S.3 Solutions to Exercises Marked with (✍) of Chapter 3

3.1 (✍) Let $x, y \in X$. We may assume that $y \neq 0$ since the case of $y = 0$ is trivial.

Sufficiency. If $x = py$ for some real $p \geqslant 0$, then we have

$$\|x+y\| = \|(1+p)y\| = (1+p)\|y\| = \|y\|+p\|y\| = \|py\|+\|y\| = \|x\|+\|y\|.$$

Necessity. Since

$$\begin{aligned}\|x+y\|^2 = \langle x+y,\ x+y\rangle &= \|x\|^2 + \|y\|^2 + 2\mathbf{Re}\,\langle x,\ y\rangle \\ &\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\|+\|y\|)^2,\end{aligned}$$

it follows that if $\|x+y\| = \|x\|+\|y\|$, then $\mathbf{Re}\,\langle x,\ y\rangle = \|x\|\|y\|$. Hence, $|\langle x,\ y\rangle| = \mathbf{Re}\,\langle x,\ y\rangle = \|x\|\|y\| \geqslant 0$ since $\mathbf{Re}\,\langle x,\ y\rangle \leqslant |\langle x,\ y\rangle| \leqslant \|x\|\|y\|$ by the Cauchy-Schwarz inequality (3.2). Noting that $y \neq 0$, let $p = \langle x,\ y\rangle/\|y\|^2$, then we have $p = \|x\|/\|y\| \geqslant 0$ and

$$\langle x-py,\ x-py\rangle = p^2\|y\|^2 + \|x\|^2 - 2p\|x\|\,\|y\| = 0.$$

Hence $x = py$ with $p = \|x\|/\|y\| \geqslant 0$.

$\|x+y\| = \|x\|+\|y\|$
$\langle x+y, x+y\rangle = \|x+y\|^2$
$= \|x\|^2+\|y\|^2+2\mathrm{Re}\langle x,y\rangle.$
$\leqslant \|x\|^2+\|y\|^2+2\|x\|\|y\|$
$=(\|x\|+\|y\|)^2$
$\Rightarrow \mathrm{Re}\langle x,y\rangle = \|x\|\|y\|$
取 $p=\frac{\|x\|}{\|y\|}$
$\langle x-py, x-py\rangle$
$= p^2\|y\|^2+\|x\|^2-2p\|x\|\|y\|$
$= 0 \Rightarrow x=py.$
根据 $py=x$ 得到.

3.2 (✍) Clearly, if $y = \lambda x + (1-\lambda)z$ for some scalar λ between 0 and 1, then

$$\|x-y\| + \|y-z\| = (1-\lambda)\|x-z\| + \lambda\|x-z\| = \|x-z\|.$$

Conversely, let $\|x-y\| + \|y-z\| = \|x-z\|$, then by Exercise 1.1 we see that there exists a real number $k \geqslant 0$ such that $x - y = k(y-z)$, then the conclusion follows by letting $\lambda = 1/(k+1)$.

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3.3 (✍) We may assume that $x \neq 0$ since the case of $x = 0$ is trivial. If the equality in the Cauchy-Schwarz inequality occurs, then $|\langle x,\ y\rangle| = \|x\|\,\|y\|$. Let $\lambda = \langle x,\ y\rangle/\|x\|^2$, similarly as in the proof of Exercise 3.1, then we get $\langle y-\lambda x,\ y-\lambda x\rangle = 0$, i.e., $y = \lambda x$ for some $\lambda \in \mathbb{F}$. Conversely, if $x = 0$ or $y = \lambda x$ for some $\lambda \in \mathbb{F}$, then the identity $|\langle x,\ y\rangle| = \|x\|\,\|y\|$ holds clearly. Therefore the required condition is $x = 0$ or $y = \lambda x$ for some $\lambda \in \mathbb{F}$.

Remark. Assume that $\langle x,\ y\rangle$ satisfies all three conditions of the inner product except that $\langle x,\ x\rangle$ may be zero for a non-zero element. Then the Cauchy-Schwartz inequality is still true.

In fact, first, the case $\langle x,y\rangle = 0$ is trivial. Second, assume that $\langle x,y\rangle \neq 0$ and set $\theta = \langle x,y\rangle/|\langle x,y\rangle|$. Let λ be a real number. We have

$$0 \leq \langle \bar{\theta}x + \lambda y, \bar{\theta}x + \lambda y\rangle = \langle x,x\rangle + \lambda\langle y,\bar{\theta}x\rangle + \lambda^2\langle y,y\rangle$$

since $\langle y,\bar{\theta}x\rangle = \theta\overline{\langle x,y\rangle} = |\langle x,y\rangle|$ and $\langle\bar{\theta}x,y\rangle = \bar{\theta}\langle x,y\rangle = |\langle x,y\rangle|$, we obtain

$$\langle x,x\rangle + 2\lambda|\langle x,y\rangle| + \lambda^2\langle y,y\rangle \geqslant 0$$

for any $\lambda \in \mathbb{R}$. This implies that

$$|\langle x,y\rangle|^2 - \langle x,x\rangle\langle y,y\rangle \leqslant 0 \quad \text{or} \quad |\langle x,y\rangle|^2 \leqslant \langle x,x\rangle\langle y,y\rangle.$$

3.4 (✍) Expanding, $\left\|x + e^{\mathrm{i}t}y\right\|^2 e^{\mathrm{i}t} = \left(\|x\|^2 + \|y\|^2\right)e^{\mathrm{i}t} + \langle x,y\rangle + \langle y,x\rangle e^{\mathrm{i}2t}$, which when integrated gives $(2\pi)^{-1}\int_0^{2\pi}\left\|x + e^{\mathrm{i}t}y\right\|^2 e^{\mathrm{i}t}\mathrm{d}t = \langle x,y\rangle$.

3.6 (✍) Consider that $(1,0),(0,2) \in \mathbb{R}^2$. Then

$$\begin{aligned}&\|(1,0)+(0,2)\|_1^2 + \|(1,0)+(0,2)\|_1^2\\ =&6 \neq 10 = 2(\|(1,0)\|_1^2 + \|(0,2)\|_1^2),\end{aligned}$$

which means that the parallelogram law dose not holds for the norm $\|\cdot\|_1$. So $\|\cdot\|_1$ is not induced by an inner product by Theorem 3.1.2.

3.7 (✍) Let $x,y \in X$ be arbitrary two vectors, then there exists a subspace $Y \subset X$ with $\dim(Y) = 2$. By the assumption, Y is an inner product space and its norm is induced by an inner product, hence the norm of Y must satisfy the parallelogram law (3.3) by Lemma 3.1.1, in particular, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. Note that the norm of Y is induced by the norm of X, it follows from Theorem 3.1.2 that the norm of X cab be induced by an inner product, that is, X is an inner product space.

3.10 (✍) Assume that the inner product space is $(X, \langle\cdot,\ \cdot\rangle)$. Note that

$$0 \leqslant \|x_n - y_n\|^2 = \|x_n\|^2 + \|y_n\|^2 - 2\mathbf{Re}\,\langle x_n,y_n\rangle \leqslant 2 - 2\mathbf{Re}\,\langle x_n,y_n\rangle \to 0$$

as $n \to \infty$ since $\lim\limits_{n\to\infty}\mathbf{Re}\,\langle x_n,y_n\rangle = 1$ by the fact $\lim\limits_{n\to\infty}\langle x_n,y_n\rangle = 1$, it follows that $\lim\limits_{n\to\infty}\|x_n - y_n\| = 0$.

3.13 (✍)

(i) It is clearly by Theorem 1.5.1.

(ii) By Corollary 2.4.2 we see that Y is closed since Y is finite-dimensional. So, Y is complete by Theorem 1.5.1.

(iii) The desired conclusions follow by Exercises 1.27.

3.16 (✍) First, H is clearly a linear space with the linear operations defined like (2.5) since $\alpha x(t)+\beta y(t)$ must be continuous on $[0,1]$ if $x(t), y(t)$ are continuous on $[0,1]$ and $\alpha, \beta \in \mathbb{F}$. Also, the function $x \mapsto \|\cdot\|_H : H \to \mathbb{R}$, defined by $\|x\|_H = \left(\int_0^1 |x(t)|^2 \,\mathrm{d}t\right)^{1/2}$, is obviously a norm on the linear space H, and it satisfies the parallelogram law since

$$\begin{aligned}
\|x+y\|_H^2 + \|x-y\|_H^2 &= \int_0^1 |x(t)+y(t)|^2 \,\mathrm{d}t + \int_0^1 |x(t)-y(t)|^2 \,\mathrm{d}t \\
&= \int_0^1 2(|x(t)|^2 + |y(t)|^2) \,\mathrm{d}t \\
&= 2\left(\|x\|_H^2 + \|y\|_H^2\right)
\end{aligned}$$

holds for all $x, y \in H$. Hence, by Theorem 3.1.2 we see that there exists an inner product on H which generates the norm $\|\cdot\|_H$, and it is clear that this inner product must be the form which is given in Example 3.1.9. So, H is an inner product space. We will show that H is not complete. Indeed, for each $n \in \mathbb{N}$ we define a continuous function $f_n(t)$ on $[0,1]$ by

$$f_n(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right), & \text{if } \frac{1}{2} - \frac{1}{n} \leqslant t < \frac{1}{2} + \frac{1}{n}, \\ 1, & \text{if } \frac{1}{2} - \frac{1}{n} \leqslant t \leqslant 1, \end{cases}$$

then $\{f_n\}$ converges to

$$f(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < 1/2 \\ 1, & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

as $n \to \infty$ since

$$\|f_n - f\|_H^2 = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} |f_n(t) - f(t)|^2 \, \mathrm{d}t < \frac{2}{n} \to 0 \quad \text{as } n \to \infty.$$

Clearly, $\{f_n\}$ is a Cauchy sequence in H since

$$\|f_n - f_m\|_H \leqslant \|f_n - f\|_H + \|f_m - f\| \to 0 \text{ as } n, m \to \infty.$$

But $f(t)$ is not continuous on $[0, 1]$ and then it does not belong to H. So H is not a Hilbert space.

3.17 (✍) Suppose that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then we have

$$\begin{aligned} 0 &= \|x + y\|^2 - (\|x\|^2 + \|y\|^2) \\ &= \langle x + y,\ x + y\rangle - (\|x\|^2 + \|y\|^2) = 2\mathbf{Re}\,\langle x,\ y\rangle. \end{aligned}$$

If X is a real inner product space, i.e., the underlying scalar field $\mathbb{F} = \mathbb{R}$, then $\mathbf{Re}\,\langle x,\ y\rangle = \langle x,\ y\rangle = 0$, so that $x \perp y$, as required.

If X is complex, i.e., the underlying scalar field $\mathbb{F} = \mathbb{C}$, the x and y may not be orthogonal. For example, let $X = \mathbb{C}$ be the unitary space which is a complex inner product space. Let $x = e^{\pi \mathrm{i}/3}, y = e^{-\pi \mathrm{i}/6} \in \mathbb{C}$. Clearly, $\|x + y\|^2 = 2 = \|x\|^2 + \|y\|^2$, but x is not orthogonal to y since $\langle x,\ y\rangle = \mathrm{i} \neq 0$.

3.18 (✍) Suppose that $\mathcal{H}$ is real and $\|x\| = \|y\|$. Then we have

$$\langle x + y,\ x - y\rangle = \|x\|^2 - \|y\|^2 - 2\mathbf{Im}\,\langle x,\ y\rangle = -2\mathbf{Im}\,\langle x,\ y\rangle = 0$$

since the inner product $\langle x,\ y\rangle$ is s real number.

If $\mathcal{H} = \mathbb{R}^2$, then $\mathcal{H}$ is a real inner product space, so $\langle x + y,\ x - y\rangle = 0$ by the above. Geometrically, it says that the diagonals of a rhombus or parallelogram in the plane $\mathbb{R}^2$ are orthogonal to each other.

If $\mathcal{H}$ is a complex inner product space, then, by the condition $\|x\| = \|y\|$ we have $\mathbf{Re}\,\langle x + y,\ x - y\rangle = 0$ since the inner product $\langle x + y,\ x - y\rangle$ keeps only its imaginary part by the above.

3.19 (✍) By the parallelogram law we have

$$2\left\|\frac{x_n - x_m}{2}\right\|^2 = \|x_n\|^2 + \|x_m\|^2 - 2\left\|\frac{x_n + x_m}{2}\right\|^2.$$

Clearly, $(x_n + x_m)/2 \in M$ since M is convex. It follows that

$$\frac{1}{2}\|x_n - x_m\|^2 = 2\left\|\frac{x_n - x_m}{2}\right\|^2 \leqslant \|x_n\|^2 + \|x_m\|^2 - 2d^2 \to 0$$

as $n, m \to \infty$, i.e., $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence of $\mathcal{H}$ so that $\{x_n\}$ is convergent by the completeness of $\mathcal{H}$.

3.20 (✍) Let $x = (\xi_1, \xi_2, \cdots, \xi_n) \in M$ and $y = (\eta_1, \eta_2, \cdots, \eta_n) \in M$ be arbitrary. For every $\alpha \in [0, 1]$ we have

$$\sum_{j=1}^{n}[\alpha\eta_j + (1-\alpha)\xi_j] = \alpha\sum_{j=1}^{n}\eta_j + (1-\alpha)\sum_{j=1}^{n}\xi_j = 1,$$

which means that $\alpha y + (1-\alpha)z \in M$, and then M is convex in $\mathbb{C}^n$ by the definition of a convex in a linear space.

Now we show that M is closed in $\mathbb{C}^n$. Indeed, let $\omega = (\omega_1, \omega_2, \cdots, \omega_n) \in \overline{M}$ be arbitrary, then there exists a sequence $\{y_m\} \subset M$ such that $y_m \to \omega$ as $m \to \infty$. For each $m \in \mathbb{N}$ we denote $y_m = (\eta_1^{(m)}, \eta_2^{(m)}, \cdots, \eta_n^{(m)})$, then we have $\sum\limits_{j=1}^{n}\eta_j^{(m)} = 1$ since each $y_m \in M$. By Example 1.3.3 we see that $\eta_j^{(m)} \to \omega_j$ in $\mathbb{C}$ as $m \to \infty$, $j = 1, 2, \cdots, n$, so that $\sum\limits_{j=1}^{n}\omega_j = 1$, meaning that $\omega \in M$. Consequently, M is closed, and then M is complete by Theorem 1.5.1 since $\mathbb{C}^n$ is complete. $y = (1/n, 1/n, \cdots, 1/n)$ has the minimum norm in M, that is, y is the solution of the system:

$$\min\left\{\sum_{j=1}^{n}|\eta_j|^2 : \eta_j \in \mathbb{C}, j = 1, 2, \cdots, n, \sum_{j=1}^{n}\eta_j = 1\right\}.$$

3.21 (✍) Let $x = x_0/\|x_0\|$, then we see that the distance from this x to x_0 is $d(x, x_0) = \|x - x_0\| = \|x_0/\|x_0\| - x_0\| = |\,\|x_0\| - 1|$, and this x achieves

<x-y, x-y>

重要等式：$\|x-y\|^2 = \|x\|^2+\|y\|^2 - 2\mathrm{Re}\langle x,y\rangle$ $\Rightarrow \|x-y\| \geqslant \big|\|x\|-\|y\|\big|$
$\geqslant \|x\|^2+\|y\|^2 - 2\|x\|\|y\|$
$= (\|x\|-\|y\|)^2$

$\min\{\|y-x_0\| : y \in X, \|y\|=1\}$ since for all $y \in X$ with $\|y\|=1$ we have

$$\begin{aligned}\|y-x_0\|^2 &= \|y\|^2+\|x_0\|^2-2\mathbf{Re}\,\langle y,\ x_0\rangle \geqslant \|y\|^2+\|x_0\|^2-2\|y\|\|x_0\| \\ &= |\,\|x_0\|-\|y\|\,|^2 = |\,1-\|x_0\|\,|^2 = \|x-x_0\|^2\end{aligned}$$

by the Cauchy-Schwarz inequality.

3.22 (✍) Denote $c_1 := c/\langle x_0, x_0\rangle$ and let $y = c_1x_0$, then y satisfies that

$$\langle y, x_0\rangle = \langle c_1x_0, x_0\rangle = c_1\langle x_0, x_0\rangle = c.$$

Now for all $x \in X$, with $\langle x, x_0\rangle = c$, we have $\langle x-y, x_0\rangle = 0$ and $\langle x-y, y\rangle = 0$, which yields that

$$\|x\|^2 = \|(x-y)+y\|^2 = \|x-y\|^2+\|y\|^2 \geqslant \|y\|^2,$$

that is, $\|y-0\| \leqslant \|x-0\|$ holds for all $x \in X$ with $\langle x, x_0\rangle = c$.

3.24 (✍) Note that for all $x, y \in X$ and $\alpha \in \mathbb{F}$ we have

$$\begin{aligned}\|x+\alpha y\|^2 &= \langle x+\alpha y,\ x+\alpha y\rangle \\ &= \|x\|^2+\overline{\alpha}\langle x,\ y\rangle+\alpha\langle y,\ x\rangle+|\alpha|^2\|y\|^2.\end{aligned} \tag{E3-3}$$

If $x\perp y$, that is, $\langle x,\ y\rangle = 0$, then by (E3-3) we get that

$$\|x+\alpha y\|^2 = \|x\|^2+|\alpha|^2\|y\|^2 \geqslant \|x\|^2,$$

as required.

Conversely, suppose that $\|x+\alpha y\| \geqslant \|x\|$ for all $\alpha \in \mathbb{F}$. Then, by (E3-3) we obtain that

$$\overline{\alpha}\langle x,\ y\rangle+\alpha\langle y,\ x\rangle+|\alpha|^2\|y\|^2 \geqslant 0. \tag{E3-4}$$

We may assume that $y \neq 0$ since otherwise we trivially have $x\perp y$. Let $\alpha = -\langle x,\ y\rangle/\|y\|^2$ in the above inequality (E3-4), then we deduce that $-|\langle x,\ y\rangle|/\|y\|^2 \geqslant 0$, and so $\langle x,\ y\rangle = 0$, i.e., $x\perp y$.

3.25 (✍) Note that for all $x, y \in X$ and $\alpha \in \mathbb{F}$ we have

$$\begin{aligned}\|x - \alpha y\|^2 &= \langle x - \alpha y,\ x - \alpha y\rangle \\ &= \|x\|^2 - \overline{\alpha}\langle x,\ y\rangle - \alpha\langle y,\ x\rangle + |\alpha|^2\|y\|^2.\end{aligned} \tag{E3-5}$$

If $x \perp y$, then, by (E3-3) and (E3-5) we obviously have $\|x + \alpha y\|^2 = \|x - \alpha y\|^2$, as required.

Conversely, if $\|x + \alpha y\|^2 = \|x - \alpha y\|^2$ for all $\alpha \in \mathbb{F}$, then, by (E3-3) and (E3-5) we obtain that $\overline{\alpha}\langle x,\ y\rangle + \alpha\langle y,\ x\rangle = 0$ for all $\alpha \in \mathbb{F}$. In particular, with $\alpha = \langle x,\ y\rangle$, we get $|\langle x,\ y\rangle| = 0$, i.e., $x \perp y$.

3.26 (✍) By the definition of the inner product in $\mathbb{R}^k$, $\langle a,\ x\rangle = \sum\limits_{j=1}^{k} a_j x_j = 0$ if and only if $a \perp x$ for every $x \in \mathbb{R}^k$. Hence

$$A^{\perp} = \left\{ (x_1, \cdots, x_k) \in \mathbb{R}^k : \sum_{j=1}^{k} a_j x_j = 0 \right\}.$$

3.27 (✍) Let $B = \{\{\xi_i\} \in \ell^2 : \xi_{2i+1} = 0 \text{ for all } i \in \mathbb{N}\}$. By the definition of the inner product in ℓ^2, we clearly have $\langle x,\ y\rangle_{\ell^2} = \sum\limits_{i=1}^{\infty} \xi_i \overline{\eta_i} = 0$ for all $x = \{\xi_i\} \in A$ and $y = \{\eta_i\} \in B$. So $B \subset A^{\perp}$.

Suppose that $y = \{\eta_i\} \in A^{\perp}$ is arbitrary. Let $\tilde{x} = \{\tilde{\xi_i}\}$ be such that $\tilde{\xi}_{2i+1} = \eta_{2i+1}$ and $\tilde{\xi}_{2i} = 0$ for all $i \in \mathbb{N}$, which means that each $\tilde{x} \in A$. Hence

$$\langle y,\ \tilde{x}\rangle_{\ell^2} = \sum_{i=1}^{\infty} |\eta_{2i+1}|^2 = 0,$$

so that $\eta_{2i+1} = 0$ for all $i \in \mathbb{N}$, i.e., $y \in B$, showing that $A^{\perp} \subset B$. This together with the above imply that $A^{\perp} = B$, that is,

$$A^{\perp} = \{\{x_n\} \in \ell^2 : x_{2n+1} = 0 \text{ for all } n \in \mathbb{N}\}.$$

3.28 (✍) The space ℓ^2 here should be changed to the space S which is given by Exercise 2.40 and is equipped with the usual ℓ^2 inner product.

We claim that B is dense in S. In fact, let $y = \{y_n\} \in S$ be arbitrary, then there exists an $N \in \mathbb{N}$ such that $y_n = 0$ for $n > N$ and $\sum\limits_{n=1}^{N} y_n = \eta$, say. For an integer K, let $x_K = \{x_{K,n}\}$ be given by

$$x_{K,n} = \begin{cases} y_n, & 1 \leqslant n \leqslant N \\ -\eta/K, & N+1 \leqslant n \leqslant N+K \\ 0, & n > N+K \end{cases}$$

Then $x_K \in B$ for each $K \in \mathbb{N}$ and $\|x_K - y\|_{\ell^2}^2 = |\eta|^2/K^2 \to 0$ as $K \to \infty$. So B is dense in ℓ_0^2, i.e., $\overline{B} = \ell_0^2$, and then $B^\perp = \{0\}$ by (f) of Lemma 3.2.1.

3.32 (✍) Obviously $\overline{A}^\perp \subset A^\perp$ by (g) of Lemma 3.2.1 since $A \subset \overline{A}$. For each $y \in A^\perp$ we shall show that $y \perp \overline{A}$, that is, $y \perp x$ for all $x \in \overline{A}$. Indeed, for each $x \in \overline{A}$ there exists a sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$. By the continuity of the inner product, we have $\langle x,\ y\rangle = \lim\limits_{n\to\infty} \langle x_n,\ y\rangle = 0$. Hence $y \perp \overline{A}$, so $A^\perp \subset \overline{A}^\perp$. This together with the above imply that $A^\perp = \overline{A}^\perp$, as required.

3.33 (✍) Since $X \subset X+Y$ and $Y \subset X+Y$, it follows from Lemma 3.2.1 that $(X+Y)^\perp \subset X^\perp$ and $(X+Y)^\perp \subset Y^\perp$, so that $(X+Y)^\perp \subset X^\perp \cap Y^\perp$.

Suppose that $z \in X^\perp \cap Y^\perp$. For every $x+y \in X+Y$ we have both $\langle x,\ z\rangle = 0$ and $\langle y,\ z\rangle = 0$ since $z \in X^\perp \cap Y^\perp$. So $\langle x+y,\ z\rangle = 0$. Hence $X^\perp \cap Y^\perp \subset (X+Y)^\perp$. This together with the above imply that $(X+Y)^\perp = X^\perp \cap Y^\perp$, as required.

3.35 (✍) Let $x \in \overline{W}$ be arbitrary. then there exists a sequence $\{x_n\} \subset W$ such that $x_n \to x$ as $n \to \infty$. By the assumption we see that x has the form of $x = x_0 + x_1$ for some $x_0 \in W$ and $x_1 \in W^\perp$. Clearly $\langle x_n,\ x_1\rangle = 0$ for each $n \in \mathbb{N}$ since each $x_n \in W$, so that

$$0 = \lim_{n\to\infty} \langle x_n,\ x_1\rangle = \langle x,\ x_1\rangle = \langle x_0 + x_1,\ x_1\rangle = \langle x_1,\ x_1\rangle = \|x_1\|^2,$$

which yields $x_1 = 0$, that is, $x = x_1 \in W$. Hence W is closed.

3.36 (✍) By Theorem 3.2.2 we see that x has the form of $x = x_0 + x_1$ for some $x_0 \in N$ and $x_1 \in N^\perp$ since N is a closed subspace of $\mathcal{H}$ by the assumption. Note that $x - z = x_1 + (x_0 - z)$ and $x_0 - z \in N$ for all $z \in N$, we obtain that $\|x_1\| \in \{\|x-z\| : z \in N\}$, $\min\{\|x-z\| : z \in N\} \geqslant \|x_1\|$ by Lemma 3.2.2, and so $\min\{\|x-z\| : z \in N\} = \|x_1\|$.

Now, for every $y \in N^{\perp}$, with $\|y\| = 1$, we have

$$|\langle x, y\rangle| = |\langle x_0 + x_1, y\rangle| = |\langle x_0,\ y\rangle + \langle x_1, y\rangle| = |\langle x_1, y\rangle| \leqslant \|x_1\|\|y\| = \|x_1\|$$

since $x_0 \in N$, i.e.,

$$\sup\{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\} \leqslant \|x_1\|. \tag{E3-6}$$

If $x_1 = 0$, then

$$\sup\{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\} = 0 = \|x_1\|. \tag{E3-7}$$

If $x_1 \neq 0$, taking $y = x_1/\|x_1\|$, then we get that $\|y\| = 1, y \in N^{\perp}$ and

$$\langle x, y\rangle = \langle x_0 + x_1, y\rangle = \|x_1\|. \tag{E3-8}$$

which implies that

$$\sup\{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\} \geqslant \|x_1\|. \tag{E3-9}$$

It follows from the inequalities (E3-6) and (E3-9) that

$$\min\{\|x - z\| : z \in N\} = \sup\{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\}.$$

By (E3-7) and (E3-8) we see that $\|x_1\| \in \{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\}$, therefore,

$$\min\{\|x - z\| : z \in N\} = \max\{|\langle x, y\rangle| : y \in N^{\perp}, \|y\| = 1\}.$$

3.38 (✍)

(i) Let $y \in A^{\perp}$ be arbitrary, then $\langle x,\ y\rangle$ for all $x \in A$. In particular, for every $n \in \mathbb{N}$ we have $\left\langle \sum\limits_{i=1}^{n} a_i x_i,\ y \right\rangle = 0$ for every $\sum\limits_{i=1}^{n} a_i x_i \in$ span$A$, where $x_i \in A$ and $a_i \in \mathbb{F}$, $i = 1, \cdots, n$. Which means that $y \in (\text{span}(A))^{\perp}$, so $A^{\perp} \subset (\text{span}(A))^{\perp}$. The inverse inclusion obviously follows from (g) of Lemma 3.2.1 since $A \subset \text{span}(A)$, thus, $A^{\perp} = (\text{span}(A))^{\perp} = (\overline{\text{span}(A)})^{\perp}$ by Exercise 3.32. Finally, we get that

$$A^{\perp\perp} = (\overline{\text{span}(A)})^{\perp\perp} = \overline{\text{span}(A)}$$

by Corollary 3.2.4 since $\overline{\operatorname{span}(A)}$ is clearly a closed linear subspace of $\mathcal{H}$. .

Another proof. $A \subset A^{\perp\perp} \Rightarrow \operatorname{span}(A) \subset \operatorname{span}(A^{\perp\perp}) = A^{\perp\perp}$ since $A^{\perp\perp}$ is a closed linear space of $\mathcal{H}$ by (h) of Lemma 3.2.1, which gives that $\overline{\text{span(A)}} \subset \overline{A^{\perp\perp}} = A^{\perp\perp}$. On the other hand, $A \subset \overline{\operatorname{span}(A)} \Rightarrow A^{\perp} \supset \overline{\operatorname{span}(A)}^{\perp}$, so that

$$A^{\perp\perp} \subset \overline{\operatorname{span}(A)}^{\perp\perp} = \overline{\operatorname{span}(A)},$$

hence the result.

(ii) $A^{\perp\perp\perp} = (A^{\perp})^{\perp\perp} = A^{\perp}$ follows by Corollary 3.2.4 since $A^{\perp}$ is a closed linear subspace of $\mathcal{H}$.

3.39 (✍) It follows from (i) of Exercise 3.38 that $M^{\perp\perp} = \overline{\operatorname{span}(M)}$. Hence, by (c) of Lemma 3.2.1 and (ii) of Exercise 3.38 we see that $\overline{\operatorname{span}(M)} = \mathcal{H}$ if and only if $M^{\perp} = \{0\}$.

3.42 (✍) The Gram-Schmidt algorithm yields

$$e_1 = \frac{\sqrt{2}}{2},\ e_2 = \frac{\sqrt{6}}{2}t,\ e_3 = \frac{\sqrt{10}}{4}(3t^2 - 1).$$

3.43 (✍) Since 9900 ordered pairs can be selected from 100 integers,

$$\left\| \sum_{n=1}^{100} x_n \right\|^2 = \sum_{n=1}^{100} \|x_n\|^2 + \mathbf{Re}\left(\sum_{n \neq m}^{100} \langle x_n, x_m \rangle \right) \leq 100 + \frac{9900}{10} = 1090,$$

consequently, $\left\| \sum\limits_{n=1}^{100} x_n \right\| \leqslant \sqrt{1090}$. To see that the estimate is sharp consider the sequences $x_n = \{x_{n,k}\} \in \ell^2$ with terms $x_{n,1} = \sqrt{0.1}$, $x_{n,n+1} = \sqrt{0.9}$, and the remaining terms equal to 0, for all $n = 1, \cdots, 100$. Then the assumptions are satisfied and $\left\| \sum\limits_{n=1}^{100} x_n \right\| = \sqrt{1090}$.

3.44 (✍) Suppose that $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$. It is clear that $\{e_{2n}\}_{n \in \mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$, too. Let $x = e_1$, then we have Hence $\sum\limits_{n=1}^{\infty} |\langle x,\ e_{2n} \rangle|^2 = 0 < 1 = \|x\|^2$, which means that the Bessel inequality holds with strict inequality.

3.45 (✍) If $\{e_n\}$ has a convergent subsequence $\{e_{n_k}\}$, then it is a Cauchy sequence, then we can choose n_{k_1} and n_{k_2}, with $n_{k_1} \neq n_{k_2}$, tending to the infinity, such that $\|e_{n_{k_1}} - e_{n_{k_2}}\| \to 0$. But $\|e_{n_{k_1}} - e_{n_{k_2}}\| = \sqrt{\|e_{n_{k_1}}\|^2 + \|e_{n_{k_2}}\|^2} = \sqrt{2}$, which is a contradiction.

3.46 (✍) Let $s_n = \sum_{i=1}^{n} x_i$ and $t_n = \sum_{i=1}^{n} \|x_i\|^2$, then for all $m, n \in \mathbb{N}$ with $n > m$ we have

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\langle \sum_{i=m+1}^{n} x_i, \sum_{i=m+1}^{n} x_i \right\rangle = \sum_{i=m+1}^{n} \langle x_i, x_i \rangle \\ &= \sum_{i=m+1}^{n} \|x_i\|^2 = t_n - t_m, \end{aligned}$$

so $\{s_n\}$ is a Cauchy sequence in $\mathcal{H}$ if and only if $\{t_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathcal{H}$ is a Hilbert space, it yields that $\{s_n\}$ is convergent in $\mathcal{H}$ if and only if $\{t_n\}$ is convergent in $\mathbb{R}$.

3.47 (✍) Applying the Riesz-Fischer theorem we obtain that

(i) $\sum_{n=1}^{\infty} \frac{e_n}{n}$ is convergent in $\mathcal{H}$ since $\sum_{n=1}^{\infty} n^{-2} < \infty$, i.e., the sequence $\{n^{-1}\} \in \ell^2$, and

(ii) $\sum_{n=1}^{\infty} \frac{e_n}{\sqrt{n}}$ is not convergent in $\mathcal{H}$ since $\sum_{n=1}^{\infty} n^{-1}$ is divergent, i.e., the sequence $\{\sqrt{n}^{-1}\} \notin \ell^2$.

3.48 (✍) Let $\{e_n\}$ be an orthonormal sequence in $\mathcal{H}$ and set $x_n = e_n/n$, $n = 1, 2, \cdots$. Then for all $m, n \in \mathbb{N}$ with $m > n$ we have

$$\left\| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=n+1}^{m} \frac{1}{k^2} \to 0 \quad \text{as } n \to \infty,$$

which means that the sequence of the partial sums of the series $\sum_{k=1}^{\infty} x_k$ is a Cauchy sequence in $\mathcal{H}$, so $\sum_{k=1}^{\infty} x_k \in \mathcal{H}$ since $\mathcal{H}$ is complete. However $\sum_{k=1}^{\infty} \|x_k\| = \sum_{k=1}^{\infty} 1/k = \infty$.

3.50 (✍) By the assumption, we know

$$\sum_{i=1}^{\infty}|\alpha_i|^2 = \|x\|^2 < \infty, \quad \sum_{i=1}^{\infty}|\beta_i|^2 = \|y\|^2 < \infty. \tag{E3-10}$$

Denote $x_n = \sum\limits_{i=1}^{n} \alpha_i e_i, y_n = \sum\limits_{i=1}^{n} \beta_i e_i, n = 1, 2, ...,$ then

$$\langle x_n, y_n\rangle = \left\langle \sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\beta_j}\langle e_i, e_j\rangle = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}.$$

Which gives that

$$\langle x, y\rangle = \lim_{n\to\infty}\langle x_n, y_n\rangle = \sum_{i=1}^{\infty}\alpha_i\overline{\beta_i}$$

by the continuity of inner product. It follows from the Cauchy inequality and (E3-10) that

$$\sum_{i=1}^{\infty}|\alpha_i\overline{\beta_i}| \leqslant (\sum_{i=1}^{\infty}|\alpha_i|^2)^{1/2}(\sum_{i=1}^{\infty}|\beta_i|^2)^{1/2} = \|x\|\|y\| < \infty.$$

3.52 (✍)

(i) By the Cauchy inequality and the Bessel inequality we have

$$\begin{aligned}\sum_{n=1}^{\infty}|\langle x, e_n\rangle\langle y, e_n\rangle| &\leqslant \left(\sum_{n=1}^{\infty}|\langle x, e_n\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|\langle y, e_n\rangle|^2\right)^{\frac{1}{2}} \\ &\leqslant \|x\|\,\|y\|.\end{aligned}$$

(ii) If $\{e_n\}$ is an orthonormal basis in $\mathcal{H}$, then for any $x, y \in \mathcal{H}$ we have

$$x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n \quad \text{and} \quad y = \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n,$$

which leads that

$$\langle x, y\rangle = \sum_{n,m=1}^{\infty}\langle\langle x, e_n\rangle e_n, \langle y, e_m\rangle e_m\rangle = \sum_{n=1}^{\infty}\langle x, e_n\rangle\langle e_n, y\rangle,$$

the desired Parseval relation.

Conversely, suppose that the above the Parseval relation for all $x, y \in \mathcal{H}$, then we clearly have

$$\|x\|^2 = \langle x,\ x\rangle = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2.$$

Hence, $\{e_n\}$ is an orthonormal basis in $\mathcal{H}$ by Theorem 3.3.3 and Proposition 3.3.1.

要证 $\{f_n\}^{\perp} = \{0\}$

3.54 (✍) For every $y \in \{f_n, n \in \mathbb{N}\}^{\perp} \subset \mathcal{H}$ we have $y = \sum\limits_{n=1}^{\infty} \langle y, e_n\rangle e_n$.

Since $\langle y, f_n\rangle = 0$, it follows that

$$y = \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n - \sum_{n=1}^{\infty}\langle y, f_n\rangle e_n = \sum_{n=1}^{\infty}\langle y, e_n - f_n\rangle e_n.$$

We claim $y = 0$. In fact, if $y \neq 0$, then we could obtian that

$$\begin{aligned}
\|y\|^2 &= \left\langle \sum_{n=1}^{\infty}\langle y, e_n - f_n\rangle e_n, \sum_{k=1}^{\infty}\langle y, e_k - f_k\rangle e_k \right\rangle \\
&= \sum_{n=1}^{\infty} |\langle y, e_n - f_n\rangle|^2 \langle e_n, e_n\rangle = \sum_{n=1}^{\infty} |\langle y, e_n - f_n\rangle|^2 \\
&\leqslant \sum_{n=1}^{\infty} \|y\|^2 \|e_n - f_n\|^2 \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \|y\|^2 \sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \|y\|^2 \quad \text{(by the assumption)},
\end{aligned}$$

a contradiction. Hence $y = 0$, so that $\{f_n, n \in \mathbb{N}\}^{\perp} = \{0\}$. By Theorem 3.3.3 we know $\{f_n\}$ is an orthonormal basis in $\mathcal{H}$.

3.xx Clearly, $\{\widehat{e}_n\}$ and $\{\widetilde{e}_n\}$ are orthonormal sequences in ℓ^2. One way to prove whether or not the sequences are bases of ℓ^2 is to check whether or not there exists a nonzero vector $x = (x_1, x_2, \ldots)$ in ℓ^2 which is orthogonal to all the vectors from the sequence.

(i) We have $x_1 + 2x_2 = 0, x_3 + 2x_4 = 0, \cdots$. The vector $x = (1, -1/2, 1/4, \cdots)$ belongs to ℓ^2 and satisfies the equalities. Hence, the sequence $\{\widehat{e}_n\}$ is not an orthonormal basis in ℓ^2.

(ii) We obtain $x_1 - x_2 = 0$ and $x_3 - x_4 = 0, \cdots$. The vector $x = (1, 1, 1/2, 1/2, 1/3, 1/3, \cdots)$ belongs to ℓ^2 and satisfies the equalities. Hence, the sequence is not an orthonormal basis in ℓ^2.