Exercise-5.1

1.a. If $A = \begin{pmatrix} 4 & -5 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. find A^T, B^T, $(AB)^T$ and show that $(AB)^T = B^T A^T$.

Solution:

We have, $A = \begin{pmatrix} 4 & -5 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

Now,

$$A^T = \begin{pmatrix} 4 & 3 \\ -5 & 6 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

Also,

$$AB = \begin{pmatrix} 4 & -5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 8+5 & 12+10 \\ 6-6 & 9-12 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 22 \\ 0 & -3 \end{pmatrix}$$

Now,

$$(AB)^T = \begin{pmatrix} 13 & 0 \\ 22 & -3 \end{pmatrix}$$

To show:

$$(AB)^T = (B^T A^T)$$

L.H.S

$$(AB)^T = \begin{pmatrix} 13 & 0 \\ 22 & -3 \end{pmatrix}$$

RHS

$$B^T . A^T = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8+5 & 6-6 \\ 12+10 & 9-12 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 0 \\ 22 & -3 \end{pmatrix}$$

$\therefore$ Lhs = RHS

proved

b. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ find A', B', $(AB)'$ and $B'A'$.

Solution:

Now,

$$A' = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} -1 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

So,

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1+6+6 & 2+0+3 & 1-2+0 \\ 0+3+8 & 0+0+4 & 0-1+0 \\ -2+9+0 & 4+0+0 & 2-3+0 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 5 & -1 \\ 11 & 4 & -1 \\ 7 & 4 & -1 \end{pmatrix}$$

Now,

$$(AB)' = \begin{pmatrix} 11 & 11 & 7 \\ 5 & 4 & 4 \\ -1 & -1 & -1 \end{pmatrix}$$

Again,

$$B'A' = \begin{pmatrix} -1 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1+6+6 & 0+3+8 & -2+9+0 \\ 2+0+3 & 0+0+4 & 4+0+0 \\ 1+(-2)+0 & 0-1+0 & 2-3+0 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 11 & 7 \\ 5 & 4 & 4 \\ -1 & -1 & -1 \end{pmatrix}$$

2a. If $A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 5 & 7 \\ 1 & -4 \end{pmatrix}$ and $k = 3$; comput

A', B' and verify that ;

i. $(A')' = A$

ii. $(A+B)' = A' + B'$

iii. $(kA)' = kA'$

Solution:

Now, $A' = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}$ and $B' = \begin{pmatrix} 0 & 5 & 1 \\ 3 & 7 & -4 \end{pmatrix}$

i. $(A')' = A$

L.H.S

$\begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}'$

$= \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}$ RHS proved

ii. $(A+B)' = A' + B'$

L.H.S

$(A+B)' = \begin{bmatrix} 1+0 & 2+3 \\ -3+5 & 6+7 \\ 0+1 & 1-4 \end{bmatrix}'$

$= \begin{pmatrix} 1 & 2 & 1 \\ 5 & 13 & -3 \end{pmatrix}$

Also,

RHS : $A' + B' = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 1 \\ 3 & 7 & -4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 1 \\ 5 & 13 & -3 \end{pmatrix}$

$\therefore$ L.H.S = R.HS proved

iii. $(kA)' = kA'$

L.H.S

$$(kA)' = \left[3\begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}\right]'$$

$$= \begin{pmatrix} 3 & 6 \\ -9 & 18 \\ 0 & 3 \end{pmatrix}'$$

$$= \begin{pmatrix} 3 & -9 & 0 \\ 6 & 18 & 3 \end{pmatrix}$$

Now,

$$\text{RHS} : kA' = 3\begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -9 & 0 \\ 6 & 18 & 3 \end{pmatrix}$$

$\therefore$ L.H.S = R.H.S proved.

b. If $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$, prove that $AA^T = A^T \cdot A = 25I$, where I is a unit matrix of order 2.

Solution:

$$A' = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Now,

$$AA^T = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 9+16 & 12+(-12) \\ 12-12 & 16+9 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$= 25 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Also,

$$A^T \cdot A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+16 & 12-12 \\ 12-12 & 16+9 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$= 25 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 25I$$

$\therefore AA^T = A^TA = 25I$ proved

c. If $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$, verify that $AA' = A'A = I$

where I is a matrix of order 3.

Solution:

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} \quad \text{then, } A' = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

Now,

$$AA' = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Again,

$$A'A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Proved

3. If $A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix}$ find A^T.

a. Show that the sum of the given matrix and its transpose is a symmetric metrix.

b. Show that the difference of the given matrix and its transpose is a skew-symmetric matrix.

c. Express the given matrix A as the sum of the symmetric and skew matrix form.

Solution:

Given,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{bmatrix} \text{ then, } A^T = \begin{bmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{bmatrix}$$

Now,

a. $$A + A^T = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 8 \\ 6 & 6 & 6 \\ 8 & 6 & 12 \end{bmatrix}$$

$\therefore$ It is a symmetric matrix.

b. $$A - A^T = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$\therefore$ It is a skew-symmetric matrix.

c. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\begin{bmatrix} 4 & 6 & 8 \\ 6 & 6 & 6 \\ 8 & 6 & 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Which is a sum of the symmetric and skew- ~~matrix~~ symmetric matrix.

4.a. If $A = \begin{bmatrix} 4 & x+2 \\ 2x-1 & 0 \end{bmatrix}$ and $A = A^T$, find the value of x.

Solution:

$$A^T = \begin{bmatrix} 4 & 2x-1 \\ x+2 & 0 \end{bmatrix}$$

By Question:

$A = A^T$

i.e. $\begin{bmatrix} 4 & x+2 \\ 2x-1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2x-1 \\ x+2 & 0 \end{bmatrix}$

Comparing the corresponding equation:

$x + 2 = 2x - 1$

or, $2x - x = 2 + 1$

$\therefore x = 3$

b. If $A = \begin{bmatrix} 0 & 2y-3 \\ 1-y & 0 \end{bmatrix}$ and $A = -A^T$, find the value of y.

Solution:

$$A^T = \begin{bmatrix} 0 & 1-y \\ 2y-3 & 0 \end{bmatrix}$$

By Question:

$A = -A^T$

i.e. $\begin{bmatrix} 0 & 2y-3 \\ 1-y & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1+y \\ 3-2y & 0 \end{bmatrix}$

Comparing the corresponding elements:

$2y - 3 = y - 1$

or, $2y - 1 = 3 - 1$

$\therefore y = 2$

c. If $A = \begin{bmatrix} 0 & -1 & 3y+5 \\ 5z-9 & 0 & y-1 \\ x-2 & 2x-3 & 0 \end{bmatrix}$ and $A = -A^T$, find the value of x, y, z.

Solution:

$A^T = \begin{bmatrix} 0 & 5z-9 & x-2 \\ -1 & 0 & 2x-3 \\ 3y+5 & y-1 & 0 \end{bmatrix}$

By question:

$A = -A^T = \begin{bmatrix} 0 & -1 & 3y+5 \\ 5z-9 & 0 & y-1 \\ x-2 & 2x-3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 9-5z & 2-x \\ 1 & 0 & 3-2x \\ 5-3y & 1-y & 0 \end{bmatrix}$

Comparing the corresponding elements:

$-1 = 9 - 5z$

or $5z = 10$

$\therefore z = 2$

$3y + 5 = 2 - x$

or, $3y + x = -3$

$\therefore x = -3 - 3y$ — (i)

Also,

$y - 1 = 3 - 2x$

or $y + 2x = 4$

or, $y = 4 - 2x$

or, $y = 4 - 2(-3 - 3y)$

$\therefore y = -2$ — (ii)

from (i) and (ii)

$x = -3 - 3(-2)$

$\therefore x = 3$