S.4 Solutions to Exercises Marked with (✍) of Chapter 4

4.1 (✍)

① $\|Tx\| = |x(1)| \leqslant \|x\| \Rightarrow \|T\| \leqslant 1$

(i) By the definition of the standard norm in $C[0,1]$, we have

$$|Tx| = |x(1)| \leqslant \max_{t\in[0,1]} |x(t)| \leqslant \|x\|_{C[0,1]} \quad \text{for all } x \in C[0,1].$$

② 找 $\{x_n(t)\}$. $\|x_n\| = 1$ 但 $\|Tx_n\| = \|x_n(1)\| \to \infty$

Hence T is bounded on $C[0,1]$, with $\|T\| \leqslant 1$.

(ii) T may not be bounded on $C[0,1]$ with respect to the norm $\|x\| = \left(\int_0^1 |f(t)|^2\,\mathrm{d}t\right)^{1/2}$. For example, for each $n \in \mathbb{N}$ let $x_n(t) = t^n$, then each $x_n \in C[0,1]$ and $\|x_n\| = 1/\sqrt{2n+1}$. Hence,

这里是指 $\{x_n\}$

$\|T\| \geqslant \dfrac{\boxed{|x_n(1)|}}{\|x_n\|}$ $\boxed{|x_n(1)|} \to$ 固定, $\|x_n\| \to$ 充分小

$$\frac{|Tx_n|}{\|x_n\|} = \frac{|x_n(1)|}{\|x_n\|} = \sqrt{2n+1} \to \infty \text{ as } n \to \infty,$$

which means that there is no positive constant M such that $|Tx_n| \leqslant M\,\|x_n\|$ for sufficiently large n.

Alternatively, we may use the above example to show that T is not be continuous on $C[0,1]$ with respect the given norm, thus T is unbounded on $C[0,1]$ since T is linear. Indeed, using the given norm, we see that $\{x_n\}$ converges to 0, the zero vector (function) of $C[0,1]$, as $n \to \infty$, but $Tx_n \not\to T0$ as $n \to \infty$ since $|Tx_n - T0| = |x_n(t) - 0| = 1 \not\to 0$ as $n \to \infty$.

4.2 (✍)

$fh \in L^2[0,1] \Leftrightarrow \int_0^1 |f(x)h(x)|^2\,dx < \infty$

(i) Since, for all $f \in L^2[0,1]$ and given $h \in L^\infty[0,1]$ we have

$$\begin{aligned}\int_0^1 |f(t)h(t)|^2\,\mathrm{d}t &\leqslant \int_0^1 |f(t)|^2 \|h\|^2_{L^\infty[0,1]}\,\mathrm{d}t \\ &\leqslant \|h\|^2_{L^\infty[0,1]} \int_0^1 |f(t)|^2\,\mathrm{d}t \\ &= \|h\|^2_{L^\infty[0,1]} \|f\|^2_{L^2[0,1]} < \infty\end{aligned}$$

$\int_0^1 |fh|^2\,dx$
$= \int_{E/E_0} |fh|^2\,dx$
$\leqslant M^2 \int_0^1 |f|^2\,dx$
$= M^2 \|f\|^2_{L^2[0,1]} < \infty$.

it gives that $hf \in L^2[0,1]$.

(ii) Clearly T is linear. For a given $h \in L^\infty[0,1]$, by part (i) we see that

$$\|Tf\|_{L^2[0,1]} = \|hf\|_{L^2[0,1]} \leqslant \|h\|_{L^\infty[0,1]} \|f\|_{L^2[0,1]}$$

holds for all $f \in L^2[0,1]$, which show that T is a bounded linear operator on $L^2[0,1]$, with $\|T\| \leqslant \|h\|_{L^\infty[0,1]}$.

4.3 (✍) Since $\|Tx\| = \|x\|$ for every $x \in X$, T is bounded and $\|T\| \leqslant 1$. Let $x \in X$, with $x \neq 0$. Then

$$\|T\| = \sup_{y\in X, y\neq 0} \frac{\|Ty\|}{\|y\|} \geqslant \frac{\|Tx\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1.$$

Hence $\|T\| = 1$.

4.4 (✍) The linearity of the operator $T : C[a,b] \to C[a,b]$, defined by $(Tx)(t) = \int_a^t x(s)\,\mathrm{d}s$, $t \in [a,b]$, is obvious. The boundedness of T follows from

$$\|Tx\|_{C[a,b]} = \max_{t\in[a,b]} \left| \int_a^t x(s)\,\mathrm{d}s \right| \leqslant \int_a^b \max_{t\in[a,b]} |x(t)|\,\mathrm{d}s = (b-a)\|x\|_{C[a,b]}$$

for all $x \in C[a,b]$. Also $\|T\| \leqslant b - a$. To obtain the value of $\|T\|$, we choose an $x_0 \in C[a,b]$ such that $x_0(t) = 1$ for all $t \in [a,b]$. Now $\|x_0\|_{C[a,b]} = 1$ and

$$\begin{aligned}\|T\| &= \sup_{x\in C[a,b], \|x\|_{C[a,b]}=1} \|Tx\|_{C[a,b]} \geqslant \|Tx_0\|_{C[a,b]} \\ &= \max_{t\in[a,b]} \left| \int_a^t \mathrm{d}s \right| = b - a,\end{aligned}$$

hence, $\|T\| = b - a$.

4.5 (✍) The linearity of the functional $f : L^1[a,b] \to \mathbb{F}$, defined by $f(x) = \int_a^b x(t)\,\mathrm{d}t$ for all $x \in L^1[a,b]$, is clear. The boundedness of f follows from

$$|f(x)| = \left| \int_a^b x(t)\,\mathrm{d}t \right| \leqslant \int_a^b |x(t)|\,\mathrm{d}t = \|x\|_{L^1[a,b]}$$

for all $x \in L^1[a,b]$. Also we have $\|f\| \leqslant 1$. To obtain the value of $\|f\|$, we choose an $x_0 \in L^1[a,b]$ such that $x_0(t) = 1/(b-a)$ for all $t \in [a,b]$. Now $\|x_0\|_{L^1[a,b]} = 1$ and

$$\|f\| = \sup_{x\in L^1[a,b], \|x\|_{L^1[a,b]}=1} |f(x)| \geqslant |f(x_0)| = \left| \int_a^b \frac{1}{b-a}\mathrm{d}s \right| = 1,$$

hence, $\|f\| = 1$.

4.6 (✍) Let $x = \{\xi_n\} \in c, y = \{\eta_n\} \in c$ and $\alpha, \beta \in \mathbb{F}$, then we have

$$f(\alpha x+\beta y) = \lim_{n\to\infty} (\alpha\xi_n+\beta\eta_n) = \alpha \lim_{n\to\infty} \xi_n+\beta \lim_{n\to\infty} \eta_n = \alpha f(x)+\beta f(y),$$

which shows the linearity of f. Note that for each $x = \{\xi_n\} \in c$ we see that $|\xi_n| \leqslant \sup_{n\in\mathbb{N}} |\xi_n| = \|x\|_c$ for all $n \in \mathbb{N}$, which gives that $|f(x)| = \left|\lim_{n\to\infty} \xi_n\right| \leqslant \|x\|_c$ for all $x = \{\xi_n\} \in c$, so that f is bounded with $\|f\| \leqslant 1$. To obtain the value of $\|f\|$, we choose an $x_0 = \{\xi_{0,n}\} \in c$ such that $\xi_{0,n} = 1$ for all $n \in \mathbb{N}$. Now $\|x_0\|_c = 1$ and

$$\|f\| = \sup_{x\in c, \|x\|_c=1} |f(x)| \geqslant |f(x_0)| = 1,$$

hence, $\|f\| = 1$.

4.7 (✍)

(i) Since, for every $x = \{x_i\} \in \ell^2$ we have

$$\|Tx\|_{\ell^2}^2 = \sum_{i=1}^{\infty}(|4x_{2i-1}|^2 + |x_{2i}|^2) \leqslant 4^2 \sum_{i=1}^{\infty} |x_i|^2 \leqslant 4^2\|x\|_{\ell^2}^2 < \infty,$$

so $Tx \in \ell^2$.

(ii) Clearly T is linear. By part (i) we actually have got $\|Tx\|_{\ell^2} \leqslant 4\|x\|_{\ell^2}$ holds for all $x \in \ell^2$, so T is a bounded linear operator on ℓ^2, with $\|T\| \leqslant 4$.

(iii) Let $e_1 = (1, 0, 0, \cdots)$, then $e_1 \in \ell^2$, with $\|e_1\| = 1$. Since

$$\|T\| = \sup_{x\in \ell^2, \|x\|_{\ell^2}=1} \|Tx\|_{\ell^2} \geqslant \|Te_1\|_{\ell^2} = 4,$$

this together with the part (ii) give that $\|T\| = 4$.

4.8 (✍) Similarly as in the proof of Example 4.1.8, we obtain that f is linear and bounded with $\|f\| \leqslant \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. To obtain the value of $\|f\|$, we choose an $x_0 = \{\xi_{0,n}\} \in \ell^\infty$ such that $\xi_{0,n} = 1$ for all $n \in \mathbb{N}$. Now $x_0 \in \ell^\infty$, with $\|x_0\|_{\ell^\infty} = 1$, and

$$\|f\| = \sup_{x\in \ell^\infty, \|x\|_{\ell^\infty}=1} |f(x)| \geqslant |f(x_0)| = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

hence, $\|f\| = \pi^2/6$.

4.9 (✍) By the Cauchy-Schwarz inequality, $|Tx| = |\langle x,\ x_0\rangle| \leqslant \|x\|\|x_0\|$ for every $x \in \mathcal{H}$, that is, T is bounded with $\|T\| \leqslant \|x_0\|$. If $x_0 = 0$, i.e., $\|x_0\| = 0$, then $Tx = 0$ for all $x \in \mathcal{H}$, which means that T is the zero operator $\mathbf{0}$, then $\|T\| = 0 = \|x_0\|$ by Example 4.1.4. If $x_0 \neq 0$, i.e., $\|x_0\| \neq 0$, then $\|T\| = \sup\limits_{x\in\mathcal{H},x\neq 0} \|Tx\|/\|x\| \geqslant |Tx_0|/\|x_0\| = \|x_0\|$. Hence $\|T\| = \|x_0\|$.

4.10 (✍) The linearity of T is clear since every inner product is linear for the first variable. Note that for given $y, z \in \mathcal{H}$, by the Cauchy-Schwarz inequality we have

$$\|Tx\| = \|\langle x,\ y\rangle z\| = |\langle x,\ y\rangle| \leqslant (\|y\|\|z\|)\|x\| \quad \text{for all } x \in \mathcal{H},$$

then T is bounded with $\|T\| \leqslant \|y\|\,\|z\|$. To obtain the value of $\|T\|$, we note that if $\|y\| = 0$, then $\|T\| = 0 = \|y\|\,\|z\|$. Hence, we may assume that $\|y\| \neq 0$, and we obtain that

$$\|T\| = \sup_{u\in X,u\neq 0} \frac{\|Tu\|}{\|u\|} \geqslant \frac{\|Ty\|}{\|y\|} = \frac{\|\langle y,\ y\rangle z\|}{\|y\|} = \frac{\|y\|^2\|z\|}{\|y\|} = \|y\|\,\|z\|,$$

so that $\|T\| = \|y\|\|z\|$.

$\|f\| = \dfrac{1}{\inf \|x\|}$

4.11 (✍) For any $x \in X$ with $x \neq 0$ we clearly have $x/f(x) \in H$ since $f\big(x/f(x)\big) = 1$, and we also know that

$$\frac{|f(x)|}{\|x\|} = \frac{1}{\left\|\frac{x}{f(x)}\right\|} = \frac{\left|f\left(\frac{x}{f(x)}\right)\right|}{\left\|\frac{x}{f(x)}\right\|} \leqslant \sup_{y\in H} \frac{|f(y)|}{\|y\|},$$

where, for every $y \in H$ we must have $y \neq 0$ since $f(y) = 0$. This implies that $\sup\limits_{x\in X,x\neq 0} \frac{|f(x)|}{\|x\|} \leqslant \sup\limits_{y\in H} \frac{|f(y)|}{\|y\|}$. The reverse inequality obviously holds. Hence,

$$\|f\| = \sup_{x\in X,x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{y\in H} \frac{|f(y)|}{\|y\|} = \sup_{y\in H} \left(\frac{1}{\|y\|}\right) = \frac{1}{\inf\limits_{y\in H} \|y\|} = \frac{1}{d}.$$

Another proof. Since f is a bounded linear function on X, we have $|f(y)| \leqslant \|f\|\,\|y\|$ for all $y \in X$. In particular, for all $x \in H$ we get that

$$1 = |f(x)| \leqslant \|f\|\|x\|, \text{ i.e., } \|f\| \geqslant 1/\|x\|,$$

where $x \neq 0$ since $f(x) = 1$. Hence

$$\|f\| \geqslant \sup_{x \in H} \left(\frac{1}{\|x\|} \right) = \frac{1}{\inf\limits_{x \in H} \|x\|} = \frac{1}{d}. \tag{E4-1}$$

By the definition of the supremum in the norm $\|f\|$ we see that for every $\varepsilon > 0$ there exists an $x_0 \in X$, with $x_0 \neq 0$, such that

$$\frac{|f(x_0)|}{\|x_0\|} > \|f\| - \varepsilon. \tag{E4-2}$$

Let $\bar{x}_0 = x_0/f(x_0)$, then we have $f(\bar{x}_0) = 1$, i.e., $\bar{x}_0 \in H$. Substituting $x_0 = f(x_0)\bar{x}_0$ into (E4-2), we arrive at

$$\|f\| - \varepsilon < \frac{|f(f(x_0)\bar{x}_0)|}{\|f(x_0)\bar{x}_0\|} = \frac{|f(x_0) f(\bar{x}_0)|}{|f(x_0)|\, \|\bar{x}_0\|} = \frac{1}{\|\bar{x}_0\|} \leqslant \frac{1}{\inf\limits_{x \in H} \|x\|} = \frac{1}{d}.$$

Letting $\varepsilon \to 0^+$, we obtain that $\|f\| \leqslant 1/d$, which together (E4-1) imply that $\|f\| = 1/d$, as required.

$\Leftarrow$

4.12 (✍) *Sufficiency.* Argue by contradiction we suppose that f is bounded on X, i.e., f is continuous on X since f is linear. Since $\mathrm{Ker}(f)$ is dense in X, that is, $\overline{\mathrm{Ker}(f)} = X$, it follows that for each $x \in X$ there exists $\{x_n\} \subset \mathrm{Ker}(f)$ such that $x_n \to x$ as $n \to \infty$, which gives that $f(x_n) = 0$ for all $n \in \mathbb{N}$ and $f(x) = \lim\limits_{n \to \infty} f(x_n) = 0$ by the continuity of f, i.e., $f(x) = 0$ for each $x \in X$. This is impossible since $f \neq 0$, hence f is unbounded (discontinuous) on X.

和 ker(f) closed $\Leftrightarrow$ f is bounded 的联系

$\Rightarrow$

Necessity. Suppose f is unbounded on X, i.e., f is discontinuous on X since f is linear. Then f must be discontinuous at $x = 0$ by Theorem 4.1.1. Hence, there exist an $\varepsilon_0 > 0$ and a sequence $\{x_n\} \subset X$, with $\|x_n\| < 1/n$, such that $|f(x_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Now let $x \in X$ be arbitrary, it is clear that for each $n \in \mathbb{N}$, $x - \frac{f(x)}{f(x_n)} x_n \in \mathrm{Ker}(f)$ since

$$f\left(x - \frac{f(x)}{f(x_n)} x_n \right) = f(x) - f\left(\frac{f(x)}{f(x_n)} x_n \right) = f(x) - \frac{f(x)}{f(x_n)} f(x_n) = 0,$$

and $x - \frac{f(x)}{f(x_n)} x_n \to x$ as $n \to \infty$ since

$$\left\| \left(x - \frac{f(x)}{f(x_n)} x_n \right) - x \right\| = \left\| - \frac{f(x)}{f(x_n)} x_n \right\| = \frac{|f(x)|}{|f(x_n)|} \|x_n\| < \frac{|f(x)|}{n \varepsilon_0} \to 0$$

as $n \to \infty$. Hence $\mathrm{Ker}(f)$ is dense in X.

4.13 (✍) Clearly, T is linear. Since $a \perp b$, it follows that $\alpha a \perp \beta b$ for all $\alpha, \beta \in \mathbb{F}$. In particular, $\langle x, b\rangle a \perp \langle x, a\rangle b$. Hence, by the Bessel inequality, for all $x \in \mathcal{H}$ we have

$$\begin{aligned}
\|Tx\|^2 &= |\langle x, b\rangle|^2 \, \|a\|^2 + |\langle x, a\rangle|^2 \, \|b\|^2 \\
&= \|a\|^2 \, \|b\|^2 \left(\left| \left\langle x, \ \frac{b}{\|b\|} \right\rangle \right|^2 + \left| \left\langle x, \ \frac{a}{\|a\|} \right\rangle \right|^2 \right) \\
&\leqslant \|a\|^2 \|b\|^2 \|x\|^2,
\end{aligned}$$

which gives that T is bounded with $\|T\| \leqslant \|a\| \, \|b\|$. To obtain the value of $\|T\|$, we note that $Ta = b\|a\|^2$, which yields that

$$\|a\|^2 \|b\| = \|Ta\| \leqslant \|T\| \, \|a\|,$$

and then $\|T\| \geqslant \|a\| \, \|b\|$. Therefore $\|T\| = \|a\| \, \|b\|$.

4.14 (✍) T is clearly a linear operator on $\mathcal{H}_1$. Since $\{e_1, e_2, \cdots, e_n\} \subset \mathcal{H}_1$ and $\{b_1, b_2, \cdots, b_n\} \subset \mathcal{H}_2$ be orthonormal systems, by the Bessel inequality we have

$$\begin{aligned}
\|Tx\|^2 &= \left\langle \sum_{i=1}^{n} \lambda_i \langle x, e_i\rangle b_i, \ \sum_{j=1}^{n} \lambda_j \langle x, e_j\rangle b_j \right\rangle = \sum_{i=1}^{n} |\lambda_i|^2 \|b_i\|^2 |\langle x, e_i\rangle|^2 \\
&= \sum_{i=1}^{n} |\lambda_i|^2 |\langle x, e_i\rangle|^2 \leqslant M^2 \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leqslant M^2 \|x\|^2,
\end{aligned}$$

where $M = \max\limits_{1 \leqslant i \leqslant n} |\lambda_i|$. Hence T is a bounded linear operator on $\mathcal{H}_1$, with $\|T\| \leqslant M$. To obtain the value of $\|T\|$, we note that for each $i = 1, 2, \cdots, n$,

$$|\lambda_i| = |\lambda_i| \|b_i\| = \|Te_i\| \leqslant \|T\| \, \|e_i\| = \|T\|,$$

which implies $M \leqslant \|T\|$. Thus $\|T\| = \max\limits_{1 \leqslant i \leqslant n} |\lambda_i|$.

4.15 (✍) The functions $\sin t$ and $\cos t$ are clearly two orthogonal nonzero vectors in the Hilbert space $L^2[0, \pi]$ since

$$\langle \sin t, \ \cos t\rangle_{L^2[0,\pi]} = \int_0^{\pi} \sin t \, \cos t \, \mathrm{d}t = 0.$$

Note that the operator T can be represented as the form of

$$(Tx)(t) = \langle x,\ \cos t\rangle_{L^2[0,\pi]}\ \sin t + \langle x,\ \sin t\rangle_{L^2[0,\pi]}\ \cos t$$

for all $x \in L^2[0,\pi]$ and $t \in [0,\pi]$. By Exercise 4.13 we see that

$$\|T\| = \|\sin t\|_{L^2[0,\pi]}\ \|\cos t\|_{L^2[0,\pi]} = \frac{\pi}{2}$$

since

$$\|\sin\|^2_{L^2[0,\pi]} = \langle \sin t,\ \sin t\rangle_{L^2[0,\pi]} = \int_0^{\pi} \sin^2 t\,\mathrm{d}t = \frac{\pi}{2} = \|\cos t\|^2_{L^2[0,\pi]}.$$

4.16 (✍) By Example 4.1.9 we see that $\|f\| = \sup_{n\in\mathbb{N}}(1/n) = 1$ and $f(e_1) = 1 = \|f\|$, where $e_1 = (1,0,0,\cdots) \in \ell^1$. This does not conflict Example 4.1.9 since now $C = \{c_n = 1/n\}$ and $\|C\|_\infty = 1 \in \{|1/n| : n \in \mathbb{N}\}$.

4.17 (✍) By change of variable $s = t^2$ we have

$$f(x) = g(x) = \frac{1}{2}\int_0^1 s^{-1/4} x(s)\,\mathrm{d}s$$

holds for all $x \in C[0,1]$ or $x \in L^2[0,1]$, hence f and g are linear functionals on the linear spaces $C[0,1]$ and $L^2[0,1]$, respectively.

(i) Since

$$|f(x)| \leqslant \frac{1}{2}\max_{0\leqslant t\leqslant 1}|x(t)|\int_0^1 s^{-1/4}\,\mathrm{d}s = \frac{2}{3}\|x\|_{C[0,1]}$$

for all $x \in C[0,1]$, and

$$|g(x)| \leqslant \frac{1}{2}\left(\int_0^1 s^{-1/2}\,\mathrm{d}s\right)^{1/2}\left(\int_0^1 |x(s)|^2\,\mathrm{d}s\right)^{1/2} = \frac{1}{\sqrt{2}}\|x\|_{L^2[0,1]}$$

for all $x \in L^2[0,1]$, we see that f and g are respectively bounded on $C[0,1]$ and $L^2[0,1]$, with $\|f\| \leqslant 2/3$ and $\|g\| \leqslant 1/\sqrt{2}$.

(ii) To obtain the value of $\|f\|$, we choose an $x_0(t) = 1$ for all $t \in [0,1]$, then $x_0 \in C[0,1]$ with $\|x_0\|_{C[0,1]} = 1$, hence

$$\|f\| = \sup_{x \in C[0,1], \|x\|_{C[0,1]}=1} |f(x)| \geqslant |f(x_0)| = \int_0^1 \sqrt{t}\,\mathrm{d}t = \frac{2}{3},$$

so that $\|f\| = 2/3$.

To obtain the value of $\|g\|$, we choose an $x_1(t) = 1/(\sqrt{2}t^{1/4})$ for all $t \in [0,1]$, then $x_1(t) \in L^2[0,1]$, with $\|x_1\|_{L^2[0,1]} = 1$, since

$$\int_0^2 |x_1(t)|^2\,\mathrm{d}t = \int_0^1 \frac{1}{2\sqrt{t}}\,\mathrm{d}t = 1.$$

Hence

$$\|g\| = \sup_{x \in L^2[0,1], \|x\|_{L^2[0,1]}=1} |g(x)| \geqslant |g(x_1)| = \int_0^1 \sqrt{t}\,x_1(t^2)\,\mathrm{d}t = \frac{1}{\sqrt{2}},$$

so that $\|g\| = 1/\sqrt{2}$.

4.25 (✍) The first norm on ℓ^∞ is the standard norm $\|x\|_{\ell^\infty} = \sup_{n \in \mathbb{N}} |\xi_n|$, where $x = \{\xi_n\} \in \ell^\infty$.

Given arbitrarily a bounded sequence $\{\eta_n\}$ of real or complex numbers such that $\eta_n \neq 0$ for all $n \in \mathbb{N}$, we will show that the function $x \mapsto \|\cdot\| : \ell^\infty \to \mathbb{R}$, defined by $\|x\| = \sup_{n \in \mathbb{N}} |\eta_n \xi_n|$ for all $x = \{\xi_n\} \in \ell^\infty$, is also a norm on ℓ^∞. In fact, for all $x = \{\xi_n\} \in \ell^\infty$ we clearly have $\|x\| \geqslant 0$, and $\|x\| = 0$ if and only if $|\eta_n \xi_n| = 0$ for all $n \in \mathbb{N}$, i.e., $\xi_n = 0$ for all $n \in \mathbb{N}$, meaning $x = 0$, since $\eta_n \neq 0$ for all $n \in \mathbb{N}$. Obviously, $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ since

$$\|\alpha x\| = \sup_{n \in \mathbb{N}} |\eta_n \alpha \xi_n| = |\alpha| \sup_{n \in \mathbb{N}} |\eta_n \xi_n| = |\alpha|\|x\|.$$

To check the triangle inequality for a norm, we observe that for every $x = \{\xi_n\}, \tilde{x} = \{\tilde{\xi}_n\} \in \ell^\infty$ it holds that

$$\begin{aligned}\|x + \tilde{x}\| &= \sup_{n \in \mathbb{N}} |\eta_n(\xi_n + \tilde{\xi}_n)| \leqslant \sup_{n \in \mathbb{N}} \Big(|\eta_n \xi_n| + |\eta_n \tilde{\xi}_n|\Big) \\ &\leqslant \sup_{n \in \mathbb{N}} |\eta_n \xi_n| + \sup_{n \in \mathbb{N}} |\eta_n \tilde{\xi}_n| = \|x\| + \|\tilde{x}\|.\end{aligned}$$

So $\|\cdot\|$ is a norm on ℓ^∞.

There are many ways to define a bounded linear functional on ℓ^∞. For example, given arbitrarily a sequence $\{\eta_k\}$ of real or complex numbers, for each $n \in \mathbb{N}$ we define a functional on ℓ^∞ by $f_n(x) = \sum\limits_{k=1}^{n} \eta_k \xi_k$ for all $x = \{\xi_k\} \in \ell^\infty$, then each f_n is clearly a bounded linear functional on both $(\ell^\infty, \|\cdot\|_{\ell^\infty})$ and $(\ell^\infty, \|\cdot\|)$. Of course, if one hope to get some more properties of the f_n, such as the convergence of f_n, etc, then one may require some more assumptions on the given sequence $\{\eta_k\}$.

4.26 (✍) Since $\mathcal{M}$ is closed linear subspace of $\mathcal{H}$ and $\mathcal{H}$ is a Hilbert space, we see that $\mathcal{M}$ is a Hilbert space by Theorem 1.5.1. If f is a bounded linear functional on $\mathcal{M}$, then by the Riesz-Fréchet theorem (Theorem 4.2.3) we have a unique $y_0 \in \mathcal{M}$ such that $f(x) = \langle x,\ y_0\rangle$ for all $x \in \mathcal{M}$ and $\|f\| = \|y_0\|$. On the other hand, the function $g : \mathcal{H} \to \mathbb{R}$, given by $g(x) = \langle x,\ y_0\rangle$ for all $x \in \mathcal{H}$, clearly defines a bounded linear functional on $\mathcal{H}$ with $\|g\| = \|y_0\|$ by Exercise 4.9. So $g(x) = f(x)$ for all $x \in \mathcal{M}$ and $\|f\| = \|y_0\| = \|g\|$.

4.27 (✍)

(i) Clearly, for all $R, S, T \in \mathscr{B}(X)$, and $\alpha \in \mathbb{F}$ we have the following equalities:

(a) $R(ST) = (RS)T$

(b) $R(S+T) = RS + RT$

(c) $(S+T)R = SR + TR$

(d) $\mathbf{I}T = T\mathbf{I} = T$

(e) $(\alpha S)T = \alpha(ST) = S(\alpha T)$.

Hence, $\mathscr{B}(X)$ is an algebra with identity $\mathbf{I}$ under the operations of addition and scalar multiplication given in (4.5), and hence a ring with identity $\mathbf{I}$ under the operation of multiplication given in Definition 4.2.3.

(ii) $\{T_n\}$ is bounded in $\mathscr{B}(X)$ since it is convergent, so there exists $K > 0$ such that $\|T_n\| \leq K$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|S_nT_n - ST\| &\leqslant \|S_nT_n - ST_n\| + \|ST_n - ST\| \\ &\leqslant K\,\|S_n - S\| + \|S\|\,\|T_n - T\|, \end{aligned}$$

which goes to zero as $n \to \infty$ since $\lim\limits_{n\to\infty} T_n = T$ and $\lim\limits_{n\to\infty} S_n = S$. Therefore, the conclusion follows immediately.

4.28 (✍)

(i) For each $n \in \mathbb{N}$ let $e_n = \{\delta_{nj}\}$, with δ_{nj} having 1 in the nth place and zeros elsewhere. Then, it is clear that each $e_n \in c_0$ with $\|e_n\|_{c_0} = 1$, and $x = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \xi_k\, e_k$ for all $x = \{\xi_n\} \in c_0$. Now for every $f \in (c_0)^*$, denote $a_n = f(e_n)$, $n = 1, 2, \cdots$, then by the continuity and linearity of f we have

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \xi_k\, f(e_k) = \sum_{n=1}^{\infty} \xi_n a_n. \tag{E4-3}$$

For each $n \in \mathbb{N}$ we consider $x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \xi_3^{(n)}, \cdots)$ defined by

$$\xi_k^{(n)} = \begin{cases} 1, & \text{if } k \leqslant n, a_k > 0, \\ -1, & \text{if } k \leqslant n, a_k < 0, \\ 0, & \text{if } k > n \text{ or } a_k = 0. \end{cases}$$

Clearly, each $x_n \in c_0$ since the sequence $\{\xi_k^{(n)}\}_{k\in\mathbb{N}}$ of real numbers has at most n nonzero terms, which must converge to 0 as $k \to \infty$. Note that for any $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} \xi_k^{(n)} a_k &\overset{\text{by (E4-3)}}{=\!=\!=} f(x_n) \\ &\leqslant \|f\|\, \|x_n\|_{c_0} = \|f\| \sup_{k\in\mathbb{N}} |\xi_k^{(n)}| = \|f\|. \end{aligned}$$

Letting $n \to \infty$, we get that $\sum\limits_{k=1}^{\infty} |a_k| \leqslant \|f\| < \infty$, which means that $\{a_k\} \in \ell^1$ and $\|\{a_k\}\|_{\ell^1} \leqslant \|f\|$. On the other hand, by (E4-3), for each $x = \{\xi_k\} \in c_0$ we have

$$|f(x)| \leqslant \sum_{k=1}^{\infty} |a_k \xi_k| \leqslant \sup_{k\in\mathbb{N}} |\xi_k| \sum_{k=1}^{\infty} |a_k| = \|x\| \|\{a_k\}\|_{\ell^1}. \tag{E4-4}$$

Therefore $\|f\| \leqslant \|\{a_k\}\|_{\ell^1}$, so that $\|f\| = \|\{a_n\}\|_{\ell^1} = \sum\limits_{k=1}^{\infty} |a_k|$.

proof.

$f \in (c_0)^*$.

$\{e_n\} \to x = \sum\limits_{n=1}^{\infty} x_n e_n$

$f(x) = \sum\limits_{n=1}^{\infty} x_n f(e_n)$

$= \sum\limits_{n=1}^{\infty} a_n x_n$

$f(x_k) = \sum\limits_{n=1}^{k} |a_n| \leq \|f\| \|x_k\|_c$

$= \|f\|$ $k \to \infty$.

$\Rightarrow \underline{a \in \ell^1}$

(ii) For given $\{a_k\} \in \ell^1$, (E4-3) clearly defines a linear functional f on c_0. By the (E4-4) in the part (i) we see that f is bounded on c_0. Thus $f \in (c_0)^*$. So $f \in (c_0)^*$.

(iii) Let $T : \ell^1 \to (c_0)^*$ defined by $T(a) = f$ for each $a = \{a_k\} \in \ell^1$, where f is defined in (E4-3). It is easy to check that T is linear. By parts (i) and (ii), T is a surjection and T preserves the norm, i.e., T is an isometry. So $(c_0)^* = \ell^1$ in the sense of isometrical isomorphism.

4.31 (✍) Clearly T is linear since the limit operation and T_n's ($n \in \mathbb{N}$) are linear. Since for each $x \in X$ the limit $\lim\limits_{n\to\infty} T_n(x)$ exists, we see that $\{T_n(x)\}$ is bounded for each $x \in X$ with a bound dependent on x. By Theorem 4.3.1 we have a constant $C > 0$, independent of x and n, such that $\|T_n\| \leqslant C$ for all $n \in \mathbb{N}$, so that $\liminf\limits_{n\to\infty} \|T_n\| \leqslant C < \infty$. Hence, we arrive at $\|Tx\| \leqslant \left(\liminf\limits_{n\to\infty} \|T_n\|\right) \|x\|$ for all $x \in X$ since

$$\|Tx\| \leqslant \|T_n x\| + \|Tx - T_n x\| \leqslant \|T_n\|\,\|x\| + \|Tx - T_n x\|$$

and $T_n x \to Tx$ as $n \to \infty$. Thus, $T \in \mathscr{B}(X, Y)$ and $\|T\| \leqslant \liminf\limits_{n\to\infty} \|T_n\|$.

4.32 (✍) It follows from Exercise 2.40 that S is a non-closed (not closed) subspace of ℓ^2, and hence S is not complete by Theorem 1.5.1 since S is not closed. For each $x = \{x_n\} \in S$ we see that there exists an $N_x \in \mathbb{N}$ such that $x_n = 0$ for all $n \geqslant N_x$, so that

$$\|T_n x\|_{\ell^2} = n|x_n| \leqslant C_{N_x} \quad \text{for all } n \in \mathbb{N},$$

where C_{N_x} is a positive constant dependent on x. On the other hand, for each $n \in \mathbb{N}$ let $e_n = \{\delta_{nj}\}$ with δ_{nj} having 1 in the nth place and zeros elsewhere. Then for each $n \in \mathbb{N}$, $\|T_n e_n\|_{\ell^2} = n$, so $\|T_n\| \geq n$, which means that the sequence $\{\|T_n\|\}$ is unbounded. Setting $X = S$ and $Y = \ell^2$, we know that this example satisfies all the hypotheses of Theorem 4.3.1 except the completeness of X. Since the conclusion of the the theorem does not hold, we see that the completeness of X is necessary.

4.33 (✍) For each $n \in \mathbb{N}$ we define $T_n(x) = \sum\limits_{i=1}^{n} a_i x_i$ for all $x = \{x_k\} \in \ell^p$, then each $\{T_n\}$ is clearly a bounded linear operator (functional) on

ℓ^p. Obviously, $T(x) = \sum\limits_{i=1}^{\infty} a_i x_i$ ($x = \{x_k\} \in \ell^p$) is well-defined since the series $\sum\limits_{k=1}^{\infty} a_k x_k$ converges by the assumption, and $\lim\limits_{n\to\infty} T_n(x) = T(x)$. By Corollary 4.3.2 we see that $T \in (\ell^p)^*$. Hence $\{a_n\} \in \ell^q$ by Theorem 4.2.5.

4.34 (✍) For each $n \in \mathbb{N}$ we define $T_n(x) = \sum\limits_{i=1}^{n} a_i x_i$ for all $x = \{x_k\} \in c_0$, then each $\{T_n\}$ is clearly a bounded linear operator (functional) on ℓ^p. Obviously, $T(x) = \sum\limits_{i=1}^{\infty} a_i x_i$ ($x = \{x_k\} \in c_0$) is well-defined since the series $\sum\limits_{k=1}^{\infty} a_k x_k$ converges by the assumption, and $\lim\limits_{n\to\infty} T_n(x) = T(x)$. By Corollary 4.3.2 we see that $T \in (c_0)^*$. Hence $\{a_n\} \in \ell^1$ by Exercise 4.28.

4.35 (✍) If $\sup\limits_{n\in\mathbb{N}} \|T_n x\| < \infty$ would hold for all $x \in X$, then for every $x \in X$ we clearly have $\|T_n x\| \leqslant \sup\limits_{n\in\mathbb{N}} \|T_n x\|$ for all $n \in \mathbb{N}$, where $\sup\limits_{n\in\mathbb{N}} \|T_n x\|$ dependents on x, but it is independent of n. By the uniform boundedness theorem (i.e., Theorem 4.3.1) we have a constant $C > 0$, independent of x and n, such that $\|T_n\| \leqslant C$ for all $n \in \mathbb{N}$, this contradicts the assumption $\sup\limits_{n\in\mathbb{N}} \|T_n\| = +\infty$. Hence there exists an $x_0 \in X$ such that $\sup\limits_{n\in\mathbb{N}} \|T_n x\| = +\infty$.

4.36 (✍) By Exercise 1.29 we see that in any metric space, a cauchy sequence is bounded, so for each $x \in X$ the Cauchy sequence $\{\|T_n x\|\}$ is bounded, i.e., there exists a constant C_x, independent of n, such that $\|T_n x\| \leqslant C_x$ for all $n \in \mathbb{N}$. Then by the uniform boundedness theorem (i.e., Theorem 4.3.1) we get that the sequence $\{\|T_n\|\}$ is bounded in $\mathbb{R}$.

4.37 (✍) Suppose that $\{T_n x\}$ ($x \in X$) is Cauchy in Y, then for each $x \in X$ the Cauchy sequence $\{T_n x\}$ is convergent in Y since Y is complete, say, $T_n x \to y \in Y$ as $n \to \infty$. Now, for each $x \in X$ we define $T : X \to Y$ by $Tx = y = \lim\limits_{n\to\infty} T_n x$, then T is well-defined, and T is a linear operator on X since $\{T_n\} \subset \mathscr{B}(X, Y)$ and for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$ we have

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty} (\alpha T_n x + \beta T_n y) \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y = \alpha T x + \beta T y. \end{aligned}$$

Moreover, T is bounded on X since

$$\|Tx\| = \left\| \lim_{n\to\infty} T_n x \right\| = \lim_{n\to\infty} \|T_n x\| \leqslant \left(\sup_{n\in\mathbb{N}} \|T_n\| \right) \|x\|$$

for all $x \in X$, where the existence of $\sup_{n\in\mathbb{N}} \|T_n\|$ was shown in Exercise 4.36. Therefore, $T \in \mathscr{B}(X, Y)$ such that $T_n x \to Tx$ for all $x \in X$ as $n \to \infty$.

4.43 (✍) Clearly, T_1T_2 is a bounded linear operator by Lemma 4.2.1, and T_1T_2 is a bijection since T_1 and T_2 are invertible. So $(T_1T_2)^{-1}$ exists such that $T_1T_2(T_1T_2)^{-1} = I$, which implies that

$$\begin{aligned} (T_1T_2)^{-1} &= [T_2^{-1}(T_1^{-1}T_1)T_2](T_1T_2)^{-1} \\ &= T_2^{-1}T_1^{-1}[T_1T_2(T_1T_2)^{-1}] = T_2^{-1}T_1^{-1}. \end{aligned}$$

4.44 (✍) By the assumption that $\|T - S\| < 1/\|T^{-1}\|$, we have

$$\|T^{-1}(T - S)\| \leqslant \|T^{-1}\| \, \|T - S\| < 1,$$

which implies that $\mathbf{I} - T^{-1}(T - S) = T^{-1}S$ is invertible by Theorem 4.4.1. Therefore $S = TT^{-1}S$ is also invertible by Lemma 4.4.1.

4.45 (✍) 不太好想 找一列$\{e_n\}$. $\|e_n\|=1$ 但$\|Te_n\|$充分小

☆

(i) For each $n \in \mathbb{N}$ let $e_n = \{\delta_{nk}\}$ with δ_{nk} having 1 in the nth place and zero elsewhere, then each $e_n \in \ell^\infty$ and $\|e_n\|_{\ell^\infty} = 1$. Since $\|T_c(e_n)\| = 1/n = (1/n)\|e_n\|$ and $1/n \to 0$ as $n \to \infty$, it follows from Theorem 4.4.5 that T_c, with $c = \{1/n\}$, is not invertible.

(ii) Since $|d_n| = |1/c_n| \leqslant 1 \Big/ \inf_{n\in\mathbb{N}} |c_n|$ for all $n \in \mathbb{N}$, we see that $\sup_{n\in\mathbb{N}} |d_n| \leqslant 1 \Big/ \inf_{n\in\mathbb{N}} |c_n|$, which gives that $d \in \ell^\infty$. Now, for every

$\frac{1}{c_n} \leq \frac{1}{\inf\{c_n\}}$

$x = \{x_n\} \in \ell^\infty$ we have

$d_n = \frac{1}{c_n}$

$$T_c T_d\{x_n\} = T_c\{d_n x_n\} = \{c_n d_n x_n\} = \{x_n\}$$

$T_d(\{x_n\}) = \{d_n x_n\}.$

$= \{\frac{x_n}{c_n}\}$

and

$$T_d T_c\{x_n\} = T_d\{c_n x_n\} = \{d_n c_n x_n\} = \{x_n\}$$

Hence $T_c T_d = T_d T_c = \mathbf{I}$.

$T_d(x) = \max\{\frac{x_n}{c_n}\}$

(iii) Suppose that $\lambda \notin \overline{\{c_n : n \in \mathbb{N}\}}$, then we have $\inf\limits_{n\in\mathbb{N}} |c_n - \lambda| > 0$ by the definition of the infimum. For each $n \in \mathbb{N}$ let $b_n = c_n - \lambda$, then $b = \{b_n\} \in \ell^\infty$ since $\{c_n\} \in \ell^\infty$. By part (ii) we see that T_b is invertible since $\inf\limits_{n\in\mathbb{N}} |b_n| > 0$ and the inverse $T_b^{-1} = T_d$ with $d = \{1/b_n\} \in \ell^\infty$. Note that for every $x = \{x_n\} \in \ell^\infty$ it holds that $T_b\{x_n\} = \{(c_n - \lambda)x_n\} = \{c_n x_n\} - \{\lambda x_n\} = (T_c - \lambda\mathbf{I})\{x_n\}$, so $T_b = T_c - \lambda\mathbf{I}$. Thus $T_c - \lambda\mathbf{I}$ is invertible.

$\leq \frac{1}{\inf\{c_n\}} \|x\|$

$\Rightarrow T_d \in B(\ell^\infty)$

4.47 (✍) The identity map $\mathbf{I} : X \to X$ is clear a bijective linear operator which maps the Banach space $(X, \|\cdot\|_2)$ into the Banach space $(X, \|\cdot\|_1)$, and vice versa, of course. By the assumption, we see that I is obviously bounded since $\|\mathbf{I}x\|_1 = \|x\|_1 \leqslant k\|x\|_2$ for all $x \in (X, \|\cdot\|_2)$. Then, the Banach theorem (i.e., Theorem 4.4.3) infers that $\mathbf{I}$ is invertible. Hence there exists a $K > 0$ such that $\|\mathbf{I}x\|_2 < K\|x\|_1$ for all $x \in (X, \|\cdot\|_1)$, so that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

4.49 (✍) Note that $T_0^{-1} \in \mathscr{B}(Y, X)$ by the Banach theorem (i.e., Theorem 4.4.3) since $T_0 \in \mathscr{B}(X, Y)$ is a bijection. For every given $y \in Y$, we define an operator $T_y : X \to X$ by $T_y x = -T_0^{-1}T_1 x + T_0^{-1}y$ for all $x \in X$. Then T_y is a contraction on X with a contractivity factor $\|T_0^{-1}\|\,\|T_1\| \in [0, 1)$ since for all $x_1, x_2 \in X$ we have

$$\|T_y(x_1) - T_y(x_2)\|_X = \|T_0^{-1}T_1(x_2 - x_1)\| \leqslant \|T_0^{-1}\|\|T_1\|\|x_1 - x_2\|_X.$$

Thus, by the Banach fixed point theorem (i.e., Theorem 1.8.1) we know that the equation $T_y x = x$ has a unique solution x_y. This x_y is also the unique solution of the equation $T_0 x = T_0(T_y x)$ since T_0 is a bijection, which means that the x_y uniquely satisfies

$$T_0(x_y) = T_0(T_y(x_y)) = -T_0(T_0^{-1}(T_1(x_y))) + T_0(T_0^{-1}y).$$

That is, for each $y \in X$ the equation $y = (T_0 + T_1)x$ has a unique solution x_y. Therefore $T_0 + T_1$ is bijective. Clearly, $T_0 + T_1 \in \mathscr{B}(X, Y)$, so $T_0 + T_1$ is invertible by the Banach theorem.

4.51 (✍) Obverse that X is a Hilbert space since X is closed in $\mathcal{H}$ and $\mathcal{H}$ is a Hilbert space. By the Riesz-Frćhet theorem (i.e., Theorem 4.2.3), there is a unique $x_f \in X$ with $\|x_f\| = \|f\|$ such that $f(x) = \langle x, x_f \rangle$ for all $x \in X$. Let g be defined by $g(x) = \langle x, x_f \rangle$ for $x \in \mathcal{H}$. Then by Exercise 4.9 we see that $g \in X^*$ with $\|g\| = \|x_f\| = \|f\|$, and clearly, $g|_X = f$.

unique

(Suppose now that $\widetilde{f}$ is another linear functional on $\mathcal{H}$ such that $\widetilde{f}|_X = f$ and $\|\widetilde{f}\| = \|f\|$. Then, by the Riesz-Frćhet theorem we have an $\widetilde{x} \in \mathcal{H}$ such that $\widetilde{f}(x) = \langle x, \widetilde{x} \rangle$ for all $x \in \mathcal{H}$, and $\|\widetilde{x}\| = \|\widetilde{f}\|$. Since $\widetilde{f}|_X = f$, we obtain that $\langle x, \widetilde{x} - x_f \rangle = 0$ $\boxed{\text{for all } x \in X}$, i.e., $\widetilde{x} - x_f \in X^{\perp}$. Hence $\|f\|^2 = \|\widetilde{f}\|^2 = \|\widetilde{x}\|^2 = \|\widetilde{x} - x_f\|^2 + \|x_f\|^2 = \|\widetilde{x} - x_f\|^2 + \|f\|^2$. Therefore $\|\widetilde{x} - x_f\|^2 = 0$, $\widetilde{x} = x_f$, and the extension is unique.)

→ 因为它们的限制是一样的

0 4.52 (✍) Clearly, for each $f \in X^*$ with $\|f\| \leqslant 1$ we have $|f(x)| \leqslant \|f\|\,\|x\| \leqslant \|x\|$, which infers that $\sup\{|f(x)| : f \in X^*, \|f\| \leqslant 1\} \leqslant \|x\|$. We may assume that $x \neq 0$ since otherwise the desired equality holds trivially. It follows from the corollary of the Hahn-Banach theorem (i.e. Corollary 4.5.2) that there exists an $f_x \in X^*$ with $\|f_x\| = 1$ such that $f_x(x) = \|x\|$, so that $\sup\{|f(x)| : f \in X^*, \|f\| \leqslant 1\} \geqslant \|x\|$. Therefore

$$\sup\{|f(x)| : f \in X^*, \|f\| \leqslant 1\} = \|x\|.$$

4.53 (✍) Clearly if $f \in X^*$ satisfies that $f(x_\nu) = \alpha_\nu$ for all $\nu = 1, 2, \cdots, k$ and $\|f\| \leqslant M$, then, for all $t_1, t_2, \cdots, t_k \in \mathbb{F}$ we have

$$\left|\sum_{\nu=1}^{k} t_\nu \alpha_\nu\right| = \left|f\left(\sum_{\nu=1}^{k} t_\nu x_\nu\right)\right| \leqslant \|f\| \left\|\sum_{\nu=1}^{k} t_\nu x_\nu\right\| \leqslant M \left\|\sum_{\nu=1}^{k} t_\nu x_\nu\right\|,$$

as required.

Conversely, denote $\mathcal{M} = \operatorname{span}(\{x_k : k \in \mathbb{N}\})$ and define a functional $\tilde{f}$ on $\mathcal{M}$ by $\tilde{f}(x) = \sum\limits_{\nu=1}^{k} t_\nu \alpha_\nu$ for every $x = \sum\limits_{\nu=1}^{k} t_\nu x_\nu \in \mathcal{M}$, where $t_1, t_2, \cdots, t_k \in \mathbb{F}$. Clearly, $\tilde{f}$ is linear, and $|\tilde{f}(x)| \leqslant M\|x\|$ on $\mathcal{M}$ by the assumption, hence $\tilde{f}$ is a bounded linear functional on $\mathcal{M}$. By the Hahn-

Banach theorem (i.e., Theorem 4.5.2), we see that there exists a linear functional f on X such that $|f(x)| \leqslant M\|x\|$ for all X, and $f(x) = \tilde{f}(x)$ on $\mathcal{M}$. Hence $f \in X^*$ and, in particular, $f(x_\nu) = \tilde{f}(x_\nu) = \alpha_\nu$ for all $\nu = 1, 2, \cdots, k$.

4.54 (✍) Without loss of generality we may suppose that Z is closed since otherwise we simply replace it by its closure, which has the same value of $d(x_0, Z)$ by (iii) of Exercise 1.45.

Let $\mathcal{M} = \mathrm{span}(\{x_0\}) + Z = \{\alpha x_0 + z : z \in Z\}$. Clearly $\mathcal{M}$ is a linear subspace of X. We claim that every $x \in \mathcal{M}$ has a unique representation of the form $x = \alpha x_0 + z_0$ for some $\alpha \in \mathbb{F}$ and $z_0 \in Z$. In fact, if there exist $\alpha_1, \alpha_2 \in \mathbb{F}$ and $z_1, z_2 \in Z$ such that $x = \alpha_1 x_0 + z_1$ and $x = \alpha_2 x_0 + z_2$, then $(\alpha_1 - \alpha_2)x_0 = z_2 - z_1 \in Z$, it gives that $\alpha_1 - \alpha_2 = 0$ since Z is a linear subspace and $z_0 \notin Z$ by assumption $d(x_0, Z) > 0$. Hence $\alpha_1 = \alpha_2$, and then $z_1 = z_2$.

$\boxed{\text{其实 ①② 给出了 norm 的构造方法。} \to \text{证③}}$

Now, we define a linear functional $f_{\mathcal{M}}$ and $g_{\mathcal{M}}$ on $\mathcal{M}$ by

$$f_{\mathcal{M}}(x) = \alpha d(x_0, Z), \quad g_{\mathcal{M}}(x) = \alpha \quad \text{for every } x = \alpha x_0 + z \in \mathcal{M}.$$

通过Hahn-Banach extension to X^*

$\boxed{\mathrm{span}\{x_0\}+Z} \xrightarrow{\text{extension}}$

Clearly, $f_{\mathcal{M}}$ is linear, and $f_{\mathcal{M}}$ is bounded since for every $x = \alpha x_0 + z \in \mathcal{M}$ we have

$$|f_{\mathcal{M}}(\alpha x_0 + z)| = |\alpha| d(x_0, Z) \leqslant |\alpha| \|x_0 + \overbrace{\alpha^{-1} z}^{\in Z}\| = \|\alpha x_0 + z\|.$$

That is, $f_{\mathcal{M}} \in \mathcal{M}^*$ with $\|f_{\mathcal{M}}\| \leqslant 1$. Identically, $g_{\mathcal{M}} \in \mathcal{M}^*$ with

$$\|g_{\mathcal{M}}\| = \|f_{\mathcal{M}}\| / d(x_0, Z) \leqslant 1/d(x_0, Z).$$

We now prove that $\|f_{\mathcal{M}}\| \geqslant 1$. Indeed, by the definition of the infimum in $d(x_0, Z)$ we obtain a sequence $\{z_n\} \subset Z$ such that $\|x_0 - z_n\| \to d(x_0, Z)$ as $n \to \infty$. Hence,

$$\|f_{\mathcal{M}}\| = \sup_{\substack{x = \alpha x_0 + z \in \mathcal{M} \\ x \neq 0}} \frac{|f_{\mathcal{M}}(x)|}{\|x\|} \geqslant \frac{f_{\mathcal{M}}(x_0 - z_n)}{\|x_0 - z_n\|} \overset{f_{\mathcal{M}}\text{的定义}}{=} \frac{d(x_0, Z)}{\|x_0 - z_n\|}.$$

Letting $n \to \infty$, we get that $\|f_{\mathcal{M}}\| \geqslant 1$, and then $\|f_{\mathcal{M}}\| = 1$. So $\|g_{\mathcal{M}}\| = 1/d(x_0, Z)$. It follows the Hahn-Banach theorem for formed spaces (i.e., Corollary 4.5.1) that $f_{\mathcal{M}}$ can be extended linearly with

4.60. 对完备的赋范空间,如果X'可分,则X亦可分.
证明:
由于X'可分,易知存在球面$\{x' \in X' : \|x'\| = 1\}$ 上可数稠密子集$\{x_n'\}_{n=1}^{\infty}$.根据$\|x'\|$的定义,应有一串点$\{x_n\}_{n=1}^{\infty} \subset X, \|x_n\| \leq 1$使

$$|x_n'(x_n)| > \tfrac{1}{2}, n = 1, 2, \cdots .$$

假设X不可分.$\{x_n'\}_{n=1}^{\infty}$张开的子空间不是X.则应有$x_0' \in X'$, 使

$$|x_0'(x_n)| = 0, n = 1, 2, \cdots .$$

且$\|x_0'\| = 1$从而

$$\|x_0' - x_n'\| \geq |x_0'(x_n) - x_n'(x_n)| = |x_n'(x_n)| > \tfrac{1}{2}, n = 1, 2, \cdots .$$

这与$\{x_n'\}_{n=1}^{\infty}$是X'之单位球面上的稠密子集矛盾.证毕.

preservation of the norm to the whole of X, and so can $g_{\mathcal{M}}$ identically. That is, $f\big|_{\mathcal{M}} = f_{\mathcal{M}}$, $g\big|_{\mathcal{M}} = g_{\mathcal{M}}$, and $\|f\| = \|f_{\mathcal{M}}\|$ and $\|g\| = \|g_{\mathcal{M}}\|$. In particular,

(i) $f(z) = 0$ and $g(z) = 0$ whenever $z \in Z$ since every $z \in Z$ can be expressed as $x = \alpha x_0 + z \in \mathcal{M}$ with $\alpha = 0$;

(ii) $f(x_0) = d(x_0, Z)$ and $g(x_0) = 1$ since $x_0 = \alpha x_0 + z \in \mathcal{M}$ with $\alpha = 1$ and $z = 0$, the zero vector of the linear space Z.

(iii) $\|f\| = \|f_{\mathcal{M}}\| = 1$ and $\|g\| = \|g_{\mathcal{M}}\| = 1/d(x_0, Z)$.

4.55 (✍) If $X \neq Y$, then we could have an $x_0 \neq 0$, $x_0 \in X \backslash Y$, and hence $d(x_0, Y) > 0$ by (iii) of Exercise 1.45 since Y is closed (i.e., $\overline{Y} = Y$). It follows from Exercise 4.54 that there would exist an $f \in X^*$ such that $f\big|_Y = 0$ but $f(x_0) = d(x_0, Y)$ and $\|f\| = 1$, this contradicts the assumption $f = 0$. Therefore, $X = Y$.

4.63
证明:必要性.
设$x_0^{***} \in X^{***}$,要证$\exists x_0^* \in X^*$,使得$< x_0^{***}, x^{**} >=< x^{**}, x_0^* >$
事实上:定义

$$J: \quad x \in X \to Jx \in X^{**}$$
$$J^*: \quad x_0^{***} \in X^{***} \to Jx_0^* \in X$$

令$x_0^* = J^* x_0^{***}$,下证x_0^*满足上式.事实上,

$$< x_0^{***}, x^{**} >=< x_0^{***}, Jx >=< J^* x_0^{***}, x >$$
$$=< x_0^*, x >=< Jx, x_0^* >=< x^{**}, x_0^* >$$

证毕
充分性:
首先,因为X是Banach空间,所以$J(X) \subset X^{**}$ 是X^{**}的子空间.其次,因为X^*自反,将X^*视为本例必要性部分中的X,即知X^{**}自反,所以$J(X) = X$.最后,又因为自反空间的闭子空间是自反空间,所以$X = J(X)$自反.

4.64 (✍) If the underlying scalar field $\mathbb{F} = \mathbb{R}$, then the result holds trivially. We may assume $\mathbb{F} = \mathbb{C}$. For each $f \in X^*$ we write f in the form of $f = u + \mathrm{i}v$ with $u = \mathbf{Re}\, f$ and $v = \mathbf{Im}\, f$. Let $g = -\mathrm{i}f$, then $g \in X^*$ and $\mathbf{Re}\, g = v$. By the assumptions we see that $\mathbf{Re}\, f(x_n) \to \mathbf{Re}\, f(x)$ as $n \to \infty$ and $\mathbf{Re}\, g(x_n) \to \mathbf{Re}\, g(x)$ as $n \to \infty$, so that $u(x_n) \to u(x)$ as $n \to \infty$ and $v(x_n) \to v(x)$ as $n \to \infty$. Therefore,

$$f(x_n) = u(x_n) + \mathrm{i}v(x_n) \to u(x) + \mathrm{i}v(x) = f(x) \quad \text{as } n \to \infty.$$

Which shows that $x_n \rightharpoonup x$ as $n \to \infty$ by the definition.

4.65 (✍) Since $\|e_m - e_n\| = \sqrt{2}$ for $n \neq m$, we see that $\{e_n\}$ is not a Cauchy sequence, so $\{e_n\}$ does not converge in $\mathcal{H}$ (actually, $\mathcal{H}$ cannot have a convergent subsequence, cf. Exercise 3.45). Let $x \in \mathcal{H}$ be an arbitrary elment, then $x = \sum\limits_{n=1}^{\infty} \lambda_n e_n$ by Theorem 3.3.3 since $\{e_n\}$ is an orthonormal basis in $\mathcal{H}$. Hence $\{\lambda_n\} \in \ell^2$ by the Riesz-Fischer theorem (i.e., Theorem 3.3.2), and then

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \langle x, e_n \rangle = 0.$$

Now, given an arbitrary bounded linear functional f on $\mathcal{H}$, we see from the Riesz-Fréchet theorem (i.e., Theorem 4.2.3) that there exists an

$\forall f \in H^*$
$f(e_n) \to 0.$
$\forall x \in H$
$\langle x, e_n \rangle \to 0.$
$\Uparrow$
$x = \sum\limits_{n=1}^{\infty} \underbrace{\langle x, e_n \rangle}_{\in \ell^2 \Rightarrow \langle x, e_n \rangle \to 0} e_n$

$x_f \in H$ such that $f(x) = \langle x, x_f \rangle$ for all $x \in \mathcal{H}$ and $\|f\| = \|x_f\|$. Then

$$\lim_{n\to\infty} f(e_n) = \lim_{n\to\infty} \langle e_n, x_f \rangle = 0,$$

and $e_n \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$.

D 4.66 (✍) By Exercise 4.9 we see that the inner product $\langle x,\ x_0 \rangle$ clearly defines a bounded linear functional f_0 on $\mathcal{H}$, that is, $f_0(x) = \langle x,\ x_0 \rangle$ for all $x \in \mathcal{H}$. Then

$$f_0(x_n) = \langle x_n,\ x_0 \rangle \to \langle x_0,\ x_0 \rangle = f_0(x_0) \quad \text{as } n \to \infty$$

since $x_0 \rightharpoonup x_0$ as $n \to \infty$, so that we also have $\langle x_0,\ x_n \rangle \to \langle x_0,\ x_0 \rangle$ as $n \to \infty$. Hence,

$f(x_n) = \langle x_n, x_0 \rangle \to f(x_0) = \langle x_0, x_0 \rangle$

$$\begin{aligned}\|x_n - x_0\|^2 &= \langle x_n - x_0,\ x_n - x_0 \rangle \\ &= \|x_n\|^2 + \|x_0\|^2 - \langle x_n,\ x_0 \rangle - \langle x_0,\ x_n \rangle \\ &\to \|x_0\|^2 + \|x_0\|^2 - 2\langle x_0,\ x_0 \rangle = 0 \quad \text{as } n \to \infty\end{aligned}$$

by the assumption $\|x_n\| \to \|x_0\|$ as $n \to \infty$, which means that $\{x_n\}$ converges strongly to x_0 as $n \to \infty$, as desired.

6 4.67 (✍) First, we see that

$$\|x - x_n\|^2 = \|x\|^2 - \langle x, x_n \rangle - \langle x_n, x \rangle + \|x_n\|^2.$$

Next, by the weak convergence of the inner product we have (cf. the proof of Exercise 4.66)

$$\lim_{n\to\infty} \langle x_n, x \rangle = \langle x, x \rangle = \|x\|^2 = \lim_{n\to\infty} \langle x, x_n \rangle.$$

Therefore

$$\begin{aligned}&\liminf_{n\to\infty} \left(\|x\|^2 - \langle x, x_n \rangle - \langle x_n, x \rangle + \|x_n\|^2\right) \\ =&\liminf_{n\to\infty} \|x_n\|^2 + \|x\|^2 - 2\|x\|^2 \geqslant 0,\end{aligned}$$

and so $\liminf\limits_{n\to\infty} \|x_n\| \geqslant \|x\| = 1$. Finally, by assumption $\limsup\limits_{n\to\infty} \|x_n\| \leqslant 1$ we get that $\lim\limits_{n\to\infty} \|x_n\| = 1 = \|x\|$, and the conclusion follows from Exercise 4.66.

4.68 (✍) We may assume that $x \neq 0$ since otherwise the conclusion holds trivially. It follows from the corollary of the Hahn-Banach theorem (i.e., Corollary 4.5.2) that there exists an $f_x \in X^*$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$. By the weak convergence, we have $f_x(x_n) \to f_x(x) = \|x\|$ as $n \to \infty$, so $|f_x(x_n)| \to |f_x(x)| = \|x\|$ as $n \to \infty$. Since $|f_x(x_n)| \leqslant \|f_x\|\|x_n\| = \|x_n\|$, we see that

$$\|x\| = \liminf_{n\to\infty} |f_x(x_n)| \leqslant \liminf_{n\to\infty} \|x_n\|$$

as desired.

S.5 Solutions to Exercises Marked with (✍) of Chapter 5

5.1 (✍) Since for all $x, w \in \mathcal{H}$

$$\begin{aligned}\langle Tx,\ w\rangle &= \langle\langle x,\ y\rangle z,\ w\rangle = \langle x,\ y\rangle\langle z,\ w\rangle \\ &= \overline{\langle w,\ z\rangle}\langle x,\ y\rangle = \langle x,\ \langle w,\ z\rangle y\rangle \\ &= \langle x,\ T^*w\rangle,\end{aligned}$$

it implies by the uniqueness of the adjoint that $T^*w = \langle w,\ z\rangle y$.

5.2 (✍) For all $f, g \in L^2(-\infty, \infty)$ the inner product of f and g is given by $\langle f,\ g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,\mathrm{d}t$, hence

$$\langle Tf,\ g\rangle = \int_{-\infty}^{\infty} f(t+1)\,\overline{g(t)}\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\tau)\overline{g(\tau-1)}\,d\tau = \langle f,\ T^*g\rangle.$$

By the uniqueness of the adjoint we see that $(T^*g)(t) = g(t-1)$.

5.3 (✍) Let $x = \{x_n\}, y = \{y_n\} \in \ell^2$ be arbitrary, then the inner product space x and y is $\langle x,\ y\rangle = x_1\overline{y_2} + x_2\overline{y_3} + x_3\overline{y_4} + \cdots$. Let $T^*y = \{z_n\} \in \ell^2$, then $\langle Tx,\ y\rangle = \langle x,\ T^*y\rangle$, thus we get an indefinite equation about z_1, z_2, $\cdots$

$$x_1\overline{z_1} + x_2\overline{z_2} + x_3\overline{z_3} + \cdots = 4x_1\overline{y_2} + x_2\overline{y_3} + 4x_3\overline{y_4} + \cdots .$$

Obviously, $(z_1 = 4y_2, z_2 = y_3, z_3 = 4y_4, \cdots)$ solves this equation, and hence by the uniqueness of the adjoint we see that

$$T^*y = (4y_2, y_3, 4y_4, \cdots).$$

5.4 (✍) The linearity of T is clearly. For all $f \in L^2[a, b]$ we have

$$\begin{aligned}\|Tf\|_{L^2[a,b]} &= \left(\int_a^b |Tf(s)|^2 \mathrm{d}s\right)^{1/2} = \left(\int_a^b \left|\int_a^b \phi(s,t)\, f(t)\right|^2 \mathrm{d}t\right)^{1/2} \\ &\leqslant \left\{\int_a^b \left(\int_a^b |\phi(s,t)|^2\,\mathrm{d}t\right)\left(\int_a^b |f(t)|^2\,\mathrm{d}t\right)\mathrm{d}s\right\}^{1/2} \\ &= \left\{\int_a^b \int_a^b |\phi(s,t)|^2\,\mathrm{d}t\, ds\right\}^{1/2} \|f\|_{L^2[a,b]}.\end{aligned}$$

Hence $T \in \mathscr{B}(L^2[a,b])$ and $\|T\| \leqslant \left\{\int_a^b \int_a^b |\phi(s,t)|^2 \,\mathrm{d}t\, ds\right\}^{1/2}$.

For all $f, g \in L^2[a,b]$ we have

$$\begin{aligned}\langle Tf,\ g\rangle &= \int_a^b Tf(t)\,\overline{g(t)}\,\mathrm{d}t = \int_a^b (\phi(s,t) f(s)\, ds)\ \overline{g(t)}\,\mathrm{d}t \\ &= \int_a^b f(s) \left(\overline{\int_a^b \overline{\phi(s,t)}\, g(t)\,\mathrm{d}t} \right) \mathrm{d}s = \langle f,\ T^* g\rangle,\end{aligned}$$

which gives that $(T^*g)(s) = \int_a^b \overline{\phi(s,t)}\, g(t)\,\mathrm{d}t$ by the uniqueness of the adjoint.

5.5 (✍) For all $x = \{x_i\}, z = \{z_i\} \in \ell^2$ we have both $Tx = \{y_j\} \in \ell^2$ and $T^*z = \{z_j^*\} \in$ satisfying

$$y_j = \sum_{i=1}^{\infty} a_{ji} x_i, \quad z_j^* = \sum_{i=1}^{\infty} a_{ji}^* z_i, \quad j = 1, 2, \cdots .$$

Since

$$\sum_{j=1}^{\infty} y_j\, \overline{z_j} = \langle Tx,\ z\rangle = \langle x,\ T^* z\rangle = \sum_{i=1}^{\infty} x_i\, \overline{z_i^*}$$

by the assumptions, we see that

$$\begin{aligned}\langle Tx,\ z\rangle &= \sum_{j=1}^{\infty} y_j\, \overline{z_j} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ji} x_i \right) \overline{z_j} \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ji} \overline{z_j} \right) x_i = \sum_{i=1}^{\infty} x_i \left(\overline{\sum_{j=1}^{\infty} \overline{a_{ji}}\, z_j} \right),\end{aligned}$$

which means that

$$z_i^* = \sum_{j=1}^{\infty} \overline{a_{ij}}\, z_j \quad i = 1, 2, \cdots ,$$

so that

$$a_{ij}^* = \overline{a_{ji}}, \quad i, j = 1, 2, \cdots$$

by the uniqueness of the adjoint.

5.6 (✍) The necessity is always true for both real and complex inner product spaces.

Sufficiency. Suppose that $\langle Tx, x\rangle = 0$ for all $x \in \mathcal{H}$. If $\mathcal{H}$ is a complex inner product space, i.e., the the underlying scalar field $\mathbb{F} = \mathbb{C}$, then for all $x, y \in \mathcal{H}$ we have

$$\begin{aligned} 0 &= \langle T(x + \alpha y),\ x + \alpha y\rangle \\ &= \langle Tx,\ x\rangle + \alpha\langle Ty,\ x\rangle + \overline{\alpha}\langle Tx,\ y\rangle + |\alpha|^2\langle Ty,\ y\rangle \\ &= \alpha\langle Ty,\ x\rangle + \overline{\alpha}\langle Tx,\ y\rangle \end{aligned}$$

Picking $\alpha = 1, \mathrm{i}$, respectively, we get that $\langle Ty,\ x\rangle + \langle Tx,\ y\rangle = 0$ and $\langle Ty,\ x\rangle - \langle Tx,\ y\rangle = 0$, respectively, which together imply that $\langle Tx,\ y\rangle = 0$ for all $x, y \in \mathcal{H}$, and then $T = \mathbf{0}$.

The sufficiency is false if $\mathcal{H}$ is a real inner product space as a ± 90 degree rotation in $\mathbb{R}^2$ shows.

In the real case the condition such that $T = \mathbf{0}$ should be $\langle x,\ y\rangle = 0$ for all $x, y \in \mathcal{H}$ since, now picking $y = Tx$ it follows that $\|Tx\|^2 = \langle Tx,\ Tx\rangle = 0$ and so $Tx = 0$ for all $x \in \mathcal{H}$.

5.7 (✍) *Necessity.* Suppose that $T + T^* = 0$, then

$$\begin{aligned} \mathbf{Re}\,\langle Tx,\ x\rangle &= \frac{1}{2}\big(\langle Tx,\ x\rangle + \langle x,\ Tx\rangle\big) \\ &= \frac{1}{2}\big(\langle Tx,\ x\rangle + \langle T^*x,\ x\rangle\big) \\ &= \frac{1}{2}\big(\langle (T + T^*)x,\ x\rangle\big) = 0. \end{aligned}$$

Sufficiency. Suppose that $\mathbf{Re}\,\langle Tx,\ x\rangle = 0$ for all $x \in \mathcal{H}$, then

$$\langle (T + T^*)x,\ x\rangle = 2\mathbf{Re}\,\langle Tx,\ x\rangle = 0 \quad \text{for all } x \in \mathcal{H}$$

by a straightforward calculation which is actually given in the proof of the necessity. Therefore, for all $x, y \in \mathcal{H}$ we have

$$\langle (T + T^*)(x + y),\ x + y\rangle = 0 \text{ and } \langle (T + T^*)(x + \mathrm{i}\,y),\ x + i\,y\rangle = 0,$$

$\Rightarrow \langle (T+T^*)x, y\rangle = 0$ for every $x, y \in H$.

$\Leftarrow T + T^* = 0$