Waves in Pipes and Strings

Tone

The sound of definite frequency is called a tone. It is produced when a body of certain mass and length is vibrated.

Note

Note of sound is the combination of many tones.

Harmonics

A harmonics is a signal or wave whose frequency is an integral multiple of some reference signal. For a wave whose fundamental frequency is f, it is called first harmonic. Then, 2nd harmonic, 3rd harmonic, 4th harmonic, ----, etc. are represented by $2f, 3f, 4f, ---$, etc respectively. The signal occuring at $2f, 4f, 6f, ---$ are called even harmonics and the signals occuring at $3f, 5f, 7f, ---$ are called odd harmonics.

Overtone

An overtone is a musical tone which is a part of the harmonic series above the fundamental tone. If a sound possess all possible harmonics $f, 2f, 3f, ----, 2f$ is called first overtone and so on. If a sound possess odd harmonics $f, 3f, 5f, ---, 3f$ is called first overtone and so on.

Organ Pipes

A hollow wooden or metallic tube used to produce musical sound is called an organ pipe. They are categorized into closed organ pipe and open organ pipe.

- Closed organ pipe

The organ pipe whose one end is open to air and another end is closed off is called closed organ pipe.

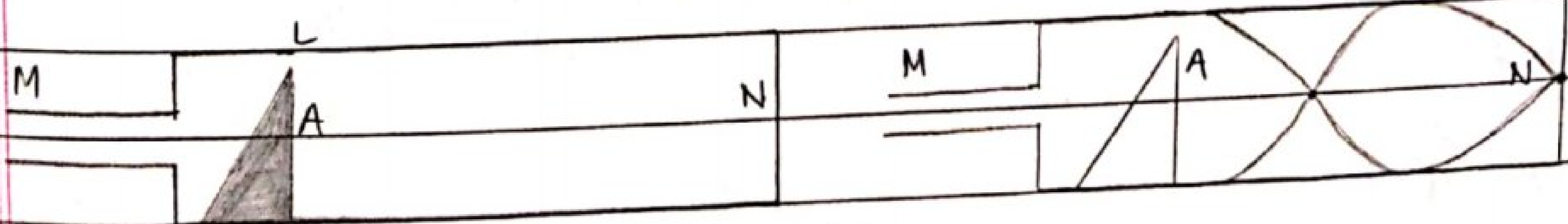


Fig: closed organ pipe

When the air is set into vibration at the open end, the longitudinal waves travel into the pipe towards the closed end. These waves reflect back towards the open end after colliding the air molecules at the closed end. In reflection, compression reflects back as the rarefaction as the phase reversal. The wave travelling from the open end when superimposed with the reflected wave, stationary wave is formed into the pipe.

Modes of vibration in closed organ pipe

In the simplest mode of vibration, there is a displacement node N at the closed end as the air is at rest there and a displacement antinode A at the open end as the air can vibrate freely. Harmonics and overtones produced in a closed organ pipe are:

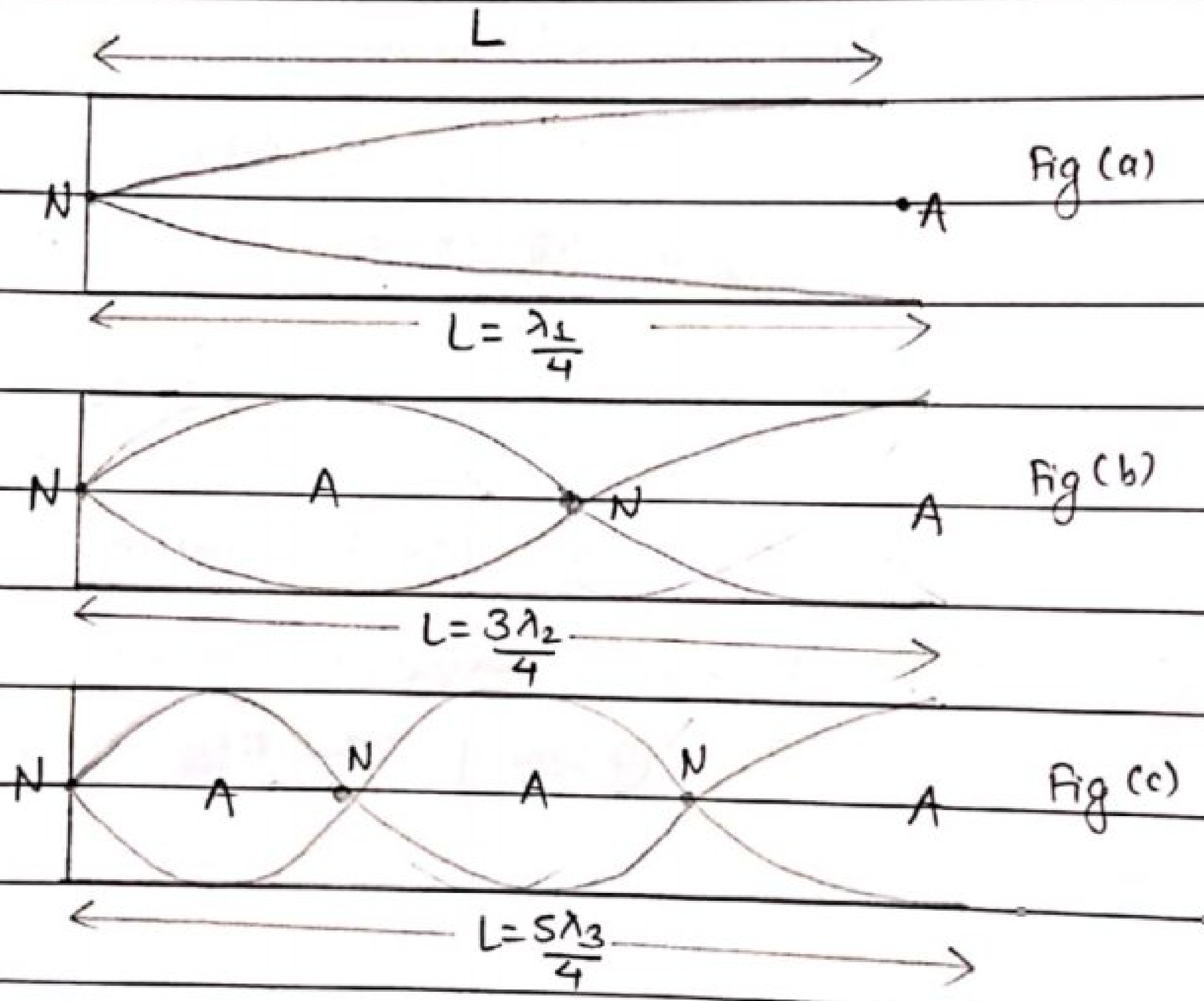


Fig: Different modes of vibration in a closed pipe.

↳ Fundamental mode

In this mode, the pipe has one node at its closed end and one antinode at its open end. As in fig (a), the length L of the pipe is equal to $\lambda_1/4$ where λ_1 is the wavelength of stationary wave. Then,

$L = \frac{\lambda_1}{4} \Rightarrow \lambda_1 = 4L$

If v is the velocity of sound and f_1 is its frequency of vibration,

$v = \lambda_1 f_1$

or $f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$

This is the lowest frequency produced in the pipe which is called fundamental frequency or first harmonic.

↳ Second mode (First overtone)

This mode contains two antinodes and two nodes as in fig (b). If λ_2 is the wavelength of the wave,

$L = \frac{3\lambda_2}{4}$

or $\lambda_2 = \frac{4L}{3}$

The frequency of vibration, f_2 is

$f_2 = \frac{v}{\lambda_2} = 3\left(\frac{v}{4L}\right) = 3f_1$

This is the frequency of first overtone or third harmonic. This frequency is three times the fundamental frequency.

↳ Third mode (second overtone)

This mode contains three antinodes and three nodes as in fig (c). If λ_3 is the wavelength of the wave,

$L = \frac{5\lambda_3}{4} \Rightarrow \lambda_3 = \frac{4L}{5}$

The frequency of vibration, f_3 is

$f_3 = \frac{v}{\lambda_3} = 5\left(\frac{v}{4L}\right) = 5f_1$

This is the frequency of second overtone or fifth harmonic. This frequency is five times the fundamental frequency.

Conclusions:

i. Only odd harmonics are possible in closed organ pipe.

ii. The even harmonics are missing and hence the sound is not pleasant to hear.

iii. In general frequency of n^{th} overtone = $(2n+1)$ times the fundamental frequency

iv. The no. of nodes and antinodes are equal in each mode.

v. For $(2n-1)^{th}$ harmonic, no. of nodes or antinodes = n but only odd harmonics are present

vi. Fundamental frequency $(f_1) \propto \frac{1}{l}$.

vii. As $f \propto v$ and $v \propto \sqrt{T}$, $f \propto \sqrt{T}$.

- Open Organ Pipe

An organ pipe whose both ends are open to air is called open organ pipe.

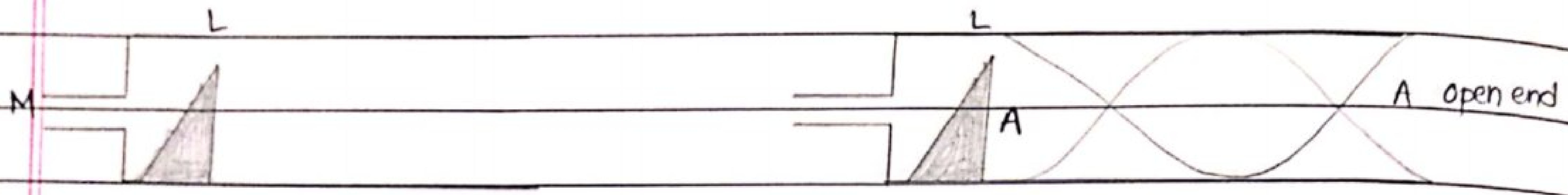


Fig: open organ pipe

When a vibration is set up at one end of the pipe, it travels towards the another end in the form of compressions and rarefactions. When compression reaches at open end, the density of air readily decreases. Hence, the rarefaction travels back to the pipe. Also, when a rarefaction reaches to the open end, compression reflects back. Finally, the stationary wave is set up into the pipe due to the superposition of incident and reflected waves.

Modes of vibration in open organ pipe

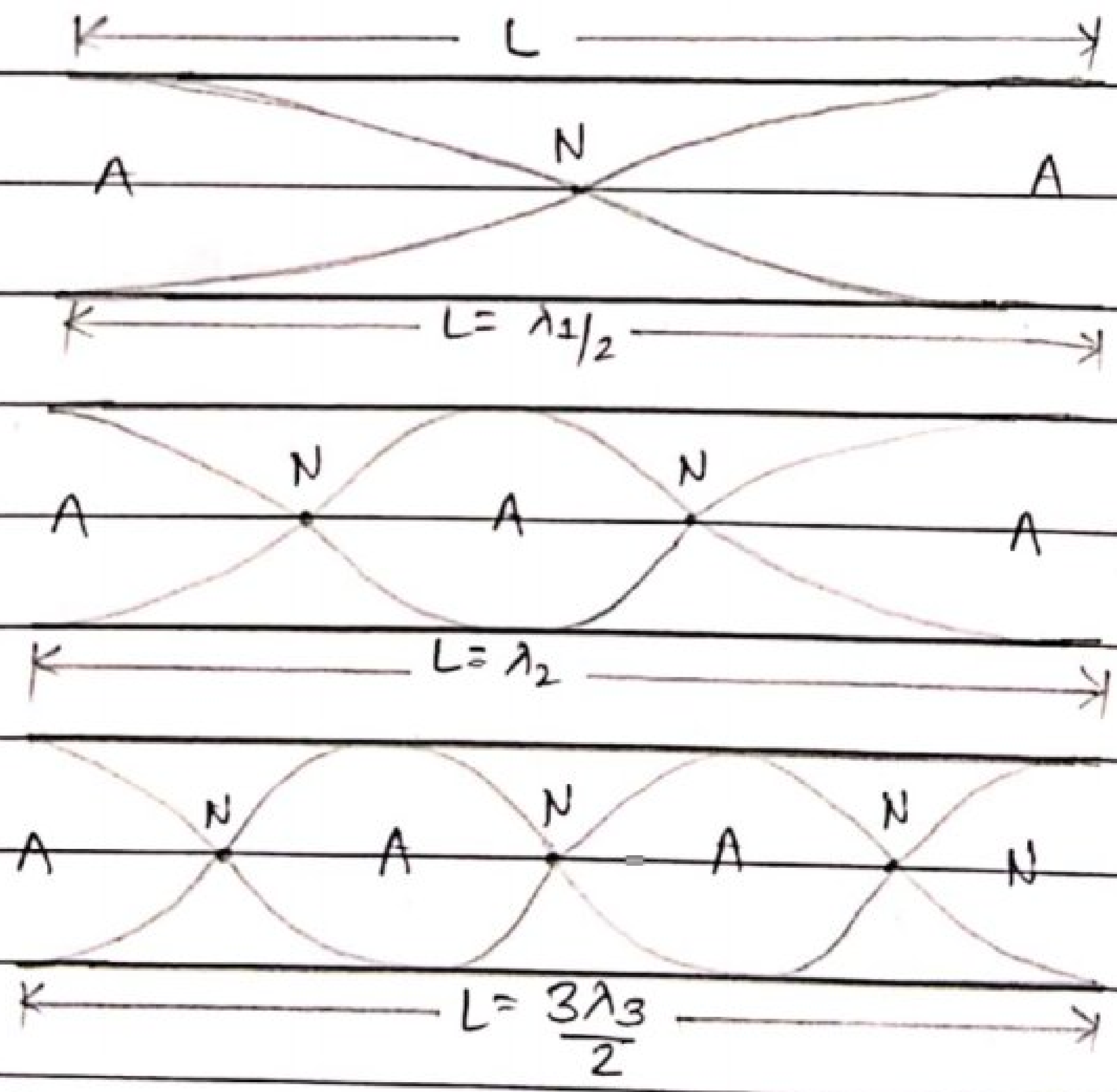


Fig: modes of vibration of open organ pipe

↳ Fundamental Mode

In this mode, the pipe has two antinodes at open end and one node at the middle as in fig (a). Let λ_1 be the wavelength of the wave, then.

$L = \lambda_1/2 \Rightarrow \lambda_1 = 2L$

The frequency $f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$

This is the lowest frequency produced by open organ pipe which is called first harmonic.

→ second mode (First Overtone).

This mode contains two antinodes at open ends and two nodes and one antinode inside the pipe as in fig (b). If λ_2 be the wavelength of the wave.

$L = \lambda_2$

Thus the frequency of first overtone is

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2\left(\frac{v}{2L}\right) = 2f_1$$

This is the frequency of first overtone or second harmonic.

→ Third mode (Second Overtone).

This mode contains three nodes and four anti-nodes as in fig(c). If λ_3 be the wavelength of the wave.

$$L = \frac{3\lambda_3}{2} \Rightarrow \lambda_3 = \frac{2L}{3}$$

The frequency of vibration is

$$f_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

This is the frequency of second overtone or third harmonic.

Conclusions

i. All harmonics are present. So, the sound is richer in quality.

ii. In general, the frequency of n^{th} overtone = $(n+1)$ times the fundamental frequency

iii. The fundamental frequency of an open pipe is twice that of a closed pipe of same length.

iv. For n^{th} harmonic, no. of nodes equals n and no. of antinodes is $(n+1)$.

End Correction of organ pipe

The end correction is defined as the distance between the real position of antinode and the open end of the pipe. It is denoted by c or e. The Rayleigh correction formula is $c = 0.6r$, where r is the radius of the tube.

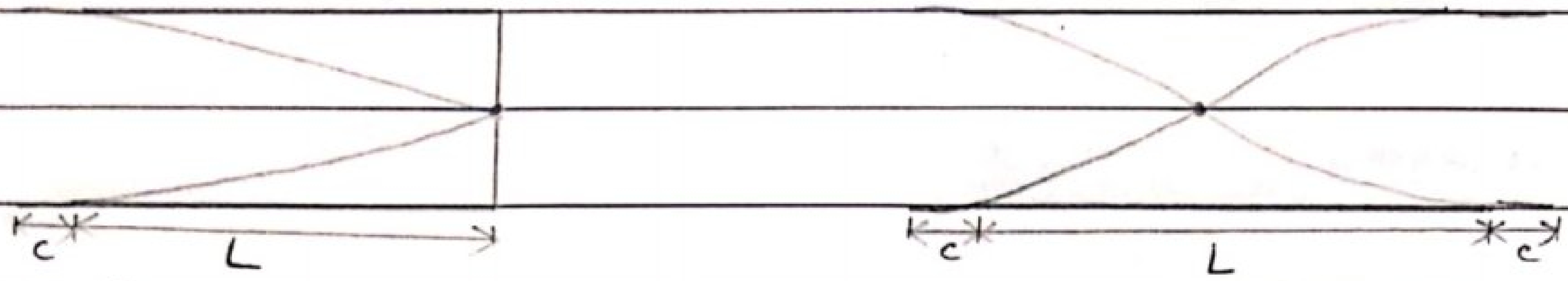


Fig (a) Fig (b)

Fig: end corrections

i. For a closed organ pipe of length L as shown in fig(a), the corrected length is

$L + c = \lambda/4$

$\lambda = 4(L+c)$

ii. For an open pipe of length l as shown in fig(b), the corrected length is.

$l + c + c = \dfrac{\lambda}{2}$

or $\lambda = 2(l + 2c)$

The temperature and end correction affect on the frequency of sound in a pipe. For example, take a closed pipe sounding in its fundamental mode. Then,

$$f = \frac{v}{\lambda} = \frac{v}{4(L+c)}$$

The velocity of sound v, at temperature θ°c in terms of velocity v_0 at 0°C is

$$\frac{v}{v_0} = \sqrt{\frac{T_\theta}{T_0}} = \sqrt{\frac{273+\theta}{273}} = \sqrt{1 + \frac{\theta}{273}}$$

$$f = \frac{v}{4(L+c)} = \frac{v_0}{4(L+c)}\sqrt{1 + \frac{\theta}{273}}$$

So, frequency of fundamental mode increases with increase in temperature. For a given temperature and length of pipe, frequency decreases as c increases.

Resonance:

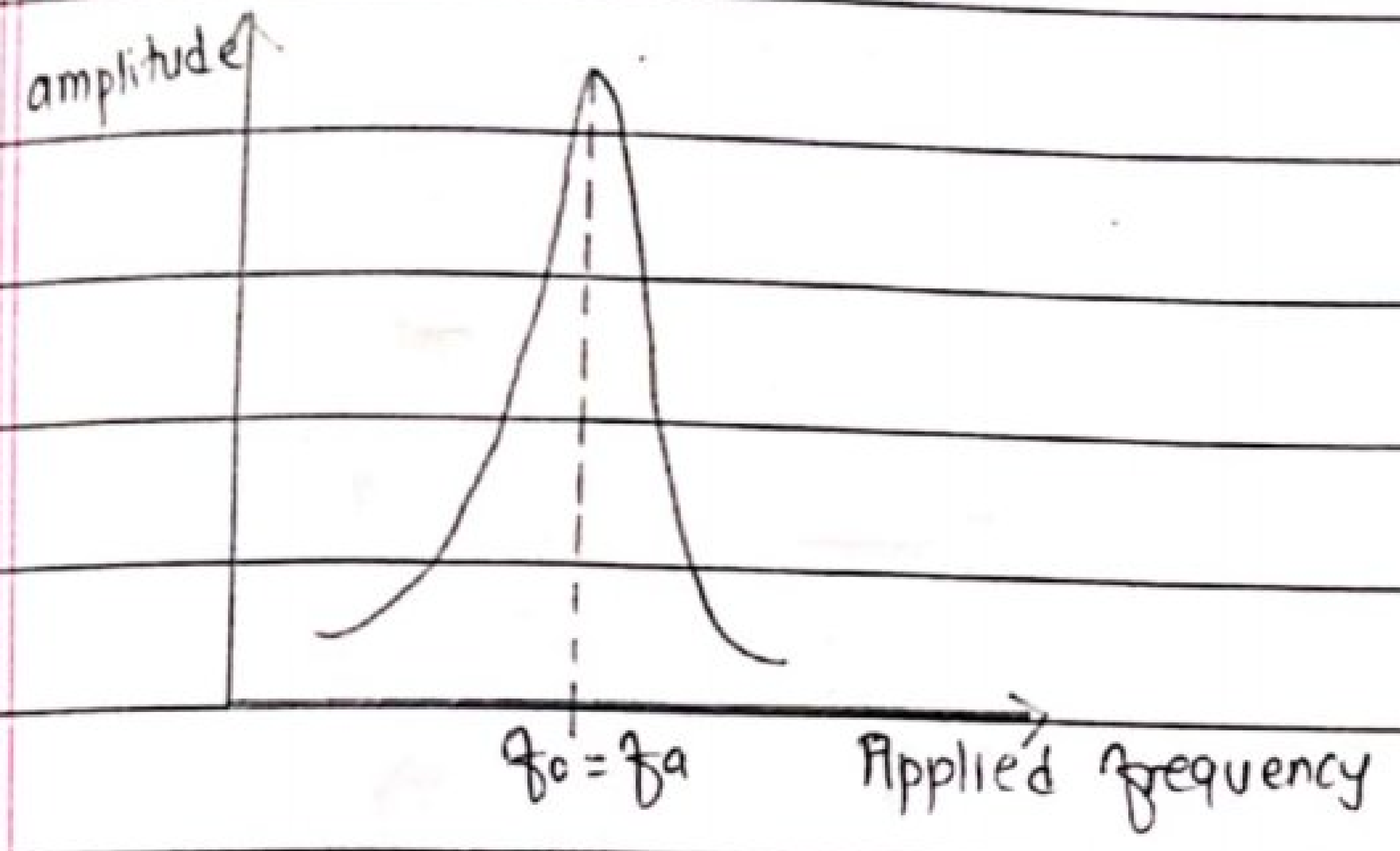
The phenomenon in which the amplitude of vibration is maximum when the frequency of vibrating system is equal to its natural frequency is called resonance and corresponding frequency is called resonant frequency.

Conditions for resonance

i) The frequency of applied force must be equal to the natural frequency of the system.

ii) The applied force must be in phase with the vibrating system.

The variation of amplitude of vibration for a body when it is given a gradually changing frequency is shown below:



where f_0 = natural frequency and f_a = frequency of forced vibration.

Consequences of Resonance

1. The soldiers are asked to break their marching steps while crossing the bridge. If the frequency of walking steps of soldier is equal to the natural frequency of vibration of bridge, the amplitude of its variation is very large. Thus, The resonance occurs. In this situation, the bridge suffers large extension and may cross the elastic limit and collapse.
2. When we set the frequency in our radio that matches the frequency broadcasted from station, the radio produces the sound.
3. A music expert can produce the musical note that may be matched to the natural frequency of oscillation of glass tumbler. In this condition, resonance occurs in the vibration of the glass and it may break.
4. Resonance can cause great damage in earthquake. If the natural frequency of a building matches the frequency of periodic oscillations present within the earth, the resonance occurs and building vibrates with large amplitude. So, the building may be collapsed.

Resonance Tube Apparatus

Resonance Tube is a metallic tube that works on the principle of resonance of the air column inside it. Resonance occurs into the tube when natural frequency of air column inside the tube is equal to the frequency of vibration supplied by tuning forks. In this condition, the air inside the tube vibrates with maximum amplitude and hence sound is produced.

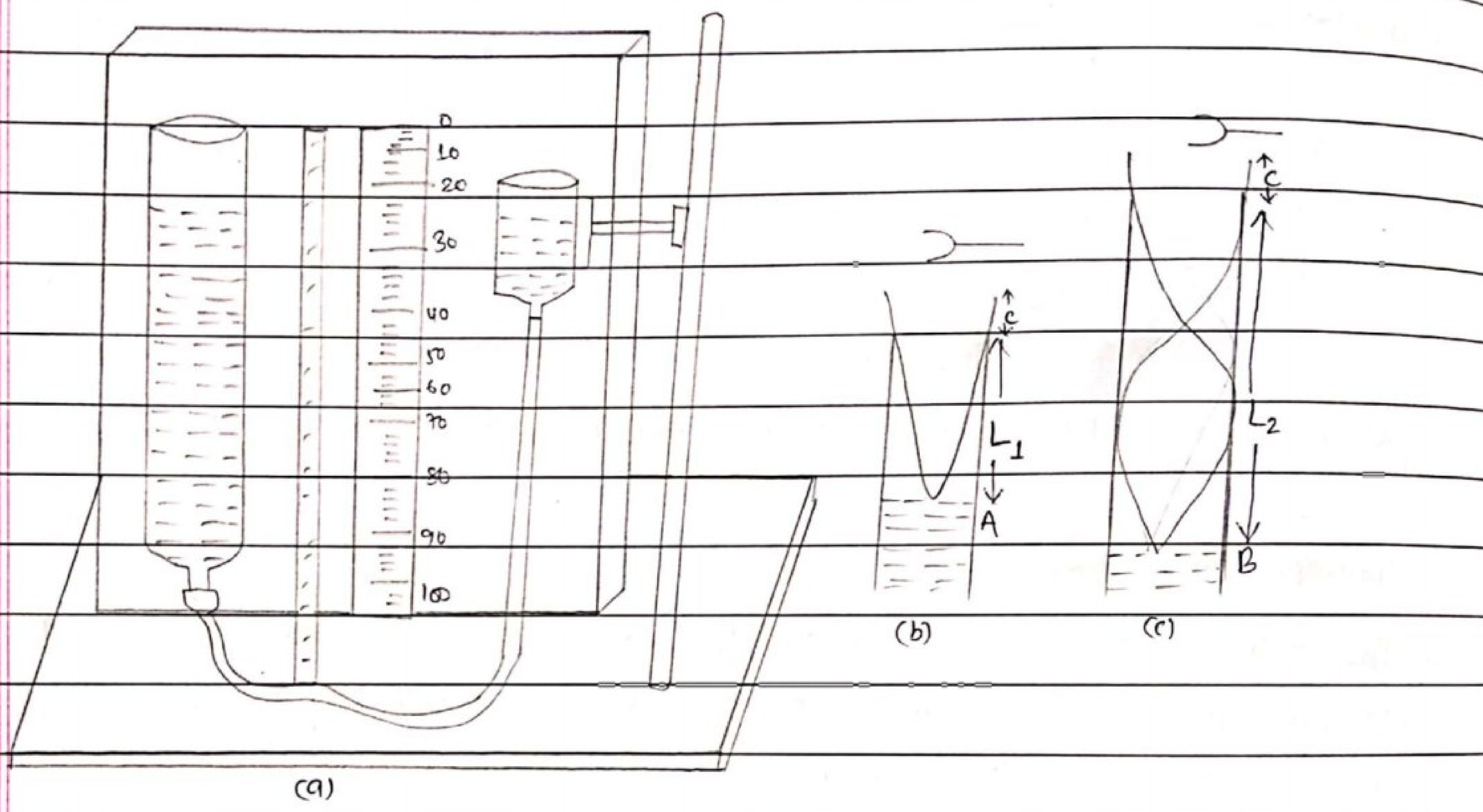


Fig: Resonance Tube apparatus.

To find the velocity of sound in air, the water level in the tube is brought close to the open end of the tube by raising the reservoir. A sounding tuning fork is held horizontally at the mouth of the tube filled with water. A wave travels in air column of the tube reflects from water surface and a stationary wave is formed as shown in b, a sound is heard, the condition is called first resonance. Let, resonance is obtained at A.

If L_1 be the length of air column in the tube for first resonance, c be the end correction of tube, λ be the wavelength of the wave. So,

$L_1 + c = \lambda/4$ ---- (i)

or $\lambda = 4(l_1 + c)$ --- (ii)

water level is further lowered from A until second resonance is obtained and sound is heard at B as in (b), If L_2 be the length of air column for second resonance. Then,

$L_2 + c = \frac{3\lambda}{4}$ ---(iii)

or $\lambda = \frac{4}{3}(L_2 + c)$ ---(iv)

Subtracting (i) from (iii),

$L_2 - L_1 = \lambda/2$

or, $\lambda = 2(L_2 - L_1)$

Knowing the frequency of tuning fork, v can be calculated as

$v = f\lambda = 2f(L_2 - L_1)$

From (ii) and (iv)

$4(L_1 + c) = \frac{4}{3}(L_2 + c)$

or $3L_1 - L_2 = c - 3c$

or $L_2 - 3L_1 = 2c$

$\therefore$ $c = \frac{L_2 - 3L_1}{2}$

Knowing L_1 and L_2, c can be calculated.

The required formula for the velocity of sound taking account of humidity correction is.

$$V_{STP} = V_\theta \sqrt{\frac{P - 0.375f}{P}} \left(1 - \frac{1}{2}\alpha\theta\right)$$

where, P = atmospheric pressure

f = saturated vapour pressure at θ°c.

α = volume coefficient

θ = room temperature.

Velocity of Transverse Wave along a Stretched String.

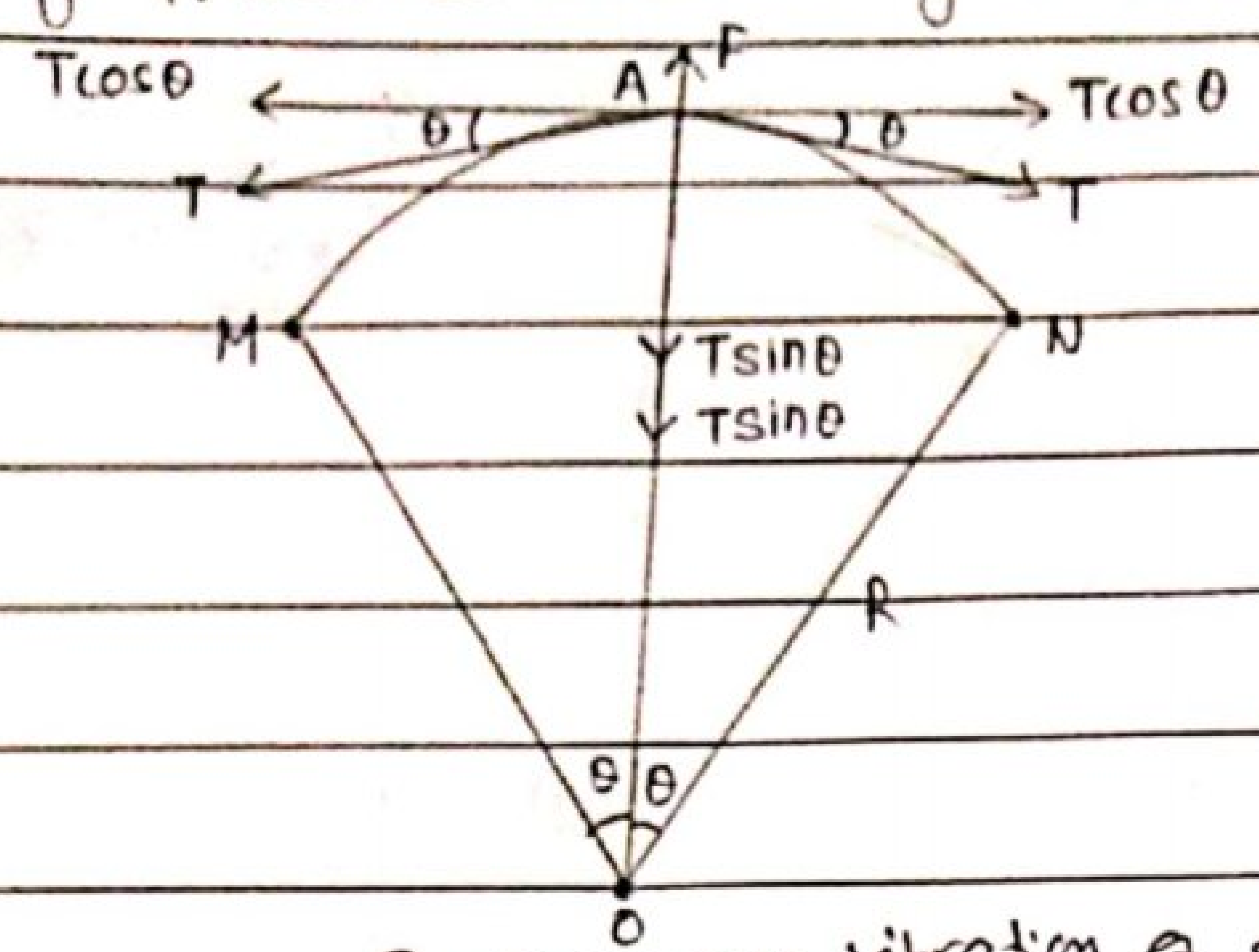


Fig: Transverse vibration of string.

Consider a string fixed at two ends M and N. The midpoint A of string is plucked up and left to oscillate. The oscillation of string produces the transverse wave. Let, T be the tension at mid point of string due to the plucking. Tension T is directed tangentially out in both sides of the point. Also θ be the angle made by tangent with its horizontal component. The point O is the center of curvature of arc MAN of the string.

Here, horizontal components of tension are equal and opposite in a line so they cancel to each other. However the vertical components are directed linearly towards the centre O of curvature. Let R be the radius of curvature of the string.

Here, tension along the centre of curvature is

$F = 2T\sin\theta$ ---- (i)

For very small angle θ, $\sin\theta \approx \theta$. so,

$F = 2T\theta$ --- (ii)

From the arc. MAN,

$$2\theta = \frac{\overset{\frown}{MAN}}{R}$$

So, $F = T \cdot \dfrac{\overset{\frown}{MAN}}{R}$ --- (iii)

This force provides centripetal force to pull the string towards the centre of curvature. So,

$F' = \dfrac{mv^2}{R}$ where m is the total mass of the string.

Also, $m = \mu \cdot \overset{\frown}{MAN}$ where μ = mass per unit length of string.

$F' = \mu \overset{\frown}{MAN} \dfrac{v^2}{R}$ ---- (iv)

From (iii) and (iv).

$$T \cdot \frac{\widehat{MAN}}{R} = \mu \, \widehat{MAN} \, \frac{v^2}{R}$$

or $v^2 = \frac{T}{\mu}$

$\therefore$ $v = \sqrt{\frac{T}{\mu}}$

Modes of Vibration of a stretched String

Consider a string, fixed at two ends A and B. If the string is plucked at a point, transverse waves are set up and travel towards the fixed ends. These waves reflect back from the fixed ends. As a result, the incident wave from the point of disturbance superimposes with reflected wave and hence produce the stationary wave as shown below.

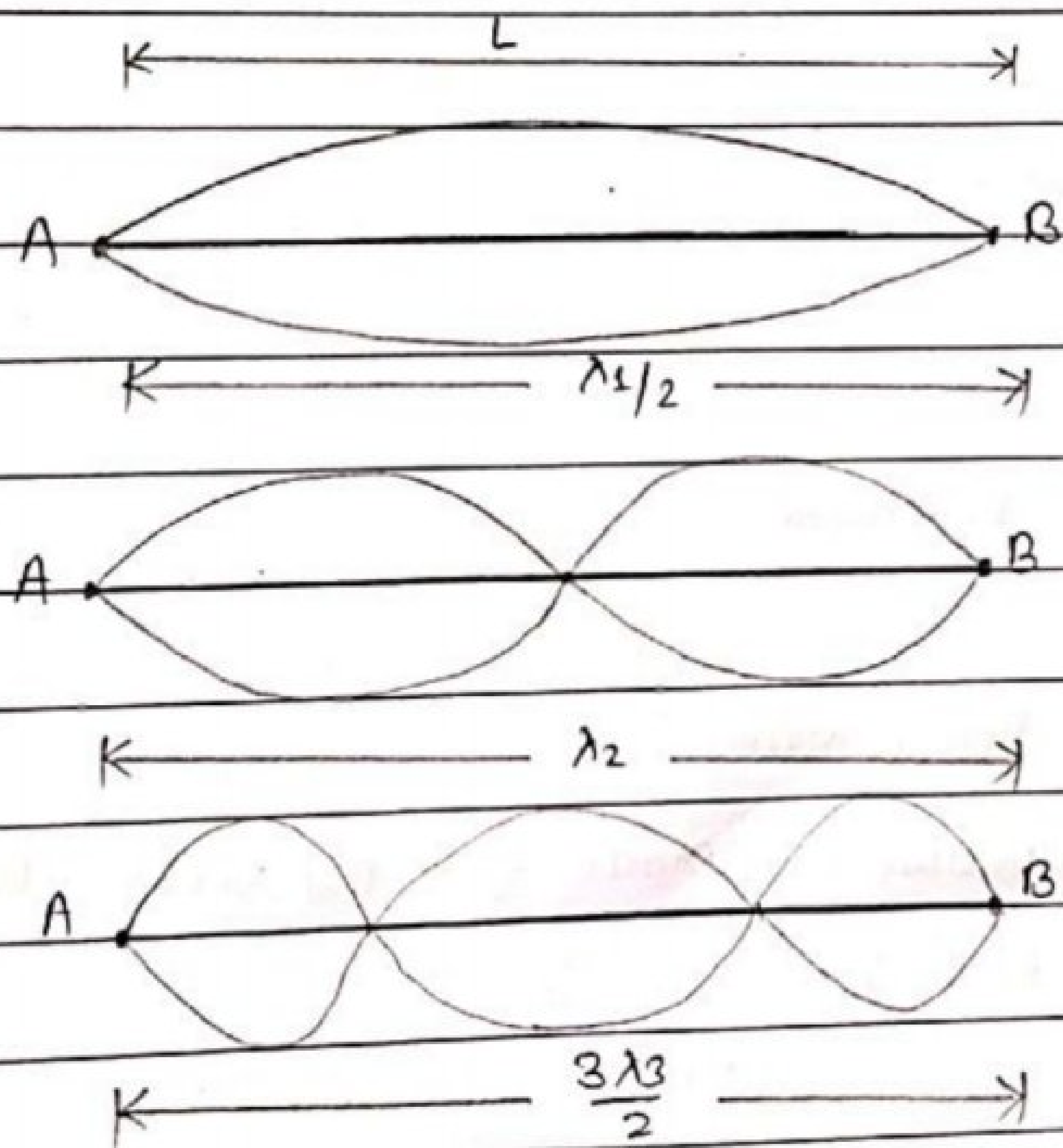


Fig: modes of vibration of stretched string.

↳ Fundamental mode.

When the string is plucked at the middle, an antinode is formed at the middle.

If λ_1 is the wavelength of the wave,

$l_1 = \lambda_1/2 \Rightarrow \lambda_1 = 2L$

The frequency of vibration is:

$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$

This is the lowest frequency produced by the vibrating string which is called fundamental frequency or first harmonic.

↳ Second mode (First overtone).

If the string is plucked at a point $1/4^{th}$ of its length from one end, three nodes and two antinodes are formed as in fig (b).

If λ_2 is the wavelength,

$L = \lambda_2$

The frequency of the wave,

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$$

This is called the second harmonic.

↳ Third mode (Second overtone).

If the string is plucked from $1/3^{rd}$ of the length from one end, additional nodes are produced in it. If λ_3 is the wavelength,

$L = 3\lambda_3/2 \Rightarrow \lambda_3 = 2L/3$

The frequency of the wave,

$$f_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

This is called third harmonic.

Laws of Vibration of Fixed string

The frequency of fundamental mode of a transverse wave in a stretched (fixed) string is given by, $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$

where T is the tension in the stretched string and μ, the mass per unit length and L is resonating length.

From this expression, it follows that there are three laws of transverse vibration of stretched string:

i) The law of length: the fundamental frequency is inversely proportional to the resonating length, L of the string,

$f \propto \frac{1}{L}$

ii) The law of tension: the fundamental frequency is directly proportional to the square root of the stretching force or tension, $f \propto \sqrt{T}$

iii) The law of mass: the fundamental frequency is inversely proportional to the square root of the mass per unit length, $f \propto \frac{1}{\sqrt{\mu}}$

Verification of the Laws of Transverse Vibration.

These laws can be verified experimentally using a sonometer. This device consists of a wire under tension which is arranged in a hollow wooden board as shown in figure. The vibrations of the wire are passed by the movable bridges to the box and then, to the air inside it.

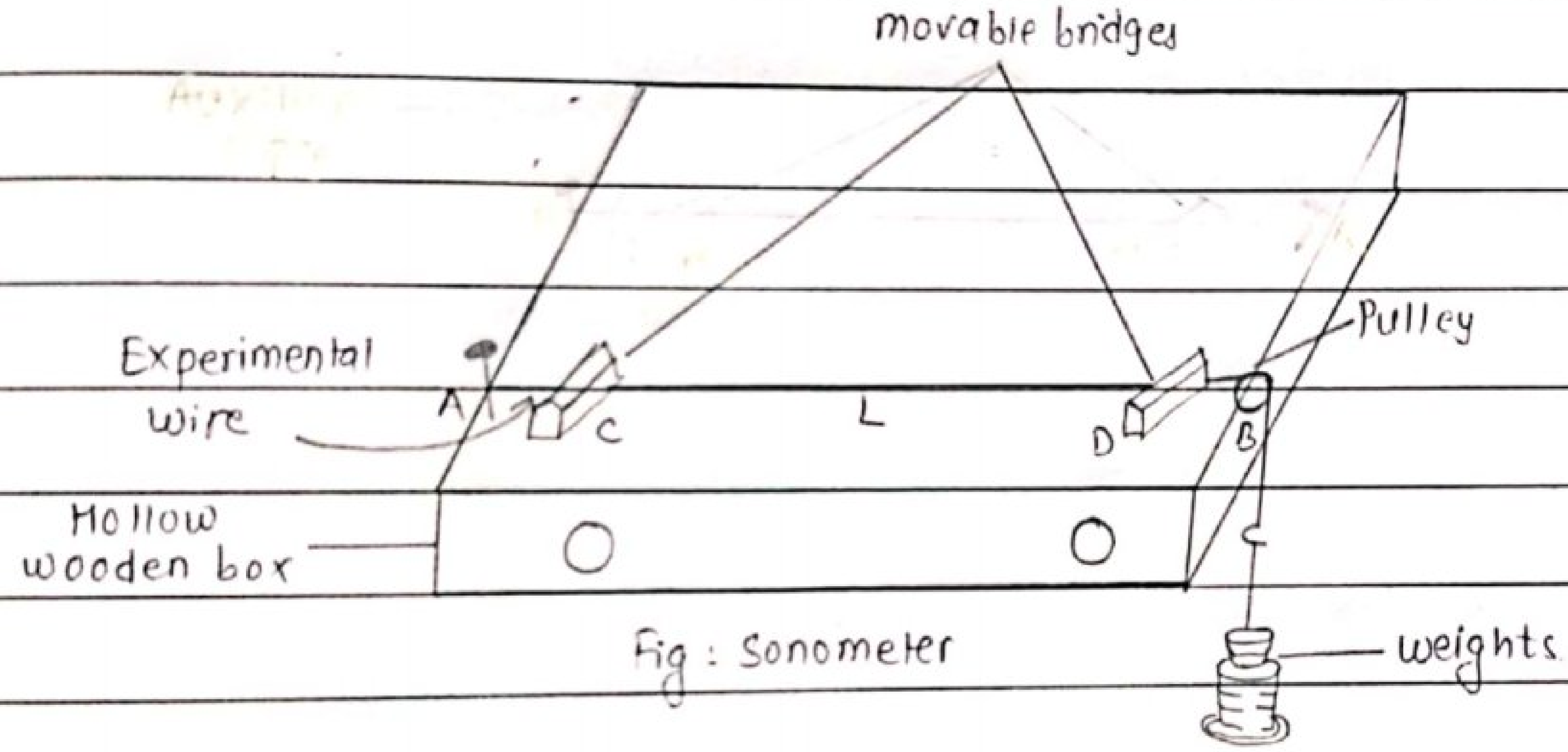


Fig : Sonometer

i. To verify $f \propto 1/L$

To verify this law, take a tuning fork of known frequency. Taking a load of 1kg on the string, find the resonating length of the wire between the bridges C and D. Let l_1 be the resonating length for this tuning fork. The same process is repeated for next tuning fork having different frequency. Let l_2 be the resonating length for the second tuning fork. It will be found that the product $f_1 \times l_1 = f_2 \times l_2$ at constant tension and mass per unit length of the string. This follows that $f \propto 1/L$

ii. To verify $f \propto 1/\sqrt{\mu}$

To verify this law, the wires of different diameters and materials are used under same vibrating length and same tension. Taking a load of 1kg on the string, find the resonating length of the wire between bridges C and D. Let, l be the resonating length for this tuning fork. Take another tuning fork of different frequency and another wire of different diameter. Taking a load of 1kg on the string, find the resonating length of the wire between the bridges C and D. On drawing a graph between frequency and $1/\sqrt{\mu}$, a straight line is obtained. This verifies $f \propto 1/\sqrt{\mu}$.

iii. To verify $f \propto \sqrt{T}$.

To verify this law, the vibrating length of the given wire and linear density 'μ' is constant. A set of tuning forks having different frequencies f_1, f_2, f_3, f_4, etc. is taken.

By adjusting the tension T, each fork is made to vibrate in unison with the fixed length of the wire, one after other. Let, the tensions corresponding to frequencies f_1, f_2, f_3, f_4, etc. be T_1, T_2, T_3, T_4, etc. respectively. Then it is found that $f \propto \sqrt{T}$.