Portfolio Theory
Solutions to Tutorial 1

1. The density of the uniform distribution on [1.02, 1.08] is

$$f(x) = \frac{1}{0.06}.$$

Therefore

$$\begin{aligned}
\mathrm{E}(A) &= \frac{1}{0.06}\int_{1.02}^{1.08} x\,dx = \frac{1}{0.12}\cdot x^2\Big|_{1.02}^{1.08} \\
&= 1.05 \\
\mathrm{Var}(A) &= \frac{1}{0.06}\int_{1.02}^{1.08} (x-1.05)^2\,dx = \frac{1}{0.18}(x-1.05)^3\Big|_{1.02}^{1.08} \\
&= \frac{1}{0.18}(0.03^3 + 0.03^3) = 3\times 10^{-4}.
\end{aligned}$$

For asset B we have

$$\begin{aligned}
\mathrm{E}(B) &= \frac{1}{4}(1.03+1.04+1.06+1.07) = 1.05, \\
\mathrm{Var}(B) &= \frac{1}{4}(0.02^2+0.01^2+0.01^2+0.02^2) = 2.5\times 10^{-4}
\end{aligned}$$

For any random level of wealth X we have:

$$\mathrm{E}(u(X)) = \mathrm{E}(X) - \alpha(\mathrm{Var}(X) + \mathrm{E}(X)^2)$$
$$\Rightarrow \mathrm{E}(u(A)) - \mathrm{E}(u(B)) = -\alpha[\mathrm{Var}(A) - \mathrm{Var}(B)] < 0$$

as A and B have the same mean. Hence the investor will choose B.

2. (a) Let w be the initial wealth, c be the certainty equivalent and X be the uncertain increase in wealth. Then

$$u(w+c) = \mathrm{E}(u(w+X))$$

i.

$$\begin{aligned}
2\sqrt{c} &= 0.5\times 2\sqrt{0} + 0.5\times 2\sqrt{2000} = \sqrt{2000} \\
c &= 500
\end{aligned}$$

ii.

$$\begin{aligned}
2\sqrt{1000+c} &= 0.5\times 2\sqrt{1000} + 0.5\times 2\sqrt{3000} \\
c &= 866.03
\end{aligned}$$

(b) In both cases, the agent is prepared to accept *with certainty* much less than the expected value of the uncertain increase in wealth (£1,000). This is because the agent is risk averse at all levels of wealth ($u''(x) < 0 \quad \forall x$). However, the individual (*i.e.* utility function) has decreasing absolute risk aversion (and constant relative risk aversion). This means that with more wealth, the individual will be prepared to take more risks and, in particular, will require a higher certainty equivalent to be persuaded not to accept the uncertain increase in wealth.

3. We know that for exponential utility function, the current wealth does not affect the decision. From the given information, we know $\mathrm{E}(u(A)) > \mathrm{E}(u(B)) > \mathrm{E}(u(C))$. Hence,

$$\begin{aligned}
\frac{1}{4}(1-\exp(-5\alpha)) + \frac{3}{4}(1-\exp(-9\alpha)) &> 1-\exp(-7\alpha) \\
1 - 4\exp(-2\alpha) + 3\exp(-4\alpha) &< 0 \\
(1-3\exp(-2\alpha))(1-\exp(-2\alpha)) &< 0 \\
\Rightarrow \quad 1 > \exp(-2\alpha) > \frac{1}{3}, & \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
1-\exp(-7\alpha) > \frac{1}{3}(1-\exp(-5\alpha)) &+ \frac{2}{3}(1-\exp(-9\alpha)) \\
1 - 3\exp(-2\alpha) + 2\exp(-4\alpha) &> 0 \\
(1-2\exp(-2\alpha))(1-\exp(-2\alpha)) &> 0 \\
\Rightarrow \quad \exp(-2\alpha) < \frac{1}{2} \text{or} \exp(-2\alpha) > 1 & \qquad \text{(rejected)} .
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{1}{2} > \exp(-2\alpha) > \frac{1}{3} \\
\frac{1}{2}\ln 3 > \alpha > \frac{1}{2}\ln 2.
\end{aligned}$$

4. (a) The expected claim on this policy is

$$100,000 \times 0.01 + 10,000 \times 0.05 = 1,500$$

Hence the insurer's expected profit is zero if the premium is £1,500. In practice the insurer will want a slightly higher premium to allow for expenses and profit.

(b) Let P be the maximum premium the homeowner is willing to pay. If the homeowner buys insurance he will definitely have final wealth of $200,000 - P$. If the homeowner does not take out insurance, then his wealth is uncertain and will be:

200,000 - 100,000 = 100,000 with probability 0.01
200,000 - 10,000 = 190,000 with probability 0.05
200,000 with probability $1 - 0.01 - 0.05 = 0.94$

To calculate the maximum price of the insurance we equate the utility of buying the insurance with the utility of not buying the insurance.

$$\begin{aligned}
u(200,000 - P) &= 0.01 \times u(100,000) + 0.05 \times u(190,000) \\
&\quad +0.94 \times u(200,000) \\
\log(200,000 - P) &= 0.115129 + 0.607739 + 11.473708 \\
P &= £1,890.24
\end{aligned}$$

(c) By selling insurance to thousands of similar homeowners, the insurer is effectively spreading the risk among all the policyholders and in order to break even the insurer can simply charge a premium equal to the expected loss because their experience should not differ much from the expected value (the insurer's utility function is therefore equivalent to $u(x) = x$ – there is no risk-aversion on the part of the insurer). However, the individual homeowner cannot do this – if he does not buy insurance he has to take on the entire risk himself and he is willing to pay over 25% more than the expected value for the insurance because of fear of a large loss. The insurer must charge at least £1,500 to break even. The insurer cannot charge more than £1,890.24, because that is the maximum homeowners are willing to pay. The actual premium will be determined by competition among insurers.

5. (a)

$$\begin{aligned}
u(x) &= \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma} \\
u'(x) &= a\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1} \\
u''(x) &= -a^2\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-2} \\
ARA(x) &= a\left(\frac{ax}{1-\gamma}+b\right)^{-1} = \frac{1}{x/(1-\gamma)+b/a}.
\end{aligned}$$

Thus, $c = (1-\gamma)^{-1}$, $d = b/a$.

(b) We change the expression of the utility function

$$u(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma} = (1-\gamma)^{1-\gamma}\frac{(ax+b(1-\gamma))^{\gamma}}{\gamma}.$$

When $\gamma \to 1$, $(ax+b(1-\gamma))^{\gamma}/\gamma \to ax$. We use the L'Hospital rule and obtain

$$\lim_{\gamma\to 1}(1-\gamma)^{1-\gamma} = \exp\left(\lim_{\gamma\to 1}\frac{\ln(1-\gamma)}{(1-\gamma)^{-1}}\right) = \exp\left(\lim_{\gamma\to 1}\frac{-(1-\gamma)^{-1}}{(1-\gamma)^{-2}}\right) = 1.$$

We have

$$\lim_{\gamma\to 1} u(x) = ax,$$

which is a linear utility function.