HW

Chapter-9
Analytical Geometry

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Exercise-9.2

1. Find the length of the perpendicular drawn from :

a. $(0,0)$ to the line $3x + y + 1 = 0$

Solution:

Given;

$(x_1, y_1) = (0,0)$

Equation of line is $3x + y + 1 = 0$.

Now,

Perpendicular distance is given by;

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

$$= \left| \frac{3.0 + 1.0 + 1}{\sqrt{3^2 + 1^2}} \right|$$

$$= \frac{1}{\sqrt{10}}$$ #

b. $(-3,0)$ to the line $3x + 4y + 7 = 0$

Solution:

Equation of line is $3x + 4y + 7 = 0$

Also, $(x_1, y_1) = (-3,0)$

Now, Perpendicular distance is given by;

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$= \left| \frac{-3\times3 + 4\times0 + 7}{\sqrt{9 + 16}} \right|$$

$$= \frac{2}{5}$$ #

Page:

c. (2,3) to the line $8x + 15y + 24 = 0$

Solution:

Equation of the line is $8x + 15y + 24 = 0$

Also, $(x_1, y_1) = (2, 3)$

Perpendicular distance is given by;

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$$

$$= \left| \frac{8\times 2 + 15\times 3 + 24}{\sqrt{8^2+15^2}} \right|$$

$$= \frac{85}{17}$$

$$= 5$$

2. If P is the length of the perpendicular dropped from the origin on the line $\frac{x}{a} + \frac{y}{b} = 1$, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$

Solution:

Given; $\frac{x}{a} + \frac{y}{b} = 1$

or, $\frac{x}{a} + \frac{y}{b} - 1 = 0$ ——(1)

By Question:

$$P = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \right|$$

$$\text{or, } P = \left| \frac{0 + 0 - 1}{\sqrt{(1/a)^2 + (1/b)^2}} \right|$$

$$\text{or } P = \left| \frac{-1}{\sqrt{1/a^2 + 1/b^2}} \right|$$

or, $\frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

S.b.s

$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ proved//

3. Find the distance between the parallel lines.

a. $3x + 5y = 11$ and $3x + 5y = -23$

Solution:

Given;

$3x + 5y = 11$ —— (i)

$3x + 5y = -23$ —— (ii)

Put $x = 2$ in equation (i)

Then y will be '1'

So, $(2, 1)$ is a point on $3x + 5y - 11 = 0$

Then,

distance from $(2,1)$ to the line $3x + 5y + 23 = 0$ is given by;

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$= \left| \frac{3 \times 2 + 5 \times 1 + 23}{\sqrt{3^2 + 5^2}} \right|$$

$$= \frac{34}{\sqrt{34}}$$

$$= \sqrt{34}$$ //

b. $2x - 5y = 6$ and $6x - 15y + 11 = 0$

Solution:

Given;

$2x - 5y = 6$ — (i)

$6x - 15y + 11 = 0$ — (ii)

Put $x = 1$ in eq (i)

Then, $y = \frac{-4}{5}$

So, $(1, -4/5)$ is a point on $2x - 5y - 6 = 0$.

Then,

distance from $(1, -4/5)$ to the line $6x - 15y + 11 = 0$ is given by;

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$= \left| \frac{6 \times 1 + (-15)(-4/5) + 11}{\sqrt{6^2 + (-15)^2}} \right|$$

$$= \frac{29}{\sqrt{261}}$$ ##

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Bisectors of the angle between two lines:

Let $L_1: A_1x + B_1y + C_1 = 0$ and $L_2: A_2x + B_2y + C_2 = 0$ are equation of two given lines then; equation of bisectors of the angle between two lines are given by;

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

4. Find the equation of the bisectors of the angles between the lines.

a. $3x - 4y + 2 = 0$ and $5x + 12y + 5 = 0$

Solution:

Given, equation of lines are;

$3x - 4y + 2 = 0$ —— (i)

$5x + 12y + 5 = 0$ —— (ii)

Now,

Bisector of angle between two lines is given by;

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

or, $$\frac{3x - 4y + 2}{\sqrt{3^2 + (-4)^2}} = \pm \frac{5x + 12y + 5}{\sqrt{5^2 + 12^2}}$$

or, $$\frac{3x - 4y + 2}{5} = \pm \frac{5x + 12y + 5}{13}$$

or, $39x - \cancel{48y}\ 52y + 26 = \pm\ 25x + 60y + 25$

Taking (+ve) Sign:

$39x - 52y + 26 = 25x + 60y + 25$

or, $14x - 112y + 1 = 0$

Taking (-ve) Sign:

$39x - 52y + 26 = -25x - 60y - 25$

or, $64x + 8y + 51 = 0$

Hence,

$14x - 112y + 1 = 0$ and $64x + 8y + 51 = 0$ are req. equations.

b. $3x - 4y + 6 = 0$ and $5x + 12y + 10 = 0$

Solution:

Given equation of lines are;

$3x - 4y + 6 = 0$ — (i)

$5x + 12y + 10 = 0$ — (ii)

Now,

Bisector of angle between two lines is given by;

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

or, $$\frac{3x - 4y + 6}{\sqrt{9+16}} = \pm \frac{5x + 12y + 10}{\sqrt{25 + 144}}$$

or, $$\frac{3x - 4y + 6}{5} = \pm \frac{5x + 12y + 10}{13}$$

or, $39x - 52y + 78 = \pm\ 25x + 60y + 50$

Taking (+ve) Sign

$39x - 52y + 78 = 25x + 60y + 50$

or, $14x - 112y + 28 = 0$

or, $x - 8y + 2 = 0$

Taking (-ve) sign:

$$39x - 52y + 78 = -25x - 60y - 50$$

or, $64x + 8y + 128 = 0$

or, $8x + y + 16 = 0$

Hence,

$x - 8y + 2 = 0$ and $8x + y + 16 = 0$ are req. equations.

C. $x - 2y = 0$ and $2y - 11x = 6$

Solution:

Given, equations of line are;

$$x - 2y = 0 \quad \text{--- (i)}$$

$$2y - 11x - 6 = 0 \quad \text{--- (ii)}$$

Now,

Bisector of the angle between the lines is given by;

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

or, $$\frac{x - 2y + 0}{\sqrt{1+4}} = \pm \frac{-11x + 2y - 6}{\sqrt{121+4}}$$

or, $$\frac{x - 2y + 0}{\sqrt{5}} = \pm \frac{-11x + 2y - 6}{5\sqrt{5}}$$

or, $$x - 2y = \pm \frac{-11x + 2y - 6}{5}$$

Taking (+ve) sign:

$$5x - 10y = -11x + 2y - 6$$

or, $16x - \cancel{8y}\ 12y + 6 = 0$

or, $8x - 6y + 3 = 0$

Taking (-ve) sign

$5x - 10y = 11x - 2y + 6$

or, $6x + 8y + 6 = 0$

or, $3x + 4y + 3 = 0$

Hence,

$8x - 6y + 3 = 0$ and $3x + 4y + 3 = 0$ are req. equations.

.a. The length of the perpendicular drawn from the point $(a, 3)$ on the line $3x + 4y + 5 = 0$ is 4. Find the value of a.

Solution:

Given, equation of line is $3x + 4y + 5 = 0$

Also, $(x_1, y_1) = (a, 3)$

Perpendicular distance $(d) = 4$

We know,

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

$$\text{or, } 4 = \left| \frac{3.a + 4.3 + 5}{\sqrt{9 + 16}} \right|$$

$$\text{or, } 4 = \left| \frac{3a + 12 + 5}{5} \right|$$

$$\text{or, } 20 = 3a + 17$$

$$\text{or, } \frac{20 - 17}{3} = a$$

$$\therefore a = 1$$

b. What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a.

Solution:

Given, equation of line is $\frac{x}{a} + \frac{y}{b} - 1 = 0$.

Perpendicular distance (d) = a

~~By Question:~~

Let the points on x-axis be (k, 0).

We know,

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

$$\text{or, } a = \left| \frac{1/a \times k + 1/b \cdot 0 + (-1)}{\sqrt{1/a^2 + 1/b^2}} \right|$$

$$\text{or, } a = \left| \frac{\frac{k}{a} - 1}{\sqrt{1/a^2 + 1/b^2}} \right|$$

$$\text{or, } a = \frac{\frac{k-a}{a}}{\sqrt{1/a^2 + 1/b^2}}$$

~~or~~ ~~a~~

$$\text{or, } a\sqrt{1/a^2 + 1/b^2} = \frac{k-a}{a}$$

Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

or, $xb + ay = ab$

or, $xb + ay - ab = 0$ —(1)

Let the point on x-axis be $(k, 0)$

We know,

$$d = \pm \left(\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right)$$

$$\text{or, } a = \pm \left(\frac{b \cdot k + a \cdot 0 - ba}{\sqrt{b^2 + a^2}} \right)$$

$$\text{or, } a = \pm \left(\frac{bk - ab}{\sqrt{a^2 + b^2}} \right)$$

$$\text{or } a \left(\sqrt{a^2 + b^2} \right) = \pm b(k - a)$$

$$\text{or } \pm \frac{a}{b} \sqrt{a^2 + b^2} = k - a$$

$$\text{or, } k = a \pm \frac{a}{b} \sqrt{a^2 + b^2}$$

$$\text{or, } k = a \left(1 \pm \frac{1}{b} \sqrt{a^2 + b^2} \right)$$

$$\text{or, } k = a \left(\frac{b \pm 1 \sqrt{a^2 + b^2}}{b} \right)$$

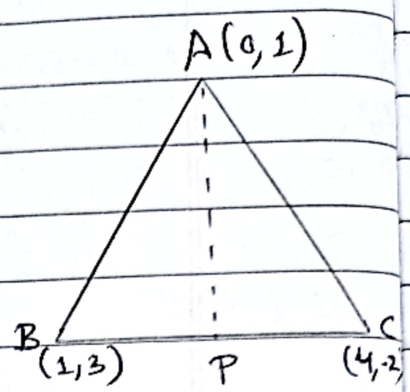
$$\therefore k = \frac{a}{b} \left(b \pm \sqrt{a^2 + b^2} \right)$$

Hence,

The req. points on the x-axis are $\left[\frac{a}{b} \left(b \pm \sqrt{a^2 + b^2} \right), 0 \right]$

8.a. Determine the equation and length of the altitude drawn from the ~~point (0, a)~~ vertex A to the opposite side of the triangle whose vertices are A(0, 1), B(1, 3), C(4, -2)

Solution:



Here,

Equation of line BC is given by;

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\text{or, } y - 3 = \frac{-2-3}{4-1}(x-1)$$

$$\text{or, } y - 3 = \frac{-5}{3}(x-1)$$

$$\text{or } 3y - 9 = -5x + 5$$

$$\text{or } 5x + 3y - 14 = 0 \quad \text{--- (1)}$$

Here

AP is the perpendicular ~~to the~~ from the vertex A

Now, Slope of BC $= \frac{-2-3}{4-1}$

$$\therefore m_1 = \frac{-5}{3}$$

Since, AP is perpendicular to BC, let its slope be m_2.

We know,

$$m_1 . m_2 = -1 \quad \text{[Perpendicular]}$$

$$\text{or } \frac{-5}{3} . m_2 = -1$$

$$\therefore m_2 = \frac{3}{5}$$

Equation of line having one coordinate A(0, 1) and slope $\frac{3}{5}$ is given by;

$$y - y_1 = m(x - x_1)$$

$$\text{or } y - 1 = \frac{3}{5}(x - 0)$$

or ~~5x - 5y~~ $5y - 5 = 3x$

or $3x - 5y + 5 = 0$ is the req. equation of line AP

The length of AP is the length of perpendicular drawn from $A(0,1)$ to the line $5x + 3y - 14 = 0$ is given by,

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

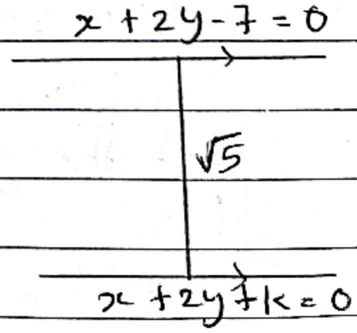
$$= \left| \frac{5 \cdot 0 + 3 \cdot 1 - 14}{\sqrt{25 + 9}} \right|$$

$$= \frac{11}{\sqrt{34}}$$

Hence, the required length and equation is $\frac{11}{\sqrt{34}}$ and $3x - 5y + 5 = 0$ respectively.

b. Find the equation of two straight lines each of which is parallel to and at a distance of $\sqrt{5}$ from the line $x + 2y - 7 = 0$.

Solution:



Equation of st. line parallel to $x + 2y - 7 = 0$ is;

$x + 2y + k = 0$ —— (i)

Also,

$x + 2y - 7 = 0$ —— (ii)

We know,

Distance between parallel lines is given by;

$$d = \pm \frac{C_2 - C_1}{\sqrt{A^2 + B^2}}$$

or, $\sqrt{5} = \pm \frac{k+7}{\sqrt{1^2+2^2}}$

or, $\sqrt{5} = \pm \frac{k+7}{\sqrt{5}}$

or, $\sqrt{5} \times \sqrt{5} = \pm\, k+7$

or, $5 = \pm(k+7)$

Now,

Taking (+ve) sign

or, $5 = k+7$

$\therefore k = -2$

Also,

Taking (-ve) sign

$5 = -k-7$

or, $k = -12$

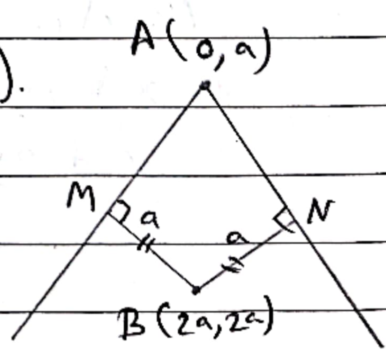
Putting the value of k in equation ① we get

$x + 2y - 2 = 0$ and $x + 2y - 12 = 0$ which are req. equations.

c. Find the equations of the two straight lines drawn through the point (0, a) on which the perpendicular drawn from the point (2a, 2a) are each of length a.

Solution:

Let the given points be $A(0,a)$ & $B(2a, 2a)$.



Now,

equation of line drawn through $A(0,a)$ is given by;

$y - y_1 = m(x - x_1)$

or, $y - a = m(x - 0)$

or, $y - a = mx$

or, $-mx + y - a = 0$ —— ①

The length of perpendicular drawn from $(2a, 2a)$ to the line $-mx + y - a = 0$ is ;

$$a = \pm \frac{-m.2a + 2a - a}{\sqrt{m^2+1}}$$

or, $a = \pm \dfrac{a - 2am}{\sqrt{m^2+1}}$

or $a\sqrt{m^2+1} = \pm (a - 2am)$

or, $a(\sqrt{m^2+1}) = \pm a(1-2m)$

or, $\sqrt{m^2+1} = \pm (1-2m)$

S.b.s

$m^2 + 1 = \pm (1 - 4m + 4m^2)$

or, $4m^2 - m^2 - 4m = 0$

or, $3m^2 - 4m = 0$

or, $m(3m-4) = 0$

Either $m = 0$ OR $m = \dfrac{4}{3}$

putting the value of 'm' in equation (i) we get ;

$-mx + y - a = 0$

or, $-0.x + y - a = 0$

$\therefore y = a$ —— (ii)

Also,

$-\frac{4}{3}x + y - a = 0$

or, $-4x + 3y - 3a = 0$

or, $4x - 3y + 3a = 0$ —— (iii)

Hence,

The req. equations are $y = a$ and $4x - 3y + 3a = 0$

d. Find the equation of the line which is at right angle to $3x + 4y - 12 = 0$, such that its perpendicular distance from the origin is equal to the length of the perpendicular from $(3,2)$ on the given line.

Solution:

The equation of line perpendicular to $3x + 4y - 12 = 0$ is

$$4x - 3y + k = 0 \quad \text{——} (1)$$

Now,

Length of perpendicular drawn from origin to $4x - 3y + k$ is;

$$d_1 = \pm\left(\frac{4.0 - 3.0 + k}{\sqrt{16+9}}\right)$$

$$\therefore d_1 = \pm\left(\frac{k}{5}\right)$$

Also,

Length of perpendicular drawn from $(3,2)$ to $3x+4y-12=0$ is given by;

$$d_2 = \pm\left(\frac{3.3 + 4.2 - 12}{\sqrt{9+16}}\right)$$

$$\text{or, } d_2 = \pm\left(\frac{5}{5}\right)$$

$$\therefore d_2 = \pm 1$$

By Question:

$$d_1 = d_2$$

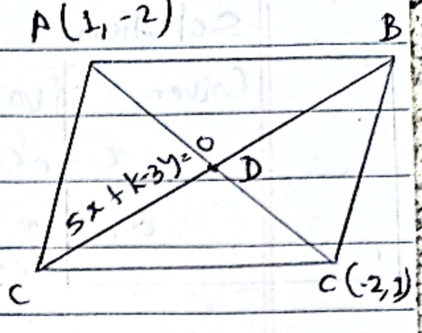
$$\text{i.e. } \frac{k}{5} = \pm 1$$

$$\therefore k = \pm 5$$

Putting the value of 'k' in eq. (i), we get;

$4x - 3y \pm 5 = 0$ which is the req. equation.

e. The equation of the diagonal of a parallelogram is $3y = 5x + k$. The two opposite vertices of a parallelogram are the points $(1, -2)$ and $(-2, -1)$. Find the value of k.



Solution:

The equation of diagonal of llgm is:

$$5x - 3y + k = 0 \quad \text{--- (1)}$$

We know,

The diagonal of llgm bisect each other which is the mid-point of $(1,-2)$ and $(-2,1)$.

Using mid-point formula:

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{1 - 2}{2}, \frac{-2 + 1}{2}\right)$$

$$= \left(\frac{-1}{2}, \frac{-1}{2}\right)$$

Since, the equation of diagonal i.e. $5x + k = 3y$ passes through the mid-point $(-1/2, -1/2)$. So it must satisfy the equation:

i.e. $5x - 3y + k = 0$

or, $5\left(-\frac{1}{2}\right) - 3\left(-\frac{1}{2}\right) + k = 0$

or, $-\frac{5}{2} + \frac{3}{2} = -k$

or, $-\frac{2}{2} = -k$

$\therefore k = 1$

$\therefore$ The value of 'k' is 1.

9. If p and p' be the length of the perpendicular from the origin upon the straight line whose equation are $x\sec\theta + y\operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, prove that $4p^2 + p'^2 = a^2$

Solution:

Given, equation of line are;

$$x\sec\theta + y\operatorname{cosec}\theta = a$$

or, $\frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a$

or, $x\sin\theta + y\cos\theta = a\sin\theta.\cos\theta$

or, $2x\sin\theta + 2y\cos\theta = a.2\sin\theta.\cos\theta$ [multiply by 2]

or, $2x\sin\theta + 2y\cos\theta = a\sin 2\theta$

or, $2x\sin\theta + 2y\cos\theta - a\sin 2\theta = 0$ —— (I)

And,

$$x\cos\theta - y\sin\theta - a\cos 2\theta = 0 \quad \text{—— (II)}$$

Now,

Length of perpendicular from (0,0) to $2x\sin\theta + 2y\cos\theta - a\sin 2\theta$ is given by;

$$P = \pm\left(\frac{2\sin\theta.0 + 2\cos\theta.0 - a\sin 2\theta}{\sqrt{4\sin^2\theta + 4\cos^2\theta}}\right)$$

or, $P = \pm\left(\frac{-a\sin 2\theta}{2}\right)$

or, $2p = \pm a\sin 2\theta$

or, $4p^2 = a^2\sin^2 2\theta$ —— (III)

Again,

Length of perpendicular from (0,0) to $x\cos\theta - y\sin\theta - a\cos 2\theta$ is given by;

$$p' = \pm\left(\frac{\cos\theta.0 - \sin\theta.0 - a\cos 2\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right)$$

or, $p' = \pm\left(\frac{-a\cos 2\theta}{1}\right)$

or, $p' = \pm a\cos 2\theta$

Sbs

or, $p'^2 = a^2\cos^2 2\theta$ —— (IV)

Now,

Adding eq (iii) and (IV), we get

$4p^2 + p'^2 = \cancel{a^2\sin^2 2\theta}\ a^2\sin^2 2\theta + a^2\cos^2 2\theta$

or, $4p^2 + p'^2 = a^2(\sin^2 2\theta + \cos^2 2\theta)$

$\therefore 4p^2 + p'^2 = a^2$

Proved

10. Show that the product of the perpendicular drawn from the two points $(\pm\sqrt{a^2-b^2}, 0)$ upon the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution:

Here, the equation of line is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

or, $bx\cos\theta + ay\sin\theta = ab$

or, $bx\cos\theta + ay\sin\theta - ab = 0$

or, $b\cos\theta x + a\sin\theta y - ab = 0$ —(i)

Now,

The length of perpendicular from $(\sqrt{a^2-b^2}, 0)$ to $b\cos\theta x + a\sin\theta y - ab = 0$ is given by;

or, $d_1 = \pm\left(\frac{b\cos\theta.\sqrt{a^2-b^2} + a\sin\theta.0 - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right)$

or, $d_1 = \pm\left(\frac{b\sqrt{a^2-b^2}.\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right)$

Again, Perpendicular drawn from $(-\sqrt{a^2-b^2}, 0)$ to $b\cos\theta x +$ $a\sin\theta y - ab = 0$ is given by;

$$d_2 = \pm \left(\frac{b\cos\theta(-\sqrt{a^2-b^2}) + a\sin\theta . 0 - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right)$$

$$\text{or, } d_2 = \pm \left(\frac{-b\sqrt{a^2-b^2}.\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right)$$

$$\text{or, } d_2 = \pm \left(\frac{b\sqrt{a^2-b^2} + ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right)$$

Now,

$$d_1 . d_2 = \pm \frac{b\sqrt{a^2-b^2}.\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \times \frac{b\sqrt{a^2-b^2} + ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \pm \frac{(b\sqrt{a^2-b^2}.\cos\theta)^2 - (ab)^2}{(\sqrt{b^2\cos^2\theta + a^2\sin^2\theta})^2}$$

$$= \pm \frac{b^2(a^2-b^2)\cos^2\theta - a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \pm \frac{b^2[a^2\cos^2\theta - b^2\cos^2\theta - a^2]}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \pm \frac{b^2[-b^2\cos^2\theta - a^2 + a^2\cos^2\theta]}{a^2\sin^2\theta + b^2\cos^2\theta}$$

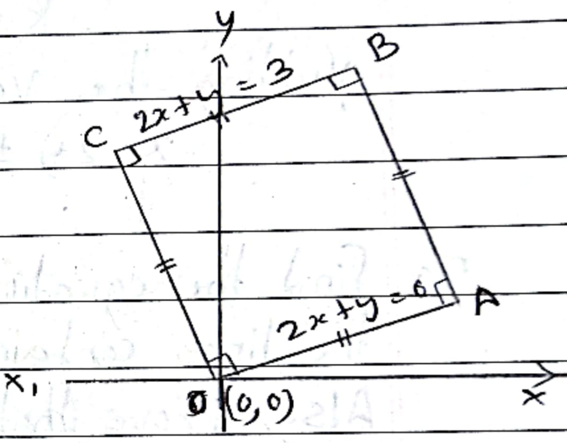
$$= \pm \frac{b^2[-b^2\cos^2\theta - a^2(1-\cos^2\theta)]}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \pm \frac{b^2 \left[-b^2\cos^2\theta - a^2\sin^2\theta\right]}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \pm \frac{b^2 \left[b^2\cos^2\theta + a^2\sin^2\theta\right]}{(b^2\cos^2\theta + a^2\sin^2\theta)}$$

$= b^2$ proved

11. The origin is a corner of a square and two of its sides are $y + 2x = 0$ and $y + 2x = 3$. Find the equation of other two sides.



Solution:

Let OABC be a square in which its two side OA and BC are $y + 2x = 0$ and $y + 2x = 3$.

Now,

The equation of OC which passes through origin and perpendicular to OA i.e. $2x + y = 0$ is given by

$$x - 2y = 0 \quad \text{—— (i)}$$

Also,

Equation of line AB which is parallel to OC i.e $x - 2y = 0$ is given by

$$x - 2y + k = 0 \quad \text{—— (ii)}$$

Now,

Length of perpendicular drawn from $(0,0)$ to $x - 2y + k = 0$ is given by;

$$d_1 = \pm\left(\frac{1.0 - 2.0 + k}{\sqrt{1+4}}\right)$$

$$\therefore d_1 = \pm \frac{k}{\sqrt{5}}$$

Again,
~~Eqn~~ Length of perpendicular drawn from (0,0) to $2x+y-3=0$ is given by;

$$d_2 = \pm\left(\frac{2\cdot 0 + 1\cdot 0 - 3}{\sqrt{2^2+1^2}}\right)$$

$$\therefore d_2 = \pm\left(\frac{-3}{\sqrt{5}}\right)$$

Since, the length of sides of square are equal i.e. $d_1 = d_2$

or, $\pm \frac{k}{\sqrt{5}} = \pm\left(\frac{-3}{\sqrt{5}}\right)$

or, $k = \pm 3$

$\therefore k = \pm 3$

Putting the value of k in equation (ii)

$x - 2y \pm 3 = 0$ is the req. equation.

5.a. Find the equations of the bisectors of the angle between the lines containing the origin in each of the following cases. Also, Prove that the bisectors of the angle are at right angle to each other.

i. $4x - 3y + 1 = 0$ and $12x - 5y + 7 = 0$

Solution:
Given equations of the line are;

$4x - 3y + 1 = 0$ —— (i)

$12x - 5y + 7 = 0$ —— (ii)

Now,
The equation of the bisector of the angle between (i) and (ii) and containing the origin is;

$$\frac{4x-3y+1}{\sqrt{16+9}} = \pm\frac{12x-5y+7}{\sqrt{144+25}}$$

or, $13(4x-3y+1) = \pm 5(12x-5y+7)$

or, $52x - 39y + 13 = \pm 60x - 25y + 35$

~~or, $8x + 14y + 22 = 0$~~

Taking (+ve) sign

$52x - 39y + 13 = 60x - 25y + 35$

or, $8x + 14y + 22 = 0$

or, $(4x + 7y + 11) = 0$ —— (iii)

Taking (-ve) sign

$52x - 39y + 13 = -60x + 25y - 35$

or, $112x - 64y + 48 = 0$

or, $28x - 16y + 12 = 0$

or, $14x - 8y + 6 = 0$

or, $7x - 4y + 3 = 0$ —— (iv)

Here,

slope of the bisector, $4x + 7y + 11$ $(m_1) = -\frac{4}{7}$

slope of another bisector, $7x - 4y + 3$ $(m_2) = \frac{7}{4}$

We know,

Inorder to be perpendicular, product of slope should be equal to -1.

i.e. $m_1 . m_2 = -1$

or, $-\frac{4}{7} \times \frac{7}{4} = -1$

$\therefore -1 = -1$ proved,

Hence, the angles of the bisector are right angle.

ii. $7x - y + 11 = 0$ and $x + y - 15 = 0$

Solution:

Given, equation of lines are;

$$7x - y + 11 = 0 \text{ and } x + y - 15 = 0$$

Now,

Making the constants positive, we have

$$7x - y + 11 = 0 \quad \text{——} \ (i)$$

$$-x - y + 15 = 0 \quad \text{——} \ (ii)$$

The equation of the bisector is given by;

$$\frac{7x - y + 11}{\sqrt{49+1}} = \pm \frac{-x - y + 15}{\sqrt{1+1}}$$

or, $\sqrt{2}\,(7x - y + 11) = \pm\, 5\sqrt{2}\,(-x - y + 15)$

Taking (+ve) sign

$$7x\sqrt{2} - y\sqrt{2} + 11\sqrt{2} = -5\sqrt{2}x - 5\sqrt{2}y + 75\sqrt{2}$$

or, $12\sqrt{2}x + 4\sqrt{2}y - 64\sqrt{2} = 0$

or, $12x + 4y - 64 = 0$

or, $3x + y - 16 = 0$ —— (iii)

Taking (-ve) sign

$$\sqrt{2}\,(7x - y + 11 = -5\sqrt{2}\,(-x - y + 15)$$

or, $7x - y + 11 = 5x + 5y - 75$

or, $2x - 6y + 86 = 0$

or, $x - 3y + 43 = 0$ —— (iv)

Here,

Slope of eq (iii) $\cancel{(m_1) = -3}$ $(m_1) = -3$

Slope of eq. (iv) $(m_2) = \frac{1}{3}$

So, $m_1 . m_2 = -3 \times \frac{1}{3} = -1$

Here, $m_1 . m_2 = -1$. So, they are perpendicular.

b. Find the equations of the bisectors of the ~~angle~~ acute angle between each of the following pairs of lines.

i. $y = x$ and $y = 7x + 4$

Solution:

The equations of the lines are;

$x - y = 0$ —— (i)

$7x - y + 4 = 0$ —— (ii)

Now,

The equation of the bisector are;

$$\frac{x-y}{\sqrt{2}} = \pm \frac{7x-y+4}{\sqrt{49+1}}$$

or, $$\frac{x-y}{\sqrt{2}} = \pm \frac{7x-y+4}{5\sqrt{2}}$$

or, $5x - 5y = \pm\, 7x - y + 4$

Taking (+ve) sign

$5x - 5y = 7x - y + 4$

or, $2x + 4y + 4 = 0$

or, $x + 2y + 2 = 0$ —— (iii)

Taking (-ve) sign

$5x - 5y = -7x + y - 4$

or, $12x - 6y + 4 = 0$

or, $6x - 3y + 2 = 0$ —— (iv)

Slope of line $x - y$, $(m_1) = 1$

Slope of bisector $x + 2y + 2 = -\frac{1}{2}$

Let θ be the angle between the line $x-y$ and the bisector $x+2y+2=0$ then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 . m_2}$$

or, $$\tan\theta = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$$

or, $\tan\theta = 3 > 1$

$\therefore \theta > 45°$

Here, $\theta > 45°$, So ~~it~~ $x + 2y + 2$ is the bisector of obtuse angle.

Hence, ~~6x~~ $6x - 3y + 2$ is the bisector of ~~act~~ acute angle.

ii. $x + 2y = 5$ and $4x + 2y + 9 = 0$

solution:

Given equation of lines are;

$x + 2y - 5 = 0$ —— (i)

$4x + 2y + 9 = 0$ —— (ii)

Now,

equation of bisector is

$$\frac{x + 2y - 5}{\sqrt{1+4}} = \pm \frac{4x + 2y + 9}{\sqrt{16+4}}$$

or, $$\frac{(x + 2y - 5)}{\sqrt{5}} = \pm \frac{(4x + 2y + 9)}{2\sqrt{5}}$$

or, $2x + 4y - 10 = \pm\ 4x + 2y + 9$

Taking (+ve) sign

or, $2x + 4y - 10 = 4x + 2y + 9$

or, $2x - 2y + 19 = 0$ —— (iii)

Taking (-ve) sign

or, $2x + 4y - 10 = -4x - 2y - 9$

or, $6x + 6y - 1 = 0$ —— (IV)

Slope of line $x + 2y - 5 = 0$, $(m_1) = -\frac{1}{2}$

Slope of bisector $2x - 2y + 19 = 0$, $(m_2) = 1$

Now,

Let θ be the angle between the line and the bisector then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 . m_2}$$

or, $$\tan\theta = \frac{-\frac{1}{2} - 1}{1 - \frac{1}{2}}$$

or, $\tan\theta = -3$

or, $|\tan\theta| = |-3| = 3 > 1$

$\therefore \theta > 45°$, which represent that is it is the bisector of obtuse angle.

Then, the bisector of obtuse angle is $6x + 6y - 1 = 0$ ##

6.a. Are the points (1, 2) and (−5, 6) on the same side or on opposite side of the line, $3x + 5y - 8 = 0$?

Solution:

Given, equation of line is;

$3x + 5y - 8 = 0$ —— ①

Now,

Put (1, 2) in eq. (i) we get;

or, $3 \times 1 + 5 \times 2 - 8$

or, $3 + 10 - 8 = 5 > 0$

Also,

Put (−5, 6) in $3x + 5y - 8$ then,

or, $-3 \times 5 + 5 \times 6 - 8$

or, $-15 + 30 - 8 = 7 > 0$

Since, the line has positive value for both the points (1, 2) and (−5, 6), so both lie on the same side of the given line.

b. On what side of the line $5x - 4y + 6 = 0$ do the points (0, 0) and (−1, 3) lie?

Solution:

Equation of given line is;

$5x - 4y + 6 = 0$ —— ①

Now, for point (0, 0)

$5 \times 0 - 4 \times 0 + 6$

$= 6 > 0$

Also,

For point (−1, 3)

or, $-5 \times 1 - 4 \times 3 + 6$

$= -11 < 0$

Since, the line has −ve value for the point (−1, 3), so the point ~~it~~ lies on the opposite side of the given line.

c. Show that two of the three points $(0,0)$, $(2,3)$ and $(3,4)$ lie on one side and the remaining on the other side of the line $x - 3y + 3 = 0$.

Solution:

Equation of the given line is;

$$x - 3y + 3 = 0 \quad \text{—— (i)}$$

Now,

For point $(0,0)$: $x - 3y + 3 = 0 - 3\cdot 0 + 3 = 3 > 0$

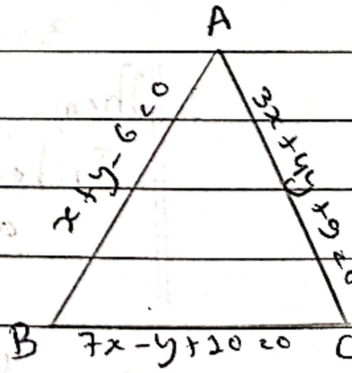
for point $(2,3)$: $x - 3y + 3 = 2 - 3\cdot 3 + 3 = -4 < 0$

for point $(3,4)$: $x - 3y + 3 = 3 - 3\cdot 4 + 3 = -6 < 0$

Since, $x - 3y + 3$ has the same sign for the points $(2,3)$ and $(3,4)$ and opposite sign for $(0,0)$. So the point $(2,3)$ and $(3,4)$ lies on the same side and $(0,0)$ lies on the other side of the given line.

12a. A triangle is formed by lines $x + y = 6$, $7x - y + 10 = 0$ and $3x + 4y + 9 = 0$. find the equation of the bisector of the angles between the first two sides.

Solution:



Given equations are;

$$x + y - 6 = 0 \quad \text{—— (i)}$$

$$7x - y + 10 = 0 \quad \text{—— (ii)}$$

$$3x + 4y + 9 = 0 \quad \text{—— (iii)}$$

Now, for side AB and BC, we have

$$x + y - 6 = 0$$

$$7x - y + 10 = 0$$

Now,

Internal bisector is given by

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{(a_2 x + b_2 y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

$$\frac{x+y-6}{\sqrt{1+1}} = -\frac{(7x-y+10)}{\sqrt{49+1}}$$

or, $$\frac{x+y-6}{\sqrt{2}} = \frac{-7x+y-10}{5\sqrt{2}}$$

or, $5x+5y-30 = -7x+y-10$

or, $12x+4y-20=0$

or, $3x+y-5=0$

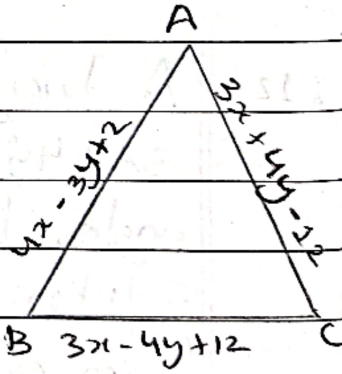
~~Again,~~

~~For side BC and AC~~

Hence, the required equation of internal bisector is

$3x+y=5$.

b. Find the equation of the internal bisectors of the angles of the triangle whose sides are $4x-3y+2=0$, $3x-4y+12=0$ and $3x+4y-12=0$. Also, find the incentre of the triangle.



Solution:

Here, For side AB ~~we have~~ and BC

$4x-3y+2=0$

$3x-4y+12=0$

Then,

Internal bisector is given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{(a_2x+b_2y+c_2)}{\sqrt{a_2^2+b_2^2}}$$

Note:- If $a_1a_2+b_1b_2 >$ 0 then (-) is us

or, $$\frac{4x-3y+2}{\sqrt{16+9}} = \frac{-3x+4y+12}{\sqrt{9+16}}$$

or, $4x-3y+2 = -3x-4y+12$

or, $7x-y=10$ —— (1)

Again for the side BC and AC

$3x - 4y + 12 = 0$

$3x + 4y - 12 = 0$

Then, Internal bisector is given by;

$$\frac{a_1x + b_1y + c_2}{\sqrt{a_1^2 + b_2^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

or, $$\frac{3x - 4y + 12}{\sqrt{9+16}} = \frac{3x + 4y - 12}{\sqrt{9+16}}$$

or, $-8y + 24 = 0$

or, $8y = 24$

or, $y - 3 = 0$ —— (ii)

Again,

for the side AB and AC, we have

$4x - 3y + 2 = 0$

$3x + 4y - 12 = 0$

Then, internal bisector ~~of~~ is given by

$$\frac{a_1x + b_1.y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

or, $$\frac{4x - 3y + 2}{\sqrt{16+9}} = \frac{3x + 4y - 12}{\sqrt{16+9}}$$

~~or,~~ $\cancel{x + 4y}$ $\cancel{x + 7y + 14 = 0}$