# 期权定价的数学模型和方法(期末)

2021421300 3　　丁一.

1. $K_1 > K_2$. (a) $\forall 0 \le \lambda \le 1$, $K_\lambda = \lambda K_1 + (1-\lambda)K_2$,

有 $P_t(K_\lambda) \le \lambda P_t(K_1) + (1-\lambda)P_t(K_2)$, $t \le \tau_1 \le T$. a.s.

(b) $0 < P_t(K_1) - P_t(K_2) < K_1 - K_2$. $t \le \tau_1 \le T$. a.s.

证: (a) 即证 $\lambda[P_t(K_1) - P_t(K_\lambda)] + (\lambda-1)[P_t(K_\lambda) - P_t(K_2)] \geqslant 0$. (*)

((b)也证了) 由作业2可知. $0 < P_t(S, K_1) - P_t(S, K_2)$　　as $\forall t < \tau_1 \le T$. $(K_1 > K_2)$.

又 $K_1 > K_\lambda > K_2$

故(*)式成立.

(b) 左边: $\Phi_1 = P(K_1)$, $\Phi_2 = P(K_2)$

若不提前, $\tau_1 = \tau_2 = T$, $V_T(\Phi_1) = (K_1 - S_T)^+ \geqslant (K_2 - S_T)^+ = V_T(\Phi_2)$

$\mathrm{Prob}\{V_T(\Phi_1) > V_T(\Phi_2)\} = \mathrm{Prob}\{S_T < K_1\} > 0$

由无套利原理知 $V_t(\Phi_1) > V_t(\Phi_2)$ a.s. $\forall t < \tau_1$

提前, $\tau_1 \le \tau_2 \le T$, $V_{\tau_1}(\Phi_1) = K_1 - S_{\tau_1} > 0$　　, 由 $K_1 > K_2$

$V_{\tau_1}(\Phi_2) = (K_2 - S_{\tau_1})^+$　　故 $V_{\tau_1}(\Phi_1) > V_{\tau_1}(\Phi_2)$

又 $\mathrm{Prob}\{V_{\tau_1}(\Phi_1) > V_{\tau_1}(\Phi_2)\} = 1$　　$\therefore P_t(K_1) > P_t(K_2)$ a.s. $\forall t < \tau_1$.

右边: $\Phi_1 = -P_t(K_1) + K_1$, $\Phi_2 = -P_t(K_2) + K_2$

则在 $\tau_1$ 时刻, $V_{\tau_1}(\Phi_1) = -(K_1 - S_{\tau_1})^+ + K_1 e^{r(\tau_1 - t)}$

$$V_{\tau_1}(\Phi_2) = -(K_2 - S_{\tau_1})^+ + K_2 e^{r(\tau_1 - t)}$$

$\therefore V_{\tau_1}(\Phi_1) \geqslant V_{\tau_1}(\Phi_2)$

又 $\mathrm{Prob}\{V_{\tau_1}(\Phi_1) > V_{\tau_1}(\Phi_2)\} \geqslant \mathrm{Prob}\{K_2 < S_{\tau_1}\} > 0$

$\therefore V_t(\Phi_1) > V_t(\Phi_2)$

$\therefore -P_t(K_1) + K_1 > -P_t(K_2) + K_2$ a.s $\forall t \le \tau_1 \le T$

2.

(a) $\bar{C}_t \begin{array}{l} \overset{p}{\nearrow} \ 1 \\ \underset{1-p}{\searrow} \ 0 \end{array}$ $\quad$ $\bar{P}_t \begin{array}{l} \overset{p}{\nearrow} \ 0 \\ \underset{1-p}{\searrow} \ 1 \end{array}$ $\quad$ $S_0 \begin{array}{l} \overset{p}{\nearrow} \ S_0u \\ \underset{1-p}{\searrow} \ S_0d. \end{array}$

$$\bar{P}_0 = \frac{1}{1+r\Delta t} E^{P}(V_{t_0+\Delta t}) = \frac{(1-p)}{1+r\Delta t}$$

令 $\Pi = \bar{P} - \Delta S$ , $0 - \Delta S_0u = 1 - \Delta S_0 d = (\bar{P}_0 - \Delta S_0) \times \underbrace{(1+r\Delta t)}_{\rho}.$

$$\therefore \Delta = \frac{-1}{(u-d)S_0}$$

$$\bar{P}_0 = \frac{u-\rho}{\rho(u-d)}$$

(b) 同上, $\bar{C}_0 = \dfrac{\rho-d}{\rho(u-d)}$

令 $S_t = \alpha \bar{C}_t + \beta \bar{P}_t$

则 $\begin{cases} S_0 = \alpha \cdot \dfrac{\rho-d}{\rho(u-d)} + \beta \dfrac{u-\rho}{\rho(u-d)} \\ S_0u = \alpha \\ S_0d = \beta \end{cases}$

$$\therefore S_t = S_0u \cdot \bar{C}_t + S_0d \cdot \bar{P}_t$$

3. $dS_i(t)/S_i = \mu_i dt + \sigma_i dW_i(t)$ , $cov(dW_i, dW_j) = \rho_{ij} dt \ (i \neq j)$

$V = V(S_1 \cdots S_n)$ , 求 $dV$.

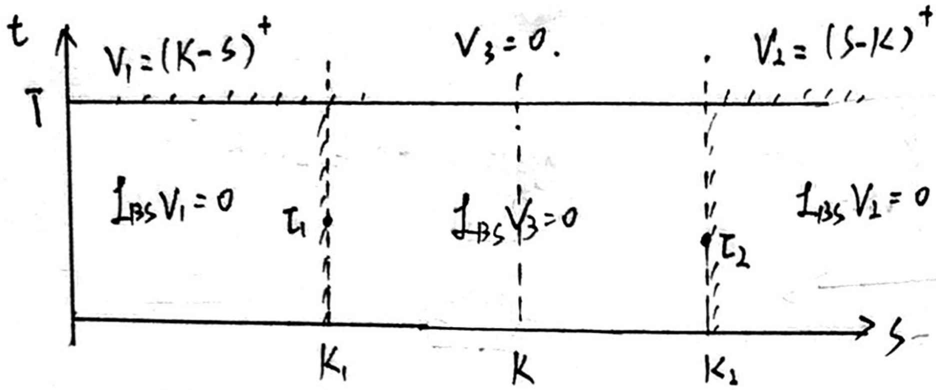
$$dV = \frac{\partial V}{\partial t} dt + \sum_i \frac{\partial V}{\partial S_i} dS_i + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 V}{\partial S_i \partial S_j} dS_i dS_j.$$

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i(t).$$

$$\therefore dV = \left[\frac{\partial V}{\partial t} + \sum_i \mu_i S_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_i (\sigma_i S_i)^2 \frac{\partial^2 V}{\partial S_i^2} + \frac{1}{2} \sum_{i \neq j} \rho_{ij} \sigma_i S_i \times \sigma_j S_j \frac{\partial^2 V}{\partial S_i \partial S_j}\right] dt$$

$$+ \sum_i \sigma_i S_i \frac{\partial V}{\partial S_i} dW_i$$

4.




$$V(S,t)=E\Big[e^{-r(T-t)}(K-S_T)^+ * I\{\tau_1<\tau_2\le T\}$$
$$+\,e^{-r(T-t)}(S_T-K)^+ * I\{\tau_2<\tau_1\le T\}\Big]$$

$$\tau_1=\inf\{\mu,\ S_\mu\le K_1\}\wedge T$$

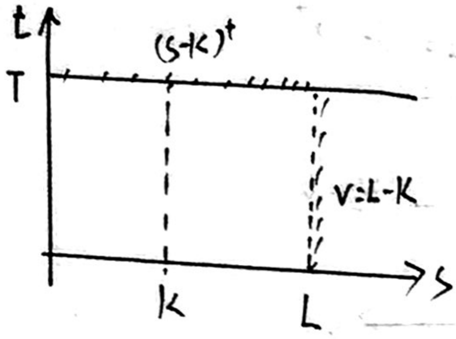
$$\tau_2=\inf\{\mu,\ S_\mu\ge K_2\}\wedge T$$

相准1
$$\begin{cases}\mathcal{L}_{BS}V_1=\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0, & 0\le S\le K_1,\ 0\le t\le T.\\ V_1(S,T)=(K-S)^+, & 0\le S\le K_1\end{cases}$$

相准2
$$\begin{cases}\mathcal{L}_{BS}V_2=0, & K_2\le S<+\infty,\ 0\le t\le T\\ V_2(S,T)=(S-K)^+, & K_2\le S<+\infty.\end{cases}$$

$$\begin{cases}\mathcal{L}_{BS}V_3=0, & K_1\le S\le K_2,\ 0\le t\le T\\ V_3(S,T)=0, & K_1\le S\le K_2\\ V_3(K_1,t)=V_1(K_1,t), & 0\le t\le T\\ V_3(K_2,t)=V_2(K_2,t), & 0\le t\le T.\end{cases}$$

5.




$$\text{Payoff} = (\min(S_t, L) - K)^+$$

$$\begin{cases} \min\{-\mathcal{L}V,\ V-(S-K)^+\}=0 & ,0\le S\le L,\ 0\le t\le T \\ V(L,t)=L-K, & 0\le t\le T \\ V(S,T)=(S-K)^+ & ,0\le S\le L \end{cases}$$

隐式差分.

$$\min\{-\delta_t^+ V_i^n - \frac{\sigma^2}{2}\delta_x^2 V_i^n - (r-q-\frac{\sigma^2}{2})\delta_x^0 V_i^n,\ V_i^n - (S_i^n-K)^+\}=0.\quad i=-M+1\cdots M-1$$
$$n=N-1\cdots 0.$$

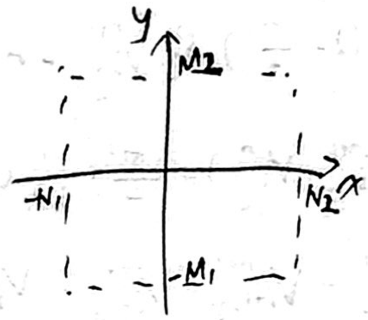
$$\therefore \min\left\{\frac{V_i^n - V_i^{n+1}}{\Delta t} - \frac{\sigma^2}{2}\times\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2} - (r-q-\frac{\sigma^2}{2})\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x},\ V_i^n - (S_i^n-K)^+\right\}=0.$$

$$\therefore \min\left\{\underbrace{(\frac{1}{\Delta t}+\frac{\sigma^2}{\Delta x^2})\Delta t}_{\beta} V_i^n + \underbrace{\Delta t\times(\frac{-\sigma^2}{2\Delta x^2}+\frac{r-q-\frac{\sigma^2}{2}}{2\Delta x})}_{\alpha} V_{i-1}^n + \underbrace{\Delta t\times(\frac{-\sigma^2}{2\Delta x^2}-\frac{r-q-\frac{\sigma^2}{2}}{2\Delta x})}_{\gamma} V_{i+1}^n - V_i^{n+1},\ V_i^n-(S_i^n-K)^+\right\}=0.$$

$$\alpha+\beta+\gamma=1$$

$$\therefore \begin{cases} V_i^n = \max\{\frac{1}{\beta}(V_i^{n+1}-\alpha V_{i-1}^n - \gamma V_{i+1}^n),\ (S_i^n-K)^+\} & ,i=-M+1\cdots M-1 \\ & n=N-1\cdots 0 \\ V_M^n = L-K, & n=0\cdots N \\ V_i^N = (S_i^n-K)^+, & i=-M\cdots M. \end{cases}$$

6. $$\begin{cases} V_t + \frac{1}{2}(\sigma_1^2 V_{xx} + \rho\sigma_1\sigma_2 V_{xy} + \sigma_2^2 V_{yy}) + (r - \frac{\sigma_1^2}{2}) V_x + (r - \frac{\sigma_2^2}{2}) V_y - rV = 0, & (x,y) \in R^2, t \le T \\ V(x,y,T) = \max(e^x, e^y). \end{cases}$$




边界: $V(-N_1, -M_1, t) = 0$

$V(-N_1, M_2, t) = e^{M_2}$

$V(N_2, -M_1, t) = e^{N_2}$

$V(N_2, M_2, t) = \max(e^{N_2}, e^{M_2})$

在 $\{-N_1 \le x \le N_2,\ -M_1 \le y \le M_2,\ 0 \le t \le T\}$ 上形成网格 $\{ih_1, jh_2, n\Delta t\}$.

离散: $\frac{\partial V}{\partial t} = \frac{1}{\Delta t}(V_{i,j}^{n+1} - V_{i,j}^{n})$

$$\frac{\partial V}{\partial x} = \frac{1}{2h_1}(V_{i+1,j} - V_{i-1,j})$$

$$\frac{\partial V}{\partial y} = \frac{1}{2h_2}(V_{i,j+1} - V_{i,j-1})$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{h_1^2}(V_{i+1,j} - 2V_{i,j} + V_{i-1,j}), \quad \frac{\partial^2 V}{\partial y^2} = \frac{1}{h_2^2}(V_{i,j+1} - 2V_{i,j} + V_{i,j-1})$$

考虑 $\rho > 0$. 待定系数法, 七点差分,

设 $\frac{\partial^2 u}{\partial x \partial y}\Big|_{(0,0)} = a_0 u_{0,0} + a_1 u_{-1,0} + a_2 u_{1,0} + a_3 u_{0,-1} + a_4 u_{0,1} + a_5 u_{-1,-1} + a_6 u_{1,1}$

对 $u = 1, x, y, x^2, xy, y^2$ 均成立.

$\Rightarrow \quad a_0 + \cdots + a_6 = 0$

$$\left.\begin{aligned} -h_1 a_1 + h_2 a_2 - h_1 a_5 + h_1 a_6 = 0 \\ -h_2 a_3 + h_2 a_4 - h_2 a_5 + h_2 a_6 = 0 \end{aligned}\right\} a_2 - a_1 = a_4 - a_3$$

$$\left.\begin{aligned} h_1^2 (a_1 + a_2 + a_5 + a_6) = 0 \\ h_2^2 (a_3 + a_4 + a_5 + a_6) = 0 \end{aligned}\right\} a_1 + a_2 = a_4 + a_3 = -\frac{1}{h_1 h_2}$$

$h_1 h_2 (a_5 + a_6) = 1 \quad \Rightarrow a_5 + a_6 = \frac{1}{h_1 h_2}$

假设 $a_1 = a_2$, 则 $a_1 = a_2 = a_3 = a_4 = -\frac{1}{2h_1 h_2}$, $a_5 = a_6 = \frac{1}{2h_1 h_2}$, $a_0 = \frac{1}{h_1 h_2}$

$$\therefore \delta_{xy}^{+} V|_{i,j} = \frac{1}{2h_1 h_2}(V_{i+1,j+1} + V_{i-1,j-1} + 2V_{i,j} - V_{i+1,j} - V_{i-1,j} - V_{i,j+1} - V_{i,j-1})$$

PDE: $\delta_t^- V_{i,j}^{n+1} + \frac{\sigma_1^2}{2}\delta_x^2 V_{i,j}^{n+1} + \frac{1}{2}\rho\sigma_1\sigma_2\delta_{xy}^+ V_{i,j}^{n+1} + \frac{\sigma_2^2}{2}\delta_y^2 V_{i,j}^{n+1} + (r-\frac{\sigma_1^2}{2})\delta_x^0 V_{i,j}^{n+1}$

$+ (r-\frac{\sigma_2^2}{2})\delta_y^0 V_{i,j}^{n+1} - rV_{i,j}^{n} = 0.$

$\Rightarrow \frac{1}{\Delta t}(V_{i,j}^{n+1} - V_{i,j}^{n}) + \frac{\sigma_1^2}{2}\times\frac{1}{h_1^2}(V_{i+1,j}^{n+1} - 2V_{i,j}^{n+1} + V_{i-1,j}^{n+1}) + \frac{1}{2}\rho\sigma_1\sigma_2\times\frac{1}{2h_1h_2}(V_{i+1,j+1}^{n+1} + V_{i-1,j-1}^{n+1}$

$+2V_{i,j}^{n+1} - V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1} - V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}) + \frac{\sigma_2^2}{2}\times\frac{1}{h_2^2}(V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1})$

$+(r-\frac{\sigma_1^2}{2})\frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2h_1} + (r-\frac{\sigma_2^2}{2})\frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2h_2} - rV_{i,j}^{n} = 0.$

$\therefore V_{i,j}^{n}(1+r\Delta t) = \alpha V_{i,j}^{n+1} + \beta_+ V_{i+1,j}^{n+1} + \beta_- V_{i-1,j}^{n+1} + \nu_+ V_{i,j+1}^{n+1} + \nu_- V_{i,j-1}^{n+1} + \mu_+ V_{i+1,j+1}^{n+1}$

$+\mu_- V_{i-1,j-1}^{n+1}$

$\alpha = (\frac{1}{\Delta t} - \frac{\sigma_1^2}{h_1^2} + \frac{\rho\sigma_1\sigma_2}{2h_1h_2} - \frac{\sigma_2^2}{h_2^2})\Delta t$

$\beta_+ = [\frac{\sigma_1^2}{2h_1^2} - \frac{\rho\sigma_1\sigma_2}{4h_1h_2} + \frac{1}{2h_1}(r-\frac{\sigma_1^2}{2})]\Delta t$

$\beta_- = [\frac{\sigma_1^2}{2h_1^2} - \frac{\rho\sigma_1\sigma_2}{4h_1h_2} - \frac{1}{2h_1}(r-\frac{\sigma_1^2}{2})]\Delta t$

$\nu_+ = [\frac{\sigma_2^2}{2h_2^2} - \frac{\rho\sigma_1\sigma_2}{4h_1h_2} + \frac{1}{2h_2}(r-\frac{\sigma_2^2}{2})]\Delta t$

$\nu_- = [\frac{\sigma_2^2}{2h_2^2} - \frac{\rho\sigma_1\sigma_2}{4h_1h_2} - \frac{1}{2h_2}(r-\frac{\sigma_2^2}{2})]\Delta t$

$\mu_+ = \frac{\rho\sigma_1\sigma_2}{4h_1h_2}\Delta t$

$\mu_- = \frac{\rho\sigma_1\sigma_2}{4h_1h_2}\Delta t.$

$$\frac{1}{2}\alpha,\ \beta_\pm,\ \nu_\pm,\ \mu_\pm \geq 0 \Rightarrow \begin{cases} \Delta t \leq \dfrac{1}{\frac{\sigma_1^2}{h_1^2} - \frac{\rho\sigma_1\sigma_2}{2h_1h_2} + \frac{\sigma_2^2}{h_2^2}} \\ \frac{1}{h_1}\sigma_1^2 - \frac{1}{2}\rho\sigma_1\sigma_2\frac{1}{h_2} \geq |r-\frac{\sigma_1^2}{2}| \\ \frac{1}{h_2}\sigma_2^2 - \frac{1}{2}\rho\sigma_1\sigma_2\frac{1}{h_1} \geq |r-\frac{\sigma_2^2}{2}| \end{cases}.$$

$$\Rightarrow \begin{cases} h_2 \leq \dfrac{\sigma_2^2(1-\frac{1}{4}\rho^2)}{|r-\frac{\sigma_1^2}{2}|\frac{\rho\sigma_2}{2\sigma_1} + |r-\frac{\sigma_2^2}{2}|} \\ h_1 \leq \dfrac{\sigma_1^2(1-\frac{1}{4}\rho^2)}{|r-\frac{\sigma_2^2}{2}|\frac{\rho\sigma_1}{2\sigma_2} + |r-\frac{\sigma_1^2}{2}|} \end{cases}$$