Chapter-20
Numerical Integration

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Elementary Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx (b-a)\,\frac{f(a)+f(b)}{2}$$

Composite Trapezoidal Rule

If a function f is continuous on the close interval $[a,b]$ then

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2n}\left[f(x_0) + 2f(x_2) + \dots + \cancel{2}f(x_n)\right]$$

where the close interval $[a,b]$ has been ~~partione~~ partitioned into n sub intervals $[x_0,x_1]$, $[x_1,x_2]$ .... $[x_{n-1},x_n]$

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Exercise-20.1

1.a From the values given in the following table, evaluate $\int_0^2 f(x)\,dx$ using trapezoidal rule.

x	0	0.50	1.00	1.50	2.0
$f(x)$	0	40	60	50	45

Solution:

Here, $x_0 = 0$ $\quad x_1 = 0.50$ $\quad x_2 = 1.00$ $\quad x_3 = 1.50$

$f(x_0) = 0$ $\quad f(x_1) = 40$ $\quad f(x_2) = 60$ $\quad f(x_3) = 50$

$x_4 = 2.0$

$f(x_4) = 45$

We know,

$$\int_0^2 f(x)\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{2-0}{2\times 4}\left[0 + 2\times 40 + 2\times 60 + 2\times 50 + 45\right]$$

$$= \frac{2}{8}\left[80 + 120 + 100 + 45\right]$$

$$= 86.25$$

b. The presented below gives the velocity v m/sec of a particle in time t sec. Find the distance covered by the particle in 6 sec. Use trapezoidal rule.

t sec	0	2	4	6
v m/s	4	6	16	34

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Solution:

Here, $x_0 = 0$ $\quad x_1 = 2$ $\quad x_2 = 4$ $\quad x_3 = 6$

$f(x_0) = 4$ $\quad f(x_1) = 6$ $\quad f(x_2) = 16$ $\quad f(x_3) = 34$

Now,

Using Trapezoidal rule:

$$\int_0^6 f(x)\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{6-0}{2\times 3}\left[4 + 2\times 6 + 2\times 16 + 34\right]$$

$$= \frac{1}{1}\left[4 + 12 + 32 + 34\right]$$

$$= 82 \text{ m}$$

2. Approximate the following integrals using composite trapezoidal rule. Give your answers correct to three places of decimals. Also, find the error.

a. $\int_1^4 5x\,dx$, $n = 3$

Solution:

Here, $x_0 = 1$ $\quad x_1 = 2$ $\quad x_2 = 3$ $\quad$ ~~$f(x_4)$~~ $x_3 = 4$

$f(x_0) = 5$ $\quad f(x_1) = 10$ $\quad f(x_2) = 15$ $\quad f(x_3) = 20$

Now,

Using Trapezoidal rule

$$\int_1^4 5x\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

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$$= \frac{4-1}{2\times 3}\left[\cancel{ft}\ 5 + 2\times 10 + 2\times 15 + 20\right]$$

$$= \frac{1}{2}\left[5 + 20 + 30 + 20\right]$$

$$= 37.5$$

Again,

Finding Actual value

$$\int_1^4 5x\,dx = \left[\frac{5x^2}{2}\right]_1^4$$

$$= \left[\frac{5\times 16}{2} - \frac{5\times 1}{2}\right]$$

$$= 37.5$$

Here,

$$\text{Error} = 37.5 - 37.5$$

$$= 0$$

b. $\int_1^5 x^2\,dx$, $n = 4$

Solution:

Here, $f(x) = x^2$

$n = 4$

So, $x_0 = 1$ $x_1 = 2$ $x_2 = 3$ $x_3 = 4$ $x_4 = 5$

$f(x_0) = 1$ $f(x_1) = 4$ $f(x_2) = 9$ $f(x_3) = 16$ $f(x_4) = 25$

Now,

Using Trapezoidal rule

$$\int_1^5 x^2\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right.$$

$$= \frac{5-1}{2\times4}\left[1+2\times4+2\times9+2\times16+25\right]$$

$$= \frac{4}{8}\left[1+8+18+32+25\right]$$

$$= 42$$

Again,

Finding Actual Value

$$\int_1^5 x^2 \cdot dx = \left[\frac{x^3}{3}\right]_1^5$$

$$= \left[\frac{5^3}{3} - \frac{1^3}{3}\right]$$

$$= 41.33$$

Here,

$$\text{Error} = 42 - 41.33$$

$$= 0.667$$

c. $\int_0^3 (3x^2-4x)\,dx,\ n=3$

Solution:

Here, $n=3$, $f(x) = 3x^2-4x$

$x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 3$

$f(x_0) = 0$ $f(x_1) = -1$ $f(x_2) = 4$ $f(x_3) = 15$

Now,

Using Trapezoidal rule

$$\int_0^3 (3x^2-4x)\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{3-0}{2\times 3}\left[0 + 2\times(-1) + 2\times 4 + 15\right]$$

$$= \frac{1}{2}\left[-2 + 8 + 25\right]$$

$$= 10.5$$

Again,

Finding Actual Value

$$\int_0^3 (3x^2 - 4x)\,dx = \left[\frac{3x^3}{3} - \frac{4x^2}{2}\right]_0^3$$

$$= \left[\frac{3\cdot 3^3}{3} - \frac{4\cdot 3^2}{2} - \frac{3\cdot 0}{3} + \frac{4\cdot 0}{3}\right]$$

$$= 9$$

Here,

$$\text{Error} = 10.5 - 9$$

$$= 1.5$$

d $\int_0^2 (2x^2+1)^2\,dx$, $n = 4$

Solution:

Here, $n = 4$, $a = 0$, $b = 2$, $f(x) = (2x^2+1)^2$

$x_0 = 0$ $x_1 = 0.5$ $x_2 = 1$ $x_3 = 1.5$ $x_4 = 2$

$f(x_0) = 1$ $f(x_1) = 2.25$ $f(x_2) = 9$ $f(x_3) = 30.25$ $f(x_4) = 81$

Now,

Using Trapezoidal rule

$$\int_0^2 (2x^2+1)^2\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right.$$

$$= \frac{1\times 2}{2\times 4}\left[1 + 2\times 2.5 + 2\times 9 + 2\times 30.25 + 81\right]$$

$$= \frac{2}{8}\left[1 + 5 + 18 + 60.5 + 81\right]$$

$$= 41.375$$

Again,

finding Actual Value

$$\int_0^2 (2x^2+1)^2\,dx = \left[\frac{2x^3}{3} + x\right]$$

e. $\int_0^{0.6} \sqrt{4-x^2}\, dx$, $n=3$

Solution:

Here, $n=3$, $a=0$, $b=0.6$

$x_0 = 0$ $\quad x_1 = 0.2$ $\quad x_2 = 0.4$ $\quad x_3 = 0.6$

$f(x_0) = 2$ $\quad f(x_1) = 1.989$ $\quad f(x_2) = 1.959$ $\quad f(x_3) = 1.907$

Now,

Using Trapezoidal Rule

$$\int_0^{0.6} \sqrt{4-x^2} = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{0.6}{2\times3}\left[2 + 2\times1.989 + 2\times1.959 + 1.907\right]$$

$$= \frac{0.6}{6}\left[2 + 3.978 + 3.918 + 1.907\right]$$

$$= 11.803 \times \frac{0.6}{6}$$

$$= 1.1803$$

3. Evaluate using Composite trapezoidal rule the following integrals.

a. $\int_1^2 \frac{dx}{x}$, $n = 4$

Solution:

Here, $n = 4$, $a = 1$, $b = 2$

$x_0 = 1$ $\quad x_1 = 1.25$ $\quad x_2 = 1.5$ $\quad x_3 = 1.75$ $\quad x_4 = 2$

$f(x_0) = 1$ $\quad f(x_1) = 0.8$ $\quad f(x_2) = 0.67$ $\quad f(x_3) = 0.57$ $\quad f(x_4) = 0.5$

Now,

Using Trapezoidal rule;

$$\int_1^2 \frac{dx}{x} = \frac{b-a}{2n}\left[\frac{f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)}{1}\right]$$

$$= \frac{2-1}{2\times4}\left[1 + 2\times0.8 + 2\times0.67 + 2\times0.57 + 0.5\right]$$

$$= \frac{1}{8}\left[1 + 2.6 + 1.34 + 1.14 + 0.5\right]$$

$$= 0.6975$$

b. $\int_1^2 \frac{2}{x^2}\,dx$, $h = \frac{1}{3}$

Solution:

Here, $a = 1$, $b = 2$, $h = 1/3$

$x_0 = 1$ $\quad x_1 = 1.33$ $\quad x_2 = 1.66$ $\quad x_3 = 2$

$f(x_0) = \cancel{6.06}\ 2$ $\quad f(x_1) = 1.5$ $\quad f(x_2) = 0.6$ $\quad f(x_3) = 1$

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Now,

Using Trapezoidal rule

$$\int_1^2 \frac{2}{x^2}\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2-1}{2\times 3}\left[6.06 + 2\times 1.5 + 2\times 0.6 + 1\right]$$

$$= \frac{1}{6}\left[6.06 + 3 + 1.2 + 1\right]$$

$$= 1.8766$$

b. $\int_1^2 \frac{2}{x^2}\,dx, \quad h = \frac{1}{3}$

Solution:

Here, ~~A~~ $a = 1$, $b = 2$, $h = 1/3$

$x_0 = 1$ $\quad x_1 = 1.3333$ $\quad x_2 = 1.6666$ $\quad x_3 = 2$

$f(x_0) = 2$ $\quad f(x_1) = 1.1250$ $\quad f(x_2) = 0.72$ $\quad f(x_3) = 0.5$

Now,

Using Trapezoidal Rule

$$\int_1^2 \frac{2}{x^2}\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2-1}{2\times 3}\left[2 + 2\times 1.1250 + 2\times 0.72 + 0.5\right]$$

$$= \frac{1}{6}\left[2 + 2.25 + 1.44 + 0.5\right]$$

$$= 1.0317$$

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c) $\int_0^1 \frac{dx}{1+x}$, $n = 5$

Solution:

Here, $a = 0$, $b = 1$, $n = 5$

$x_0 = 0$ $\quad x_1 = 0.2$ $\quad x_2 = 0.4$ $\quad x_3 = 0.6$ $\quad x_4 = 0.8$

$f(x_0) = 1$ $\quad f(x_1) = 0.8333$ $\quad f(x_2) = 0.7142$ $\quad f(x_3) = 0.625$ $\quad f(x_4) = 0.5555$

$x_5 = 1$

$f(x_5) = 0.5$

Now,

Using Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right]$$

$$= \frac{1}{2\times5}\left[1 + 2\times0.8333 + 2\times0.7142 + 2\times0.625 + 2\times0.5555 + 0.5\right]$$

$$= \frac{1}{10}\left[1 + 1.6666 + 1.4284 + 1.25 + 1.111 + 0.5\right]$$

$$= 0.69562$$

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d. $\int_0^1 \frac{dx}{1+x^2}$, $n=4$

Solution

Here, $n=4$, $a=0$, $b=1$

$x_0 = 0$ $\quad x_1 = 0.25$ $\quad x_2 = 0.5$ $\quad x_3 = 0.75$

$f(x_0) = 1$ $\quad f(x_1) = 0.941$ $\quad f(x_2) = 0.8$ $\quad f(x_3) = 0.64$

$x_4 = 1$

$f(x_4) = 0.5$

Now,

Using Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{2\times4}\left[1 + 2\times0.941 + 2\times0.8 + 2\times0.64 + 0.5\right]$$

$$= \frac{1}{8}\left[1 + 1.882 + 1.6 + 1.28 + 0.5\right]$$

$$= \frac{1}{8}\left[5.262\right]$$

$= 0.$

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4. Using Composite trapezoidal rule, evaluate the following integrals.

a. $\int_0^{\pi} (3\cos x + 1)\, dx$, $n = 3$

Solution:

Here, $a = 0$ $b = 180$, $n = 3$

$x_0 = 0$ $x_1 = 60$ $x_2 = 120$ $x_3 = 180$

$f(x_0) = 4$ $f(x_1) = 2.5$ $f(x_2) = -0.5$ $f(x_3) = -2$

Now,

Using Trapezoidal Rule

$$\int_0^{\pi} (3\cos x + 1)\, dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{180}{2\times 3}\left[4 + 2\times 2.5 + 2(-0.5) + (-2)\right]$$

$$= \frac{180}{6}\left[4 + 5 - 1 - 2\right]$$

$$= \pi$$

$$= 3.1416$$