Chapter 3

Stellar Interiors

Our observations of stars are only skin-deep. The mass of the Sun's photosphere, chromosphere, and corona (the portions of the Sun we can see directly) only adds up to 10^{-10} of the Sun's total mass. We are not entirely ignorant of the 99.99999999% of the Sun that is opaque, however. Because the structure of the Sun, and other stars, is dictated by well-understood laws of physics, we can make models of stellar interiors, using the observed surface properties of stars as our boundary conditions.

3.1 Equations of Stellar Structure

The internal structure of a spherical star in equilibrium is dictated by a few basic **equations of stellar structure**. The first equation of stellar structure is the familiar **equation of hydrostatic equilibrium**:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \ . \tag{3.1}$$

Make note of the assumptions that have gone into this equation: the star is spherical and non-rotating, the star is neither expanding nor contracting, and gravity and pressure gradients provide the only forces. Equation (3.1) is a single equation with three unknowns – $P(r)$, $M(r)$, and $\rho(r)$ – so even with known boundary conditions, we can't solve it to find a unique solution for the pressure and density inside the star. However, we can still extract interesting information from the equation of hydrostatic equilibrium. For instance, we can make a very crude estimate of the central pressure of the Sun.

A rough approximation to the equation of hydrostatic equilibrium is

$$\frac{\Delta P}{\Delta r} \sim -\frac{G\langle M\rangle\langle\rho\rangle}{\langle r\rangle^2} \ , \tag{3.2}$$

where ΔP is the difference in pressure between the Sun's photosphere and its center, Δr is the difference in radius between the Sun's photosphere and its center, and $\langle M\rangle$, $\langle\rho\rangle$, and $\langle r\rangle$ are typical values of mass, density, and radius in the Sun's interior. As a rough guess, we can set $\langle\rho\rangle \sim \rho_\odot = 1400\,\mathrm{kg\,m^{-3}}$, the average density of the Sun. We can also guess that $\langle M\rangle \sim M_\odot/2 = 1.0 \times 10^{30}\,\mathrm{kg}$ and $\langle r\rangle \sim r_\odot/2 \sim 3.5 \times 10^8\,\mathrm{m}$. The pressure at the photosphere will be much less than the central pressure, so we can rewrite equation (3.2) as

$$\frac{0 - P_c}{r_\odot - 0} \sim -\frac{G(M_\odot/2)\rho_\odot}{(r_\odot/2)^2} \sim -\frac{2GM_\odot\rho_\odot}{r_\odot^2} \ . \tag{3.3}$$

With the numerical values of $M_\odot$, $\rho_\odot$, and $r_\odot$ inserted, we find that

$$P_c \sim \frac{2GM_\odot\rho_\odot}{r_\odot} \sim 5 \times 10^{14}\,\mathrm{N\,m^{-2}} \sim 5 \times 10^9\,\mathrm{atm} \ . \tag{3.4}$$

When we compare this to the pressure $P_{\rm phot} \sim 10^{-3}\,\mathrm{atm}$ in the Sun's photosphere (as computed in section 2.3), we see that the center of a star is a high-pressure place.

The second equation of stellar structure is the **equation of mass continuity**:

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \ . \tag{3.5}$$

This simply tells us that the total mass of a spherical star is the sum of the masses of the infinitesimally thin spherical shells of which it is made. It tells us the relation between $M(r)$, the mass enclosed within a radius r, and $\rho(r)$, the local mass density at r. Combining equations (3.1) and (3.5) gives us two equations in the three unknowns, $P(r)$, $M(r)$, and $\rho(r)$. We need further information before we can compute unique solutions for P, M, and ρ.

The third equation of stellar structure is the **equation of state**, which tells us the relation between pressure and density within a star. For most stars, the appropriate equation of state is the perfect gas law:

$$P(r) = \frac{\rho(r) k T(r)}{\mu m_p} \ . \tag{3.6}$$

Strictly speaking, we should also include the radiation pressure exerted by the photons,

$$P_{\rm rad}(r) = \frac{a}{3} T(r)^4 \ , \tag{3.7}$$

where $a = 4\sigma/c = 7.56 \times 10^{-16}\,{\rm J\,m^{-3}\,K^{-4}}$ is the radiation constant. However, for all but the hottest stars, the radiation pressure is negligibly tiny compared to the gas pressure. Generally, the mean molecular weight μ in equation (3.6 is a function of r, since the chemical composition and ionization state change with radius inside a star. Most of the Sun is almost completely ionized, and the chemical composition is nearly constant outside the central regions where hydrogen is fused to helium; thus, for most of the Sun's radius, the mean molecular weight is $\mu \approx 0.60$ (see section 2.1). If we approximate the mean molecular weight as being constant in a star, we have three equations in four unknowns – T, P, M, and ρ. Although we can't find a complete solution for the solar interior, we can make a crude estimate of the central temperature of the Sun, using the perfect gas law as the equation of state:

$$T_c \sim P_c \frac{\mu_\odot m_p}{\rho_\odot k} \sim \frac{2GM_\odot \mu_\odot m_p}{r_\odot k} \ . \tag{3.8}$$

With $\mu_\odot = 0.60$, this yields

$$T_c \sim 3 \times 10^7\,{\rm K} \ . \tag{3.9}$$

Careful computer models of the Sun's interior yield a central temperature $T_c = 1.47 \times 10^7$, so our crude guesstimate is off by a factor of two. Note that we are able to guess the central temperature of the Sun without knowing *anything* about how energy is generated in the Sun. The central temperature of a star of mass M and radius r is dictated by the fact that it is a sphere made of perfect gas in hydrostatic equilibrium. We have used the central temperature of the Sun as our example, but note that

$$T_c \propto \frac{M\mu}{r} \tag{3.10}$$

for any sphere of perfect gas in hydrostatic equilibrium. In section 1.5, we found that main sequence stars with $M > 1.3 M_\odot$ have $M \propto r$. Since all main sequence stars have similar mean molecular weights ($\mu \sim 1$), this implies that massive main sequence stars have

$$T_c \propto \frac{M}{r} \approx \text{constant} \ . \tag{3.11}$$

Thus, even the most massive main sequence stars shouldn't have central temperatures very much higher than that of the Sun.

One of the defining characteristics of stars (and one that's been ignored so far in this chapter) is that they glow in the dark. A basic question about stars – one so simple that a child might ask it – is "Why do stars shine?" The basic answer to that question is "Stars shine because they are hot." If you place a hot, bright object in the middle of cool, dark space, then energy (in the form of photons) will flow away from the hot object. Not only do stars shine because they are hot, but within the star, energy flows from the very hot center ($T_c \approx 14{,}700{,}000\,\mathrm{K}$) to the not-so-hot photosphere ($T_{\rm phot} \approx 5800\,\mathrm{K}$). The rate at which thermal energy flows outward is dictated by the fourth equation of stellar structure, the **equation of energy transport**.

In general, thermal energy can be transported by one of three methods: conduction, convection, and radiation (Figure 3.1). Conduction is only effec-

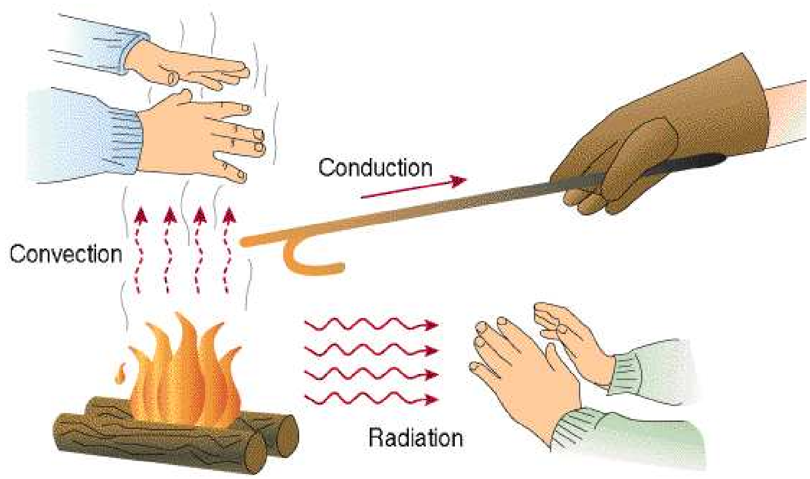

Figure 3.1: Transport of thermal energy from a campfire: conduction, convection, and radiation.

tive in solids (stick a poker into a fire, and the heat that eventually reaches the handle will be brought there by conduction through the solid iron of the poker); thus, conduction can be disregarded in gaseous stars. In a star, heat will be carried from the center to the surface by **convection** (hot blobs of gas move upward, while cooler blobs sink downward) or by **radiation** (photons carry quanta of energy upward). In the Sun, it happens that most of the energy is transported by photons rather than hot gas, so we'll look first at radiative energy transport.[1]

Consider a thin spherical shell centered on a star's center. The inner radius of the shell is r; the outer radius is $r+dr$, with $dr \ll r$. The temperature at the inner surface of the shell is T; the temperature at the outer surface is $T + dT$, where $|dT| \ll T$. Typically, stars have $dT < 0$, meaning that the temperature drops as you move away from the center. The radiation pressure at the inner surface of the shell is

$$P_{\rm rad}(r) = \frac{a}{3}T^4 \ , \tag{3.12}$$

while the radiation pressure at the outer surface is

$$P_{\rm rad}(r+dr) = \frac{a}{3}[T+dT]^4 = \frac{a}{3}T^4\left[1+\frac{dT}{T}\right]^4 \approx \frac{a}{3}T^4\left[1+4\frac{dT}{T}\right] \ , \tag{3.13}$$

using the mathematical relation $(1+\epsilon)^n \approx 1 + n\epsilon$ for $|\epsilon| \ll 1$. The net radiation force acting on the shell will be the pressure difference between the inside and outside, multiplied by the shell's area:

$$\begin{aligned} F_{\rm rad} &= [P_{\rm rad}(r) - P_{\rm rad}(r+dr)]\,4\pi r^2 && (3.14)\\ &\approx -\frac{a}{3}4T^4\frac{dT}{T}4\pi r^2 = -\frac{16\pi}{3}ar^2T^3dT \ . && (3.15)\end{aligned}$$

Thus, a temperature gradient across a thin shell is accompanied by a radiation force, caused by photons shoving on the material inside the shell. We've already seen that the optical depth of a thin spherical shell is (equation 2.24)

$$d\tau = -\rho(r)\kappa(r)dr \ , \tag{3.16}$$

where κ is the opacity, in units of $\mathrm{m^2\,kg^{-1}}$. If $d\tau \ll 1$, the probability that a photon will be absorbed while crossing the shell is $dP \approx d\tau$. The total rate at which photons carry energy through the shell is just the luminosity, $L(r)$. Since a photon has a momentum $p = E/c$, where E is the photon energy, the rate at which photons carry momentum through the shell is $L(r)/c$. Thus, the rate at which photon momentum is transferred to the shell (in other words, the *force* on the shell) is

$$F_{\rm rad}(r) = \frac{L(r)}{c}d\tau = -\frac{L(r)}{c}\rho(r)\kappa(r)dr \ . \tag{3.17}$$

[1] To avoid accusations of helio-chauvinism, we'll also look at convective energy transport a little later on.

Setting equations (3.15) and (3.17) equal to each other, we have an equation that relates the temperature, luminosity, and opacity of a star:

$$-\frac{16\pi}{3} a r^2 T(r)^3 dT = -\frac{\rho(r)\kappa(r)L(r)dr}{c} \ . \tag{3.18}$$

With a bit of rearrangement, this becomes the **equation of radiative energy transfer**:

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{16\pi a c T(r)^3 r^2} \ . \tag{3.19}$$

This equation can be written in alternate forms. For instance, Zeilik and Gregory take advantage of the equality $a = 4\sigma/c$, where σ is the Stefan-Boltzmann constant, to write the equation of radiative energy transfer in the form

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{64\pi\sigma T(r)^3 r^2} \ . \tag{3.20}$$

Equation (3.20) links the luminosity to the temperature gradient of the star, in much the same way that the equation of hydrostatic equilibrium (equation 3.1) links the mass to the pressure gradient of the star. A perfectly transparent star ($\kappa = 0$) would have no temperature gradient, since it wouldn't absorb any of the gamma rays generated by fusion reactions in the star's central core. The actual temperature gradient in the Sun, between the center and the photosphere, averages to

$$\frac{\Delta T}{\Delta r} \approx \frac{T_{\rm phot} - T_c}{r_\odot - 0} \approx \frac{5800\,{\rm K} - 1.47 \times 10^7\,{\rm K}}{6.96 \times 10^5\,{\rm km}} \approx -20\,{\rm K\,km^{-1}} \ . \tag{3.21}$$

For every kilometer you move outward in the Sun, on average, you cool down by 20 degrees.

The equation of radiative energy transfer (equation 3.20) can be used to make a crude estimate of the Sun's luminosity:

$$\frac{\Delta T}{\Delta r} \sim \frac{-T_c}{r_\odot} \sim -\frac{3\langle\kappa\rangle\rho_\odot L_\odot}{64\pi\sigma(T_c/2)^3(r_\odot/2)^2} \ , \tag{3.22}$$

and so

$$L_\odot \sim \frac{2\pi\sigma T_c^4 r_\odot}{3\langle\kappa\rangle\rho_\odot} \ , \tag{3.23}$$

where $\langle\kappa\rangle$ is the typical opacity inside the Sun. With $T_c = 1.47 \times 10^7\,\mathrm{K}$, $r_\odot = 6.96 \times 10^8\,\mathrm{m}$, and $\rho_\odot = 1400\,\mathrm{kg\,m^{-3}}$, we find

$$L_\odot \sim 3 \times 10^{27}\,\mathrm{W} \left(\frac{\langle\kappa\rangle}{1\,\mathrm{m^2\,kg^{-1}}} \right)^{-1} . \tag{3.24}$$

If we assumed that the opacity throughout the Sun had the same value as in the photosphere, $\kappa = 3\,\mathrm{m^2\,kg^{-1}}$, we would compute a solar luminosity of $L_\odot \sim 10^{27}\,\mathrm{W}$, about 2.5 times the actual value of $L_\odot = 3.90 \times 10^{26}\,\mathrm{W}$. This implies that the average opacity inside the Sun is actually

$$\langle\kappa\rangle \approx 8\,\mathrm{m^2\,kg^{-1}} . \tag{3.25}$$

Since opacity depends on temperature and density, its value varies with radius in the Sun; however, $\kappa \approx 8\,\mathrm{m^2\,kg^{-1}}$ is not absurd as an average value for the Sun as a whole.

Suppose, as a thought experiment, you inserted a layer into the Sun that was perfectly opaque, with $\kappa = \infty$. Equation (3.20) seems to imply that an infinite temperature gradient would occur across that layer, with the photons absorbed at the bottom of the layer driving the temperature at the bottom arbitrarily high. In reality, this does not happen; as the bottom of the layer absorbed energy, the hot gas would become buoyant, and rise upward. Energy would then be transported across the opaque layer by hot rising blobs of gas, rather than by photons. Generally, in stars with high opacity, energy is transported outward by convection, with hot gas rising and cooler gas sinking to take its place.

Convection is a chaotic, turbulent process, as you can verify by watching a pot of simmering soup on a stove, so the detailed physics tends to be gruesome. Rather than go into the details of hydrodynamic turbulence, we'll simply quote the results of a simple calculation, assuming that the star is made of perfect gas with an adiabatic index γ.[2] In this case, the **equation of convective energy transfer** is

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T(r)}{P(r)} \frac{dP}{dr} . \tag{3.26}$$

Thus, when energy is transported by convection, the temperature gradient is proportional to the pressure gradient.

[2]The adiabatic index is $\gamma = 5/3$ for a fully ionized gas.

In general, energy is carried outward in a star either by radiation or by convection, whichever is more efficient at shuttling joules toward the photosphere. The more efficient process is the one which leads to the smaller temperature gradient (equations 3.20 and 3.26). Within a single star, energy can be carried by radiation in one region and by convection in another. In the Sun, for example, radiation is the more efficient process out to $r = 0.7r_\odot$ (Figure 3.2). In the outer 30% of the Sun (by radius), convection is the

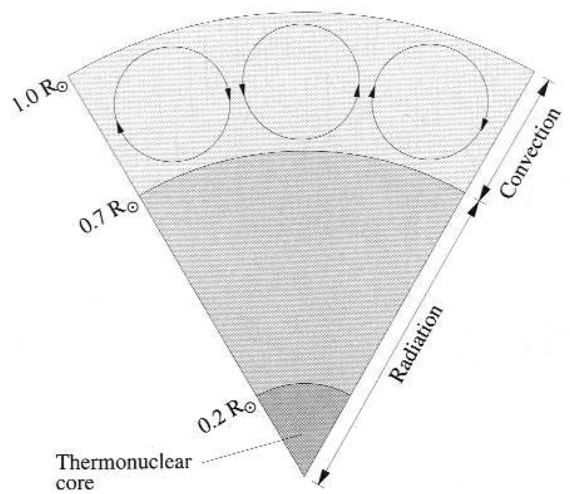

Figure 3.2: Radiative and convective energy transport within the Sun.

dominant means of energy transport.

So far, we've talked about how energy is carried to the photosphere of the star, but not about how it is generated in the star's interior. The energy that a star tosses away into space must come from some source inside the star. The generation of energy within a star is described by the last of the equations of stellar structure, the **equation of energy generation**. Consider the usual thin spherical shell of inner radius r and outer radius $r + dr$, centered on the star's center. A luminosity L flows through the inner surface, and a luminosity $L + dL$ flows through the outer surface. Where does the extra bit of power dL come from? We don't know yet, but we can express it in terms of the rate of energy production ϵ (the units of ϵ are watts per kilogram). The equation of energy generation can be written as

$$dL = (4\pi r^2 dr)\rho\epsilon \ , \tag{3.27}$$

or

$$\frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r) \ . \tag{3.28}$$

All we need do now is find the physical process by which energy is generated, and determine how ϵ depends on the temperature, density, and chemical composition within a star.

3.2 Energy Generation in Stars

The answer to the question "Why do stars shine?" is "Stars shine because they are hot." The obvious follow-up question is "Why don't they cool down?" There are several possible answers to this question. One possible source of energy for stars is **gravitational potential energy**. The current gravitational potential energy of the Sun is

$$U_\odot = -q \frac{GM_\odot^2}{r_\odot} \ , \tag{3.29}$$

where q is a factor of order unity. For a sphere of uniform density, $q = 3/5$. Stars, however, are centrally concentrated, and have $q \approx 1.5$. For the Sun, then,

$$U_\odot = -1.5 \frac{GM_\odot^2}{r_\odot} \approx -5.7 \times 10^{41}\,\mathrm{J} \ . \tag{3.30}$$

Since the Sun started as a gas cloud with $r \gg r_\odot$, in collapsing to its present size, it lost 5.7×10^{41} J of gravitational potential energy. If all this energy was converted to photons, it would keep the Sun at its present luminosity for a time equal to the **Kelvin-Helmholtz time**:

$$t_{\mathrm{KH}} \equiv \frac{|U_\odot|}{L_\odot} = \frac{5.7 \times 10^{41}\,\mathrm{J}}{3.9 \times 10^{26}\,\mathrm{J\,s^{-1}}} \approx 1.5 \times 10^{15}\,\mathrm{s} \approx 50\,\mathrm{Myr} \ . \tag{3.31}$$

Kelvin and Helmholtz proposed, in the late nineteenth century, that the Sun was powered by gravitational potential energy. However, when they stated that the Sun had to be ~ 50 million years old, geologists were dubious. They pointed out, quite rightly, that the fossil record implies that the Sun has been shining at a roughly constant luminosity for a time much longer than 50 million years.[3]

[3] The Kelvin-Helmholtz time for brown dwarfs is hundreds of times longer than the Sun's Kelvin-Helmholtz time. Thus, the dim light from brown dwarfs is powered by gravitational potential energy.

Until the 1930s, astronomers were ignorant of the source of the stars' energy. What keeps stars going and going and going? Chemical energy is far too feeble a source: if the Sun were made of coal, burning it would only maintain the Sun's luminosity for 50 thousand years or so. Gravitational potential energy falls short, too: it would only maintain the Sun's luminosity for 50 million years. In 1938 came the grand realization: **nuclear fusion** provides the necessary energy to keep the stars hot.

The Sun, like other main sequence stars, fuses hydrogen into helium:

- Mass of 4 hydrogen atoms = $6.692 \times 10^{-27}\,\mathrm{kg}$
- Mass of 1 helium atom = $6.644 \times 10^{-27}\,\mathrm{kg}$
- Mass difference = $0.048 \times 10^{-27}\,\mathrm{kg}$

When four hydrogen atoms fuse to form one helium atom, the lost mass, $\Delta m = 4.8 \times 10^{-29}\,\mathrm{kg}$, is converted to energy. The conversion rate is given by Einstein's formula:

$$\Delta E = (\Delta m)c^2 = 4.3 \times 10^{-12}\,\mathrm{J}\ . \tag{3.32}$$

Fusing together four hydrogen atoms doesn't create a lot of energy: 4.3×10^{-12} J is about enough to lift a nickel through a height of 1 Ångstrom against the Earth's gravity at sea level. However, there are a whole lotta hydrogen atoms inside a star. If the Sun had started out made entirely of hydrogen, it would have contained N_{H} hydrogen atoms, where

$$N_{\mathrm{H}} = \frac{M_\odot}{m_{\mathrm{H}}} \approx \frac{1.99 \times 10^{30}\,\mathrm{kg}}{1.67 \times 10^{-27}\,\mathrm{kg}} \approx 1.2 \times 10^{57}\ . \tag{3.33}$$

Fusing all the hydrogen atoms into $N_{\mathrm{H}}/4$ helium atoms would release an amount of energy

$$E_{\mathrm{fus}} = \frac{N_{\mathrm{H}}}{4}\Delta E = \frac{1.2 \times 10^{57}}{4}(4.3 \times 10^{-12}\,\mathrm{J}) = 1.3 \times 10^{45}\,\mathrm{J}\ . \tag{3.34}$$

This is about 2000 times the magnitude of the Sun's gravitational potential energy. Thus, fusion can keep the Sun shining at a constant rate for a time

$$t_{\mathrm{fus}} = \frac{E_{\mathrm{fus}}}{L_\odot} \approx 3.3 \times 10^{18}\,\mathrm{s} \approx 100\,\mathrm{Gyr}\ . \tag{3.35}$$

(The "gigayear", abbreviated Gyr, is equal to one billion years, and is a time scale very useful when discussing the lifetimes of stars.) In truth, the Sun wasn't pure hydrogen when it started out, and the conversion of hydrogen to helium in the Sun isn't totally efficient. The lifetime of the Sun, as a consequence, is only $\sim 10\,\mathrm{Gyr}$ instead of $\sim 100\,\mathrm{Gyr}$. It is still comfortably longer than the Kelvin-Helmholtz time, though.

All main sequence stars are powered by the fusion of hydrogen into helium in their central regions. The Sun's main sequence lifetime is $\tau_\odot \approx 10\,\mathrm{Gyr}$. Since the main sequence lifetime is $\tau \propto M/L$ and since $L \propto M^4$ for stars with $M > 0.43 M_\odot$ (section 1.5), the main sequence lifetime of a massive star is

$$\tau \approx 10\,\mathrm{Gyr} \left(\frac{M}{1 M_\odot} \right)^{-3} . \tag{3.36}$$

The lifetime of a $20 M_\odot$ star (with spectral type 09) will be only 1 Myr. The lifetime of a $0.5 M_\odot$ star (spectral type M0) will be 80 Gyr, longer than the age of our galaxy. Every M dwarf ever made is still fusing hydrogen into helium; they aren't going to run out of fuel any time soon.

3.3 Nuclear Fusion Reactions

Fusion of hydrogen into helium occurs by a series of two-body collisions, instead of a single grand four-body collision. For stars with central temperatures less than 18 million Kelvin (this includes the Sun), helium is created from hydrogen via the **PP chain**, illustrated in Figure 3.3. In the first step of

$$\boxed{\mathrm{p}} + \boxed{\mathrm{p}} \rightarrow {}^{2}\mathrm{H} + e^{+} + \nu_e \;\text{(twice)}$$
$${}^{2}\mathrm{H} + \mathrm{p} \rightarrow {}^{3}\mathrm{He} + \gamma \;\text{(twice)}$$
$${}^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow \boxed{{}^{4}\mathrm{He}} + \mathrm{p} + \mathrm{p}$$

Figure 3.3: The PP chain, dominant form of hydrogen fusion at $T_c < 1.8 \times 10^7\,\mathrm{K}$. [Image credit: Richard Pogge]

the chain, two protons (p) fuse together to form a deuteron (^{2}H). A deuteron is the nucleus of a deuterium (or "heavy hydrogen") atom, and consists of a proton and neutron held together by the strong nuclear force. When one of the protons is converted to a neutron, a positron (e^+) is emitted to conserve charge, and an electron neutrino (ν_e) is emitted to conserve electron quantum number. Because the positively charged protons repel each other, and because they have to come very close in order to fuse, the first step of the PP chain is the slowest one. During the past 4.6 billion years, only half the protons in the Sun's core have undergone fusion. In the second step of the PP chain, the deuteron (^{2}H) fuses with a proton to form light helium (^{3}He), which contains two protons and only one neutron. The excess energy from the fusion is carried away by a gamma-ray photon (γ). In the final step of the PP chain, two light helium nuclei fuse together to form ordinary helium (^{4}He), which contains two protons and two neutrons. The excess protons are spat out, ready to begin a new PP chain.

The net result of the PP chain is:

$$4p \rightarrow^4 \mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e} + 2\gamma \; . \tag{3.37}$$

The positrons quickly annihilate with electrons to form additional gamma rays. The neutrinos carry away only 2% of the energy released in the PP chain; gamma rays take away the rest. The neutrinos, because of their extremely tiny cross-sections for interactions, stream freely through the Sun. In other words, although the Sun is opaque to photons, it is transparent to neutrinos. The Sun emits about 2×10^{38} neutrinos per second, of which roughly 10^{15} pass through your body.

In main sequence stars with central temperatures greater than 18 million Kelvin (this includes O, B, A, and F stars), hydrogen is fused into helium via the **CNO cycle**, illustrated in Figure 3.4. In the CNO cycle, carbon (C), nitrogen (N), and oxygen (0) act as catalysts to speed the fusion of hydrogen. The net result of the CNO cycle is:

$$4p \rightarrow^4 \mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e} + 3\gamma \; . \tag{3.38}$$

Again, the positrons annihilate with electrons to form additional gamma rays.

Fusion of hydrogen into helium is a reasonably efficient form of energy; about 0.7% of the hydrogen's mass is converted into energy in the process. However, still more energy can be squeezed out of a star if the helium is

$$\begin{aligned}
\boxed{{}^{12}C} + \boxed{p} &\rightarrow {}^{13}N + \gamma \\
{}^{13}N &\rightarrow {}^{13}C + e^{+} + \nu_e \\
{}^{13}C + \boxed{p} &\rightarrow {}^{14}N + \gamma \\
{}^{14}N + \boxed{p} &\rightarrow {}^{15}O + \gamma \\
{}^{15}O &\rightarrow {}^{15}N + e^{+} + \nu_e \\
{}^{15}N + \boxed{p} &\rightarrow \boxed{{}^{12}C} + \boxed{{}^{4}He}
\end{aligned}$$

Figure 3.4: The CNO cycle, dominant form of hydrogen fusion at $T_c > 1.8 \times 10^7$ K. [Image credit: Richard Pogge]

fused into heavier and heavier elements, until iron is reached. Iron has the lowest mass per nucleon of any element, so it is the end of the line as far as fusion is concerned.[4] The process by which stars convert helium to carbon is the **triple alpha process**, shown in Figure 3.5. In nuclear physics, a ^{4}He

$$\begin{aligned}
{}^{4}He + {}^{4}He &\rightarrow {}^{8}Be + \gamma \\
{}^{4}He + {}^{8}Be &\rightarrow {}^{12}C + \gamma
\end{aligned}$$

Figure 3.5: The triple alpha process, by which helium is fused to heavier elements.

nucleus is also called an "alpha particle", which helps to explain the odd name "triple alpha".[5]

[4]Elements more massive than iron can release energy by fission, splitting into lower mass nuclei. Stars are powered by fusion rather than fission because practically all the nuclei in the universe are lower in mass than iron.

[5]Electrons are "beta particles", and high-energy photons are "gamma particles", or

In the first step of the triple alpha process, two ordinary helium nuclei (^{4}He) fuse to form a beryllium nucleus (^{8}Be). However, the ^{8}Be nucleus is extremely unstable; it decays back into a pair of helium nuclei with a half-life of only $t_{1/2} \sim 2 \times 10^{-16}\,$s.[6] However, if the ^{8}Be nucleus encounters a ^{4}He nucleus during the brief period before it decays, the two nuclei can fuse to form a stable ^{12}C nucleus. Because the beryllium nucleus has such a brief life, it will only encounter a helium nucleus and fuse if the surroundings are very dense (which increases the number density of helium nuclei) and very hot (which increases the average speed of the helium nuclei). In practice, the triple alpha process only occurs at a significant rate when $T_c > 10^8\,$K. The Sun isn't currently fusing helium into carbon because it's not hot enough.

3.4 Modeling Stellar Interiors

We have the five basic equations that govern the structure of stellar interiors. The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \ . \tag{3.39}$$

The equation of mass continuity is

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \ . \tag{3.40}$$

The equation of state is

$$P(r) = \frac{k\rho(r)T(r)}{\mu(r)m_p} \ .labeleq:s3 \tag{3.41}$$

The equation of energy transport is

$$\frac{dT}{dr} = \begin{cases} -\frac{3\kappa(r)\rho(r)L(r)}{64\pi\sigma r^2 T(r)^3} \\ (1-1/\gamma)\frac{T(r)}{P(r)}\frac{dP}{dr} \ , \end{cases} \tag{3.42}$$

choosing whichever form of transport (radiative or convective) gives the smaller temperature gradient. Finally, the equation of energy generation is

$$\frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r) \ . \tag{3.43}$$

"gamma rays".

[6] The stable isotope of beryllium – the kind found in emeralds – is ^{9}Be.

To solve this set of equations, we need boundary conditions at the photosphere. We also need to know how the mean molecular weight $\mu(\rho, T)$, opacity $\kappa(\rho, T)$, and energy generation rate $\epsilon(\rho, T)$ depends on density and temperature within the star. The energy generation rate ϵ, in particular, is extremely sensitive to temperature. For the PP chain, $\epsilon \propto T^4$, and for the CNO cycle, $\epsilon \propto T^{20}$.

Given all this information, models of stellar interiors can be built up by numerically solving the five equations of stellar structure. In the mid-20th century, back in the time of slide rules and mechanical calculating machines, you could earn a PhD thesis by modeling a single star. Nowadays, of course, computers can crank out stellar models on an assembly line. The result of a model of the Sun's interior is shown in Figure 3.6. Note in particular that

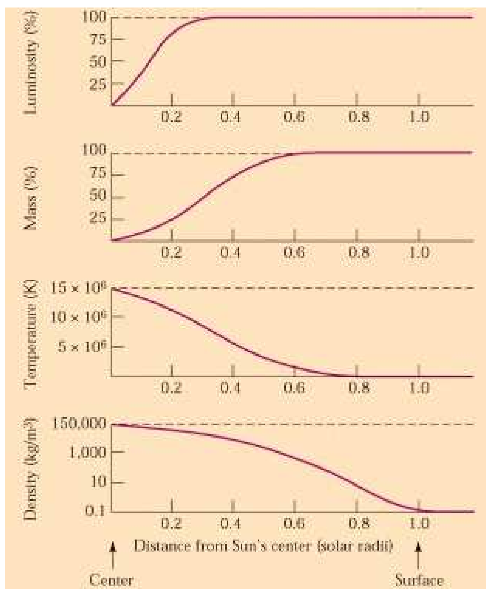

Figure 3.6: $L(r)$, $M(r)$, $T(r)$, and $\rho(r)$ for the solar interior.

most of the Sun's luminosity comes from $r < 0.2 r_\odot$. Because of the strong dependence of ϵ on temperature, as the temperature T gradually drops with r, the energy generation rate ϵ plummets. It is also interesting to note that the Sun's central density is roughly 150 times the density of water. The high temperatures in the core keep the material in ionized gaseous form, despite its high density.

How do we know that our models of the solar interior are correct? The boundary conditions are determined by the well-observed properties of the photosphere. The equations of stellar structure (equations 3.39 through 3.43) are based on well-understood physics. Nevertheless, it is a good thing to verify our models by comparison with observations. Although we cannot see directly into the Sun's interior, there are indirect methods by which we can deduce the Sun's interior structure. For instance, **helioseismology** (the study of seismic waves in the Sun's interior) can tell us the sound speed inside the Sun. In the interior of the Earth, as mentioned in section 10.1 of *BA*, both S-waves (shear waves) and P-waves (pressure waves) can propagate. In the interior of the Sun, S-waves, which can only propagate through solids, are not found. However, P-waves, which are essentially sound waves, are free to move throughout the Sun's interior. Because the sound speed is $c_s \propto T^{1/2}$, it increases as you go further into the Sun's interior. This causes P-waves, or sound waves, to be refracted upward.

Although it is impractical to put a seismometer in the Sun's photosphere, we can see the vertical motions of the photosphere in a "dopplergram" (Figure 3.7), which shows the Doppler shift as a function of position on the visible hemisphere of the Sun. The rotation of the Sun can be seen in Fig-

Figure 3.7: Dopplergram of the Sun's surface. [Image credit: SOHO MDI]

ure 3.7, as well as the upward and downward motions due to P-waves reaching the photosphere. The sound oscillations in the Sun can be decomposed into different modes using spherical harmonics (in much the same way that the sound from a piano can be decomposed into different frequencies using Fourier transforms). The observed modes of oscillation can be used to determine the sound speed as a function of radius within the Sun. The sound speeds measured in this way agree with those predicted by the best solar models to an accuracy of $< 0.1\%$.

Another source of information about the Sun's interior is **solar neutrinos**. The PP chain that provides most of the fusion energy in the Sun's core produces two electron neutrinos for every helium nucleus created. These electron neutrinos fly straight through the Sun with a very tiny chance of interaction. Thus, if we could manage to capture a few of the neutrinos, we would have a direct window on the fusion reactions at the Sun's center. Although neutrinos have small cross-sections for interaction with "ordinary" matter, there are capable of undergoing reactions such as

$$\nu_e +^{37}\mathrm{Cl} \rightarrow^{37}\mathrm{Ar} + e^- \tag{3.44}$$

and

$$\nu_e +^{71}\mathrm{Ga} \rightarrow^{71}\mathrm{Ge} + e^- \ . \tag{3.45}$$

Over the past three or four decades, typical solar neutrino experiments have involved filling large tanks with carbon tetrachloride or gallium and waiting for the infrequent neutrino reactions. Numerous hunts for solar neutrinos all found that only *half* the expected number of electron neutrinos were detected. Astronomers and physicists scrambled to solve the "solar neutrino problem", as it was called. One suggested solution was that the solar interior was slightly cooler than the standard solar models suggested; this would drive down the rate of energy generation $\epsilon \propto T^4$. However, helioseismology confirmed that the temperatures predicted by the standard solar model was correct.

The ultimate resolution of the solar neutrino problem came from particle physics. There are three types, or flavors, of neutrinos. In addition to the electron neutrinos, ν_e, there are also muon neutrinos, ν_μ, and tau neutrinos, ν_τ. Although nuclear fusion only produces electron neutrinos, if neutrinos have mass and if the masses of the three types differ, then electron neutrinos can spontaneous convert into muon or tau neutrinos. Recent neutrino experiments indicate that the three types of neutrino really do have (small) masses. In addition, the Sudbury Neutrino Observatory, located in a former nickel mine in Ontario, has searched for neutrinos of all three types. The heart of the Sudbury detector is a large tank filled with heavy water (D_20). The deuterium (or "heavy hydrogen") has a small probability of being split by a neutrino:

$$\nu + \mathrm{D} \rightarrow \mathrm{p} + \mathrm{n} + \nu \ . \tag{3.46}$$

This reaction occurs for any time of neutrino, and thus doesn't distinguish among ν_e, ν_μ, and ν_τ. The total number of all neutrinos detected by Sudbury is consistent with the number predicted by the standard solar model. This indicates that some of the electron neutrinos have been converted to muon and tau neutrinos during their flight from the Sun to Ontario.