$

b. $f(0) = 2|0| + 3\times 0$
$= 0$

c. $f(1) = 2|1| + 3\times 1$
$= 5$

d. $f(2) = 2|2| + 3\times 2$
$= 10$

e. $f(-2)$ = function is not defined as -2 doesn't belong to domain of definition

f. $f(3)$ = function is not defined as 3 doesn't belong to domain of definition.

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4. i. Let the function $f: R \to R$ be defined by

$$f(x) = \begin{cases} 3 + 2x & \text{for } -1/2 \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < 1/2 \\ -3 - 2x & \text{for } x \ge 1/2 \end{cases}$$

Find;

a. $f(-1/2)$

Solution:

Since, $-1/2 \in [-1/2, 0)$ or $-1/2 \le x < 0$, we use the formula;

$f(x) = 3 + 2x$

Now, $f(-1/2) = 3 + 2(-1/2)$
$= 2$

b. $f(0)$

Solution:

Since, $0 \in [0, 1/2)$ or $0 \le x < 1/2$, we use the formula

$f(x) = 3 - 2x$

Now, $f(0) = 3 - 2\times 0 = 3$

c. $f(1/2)$

Solution:

Since, $1/2 \in [1/2, \infty)$ or $x \geq 1/2$, we use the formula

$$f(x) = -3 - 2x$$

Now, $f(1/2) = -3 - 2 \times \frac{1}{2}$

$$= -4$$

d. $\frac{f(h) - f(0)}{h}$ for $0 \leq h < 1/2$

Solution:

Since, for $0 \leq h < 1/2$, we use the formula

$$f(x) = 3 - 2x$$

Now, $\frac{f(h) - f(0)}{h} = \frac{3 - 2h - f(0)}{h}$

$$= \frac{3 - 2h - 3}{h} \quad [\because f(0) = 3]$$

$$= -2$$

ii. Let the function $f: R \to R$ be defined by $f(x) = \begin{cases} 4x - 2 & \text{for } x \geq 1 \\ 2x & \text{for } x < 1 \end{cases}$

find.

a. $f(2)$

Solution:

Since, $2 \in (\infty, 1]$ or $x \geq 1$, we use the formula

$$f(x) = 4x - 2$$

Now, $f(2) = 4 \times 2 - 2$

$$= 6$$

b. $f(1)$

Solution:

Since $1 \in (x \geq 1)$ we use the formula

$$f(x) = 4x - 2$$

Now, $f(1) = 4 \times 1 - 2$

$= 2$

c. $f(0)$

Solution:

Since, $0 \in (x < 1)$ we use the formula

$f(x) = 2x$

Now, $f(0) = 2 \times 0$

$= 0$

d. $f(-1)$

Solution:

Since, $-1 \in (x < 1)$, we use the formula,

$f(x) = 2x$

Now, $f(-1) = 2 \times (-1)$

$= -2$

e. $\dfrac{f(h) - f(1)}{h}$ for $1 < h$

Solution:

for $1 < h$, we use the formula

$f(x) = 4x - 2$

Now, $\dfrac{f(h) - f(1)}{h} = \dfrac{4h - 2 - f(1)}{h}$

$= \dfrac{4h - 2 - 2}{h}$ $[\because f(1) = 2]$

$= \dfrac{4h - 4}{h}$

$= \dfrac{4(h-1)}{h}$

5. Let $A = \{-1, 0, 2, 4, 6\}$ and a function $f: A \to R$ is defined by (i) $y = f(x) = \frac{x}{x+2}$ ii) $y = f(x) = \frac{x(x+1)}{x+2}$

Find the range of f.

i. $y = f(x) = \frac{x}{x+2}$

Solution:
Given; $f(x) = \frac{x}{x+2}$ and $A = \{-1, 0, 2, 4, 6\}$

When $x = \{-1, 0, 2, 4, 6\}$ the value of $f(x)$ i.e. $f(-1), f(0), f(2), f(4), f(6)$ are $-1, 0, \frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively.

$\therefore$ range of $f = \{f(-1), f(0), f(2), f(4), f(6)\}$

$\therefore f(A) = \left\{-1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$

ii. $y = f(x) = \frac{x(x+1)}{x+2}$

Solution:
When $x = \{-1, 0, 2, 4, 6\}$ the value of $f(x)$ i.e. $f(-1), f(0), f(2), f(4), f(6)$ are $0, \frac{3}{2}, \frac{10}{3}$ and $\frac{21}{4}$

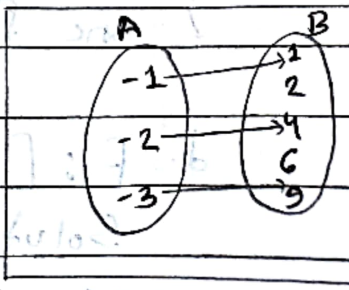
Now,

Range, $R(f) = \left\{0, \frac{3}{2}, \frac{10}{3}, \frac{21}{4}\right\}$

6. Determine whether the functions of f and g defined below are equal or not.

a. $f(x) = x^2$ where $A = \{x : x = -1, -2, -3\}$

b. Venn diagram.



Solution:
Here, Given function $g(x) = \{(-1, 1)(-2, 4), (-3, 9)\}$

Also,

The function f is defined by;

$f(x) = x^2$ where $A = \{x : x = -1, -2, -3\}$

Now,

$f(x) = \{(-1, 1)\ (-2, 4)\ (-3, 9)\}$

Since, $f(x) = g(x)$. The function are equal.

7. Determine whether or not each of the following functions are one to one.

a. Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{4, 0, 1\}$. A function $f : A \to B$ is defined by;

$$\left.\begin{matrix} f(-2) \\ f(2) \end{matrix}\right\} = 4, \quad f(0) = 0, \quad \left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right\} = 1$$

Solution:

No, since $-2 \neq 2$ but ~~$f(x) = f(y)$~~ $= f(-2) = f(2) = 4$. so f is not one to one.

b. Let A be the set of positive integers and B the set of square of positive integers. A function $f : A \to B$ is defined by $f(x) = x^2$.

Solution

~~Yes,~~ Since $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$. So f is one to one function.

c. $f : [-2, 2] \to R$ is defined by $f(x) = x^2$

Solution

Since, $2 \neq 2$ but $f(-2) = f(2)$. So, f is not one to one function.

d. $f : [0, 3] \to R$ is defined by $f(x) = x^2$

Solution:

Since, $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$. So f is one to one function.

8. Examine whether the following functions are one to one, onto, both or neither.

i. $f : A \to B$ defined by $f(x) = x^2$ where, $A = \{1, -3, 3\}$ and $B = \{1, 9\}$

Solution:

Here, $f(x) = x^2$ then

$f(1) = (1)$

$f(-3) = (9)$

$f(3) = (9)$

Since, All the element on set B has pre-image on set A.

It is onto function.

ii. $f : N \to N$ defined by $f(x) = 2x$

Solution:

$f(x) = 2x$

Let $x_1, x_2 \in N$

then $f(x_1) = f(x_2) \to 2x_1 = 2x_2 \to x_1 = x_2$

$\therefore$ f is one to one function

Also,

Since ~~all~~ for all $y \in N$, $x \notin N$; so f is not onto function

iii. $f : (-2, 2) \to R$ defined by $f(x) = x^2$

Solution:

Here, $f(x) = x^2$

$f(-1) = 1$

$f(0) = 0$

$f(1) = 1$

Since, $(-1) \neq (1)$ but $f(-1) = f(1)$. It is not one to one function.

Again,

Let $f(x) = y$ such that $y \in R$

$\rightarrow y = x^2$

$\rightarrow x = \sqrt{y}$, If $y = -1$, as $y \in R$

Then x will be undefined. Hence, $f(x)$ is not onto function.

iv. $f : Q \rightarrow Q$ defined by $f(x) = 6x + 5$

Solution:

Let x_1 and $x_2 \in Q$ then,

$f(x_1) = f(x_2) \rightarrow 6x_1 + 5 = 6x_2 + 5$

$\rightarrow 6x_1 = 6x_2$

$\rightarrow x_1 = x_2$

$\therefore$ f is one to one function.

Again,

Let y be any element $\in Q$ then,

$f(x) = y = 6x + 5$

or $y - 5 = 6x$

or, $\frac{y-5}{6} = x$

or $x = \frac{y-5}{6} \in Q$ for all $y \in Q$

And, $f\left(\frac{y-5}{6}\right) = 6\left(\frac{y-5}{6}\right) + 5 = y$

$\therefore$ y is the image of $\frac{y-5}{6}$

$\therefore$ f is onto function.

v. $f : R \rightarrow R$ defined by $f(x) = x^3$

Solution:

Let $x_1, x_2 \in R$ then,

$f(x_1) = f(x_2) \rightarrow (x_1)^3 = (x_2)^3$

$\rightarrow x_1 = x_2$

$\therefore$ f is one to one function.

Again,
Let y be any element $\in R$ then,

$f(x) = y = x^3$

or, $y = x^3$

or, $y^{1/3} = x$

or, $x = y^{1/3} \in R$

Now,

$f(y^{1/3}) = (y^{1/3})^3$

$= y$

Here, y is the image of $y^{1/3}$

$\therefore f$ is onto function.

9.a. Let a function $f: A \rightarrow B$ be defined by $f(x) = \frac{x^2}{6}$ with $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, \frac{1}{6}, \frac{2}{3}\}$. find the range of f. Is the function one to one and onto?

Solution:

Here, $f(x) = \frac{x^2}{6}$

$f(-2) = \frac{(-2)^2}{6} = \frac{2}{3}$

$f(-1) = \frac{1}{6}$

$f(0) = 0$

$f(1) = \frac{1}{6}$

$f(2) = \frac{2}{3}$

Since, $-2 \neq 2$ but $f(-2) = f(2)$. So, it is not one to one function.

Again,

Since all element of set B has pre image on set A. So, it is onto function.

Also,

$R(f) = \left\{0, \frac{1}{6}, \frac{2}{3}\right\}$

b. Let a function $f: A \to B$ be defined by $f(x) = \frac{x^2+1}{2x-1}$ with $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \{-1, 0, 4/5, 5/7, 1, 2, 3\}$. Find the range of f. Is the function f one to one and onto both? If not, how can the function be made one-to-one and onto both?

Solution:

Here, $A = \{-1, 0, 1, 2, 3, 4\}$

$B = \{-1, 0, 4/5, 5/7, 1, 2, 3\}$

we know,

$$f(x) = \frac{x^2+1}{2x-1}$$

$f(-1) = 0$

$f(0) = -1$

$f(1) = 2$

$f(2) = 1$

$f(3) = 4/5$

$f(4) = 5/7$

Here,

$x_1 \neq x_2 \to f(x_1) \neq f(x_2)$. So, the function f is one to one function.

Again,

The element '3' in set B doesn't have pre-image in set A. So it is not onto function.

Also,

$R(f) = \{-2, 0, 4/5, 5/7, 1, 2\}$