S.2 Solutions to Exercises Marked with (✍) of Chapter 2

2.1 (✍) The plane in $\mathbb{R}^3$ which passes through the points $(0,0,0)$, $(1,1,1)$ and $(0,0,2)$. Explicitly, the plane is the set $\{x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2\}$.

2.2 (✍) Since $x \in X \setminus \{0\}$, we see that $\|x\| \neq 0$. Let $c = r/\|x\|$, then $\|cx\| = c\|x\| = r$.

2.3 (✍) It is a consequence of the definition of a bounded set in a metric space since we can find an open ball $B(x_0, r)$ such that $M \subset B(x_0, r)$, that is, $d(x, x_0) = \|x - x_0\| < r$ for all $x \in M$. Let $c = r + \|x_0\|$, then for all $x \in M$ we have

$$\|x\| = d(x, 0) \leqslant d(x, x_0) + d(x_0, 0) = \|x - x_0\| + \|x_0\| < c.$$

2.4 (✍)

(i) For every $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ and $\alpha, \beta \in \mathbb{F}$ we have

(a)
$$\begin{aligned}
(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\
&= (x_2 + x_1, y_2 + y_1) \\
&= (x_2, y_2) + (x_1, y_1), \\
(x_1, y_1) + ((x_2, y_2) + (x_2, y_2)) &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\
&= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\
&= ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3);
\end{aligned}$$

(b) $(x_1, y_1) + (0, 0) = (x_1 + 0, y_1 + 0) = (x_1, y_1)$;

(c) $(x_1, y_1) + (-x_1, -y_1) = (x_1 + (-x_1), y_1 + (-y_1)) = (0, 0)$;

(d) $1(x_1, y_1) = (1x_1, 1y_1) = (x_1, y_1)$,
$\alpha(\beta(x_1, y_1)) = \alpha(\beta x_1, \beta y_1) = (\alpha\beta x_1, \alpha\beta y_1) = (\alpha\beta)(x_1, y_1)$;

(e)
$$\begin{aligned}\alpha((x_1,y_1)+(x_2,y_2)) &= \alpha(x_1+x_2,y_1+y_2)\\ &= (\alpha x_1+\alpha x_2,\alpha y_1+\alpha y_2)\\ &= \alpha(x_1,y_1)+\alpha(x_2,y_2),\\ (\alpha+\beta)(x_1,y_1) &= ((\alpha+\beta)x_1,(\alpha+\beta)y_1)\\ &= (\alpha x_1+\beta x_1,\alpha y_1+\beta y_1)\\ &= (\alpha x_1,\alpha y_1)+(\beta x_1,\beta y_1)\\ &= \alpha(x_1,y_1)+\beta(x_1,y_1).\end{aligned}$$

Hence $X\times Y$ is a linear space over $\mathbb{F}$.

(ii) We will show $\|\cdot\|$ satisfies the axioms 1°-3° for a norm. For every $(x_1,y_1),(x_2,y_2)\in X\times Y$ and $\alpha\in\mathbb{F}$ we have

(1°) $\|(x_1,y_1)\| = \|x_1\|_X+\|y_1\|_Y \geqslant 0$, and $\|(x_1,y_1)\|=0$ if and only if $\|x_1\|_X=0$ and $\|y_1\|_Y=0$ if and only if $x_1=y_1=0$, i.e., $(x_1,y_1)=(0,0)$,

(2°)
$$\begin{aligned}\|\alpha(x_1,y_1)\| &= \|(\alpha x_1,\alpha y_1)\| = \|\alpha x_1\|_X+\|\alpha y_1\|_Y\\ &= \alpha\|x_1\|_X+\alpha\|y_1\|_Y = \alpha(\|x_1\|_X+\|y_1\|_Y)\\ &= \alpha\|(x_1,y_1)\|,\end{aligned}$$

(3°)
$$\begin{aligned}\|(x_1,y_1)+(x_2,y_2)\| &= \|(x_1+x_2,y_1+y_2)\|\\ &= \|x_1+x_2\|_X+\|y_1+y_2\|_Y\\ &\leqslant \|x_1\|_X+\|x_2\|_X+\|y_1\|_Y+\|y_2\|_Y\\ &= \|(x_1,y_1)\|+\|(x_2,y_2)\|,\end{aligned}$$

it follows that $\|\cdot\|$ is a norm on $X\times Y$.

(iii) Clearly, a sequence $\{(x_n,y_n)\}$ of $X\times Y$ converges to $(x,y)\in X\times Y$ as $n\to\infty$ if and only if both $\|x_n-x\|_X\to 0$ and $\|y_n-y\|_Y\to 0$ as $n\to\infty$, since

$$\|(x_n,y_n)-(x,y)\| = \|(x_n-x,y_n-y)\| = \|x_n-x\|_X+\|y_n-y\|_Y.$$

Which implies that the sequence $\{(x_n,y_n)\}$ converges to $(x,y)\in X\times Y$ as $n\to\infty$ if and only if $\{x_n\}$ converges to $x\in X$ and $\{y_n\}$ converges to $y\in Y$ as $n\to\infty$.

(iv) Let $\{(x_n, y_n)\}$ be a sequence in $X \times Y$, then $\{(x_n, y_n)\}$ is Cauchy in $X \times Y$ if and only if $\|(x_n, y_n) - (x_m, y_m)\| \to 0$ as $n, m \to \infty$ if and only if both $\|x_n - x_m\|_X \to 0$ and $\|y_n - y_m\|_Y \to 0$ as $n, m \to \infty$ since

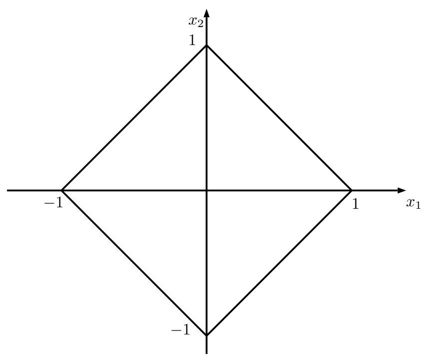
$$\begin{aligned}\|(x_n, y_n) - (x_m, y_m)\| &= \|(x_n - x_m, y_n - y_m)\| \\ &= \|x_n - x_m\|_X + \|y_n - y_m\|_Y.\end{aligned}$$

Equivalently, $\{x_n\}$ is Cauchy in X and $\{y_n\}$ is Cauchy in Y by the definition.

2.5 (✍)

(i) Since the absolute value $|\cdot|$ is the usual norm on $\mathbb{R}$, by Exercise 2.4 we see that $|\cdot|_1$ is a norm on $\mathbb{R}^2$

(ii) The unit circle $\{x \in \mathbb{R}^2 : \|x\|_1 = 1\}$ is the parallelogram or rhombus in the plane $\mathbb{R}^2$, with vertices $(0, 1)$, $(1, 0)$, $(-1, 0)$ and $(0, -1)$ as the following figure:



2.6 (✍) Suppose that X is a vector space on $\mathbb{F}$ and that d is the discrete metric on X. Since $X \neq \{0\}$, there exists a vector v such that $v \neq 0$. And this v satisfies that $d(2v, 0) = 1 \neq 2 = 2d(v, 0)$, so that d cannot be obtained from a norm by Remark 2.2.2.

2.7 (✍) Clearly, $\tilde{d}$ satisfies the axioms (M1) and $(M2)$ for a metric. To show that $\tilde{d}$ satisfies the triangle inequality (M3), we let $x, y, z \in X$ be arbitrary and we have

$$\begin{aligned}\tilde{d}(x, y) &= \|x - y\| + 1 \leqslant \|x - z\| + 1 + \|y - z\| \\ &\leqslant \tilde{d}(x, z) + \tilde{d}(y, z)\end{aligned}$$

if $x \neq y$. Also the above inequality obviously holds for $x = y$, hence $\tilde{d}$ is a metric on X. Since $\tilde{d}(\alpha x, 0) \neq |\alpha|\tilde{d}(x, 0)$ in general, we see that $\tilde{d}$ is not the metric induced by the norm $\|\cdot\|$.

2.12 (✍) Let $x, y \in L^p(E)$ $(1 \leqslant p < \infty)$, then both

$$\int_E |x(t)|^p \,\mathrm{d}t < \infty \quad \text{and} \quad \int_E |y(t)|^p \,\mathrm{d}t < \infty.$$

We will show that $x + y \in L^p(E)$ and $\alpha x \in L^p(E)$ for all $\alpha \in \mathbb{F}$, where the sum of x and y and the product α and x are defined like (2.5). Obviously, $\alpha x \in L^p(E)$ for all $\alpha \in \mathbb{F}$ since

$$\int_E |\alpha x(t)|^p \,\mathrm{d}t = |\alpha|^p \int_E |x(t)|^p \,\mathrm{d}t < \infty.$$

Obverse that

$$\begin{aligned}\int_E |x(t) + y(t)|^p \,\mathrm{d}t &\leqslant \int_E (|x(t)| + |y(t)|)^p \,\mathrm{d}t \\ &\leqslant 2^p \int_E (\max\{|x(t)|, |y(t)|\})^p \,\mathrm{d}t \\ &\leqslant 2^p \left(\int_E |x(t)|^p \,\mathrm{d}t + \int_E |y(t)|^p \,\mathrm{d}t \right) < \infty,\end{aligned}$$

we see that $x + y \in L^p(E)$, so that $L^p(E)$ is a linear space.

2.13 (✍) Suppose that $\{x_i\} \subset L^\infty(E)$ such that $\sum\limits_{i=1}^{\infty} \|x_i\|_{L^\infty(E)} < \infty$. By the Example 2.3.7 we see that for each $i \in \mathbb{N}$ there exists a set $E_i \subset E$ such that $\|x_i\|_{L^\infty(E)} = \sup_{E \setminus E_i} |x_i(t)|$ and $\operatorname{meas}(E_i) = 0$. Let $E_0 = \bigcup\limits_{i \in \mathbb{N}} E_i$, then $E_0 \subset E$ and $\operatorname{meas}(E_0) = 0$. By the definition of $\|\cdot\|_{L^\infty(E)}$ we have

$$\begin{aligned}\left\| \sum_{i=1}^{\infty} x_i \right\|_{L^\infty(E)} &\leqslant \sup_{t \in E \setminus E_0} \left| \sum_{i=1}^{\infty} x_i(t) \right| \leqslant \sum_{i=1}^{\infty} \sup_{t \in E \setminus E_0} |x_i(t)| \\ &\leqslant \sum_{i=1}^{\infty} \sup_{t \in E \setminus E_i} |x_i(t)| = \sum_{i=1}^{\infty} \|x_i\|_{L^\infty(E)} < \infty,\end{aligned}$$

which means that the series $\sum\limits_{i=1}^{\infty} x_i$ is convergent in $L^\infty(E)$. Thus $L^\infty(E)$ is complete by Theorem 2.2.2, and then it is a Banach space.

Another Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(E)$. By the definition of the norm $\|\cdot\|_{L^\infty(E)}$ we see that there exist subsets A_k and $B_{m,n}$ of E, with $\text{meas}(A_k) = 0 = \text{meas}(B_{m,n})$, such that $|f_k(x)| > \|f_k\|_{L^\infty(E)}$ for all $x \in A_k$ and $|f_n(x) - f_m(x)| > \|f_n - f_m\|_{L^\infty(E)}$ for all $x \in B_{m,n}$, $k, m, n = 1, 2, \cdots$ Let E_0 be the union of these sets, for $k, m, n = 1, 2, \cdots$. Then $E_0 \subset E$ with $\mu(E_0) = 0$, and on the complement of E the sequence $\{f_n(x)\}$ converges uniformly to a bounded function f since

$$\sup_{x \in E \setminus E_0} |f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_{L^\infty(E)} \to 0 \quad \text{as } m, n \to \infty$$

by the preassumption that $\{f_n\}$ is Cauchy in $L^\infty(E)$, it follows that $\{f_n(x)\}$ converges uniformly to a function $f(x)$ for all $x \in E \setminus E_0$, this function f must satisfy $\sup\limits_{x \in E \setminus E_0} |f(x)| \leqslant \sup\limits_{n \in \mathbb{N}} \|f_n\|_{L^\infty(E)} < \infty$ by the boundedness of the Cauchy sequence. Define $f(x) = 0$ for $x \in E_0$. Then $f \in L^\infty(E)$ and $\|f_n - f\|_{L^\infty(E)} \to 0$ as $n \to \infty$.

2.15 (✍) p must satisfy that $1/\beta < p < 1/\alpha$. Since

$$\begin{aligned}\int_0^\infty \frac{dx}{|x^\alpha + x^\beta|^p} &= \int_0^1 \frac{dx}{|x^\alpha + x^\beta|^p} + \int_1^\infty \frac{dx}{|x^\alpha + x^\beta|^p} \\ &= \int_0^1 \frac{dx}{x^{\alpha p}(1 + x^{\beta - \alpha})^p} + \int_1^\infty \frac{dx}{x^{\beta p}(1 + x^{\alpha - \beta})^p},\end{aligned}$$

both integrals in the right-hand side converge if and only if $\beta p > 1$ and $\alpha p < 1$, equivalently, $1/\beta < p < 1/\alpha$.

2.16 (✍) The famous Riemann function g defined on $[a, b] = [0, 1]$ is essentially bounded on $[0, 1]$ since it is Riemann integrable on $[a, b]$, that is, $g \in L^\infty[0, 1]$, but it is discontinuous at each rational point of $[0, 1]$, which means that $g \notin C[0, 1]$. Hence $C[0, 1] \subsetneqq L^\infty[0, 1]$.

Clearly, every $h \in L^\infty[0, 1]$ infers $h \in L^p[0, 1]$ for all $1 \leqslant p < \infty$ since

$$\int_0^1 |h(t)|^p \,\mathrm{d} \leqslant \int_0^1 \operatorname*{ess\,sup}_{t \in [0,1]} |h(t)|^p \,\mathrm{d}t = \|h\|^p_{L^\infty[0,1]} < \infty.$$

Let $f(t) = \begin{cases} \ln(1/t), & \text{if } t \in (0, 1] \\ 0, & \text{if } t = 0, \end{cases}$ then $\int_0^1 (f(t))^p \,\mathrm{d}t < \infty$ since for any given $0 < q < 1$

$$\lim_{t\to 0^+} \frac{(f(t))^p}{(1/t)^q} = \lim_{t\to 0^+} \frac{(\ln(1/t))^p}{(1/t)^q} = \lim_{s\to+\infty} \frac{(\ln s)^p}{s^p} = 0,$$

which means that $f \in L^p[0,1]$. For an arbitrary $M > 0$ we see that

$$\begin{aligned}\operatorname{meas}(\{t \in [0,1] : |f(t)| > M\}) &= \operatorname{meas}(\{t \in (0,1) : |\ln(1/t)| > M\}) \\ &= \operatorname{meas}(\{t \in (0,1) : 0 < t < e^{-M}\}) \\ &= e^{-M} > 0,\end{aligned}$$

which means that $f \notin L^\infty[0,1]$, so that $L^\infty[0,1] \subsetneqq L^p[0,1]$.

2.17 (✍) For simplicity we shall consider the case of $E \subset \mathbb{R}$ with $\operatorname{meas}(E) = \infty$. For example, let $E = (0, +\infty)$ and $1/p > k > 1/q$. Then $kp < 1$ and $kq > 1$. Set

$$x_q(t) = \begin{cases} t^{-k}, & \text{if } t \geqslant 1, \\ 0, & \text{if } 0 < t < 1. \end{cases}$$

We have $\displaystyle\int_0^{+\infty} |x_q(t)|^q \,\mathrm{d}t = \int_1^{+\infty} \frac{\mathrm{d}t}{t^{kq}} < \infty$, hence $x_q \in L^q(0,+\infty)$, but $x_q \notin L^p(0,+\infty)$ since $\displaystyle\int_0^{+\infty} |x_q(t)|^p \,\mathrm{d}t = \int_1^{+\infty} \frac{\mathrm{d}t}{t^{kp}} = \infty$. A similar function

$$x_p(t) = \begin{cases} t^{-k}, & \text{if } 0 < t \leqslant 1, \\ 0, & \text{if } t > 1 \end{cases}$$

also shows that $x_p \in L^p(0,+\infty)$ but $x_p \notin L^q(0,\infty)$.

2.19 (✍) Let $x \in L^\infty(E)$ be arbitrary, then we have

$$\|x\|_{L^\infty(E)} = \operatorname*{ess\,sup}_{t\in E} |x(t)| = \inf_{\substack{\operatorname{meas}(E_0)=0 \\ E_0 \subset E}} \sup_{t \in E\setminus E_0} |x(t)| < \infty.$$

Hence, for all $p \geqslant 1$ we get that

$$\begin{aligned}\int_E |x(t)|^p dt &\leqslant \int_E \operatorname*{ess\,sup}_{t\in E} |x(t)|^p \,\mathrm{d}t = \left(\operatorname*{ess\,sup}_{t\in E} |x(t)|\right)^p \int_E \mathrm{d}t \\ &= \|x\|^p_{L^\infty(E)} \operatorname{meas}(E) < \infty\end{aligned}$$

which means that $x \in L^p(E)$, so that $L^\infty(E) \subset L^p(E)$ and

$$\|x\|_{L^p(E)} \leqslant \|x\|_{L^\infty(E)}(\text{meas}(E))^{1/p}. \tag{E2-6}$$

For every $0 < \varepsilon < \|x\|_{L^\infty(E)}$, we claim that the measure of subset $A = \{t \in E : |x(t)| > \|x\|_{L^\infty(E)} - \varepsilon\}$ of E is positive, i.e. $\text{meas}(A) > 0$. Indeed, if $\text{meas}(A) = 0$ for some $0 < \varepsilon_0 < \|x\|_{L^\infty(E)}$, then by the definition of the infimum in $\|x\|_{L^\infty(E)}$ we see that

$$\begin{aligned}\|x\|_{L^\infty(E)} &= \inf_{\substack{\text{meas}(E_0)=0\\ E_0\subset E}} \sup_{t\in E\setminus E_0} |x(t)| \leqslant \sup_{t\in E\setminus A} |x(t)| \\ &\leqslant \|x\|_{L^\infty(E)} - \varepsilon_0 < \|x\|_{L^\infty(E)},\end{aligned}$$

since $\varepsilon_0 > 0$, this contradiction leads that $\text{meas}(A) > 0$. Thus, for every $p > 1$ we obtain that

$$\begin{aligned}\|x\|_{L^p(E)} &= \left(\int_E |x(t)|^p \,\mathrm{d}t\right)^{1/p} \geqslant \left(\int_A |x(t)|^p \,\mathrm{d}t\right)^{1/p} \\ &\geqslant \left(\|x\|_{L^\infty(E)} - \varepsilon\right)(\text{meas}(A))^{1/p},\end{aligned}$$

which deduces that $\liminf_{p\to\infty} \|x\|_{L^p(E)} \geqslant \|x\|_{L^\infty(E)} - \varepsilon$. Letting $\varepsilon \to 0^+$, we have $\liminf_{p\to\infty} \|x\|_{L^p(E)} \geqslant \|x\|_{L^\infty(E)}$. On the other hand, it follows from (E2-6) that $\limsup_{p\to\infty} \|x\|_{L^p(E)} \leqslant \|x\|_{L^\infty(E)}$, so that

$$\lim_{p\to\infty} \|x\|_{L^p(E)} = \|x\|_{L^\infty(E)}.$$

2.21 (✍) if $n \geqslant 2$, then the triangle inequality 3° for a norm may not hold. For example, let $x = (1, 0, 0, \cdots, 0)$ and $y = (0, 1, 0, \cdots, 0)$, then $x + y = (1, 1, 0, \cdots, 0)$. Now $\|x\|_p = 1 = \|y\|_p$ and

$$\|x + y\|_p = 2^{1/p} > 2 = \|x\|_p + \|y\|_p$$

since $1/p > 1$.

2.23 (✍) The proof of $\ell^r \subset \ell^p$ is similar to Example 2.3.8. Also, for each $x = \{x_n\} \in \ell^p$ $(p > 1)$ we see that $\{x_n\}$ is bounded since the series $\sum\limits_{n=1}^{\infty} |x_n|^p$ is convergent, which means that $x \in \ell^\infty$. Moreover, $\|x\|_{\ell^\infty} = \sup\limits_{n\geqslant 1} |x_n| = |x_{n_0}|$ for some $n_0 \in \mathbb{N}$ since $|x_n| \to 0$ as $n \to \infty$. Hence

$$\|x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \geqslant |x_{n_0}| = \|x\|_{\ell^\infty},$$

which yields that $\liminf\limits_{p\to\infty} \|x\|_{\ell^p} \geqslant \|x\|_{\ell^\infty}$. On the other hand, for each fixed $p > 1$, by the convergence of the series $\sum\limits_{n=1}^{\infty} |x_n|^p$ we see that there exists $N \in \mathbb{N}$ such that $\sum\limits_{n=N+1}^{\infty} |x_n|^p < \|x\|_{\ell^\infty}^p$. So,

$$\begin{aligned}\|x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} &= \left(\sum_{n=1}^{N} |x_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p\right)^{1/p} \\ &\leqslant \left(N|x_{n_0}|^p + \|x\|_{\ell^\infty}^p\right)^{1/p} = (N+1)^{1/p}\, \|x\|_{\ell^\infty},\end{aligned}$$

this infers that $\limsup\limits_{p\to\infty} \|x\|_{\ell^p} \leqslant \|x\|_{\ell^\infty}$. In summary, we get the conclusion.

Remark. Let $p > 1$. As a consequence of Example 2.3.8 we have $\ell^p \subsetneqq \ell^q$ if and only if $p < q$. In fact, if $p < q$, then $\ell^p \subsetneqq \ell^q$ holds by Example 2.3.8. Conversely, if $\ell^p \subsetneqq \ell^q$, then we must have $p < q$ since otherwise we obtain $\ell^q \subset \ell^p$, which leads a contradiction $\ell^p = \ell^q$.

2.26 (✍) Let $1 \leqslant p < \infty$ and $x = \{x_n\} \in \ell^p$, i.e., $\sum\limits_{n=1}^{\infty} |x_n|^p$ converges. Then $|x_n| \to 0$ as $n \to \infty$ and therefore $x = \{x_n\} \in c_0$. Choose $x = (1, 1/\ln 2, \cdots, 1/\ln n, \cdots)$. Obviously, $x \in c_0$ since $1/\ln n \to 0$ as $n \to \infty$. We claim that this x does not belong any ℓ^p for $1 \leqslant p < \infty$. Indeed, if there would exist $1 \leqslant p < \infty$ such that $x \in \ell^p$, then we could have $1 + \sum\limits_{n=2}^{\infty} 1/(\ln n)^p < \infty$. From the well-known Cauchy test we could obtain that the series $\sum\limits_{n=2}^{\infty} 2^n/\left(\ln 2^n\right)^p = \sum\limits_{n=2}^{\infty} 2^n/(n \ln 2)^p$ converges, which is false, since $\lim_{n\to\infty} 2^n/n^p = \infty$ if $1 \leq p < \infty$.

2.30 (✍) By the Exercise 2.4, $(Z, \|\cdot\|)$ is a normed linear space. For every Cauchy sequence $\{(x_n, y_n)\}$ in Z, it is easy to see that $\{x_n\}$ is Cauchy in X and $\{y_n\}$ is Cauchy in Y by the definition of the norm $\|\cdot\|$. Since $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are Banach spaces, we have $\{x_n\}$ and $\{y_n\}$ are convergent to some $x \in X$ and some $y \in Y$ as $n \to \infty$, respectively. Hence $\{(x_n, y_n)\}$ is convergent to (x, y) again by the definition of the norm $\|\cdot\|$, so that Z is a Banach space.

2.33 (✍) If $y = 0$, then $\varphi(t) = \|x\|$ for all $t \in \mathbb{R}$ and $\inf_{t\in\mathbb{R}} \varphi(t) = \varphi(0)$. We now assume that $y \neq 0$. Let $m = \inf_{t\in\mathbb{R}} \varphi(t)$, then $m \in [0, \infty)$. By the definition of the infimum we obtain a sequence $\{t_n\} \subset \mathbb{R}$ of real numbers such that $\varphi(t_n) \to m$ as $n \to \infty$. Then, for all $n \in \mathbb{N}$

$$\|t_n y\| \leqslant \|x - t_n y\| + \|x\| = \varphi(t_n) + \|x\| \leqslant 2m + \|x\|,$$

which gives that $|t_n| \leqslant (2m + \|x\|)/\|y\|$ for all $n \in \mathbb{N}$, i.e., $\{t_n\}$ is bounded in $\mathbb{R}$. Consequently, by the Balzano-Weierstrass theorem, $\{t_n\}$ has a convergent subsequence $\{t_{n_j}\}$, say, $t_{n_j} \to t_0$ for some $t_0 \in \mathbb{R}$, as $j \to \infty$. Hence $\varphi(t_{n_j}) \to \varphi(t_0)$ as $j \to \infty$ since φ is clearly continuous, so that $m = \varphi(t_0)$.

2.34 (✍) For every $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, since

$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} \{|x_i|\} \leqslant n \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} = n\|x\|_2$$

and

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \leqslant n^{1/2} \max_{1\leqslant i\leqslant n} \{|x_i|\} \leqslant n\|x\|_\infty,$$

we get that $(1/n)\|x\|_2 \leqslant \|x\|_\infty \leqslant n\|x\|_2$, which shows that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

2.35 (✍)

(i) Consider $f_n(t) = t^n$ for all $t \in [0, 1]$ and $n \in \mathbb{N}$. Clearly each $f_n \in P[0, 1]$ and

$$\|f_n\|_\infty = \sup_{t\in[0,1]} |f_n(t)| = 1, \quad \|f_n\|_1 = \int_0^1 |f_n(t)| \,\mathrm{d}t = 1/(n + 1),$$

we see that $\inf_{n\in\mathbb{N}} (\|f_n\|_1/\|f_n\|_\infty) = 0$ since

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \|f_n\|_1 = \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Which means that there is no positive constant k such that

$$\|f_n\|_1/\|f_n\|_\infty \geqslant k, \quad \text{or} \quad \|f_n\|_1 \geqslant k\|f_n\|_\infty \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\|f\|_\infty$ and $\|f\|_1$ are not equivalent.

(ii) By Theorem 2.4.3 we know that all norms on a finite-dimensional linear space are equivalent. Hence, $P[0,1]$ must be infinite-dimensional since otherwise the above norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ would be equivalent.

2.36 (✍) Clearly, the set $Z = \{A = (a_{jk}) : j = 1, \cdots, m, k = 1, \cdots, n\}$ constitute a linear space under the two algebraic operations of matrix addition and matrix multiplication by scalars. For $j = 1, \cdots, m$, $k = 1, \cdots, n$ let $E_{jk} = (\xi_{jk})$ with ξ_{jk} having 1 in the place of jth row and kth column and zeros elsewhere. Clearly, these mn matrices E_{jk} are linearly independent in Z, hence $\dim(Z) \geqslant mn$. Since any matrix $A = (a_{jk})_{m\times n}$ in Z has a unique representation as a linear combination of the matrices E_{jk}: $A = \sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n} a_{jk}E_{jk}$ by the linear independence of E_{jk}'s, we see that any set of $mn+1$ or more matrices of Z is linearly dependent. Therefore, $\dim(Z) \leqslant mn$, so that $\dim(Z) = mn$, thus Z is an mn-dimensional linear space with a basis $\{E_{jk} : j = 1, \cdots, m, k = 1, \cdots, n\}$.

Since Z is a finite-dimensional linear space, we see form Theorem 2.4.3 that all norms on Z are equivalent. Moreover,

$$\|A\|_1 = \sum_{j=1}^{m}\sum_{k=1}^{n} |a_{jk}|, \quad \|A\|_2 = \left(\sum_{j=1}^{m}\sum_{k=1}^{n} |a_{jk}|^2\right)^{1/2}$$

and $\|A\|_\infty = \max\{|a_{jk}| : j = 1, \cdots, m, k = 1, \cdots, n\}$.

2.37 (✍)

i) Let $\delta = d(x, Y) = \inf\limits_{y\in Y} \|x - y\|$. Then, by the definition of the infimum we see that there exists a sequence $\{y_n\} \subset Y$ such that $\|x - y_n\| \to \delta$ as $n \to \infty$. It follows that the sequence $\{x - y_n\}$ is bounded in X, and therefore, $\{y_n\} \subset Y$ is bounded, too, by the triangle inequality for the norm of X, where we recognize the linear space Y as a normed linear subspace of X. Since Y is finite-dimensional, by the Bolzano-Weierstrass theorem, we obtain that there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to t_0 \in Y$ as $k \to \infty$, thus $\|x - y_{n_k}\| \to \|x - y_0\|$ as $k \to \infty$. Hence $\delta = \|x - y_0\|$.

ii) The answer is no. Indeed, for every $n \in \mathbb{N}$ set

$$E_n = \operatorname{span}(\{e_1, \cdots, e_n\})$$

where $e_n = \{\delta_{nj}\}$ having 1 in the nth place and zeros elsewhere. Let $x_n = e_{n+1}$ for each $n \in \mathbb{N}$, then each $E_n \subset \ell^\infty$ and $x_n \in \ell^\infty$. For each $n \in \mathbb{N}$ we choose numbers $\alpha_1, \cdots, \alpha_n$ such that $|\alpha_i| \leqslant 1$, $i = 1, 2, \cdots, n$, and let $y_n = (\alpha_1, \cdots, \alpha_n, 0, 0, \cdots)$, then each $y_n \in E_n$. Clearly, the choice of y_n ia not unique. However, it is easy to verify that $d(x_n, E_n) = 1 = \|x_n - y_n\|_{\ell^\infty}$ for each $n \in \mathbb{N}$.

2.39 (✍) For every $n \in \mathbb{N}$ we have $x_n \in X$ since $\|x_n\| = \left(\int_0^1 |x_n(t)|^2 \,\mathrm{d}t|\right)^{1/2} = 1 - 2/3n < 1$. By the definition we see that, each $x_n(0) = 0$, and so, each $x_n \in S$.

2.40 (✍) For each $n \in \mathbb{N}$ we consider $\xi_n = \{x_{n,i}\}$ such that $x_{n,i} = i^{-4}$ for $i \leqslant n$ and $x_{n,i} = 0$ for $i > n$. Clearly, each $\xi_n \in \ell_0^2$ and $\xi_n \to \xi = \{i^{-4}\} \in \ell^2$ in the ℓ^2-norm as $n \to \infty$, but $\xi = \{i^{-4}\} \notin \ell_0^2$, so that ℓ_0^2 is not closed in ℓ^2.

2.41 (✍) This is a consequence of Example 1.5.7 and Theorem 1.5.1.

2.44 (✍)

(i) Since $\|\eta x/2\|x\|\| = \eta/2 < \eta$, it follows that $\eta x/2\|x\| \in \{y \in X : \|y\| < \eta\} \subset Y$.

(ii) Clearly that $0 \in Y$. Since Y is open in X, we see that there exists a real number $\eta > 0$ such that $B(0, \eta) = \{y \in X : \|y - 0\| < \eta\} \subset Y$, so that $\dfrac{\eta x}{2\|x\|} \in Y$ for every $x \in X \setminus \{0\}$ by (i). Clearly, Hence, for each for every $x \in X \setminus \{0\}$ we have $x \in \operatorname{span}(\{\eta x/2\|x\|\}) \subset \operatorname{span}(Y) = Y$ since Y is a linear space. Trivially, if $x \in X$ with $x = 0$, then $x \in Y$. Thus, $X \subset Y$, so that $X = Y$ by the assumption that $Y \subset X$.

2.45 (✍)

(i) Since $\|\cdot\|$ is a continuous mapping on X and $[0, 1]$ is closed, we obtain that $T = \{x \in X : \|x\| \leqslant 1\} = \{x \in X : \|x\| \in [0, 1]\}$ is closed.

(ii) Since $\|x_n - x\| = \|-x/n\| \leqslant 1/n \to 0$ as $n \to \infty$, it follows that $x_n \to x$ as $n \to \infty$. Note that for each $n \in \mathbb{N}$ we have $\|x_n\| = \|(1 - 1/n)x\| \leqslant (1 - 1/n) < 1$, which means that the sequence $\{x_n\} \subset S$, and the x is a limit point of S. Hence $T \subset \overline{S}$, so that $\overline{S} = T$ since $S \subset T$ and T is closed by (i).

2.46 (✍) *Necessity.* If X is complete, then the completeness of S follows from the closedness of S in X by Theorem 1.5.1.

Sufficiency. Suppose that S is complete. Let $\{x_n\}$ be an arbitrary Cauchy sequence of X. Since $|\|x_n\| - \|x_m\|| < \|x_n - x_m\|$ for every $n, m \in \mathbb{N}$, we get that $\{\|x_n\|\}$ is a Cauchy sequence in $\mathbb{R}$, hence $\{\|x_n\|\}$ is convergent to some nonnegative number c as $n \to \infty$.

If $c = 0$, then $x_n \to 0$ by the continuous of the norm $\|\cdot\|$, so that the Cauchy sequence $\{x_n\}$ converges, and hence X is complete.

If $c > 0$, then there exists an $N_1 > 0$ such that $c/2 < \|x_n\| < 3c/2$ for all $n > N_1$. Since $\{x_n\}$ is a Cauchy sequence of X, for each $\varepsilon > 0$ there exists an $N > N_1$ such that $\|x_n - x_m\| < \varepsilon$ for all $m, n > N$. Therefore

$$\begin{aligned}
&\left\| \frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|} \right\| = \left\| \frac{\|x_m\| x_n - \|x_n\| x_m}{\|x_n\|\|x_m\|} \right\| \\
=&\frac{4}{c^2} \|(x_n - x_m)\|x_m\| + x_m(\|x_m\| - \|x_n\|)\| \\
\leqslant&\frac{4}{c^2} (\|x_n - x_m\|\|x_m\| + \|x_m\||\|x_m\| - \|x_n\||) \leqslant \frac{4}{c^2}\frac{3c}{2}\varepsilon = \frac{6}{c}\varepsilon
\end{aligned}$$

for all $m, n > N$, which means that $\{x_n/\|x_n\|\}$ is a Cauchy sequence in S, hence it must converge to some $x \in S$ as $n \to \infty$ since S is complete. Equivalently $x_n \to c\,x \in X$ as $n \to \infty$, showing the completeness of X.

2.47 (✍) Suppose that Y is an n-dimensional linear subspace of X, then Y is closed in X by Corollary 2.4.2. Similarly as in the proof of the Riesz lemma, we have an $x \in X \backslash Y$ such that $d(x, Y) > 0$. By Exercise 2.37 we see that there exists a $y_0 \in Y$ such that $\|x - y_0\| = d(x, Y) > 0$. Let $x_1 = \dfrac{x - y_0}{\|x - y_0\|}$, then $x_1 \in X$ with $\|x_1\| = 1$ and for $y \in Y$ we obtain that

$$\begin{aligned}
\|x_1 - y\| &= \frac{1}{\|x - y_0\|} \|x - y_0 - \|x - y_0\| y\| \\
&= \frac{1}{d(x, Y)} \|x - (y_0 + d(x, Y) y)\| \geqslant \frac{1}{d(x, Y)} d(x, Y) = 1
\end{aligned}$$

since $y_0 + d(x, Y) y \in Y$, that is, $d(x_1, Y) \geqslant 1$. In particular, $d(x_1, Y) = 1$ since $d(x_1, Y) \leqslant d(x_1, 0) = \|x_1\| = 1$.

2.56 (✍) X/N is set of all lines parallel to the ξ_1-axis since for every $x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$ we have

$$\pi(x) = x + N = \{x + y : y \in N\} = \{(z_1, x_2, x_3) : z_1 \in \mathbb{R}\},$$

maening that every element $\pi(x)$ of X/N is a line in $\mathbb{R}^3$ which passes point (x_1, x_2, x_3) and is parallel to ξ_1-axis. Similarly, if M is a plane of $X = \mathbb{R}^3$ passing the origin (by the way, a subspace of $\mathbb{R}^3$ must be one of three cases: a line passing the origin, a plane passing the origin and $\mathbb{R}^3$ itself), we see that X/M is the set of all planes parallel to the plane M. Therefore, $X/X = \{0\}$ and $X/\{0\} = X$.

2.57 (✍) Note that N is itself a linear space, so for example $N+N = N$ and $\alpha N = N$ for all $\alpha \neq 0$. Suppose that $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. Then there are $x_0, y_0 \in N$ such that $x - x' = x_0$ and $y - y' = y_0$ by (2) of Remark 2.6.1. Hence, for each $z \in \pi(x)+\pi(y)$ we have a $z_0 \in N$ such that $z = x+y+z_0$ since $x+N+y+N = x+y+N$, which implies that $z = x'+x_0+y'+y_0+z_0 \in \pi(x')+\pi(y')$, thus $\pi(x)+\pi(y) \subset \pi(x')+\pi(y')$, and vice versa, of course. Therefore, the first equality in (2.15) holds. Similarly, the second equality also holds since $\alpha N = N$ for all $\alpha \neq 0$.

2.58 (✍) The sketch maps are in the following.