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# Chapter-1
## Rotational Dynamics

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i. **Centre of Mass (C.M)** : It is a point inside a body or substance such that whole mass of the body is supposed to concentrate on it.

ii. **Centre of gravity (C.G)** : It is a point where gravitational force acts on a body and total gravitational torque about the point equal to zero.

iii. **Moment of Inertia (I)** : Sum of product of mass and square of the distance from masses and axis of rotation. It's unit is $kgm^2$.

# Difference / Comparision

| | Translational motion | | Rotational Motion |
|---|---|---|---|
| i. | Distance travelled = $s$ | i. | Angular displacement = $\theta = \frac{s}{r}$ |
| ii. | Velocity, $u = s/t$ , $v = s/t$ | ii. | Angular Velocity = $\omega_0$, $\omega = \frac{d\theta}{dt}$ |
| iii. | Acceleration, $a = \frac{dv}{dt}$ | iii. | Acceleration, $\alpha = \frac{\omega}{t}$ |
| iv. | force = $f = ma$ | iv. | Torque = $\tau = I\alpha = \frac{dL}{dt}$ |
| v. | Kinetic Energy = $\frac{1}{2}mv^2$ | v. | Rotational K.E = $\frac{1}{2}I\omega^2$ |
| vi. | Momentum = $P = m.v$ | vi. | Angular momentum = $L = I\omega$ |
| vii. | $S = ut + \frac{1}{2}at^2$ | vii. | $\theta = \omega_0 + \frac{1}{2}\alpha\theta$ |
| viii. | $V^2 = u^2 + 2as$ | viii. | $\omega^2 = \omega_0^2 + 2\alpha\theta$ |

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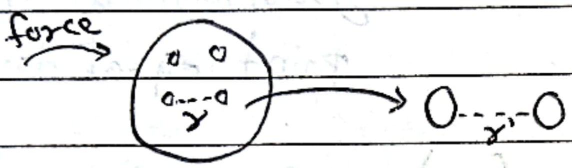
| | | | |
|---|---|---|---|
| ix. | $V = u + at$ | ix. | $\omega = \omega_0 + \alpha \cdot t$ |

# Rigid Rotation

- Rigid Body : The body or substance in which inter-molecular distance is not affected by any external force. Eg: quartz crystal




Here,

$r$ = distance between two molecules before applying external force.

$r'$ = distance between two molecules after applying external force.

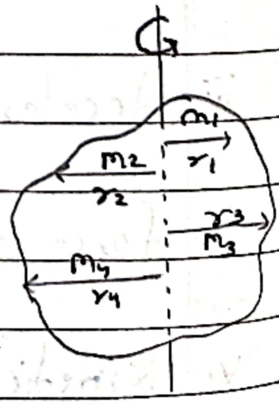
For a rigid body, $r = r'$

# Radius of gyration

$$k = \sqrt{\frac{r_1^2 + r_2^2 + \ldots\ldots + r_n^2}{n}}$$

= rms distance

= square root of mean of square of distance between axis of rotation and mass of particles of a body.




Here,

$k$ = radius of gyration

# Moment of Inertia

or, $I = mr^2$

$I = mk^2$

where, $k = \sqrt{\dfrac{r_1^2 + r_2^2 + \dots + r_n^2}{n}}$

# Theorem of Moment of Inertia

1. Theorem of Parallel Axis

It states that moment of Inertia about any given axis is equal to moment of inertia about the axis passing through centre of mass plus product of mass of the body and square of distance between two parallel axis.

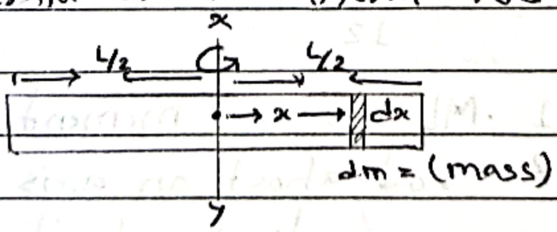
i.e. $I = I_{cm} + Mr^2$

2. Theorem of Perpendicular Axis

It states that the sum of moment of inertia of a laminar body about any two mutually perpendicular axis in its plane is equal to its moment of inertia about an axis perpendicular to its plane and passing through the point of intersection of two axis.

i.e. $I = I_x + I_y$

# Moment of Inertia of a Uniform rod




Let,

L = Length of Uniform rod

m = Mass of the rod

xy = axis of rotation about C.M

dm = It is the mass of 'dx' length of rod.

$\because$ Mass of 'L' Length of uniform rod = M

$\therefore$ Mass of '1' length of uniform rod = $\frac{M}{L}$

$\therefore$ Mass of 'dx' Length of uniform rod = $\frac{M}{L}.dx$

i.e. $dm = \frac{M}{L}.dx$

a. Now, Moment of inertia of mass 'dm' about an axis passing through C.M and perpendicular to its length is $dmx^2$ ($I = mr^2$)

Total moment of Inertia of the rod;

$$I = \int_{-L/2}^{L/2} dm x^2$$

$$= \int_{-L/2}^{L/2} \frac{m}{L}.dx.x^2$$

$$= \frac{m}{L}\int_{-L/2}^{L/2} x^2.dx$$

$$= \frac{m}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} + c$$

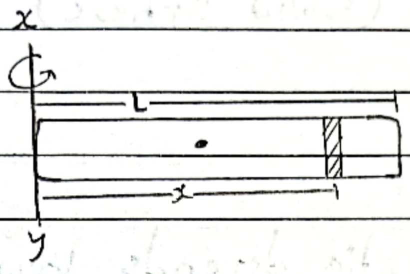
$$= \frac{m}{L}\left[\frac{(L/2)^3}{3} - \left(\frac{(-L/2)^3}{3}\right)\right] + c$$

$$= \frac{M}{L}\left[\frac{2L^3}{24}\right]$$

$$= \frac{ML^2}{12}$$

$\therefore$ $\frac{1}{12}.ML^2$ is the moment of inertia of a uniform rod about an axis passing through C.M and perpendicular to its length

b. Moment of inertia of a uniform rod about an axis passing through one end of the rod and perpendicular to its Length.




Here,

$$I = I_{cm} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$= \frac{ML^2 + 3ML^2}{12}$$

$$= \frac{1}{3} ML^2$$

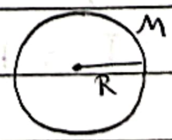
c. $I = \frac{1}{2} Mr^2$ (Passing horizontally)

# Moment of inertia of a ring

$I = MR^2$

# Moment of Inertia of a disc.

$I = \frac{1}{2} MR^2$




# Moment of inertia of a cylinder

$I = MR^2$ (Hollow cylinder)

$I = \frac{1}{2} MR^2$ (solid cylinder)

# Moment of Inertia of a sphere

$I = \frac{2}{3} MR^2$ (Hollow sphere)

$I = \frac{2}{5} MR^2$ (solid sphere)

Note :

i. Moment of inertia depends upon axis of rotation
ii. Moment of inertia increases when axis of rotation goes away from the centre of mass and vice versa.

# Conservation of Angular Momentum

$I\omega$ = Constant

$\omega$ → Angular Velocity

$I$ → Moment of Inertia

# Radius of Gyration

The perpendicular distance between centre of mass and axis of rotation of a rigid body is called radius of gyration.

If 'm' be the mass of body and k is it's radius of gyration, then moment of inertia of a body is given by;

$$I = mk^2$$

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# Some Give Reasons

1. Raw and boiled egg are made to spin on a smooth table by applying the same Torque. Which one spin faster?

Ans: The boiled egg acts like rigid body while raw egg contains liquid in it. The mass in raw egg is distributed farther from the axis as compared to the boiled egg. So, moment of inertia for raw egg is more and spin with less angular acceleration. Therefore, boiled egg will spin faster than raw egg.

2. If the polar ice caps melt and spread uniformly, how will the length of the day be affected?

Ans: If polar ice caps will melt, water should spread on earth and so its moment of inertia would increase. But by the law of conservation of momentum, speed of rotation would decrease so that angular momentum remains constant. As the speed of rotation decreases, length of day would increase.

3. Why it is easier to revolve a stone using shorter string than a longer string?

Ans: We know, Torque ($\tau$) = Force x Perpendicular distance. Now, the longer string has more perpendicular distance as compared to the shorter one. Because of this, the longer string needs more torque to rotate the stone. So, we can conclude that tying a stone to a shorter string makes it easier to spin.

# Torque and Angular momentum of a rigid body

- Torque is the rotational effect of the force and it is represented by a letter $\tau$ (Tau). It is a vector quantity.

i.e. $\vec{\tau} = \vec{r} \times \vec{F}$

or, $\tau = rF\sin\theta\,\hat{n}$ ($\hat{n}$ = unit vector along $\tau$)

We know, $\vec{\tau} = \vec{r} \times \vec{F}$

Taking magnitude only $\vec{r}$ and $\vec{F}$ are perpendicular

So, $\tau = rF$

or, $\tau = rma$

or, $\tau = rmr\alpha$ ($a = r\alpha$, $a$ = linear Acc., $\alpha$ = angular acc.)

or, $\tau = mr^2\alpha$

SI unit of $\tau$ is Nm (Newton metre) or Joule

# Angular Momentum

It is the product of perpendicular distance and linear momentum.

$\vec{L} = \vec{r} \times \vec{P}$

$L$ — Angular momentum; $P$ — Linear Momentum

Now,

Taking magnitude only

$L = rp = rmv = rmr\omega$

$\therefore L = mr^2\omega$

$v$ — Linear Velocity; $\omega$ — Angular velocity

The SI unit of Linear momentum is kg m/s.

#1. A Wheel of moment of Inertia $2\times10^3\ kgm^2$ rotating at Uniform angular speed of 4 rad/sec. What is the torque required to stop it in 1 sec.

Solution:

Given, Moment of Inertia (I) = $2\times10^3\ kgm^2$

Initial Angular Velocity ($\omega_1$) = 4 rad/sec

Final Angular Velocity ($\omega_2$) = 0 rad/sec

Time (t) = 1 sec.

Torque ($\tau$) = ?

We know,

$$\omega_2 = \omega_1 + \alpha t$$

or, $0 = 4 + \alpha \cdot 1$

$\therefore\ \alpha = -4\ rad/sec^2$ (-ve is just for direction)

Again,

$$\tau = I\alpha$$

$$= 2\times10^3 \times 4$$

$$= 8\times10^3\ Nm$$

#2. A constant Torque of 1000 Nm turns a wheel of moment of inertia 200 kgm² about an axis through its Centre. What is it's angular Velocity after 3 sec?

Solution:

Given, Torque ($\tau$) = 1000 Nm

Moment of inertia of wheel (I) = 200 kgm²

Time (t) = 3 sec

Angular Velocity ($\omega$) = ?

We know,

$$\tau = I\alpha$$

or, $1000 = 200 \times \alpha$

$\therefore\ \alpha = 5\ rad/sec^2$

Again,

$$\alpha = \omega / t$$

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or, $\frac{5}{\cancel{3}}$ or, $5 = \frac{\omega}{3}$

$\therefore \omega = 15$ rad/sec

#3. A ballet dancer spins with 2.4 rev/sec with her arm outstreched, when the moment of inertia about the axis of rotation is I with her arms folded, the moment of inertia about the same axis is 0.6I. calculate the new rate of revolution.

Solution:

When hands are outstreched

Moment of Inertia $(I_1) = I$

Frequency $(f_1) = 2.4$ rev/sec

Armed folded

Moment of inertia $(I_2) = 0.6I$

frequency $(f_2) = ?$

From principle of conservation of angular momentum

$$I_1 \omega_1 = I_2 \omega_2$$

or, $I \times 2\pi f_1 = 0.6 I \times 2\pi F_2$

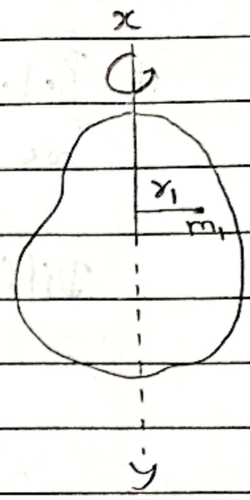
or, $F_1 = 0.6 F_2$

or, $\frac{2.4}{0.6} = F_2$

$\therefore F_2 = 4$ rev/sec

# Torque acting on rigid body

Let us suppose a rigid body of mass '$m_1$' is rotating with acceleration '$a_1$'.




Then,

$F_1 = m_1 a_1$

$= m_1 r_1^2 \alpha$ [∵ Angular acc. of the body is constant]

Again,

$I_1 = r_1 F_1 = m_1 r_1^2 \alpha$

similarily,

$I_2 = m_2 r_2^2 \alpha$

$\vdots$

$I_n = m_n r_n^2 \alpha$

$\therefore \Gamma = I_1 + I_2 + \dots\dots + I_n$

or, $\Gamma = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + \dots\dots + m_n r_n^2 \alpha$

or, $\Gamma = (m_1 r_1^2 + m_2 r_2^2 + \dots\dots + m_n r_n^2)\alpha$

or, $\Gamma = \left(\sum_{i=1}^{n} M_i r_i^2\right)\alpha$

$\therefore \Gamma = I\alpha$

# For Angular Momentum

$L = rp$

Suppose linear momentum of the particle $m_1$ is $m_1 v_1$ where '$v_1$' is the linear velocity of the particle '$m_1$'

$\therefore P = m_1 v_1 = m_1 r_1 \omega$

Now,

$L_1 = r_1 P_1 = r_1 m_1 r_1 \omega$

or, $L_1 = m_1 r_1^2 \omega$

$\vdots$

$L_n = m_n r_n^2 \omega$

$\therefore L = L_1 + L_2 + \dots\dots + L_n$

$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \dots\dots + m_n r_n^2 \omega$

$$\text{or, } L = \left( \sum_{i=1}^{n} m_i r_i^2 \right) \omega$$

$$\therefore L = I\omega$$

# Relation between Angular Momentum and Torque

We have,

$$L = I\omega \quad \text{——— (i)}$$

Differentiating equation (i) w.r.t time, we get

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt}$$

$$\text{or, } \frac{dL}{dt} = I \frac{d\omega}{dt}$$

$$\text{or, } \frac{dL}{dt} = I\alpha \quad \left( \because \alpha = \frac{d\omega}{dt}, \; a = \frac{dv}{dt} \right)$$

$$\therefore \frac{dL}{dt} = I\alpha \quad \text{——— (ii)}$$

$$\text{But, } \tau = I\alpha \quad \text{——— (iii)}$$

From equation (ii) and (iii), we get,

$$\tau = \frac{dL}{dt} \quad \text{——— (iv)}$$

Comparing with Linear motion, $F = \frac{dp}{dt}$

~~That is~~

i.e. time rate of change of angular momentum is the torque.

# Principle of Conservation of Angular Momentum

It state that if no external torque acts on a system, the total angular momentum of the system remains conserved.

i.e. $I\omega$ = constant

Proof:

We have, $\frac{dL}{dt} = \Gamma$ ——— (1)

if $\Gamma = 0$, $\frac{dL}{dt} = 0$ from eq (i)

Here, $L =$ Constant

$L = I\omega =$ Constant

Proved

# Give reasons

1. Why does a figure skater spin faster when they pull their arms closer to their body during a spin?

Ans: When a figure skater pulls their arms closer to their body during a spin, their moment of inertia decreases due to the redistribution of mass. According to the law of conservation of angular momentum, $I\omega = L =$ constant. As the moment of Inertia decreases, the angular velocity must increase to maintain the same angular momentum. So, the skater spin faster.

2. A solid disk and a ring of the same mass and radius, when released from rest at the top of an incline will have different rolling accelerations. Why?

Ans: The solid disk and ring have different distribution of mass w.r.t their ~~axes~~ axis of rotation. Due to this mass distribution difference, they have different moment of inertia (Inertia of disk < inertia of ring), resulting in varying rolling acceleration down the incline.

3. If earth shrinks, how will the duration of a day affected.

Ans: If the earth shrinks, the duration of a day will get decreased as the moment of inertia will decrease on shrinking. By the conservation of angular momentum, angular velocity must increase and it result in decreasing the length of the day.

4. A handle or knob is fixed at the free end of the door. Why?

Ans: We know, Moment of force = force x Perpendicular distance
It means, the larger the distance from the hinge, the lesser the force will be required to open the door.
So, a handle or knob is fixed at the free end of the door to minimize the req. force.

5. Why spokes are fitted in the cycle wheel?

Ans: The spokes of the cycle wheel helps to distribute the load between the hub and the rim. It increases moment of inertia, due to which angular acceleration decreases which provide smooth ride experience.

6. A fan with blade takes longer time to come to rest than without blades. Why?

Ans: It is because the mass of the fan with blades is greater than the one without blades. It means the moment of inertia of fan with blades will be more than the one without blades. So, it takes longer time to come to rest.

# Numerical Problems

1. An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4 seconds. Find the angular acceleration and the number of revolutions made by the motor in the 4 sec. interval.

Solution:

Initial frequency $(f_1) = \frac{500}{60}$ rev/sec

Final frequency $(f_2) = \frac{200}{60}$ rev/sec.

Now,

Initial Angular velocity $(\omega_1) = 2\pi f_1$

$= \frac{2 \times 500\pi}{60}$ rad/sec

Final Angular velocity $(\omega_2) = 2\pi f_2$

$= \frac{2 \times 200\pi}{60}$ rad/sec

We know,

$$\omega_2 = \omega_1 + \alpha t$$

or, $\frac{20\pi}{3} = \frac{50\pi}{3} + \alpha \cdot 4$

or, $\alpha = -\frac{30\pi}{3}$

$\therefore \alpha = -7.86$ rad/sec$^2$

Again,

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta$$

or, $\left(\frac{20\pi}{3}\right)^2 - \left(\frac{50\pi}{3}\right)^2 = 2(-7.86)\theta$

or, $\frac{400\pi^2 - 2500\pi^2}{9} = -15.72\theta$

$\therefore \theta = 146.62$ rad

No. of revolution in 4 sec. interval,

$$n = \frac{\theta}{2\pi}$$

$$= \frac{146.62}{2\pi}$$

$$= 23.32 \text{ rev/sec}$$

2. A disc of radius 1m and mass 5kg is rolling along a horizontal plane. It's moment of inertia about the centre is $2.5 \text{ kg m}^2$. If it's velocity along the plane is 2m/s. Find it's angular velocity and total K.E.

Solution:

Given, radius $(r) = 1m$

Mass $(m) = 5kg$

Moment of Inertia $(I) = 2.5 \text{ kg m}^2$

Velocity $(v) = 2 \text{ m/s}$

Angular Velocity $(\omega) = ?$

Total K.E $(K.E_T) = ?$

We know,

$$v = r.\omega$$

or, $2 = 1.\omega$

$\therefore \omega = 2 \text{ rad/sec}$

Now,

Total Energy = P.E + K.E

$= 0 + K.E$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}.5.2^2 + \frac{1}{2}.(2.5).2^2$$

$= 15$ Joule

3. A disc of moment of inertia $5 \times 10^{-4}$ kgm$^2$ is rotating freely about the axis through its centre at 40 rpm. Calculate the new revolution per minute of some wax of mass 0.02 kg is dropped gently on to the disc 0.08 m from the axis.

Solution:

Moment of inertia $(I_1) = 5 \times 10^{-4}$ kgm$^2$

Init. frequency $(f_1)$ = 40 rev/min

$= \frac{2}{3}$ rev/sec

Mass of wax $(m_1)$ = 0.02 kg

radius $(r)$ = 0.08 m

Now,

$$\omega_1 = 2\pi f_1$$
$$= 2\pi \cdot \frac{2}{3}$$
$$= 4.18 \text{ rad/sec}$$

Final frequency $(f_2)$ = ?

We know,

$$I_2 = I_1 + mr^2$$
$$= 5 \times 10^{-4} + 0.02(0.08)^2$$
$$= 6.28 \times 10^{-4} \text{ kgm}^2$$

Here,

from Law of Conservation of angular momentum

$$I_1 \omega_1 = I_2 \omega_2$$

or, $5 \times 10^{-4} \times 4.18 = 6.28 \times 10^{-4} \times \omega_2$

$\therefore \omega_2 = 3.32$ rad/sec

Also,

Calculating RPM

$$\omega_2 = 2\pi f$$

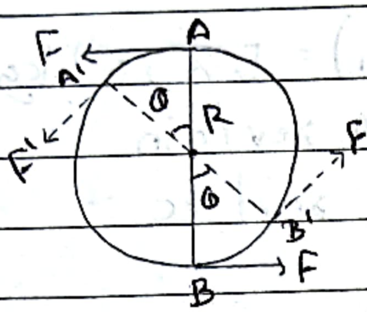
or, $\frac{3.32}{2\pi} \times$ [illegible] $= f$

or, $F = 0.52$ rev/sec

or, $F = 0.52 \times 60$ rev/min

$\therefore F = 31.7$ rpm

# Work done by a couple of force



Let, $F$ = force

$R$ = Radius of wheel

$\theta$ = angle of rotation

Now,

$AA' = R\theta$

$BB' = R\theta$

Work done for $AA' = F \cdot AA' = FR\theta$

Work done for $BB' = F \cdot BB' = FR\theta$

Total work $= FR\theta + FR\theta$

$= F\theta(R+R)$

$= [2R \times F]\theta$ [2R = Total distance]

$W = \Gamma\theta$ ($\because$ Compare, $W = f \times d$)

This is the work done by couple of force.

Now,

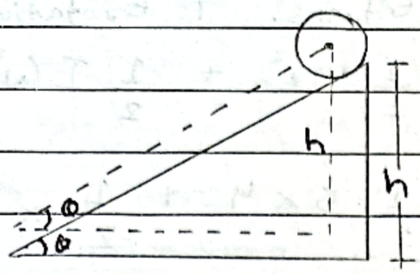
$$\text{Power} = \frac{W}{t} = \frac{\Gamma\theta}{t}$$

$$= \Gamma\omega$$

$$\therefore P = \Gamma\omega$$

($\because$ Compare $P = \frac{w}{t} = f\frac{d}{t} = fv$)

# Kinetic Energy of a rolling body and its acceleration




$(K.E)_{\text{rolling body}} = E_{rot.} + E_{trans.}$

$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$

Here, $V = R\omega$, ($R$ = radius of rolling body)

Now, $= \frac{1}{2} I\omega^2 + \frac{1}{2} mR^2\omega^2$

$\therefore (K.E)_{\text{rolling body}} = \frac{1}{2}\left(I + mR^2\right)\omega^2$

# A disc rolling along a horizontal plane has a moment of Inertia $4\ kgm^2$ about its centre and mass of $5\ kg$. The velocity along the plane is $2\ m/s$. If the radius of the disc is $2m$ find. (i) Angular velocity
(ii) The total energy of the disc

Solution:

Moment of Inertia of disc ($I$) = $4\ kgm^2$

Mass of disc ($m$) = $5\ kg$

Velocity ($v$) = $2\ m/s$

radius of the disc ($r$) = $2m$

Angular velocity ($\omega$) = ?

Total energy ($E_{Total}$) = ?

We know,

$V = r\omega$

$2 = 2\omega$

$\therefore \omega = 1\ rad/sec$

Now,

$$\text{Total Energy} = E_{trans.} + E_{rotational}$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \cdot 5 \times 4 + \frac{1}{2} \times 4 \times 1$$

$$= 12 \text{ Joule}$$

# A constant torque of 500 Nm turns a wheel which has a moment of Inertia 20 kgm² about its centre find the angular velocity gained in 2 sec and kinetic energy gained.

Solution:

Given, Torque ($\tau$) = 500 Nm

Moment of inertia of wheel ($I$) = 20 kgm²

Time ($t$) = 2 sec

Angular velocity ($\omega$) = ?

Kinetic energy (K.E) = ?

We know,

$$\tau = I\alpha$$

$$\text{or, } 500 = 20\alpha$$

$$\therefore \alpha = 25 \text{ rad/sec}^2$$

Now,

$$\alpha = \frac{\omega}{t}$$

$$25 = \frac{\omega}{2}$$

$$\therefore \omega = 50 \text{ rad/sec}$$

Again,

$$K.E = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \times 20 \times (50)^2 = 25000 \text{ Joule}$$