Portfolio Theory
Solutions to Tutorial 3

1. Set $X^* := X$, $Y^* := F_Y^{-1}(F_X(X^*))$, then for all x,

$$\begin{aligned} P(Y^* \le x) &= P(F_Y^{-1}(F_X(X^*)) \le x) \\ &= P(F_X(X^*) \le F_Y(x)) \\ &= P(X^* \le F_X^{-1}(F_Y(x))) \\ &= F_X(F_X^{-1}(F_Y(x))) \\ &= F_Y(x) = P(Y \le x). \end{aligned}$$

Hence, X and Y have the same distributions as X^* and Y^*. Moreover, we have

$$\begin{aligned} P(X^* \ge Y^*) &= P(X^* \ge F_Y^{-1}(F_X(X^*))) \\ &= P(F_Y(X^*) \ge F_X(X^*)) \quad = \quad 1. \end{aligned}$$

2. Let $X^* = aX + Z$, $Y^* = aY + Z$. As X, Y and Z are independent, for all x,

$$\begin{aligned} \mathrm{P}(X^* \le x) &= \mathrm{P}(aX + Z \le x) \\ &= \mathrm{E}(\mathrm{P}(aX + Z \le x|Z)) \\ &= \mathrm{E}\left(\mathrm{P}\left(X \le \frac{x-Z}{a}\Big|Z\right)\right) \\ &\le \mathrm{E}\left(\mathrm{P}\left(Y \le \frac{x-Z}{a}\Big|Z\right)\right) \\ &= \mathrm{E}(\mathrm{P}(aY + Z \le x|Z)) \quad = \quad \mathrm{P}(Y^* \le x). \end{aligned}$$

Thus, we have $X \ge_{sd} Y \Rightarrow (aX + Z) \ge_{sd} (aY + Z)$.

3. (a) From the given condition, $f^{-1}(x)$ exists and is well defined. For all x, we have

$$\begin{aligned} \mathrm{P}(f(X) \le x) &= \mathrm{P}(X \le f^{-1}(x)) \\ &\le \mathrm{P}(Y \le f^{-1}(x)) \\ &= \mathrm{P}(f(Y) \le x). \end{aligned}$$

Therefore, $X \ge_{sd} Y \Rightarrow f(X) \ge_{sd} f(Y)$.

(b) Utility function must be a strictly increasing function, hence,

$$X \ge_{sd} Y \Rightarrow u(X) \ge_{sd} u(Y) \Rightarrow \mathrm{E}(u(X)) > \mathrm{E}(u(Y)).$$

4. Note that the distribution is symmetric about its mean, which is 100.

(a)

$$\begin{aligned} \mathrm{SV}(X) &= \int_{20}^{100} (x-100)^2(x-20)/80^2\,dx \\ &= \frac{1}{80^2}\int_{20}^{100} (x^3 - 220x^2 + 14000x - 200000)\,dx = 533.33 \end{aligned}$$

(b) By symmetry, $\mathrm{Var}(X) = 2\mathrm{SV}(X) = 1066.66$.

(c)

$$P(X \leq 30) = \int_{20}^{30} (x-20)/80^2\, dx = 0.0078.$$

(d)

$$\begin{aligned} \text{ESF} &= \int_{20}^{30} (30-x)(x-20)/80^2\, dx \\ &= \frac{1}{80^2} \int_{20}^{30} (-x^2+50x-600)\, dx = 0.026 \end{aligned}$$

(e) The loss is $Z = 90 - X$, so the 95% VaR is the number z so that $P(Z > z) = P(X < 90 - z) = 0.05$. Putting $y = 90 - z$ and solving

$$0.05 = \int_{20}^{y} (x-20)/80^2\, dx = \frac{1}{80^2}\left[\frac{x^2}{2} - 20x\right]_{20}^{y}$$

gives

$$\frac{y^2}{2} - 20y - 120 = 0 \quad \Rightarrow y = 45.3$$

Hence 95% VaR is $z = 90 - y = 44.7$. The CTE is then given by

$$\begin{aligned} E[90-X|90-X > 44.7] &= E[90-X|X < 45.3] \\ = \frac{1}{0.05}\int_{20}^{45.3} &(90-x)(x-20)/80^2\, dx = 53.14 \end{aligned}$$

5. The density function of A is $f_A(x) = 1/0.06$ for $1.04 < x < 1.1$.

(a) Clearly $E(A) = 1.07$.

$$E(B) = (1.12 + 1.1 + 1.08)/5 + 2 \times 1.03/5 = 1.072$$

(b)

$$\begin{aligned} \text{Var}(A) &= \int_{1.04}^{1.1} x^2/0.06dx - 1.07^2 = \left[\frac{x^3}{0.18}\right]_{1.04}^{1.1} - 1.07^2 = 0.0003 \\ \text{Var}(B) &= (1.12^2 + 1.1^2 + 1.08^2)/5 + 2 \times 1.03^2/5 - 1.072^2 = 0.001336 \end{aligned}$$

(c) Since A has a symmetric distribution about the mean of 1.07, $\text{SV}(A) = \text{Var}(A)/2 = 1.5 \times 10^{-4}$.

$$\text{SV}(B) = 2(1.03 - 1.072)^2/5 = 7.056 \times 10^{-4}$$

(d) $P(A < 1.06) = (1.06 - 1.04)/0.06 = 1/3$, $P(B < 1.06) = 2/5$

(e)

$$\begin{aligned} \text{ESF}(A) &= \int_{1.04}^{1.05} (1.05-x)/0.06dx = \left[-\frac{(1.05-x)^2}{0.12}\right]_{1.04}^{1.05} = 0.00083 \\ \text{ESF}(B) &= 2(1.05 - 1.03)/5 = 0.008 \end{aligned}$$