Portfolio Theory
Solutions to Tutorial 5

1. (a) Assets A and B have the same expected return, 60. Hence all portfolios constructed from A and B must also have expected return 60. From Tutorial 4, Question 5(c), we know that the lowest standard deviation is achieved by the portfolio with $\pi_A = 4/5$ and $\pi_B = 1/5$, giving $\sigma = \sqrt{120} = 10.95$. The largest standard deviation without short sales must occur if we invest in B, i.e. $\sqrt{600} = 24.49$. Hence the opportunity set is the straight line segment:

$$\mu = 60 \qquad \text{for} \qquad 10.95 \leq \sigma \leq 24.49$$

The efficient frontier in this unusual case is simply a single point where $\mu = 60$ and $\sigma = 10.95$.

(b) The efficient frontier with a risk free asset is the straight line with the steepest gradient that passes through the risk-free asset and touches the risky efficient frontier. The line must pass through $\mu = 50$, $\sigma = 0$ (risk-free asset) and $\mu = 60$, $\sigma = 10.95$ (the ONLY point on the risky efficient frontier). The line cannot extend beyond these 2 points as short sales and borrowing are not allowed. The equation of this line is:

$$\mu = 50 + \frac{10}{10.95}\sigma \qquad \text{for} \qquad 0 \leq \sigma \leq 10.95$$

(c) If short sales are allowed we can create an asset with infinite variance from A and B. However we cannot improve on the minimum variance. Hence the opportunity set in (a) is a half-line:

$$\mu = 60 \qquad \text{for} \qquad 10.95 \leq \sigma$$

The efficient frontier in (a) is unchanged.

We can now borrow risk-free to invest in risky assets. The efficient frontier in (b) is now extended as a straight line out to infinity:

$$\mu = 50 + \frac{10}{10.95}\sigma \qquad \text{for} \qquad 0 \leq \sigma$$

(d) All investors will invest in the same risky portfolio $R = 0.8A + 0.2B$, *i.e.* everyone invests four times as much in A as B. Investors' preference for risk is determined by the proportion they invest in the risk free asset.

2. Consider any two portfolios A and B on the efficient frontier. Let B be the portfolio with higher mean ($\mathrm{E}[B] > \mathrm{E}[A]$). By definition, no portfolio on the efficient frontier can have lower mean *and* higher variance than any other portfolio. Hence B also has higher variance.

   We can calculate the mean and standard deviation of a new portfolio, P, which has weight π_A invested in A and weight $1-\pi_A$ invested in B, where $0 \leq \pi_A \leq 1$.

$$\mathrm{E}[P] = \pi_A \mathrm{E}[A] + (1 - \pi_A)\mathrm{E}[B]$$
$$\sigma_P = \sqrt{\pi_A^2 \sigma_A^2 + 2\pi_A(1 - \pi_A)\rho_{AB}\sigma_A\sigma_B + (1 - \pi_A)^2\sigma_B^2}$$

   We can see that the highest standard deviation for P occurs when $\rho_{AB} = 1$, so that:

$$\sigma_P = \pi_A\sigma_A + (1 - \pi_A)\sigma_B$$

   In this worst case scenario, P lies on the chord joining A and B.

   But by definition, the efficient frontier contains the portfolios with the lowest variance for any given level of expected return. Hence the efficient frontier must contain a portfolio with expected return $E[P]$ and standard deviation no greater than $\pi_A\sigma_A + (1 - \pi_A)\sigma_B$. Hence, the efficient frontier (for the section between A and B) must lie on or above the chord joining A and B (i.e. the efficient frontier is concave).

3. (a) $r = 5\pi + 10(1 - \pi) = 10 - 5\pi$.

   (b) Recall the equation for calculating variance of a portfolio,

$$\begin{aligned} \sigma^2 &= 100\pi^2 + 900(1 - \pi)^2 + 2(0.5)(10)(30)\pi(1 - \pi) \\ \sigma &= (700\pi^2 - 1500\pi + 900)^{1/2}. \end{aligned}$$

   (c) By result of (a), we have $\pi = (10 - r)/5$, so

$$\begin{aligned} \sigma &= (700\frac{(10 - r)^2}{5^2} - 1500\frac{10 - r}{5} + 900)^{1/2} \\ &= (28r^2 - 260r + 700)^{1/2}. \end{aligned}$$

   (d) Efficient frontier is the opportunity set with the highest gradient or slope, which can be found by equation

$$g(r) \;=\; \frac{r - r_f}{\sigma} \;=\; \frac{r - 3}{(28r^2 - 260r + 700)^{1/2}}.$$

To find the efficient portfolio C, we differentiate the gradient with respect to r,

$$\begin{aligned}
\frac{dg}{dr} &= (28r^2 - 260r + 700)^{-1}\Big((28r^2 - 260r + 700)^{1/2} - \\
&\qquad (r-3)(1/2)(28r^2 - 260r + 700)^{-1/2}(56r - 260)\Big) \\
&= (28r^2 - 260r + 700)^{-3/2}\Big(28r^2 - 260r + 700 - (r-3)(28r - 130)\Big) \\
&= (28r^2 - 260r + 700)^{-3/2}(-46r + 310).
\end{aligned}$$

[You can alternatively try to maximise $g(r)^2$.]

Consider the point C, we have

$$\left.\frac{dg}{dr}\right|_C = 0 \quad \Rightarrow \quad r_C = \frac{155}{23} = 6.7391, \quad \pi_C = \frac{15}{23} = 0.6522.$$

4. Let π_i be the portfolio of asset i in the minimum variance portfolio. As the n assets are uncorrelated, therefore, the variance of the portfolio is $\sum_{i=1}^n \pi_i^2 \sigma_i^2$. We know that there is a composition constraint for a portfolio,

$$\sum_{i=1}^{n} \pi_i = 1 \quad \Rightarrow \quad \pi_n = 1 - \sum_{j=1}^{n-1} \pi_j.$$

Substitute it into the variance equation, and differentiate the variance with respect to π_i $(i = 1, \ldots, n-1)$, we have

$$\begin{aligned}
\frac{d}{d\pi_i}\left(\sum_{j=1}^{n-1} \pi_j^2 \sigma_j^2 + \Big(1 - \sum_{j=1}^{n-1} \pi_j\Big)^2 \sigma_n^2\right) &= 2\pi_i \sigma_i^2 - 2\Big(1 - \sum_{j=1}^{n-1} \pi_j\Big)\sigma_n^2 = 0 \\
\Rightarrow \qquad \pi_i &= \sigma_i^{-2}\Big(1 - \sum_{j=1}^{n-1} \pi_j\Big)\sigma_n^2.
\end{aligned}$$

Define $p := \Big(1 - \sum_{j=1}^{n-1} \pi_j\Big)\sigma_n^2$, then $\pi_i = p/\sigma_i^2$ for all $i = 1, \ldots, n-1$.

This, of course, is self-referential: π_i depends on p which depends on π_j for $j = 1, \ldots, n-1$. So we need to find p.

By definition of p, we have

$$\frac{p}{\sigma_n^2} = 1 - \sum_{i=1}^{n-1} \pi_i = 1 - \sum_{i=1}^{n-1} \frac{p}{\sigma_i^2} \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{p}{\sigma_i^2} = 1 \quad \Rightarrow \quad p = \Big(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\Big)^{-1}.$$

Therefore, for all $i = 1, \ldots, n$ (note: including $i = n$, and you can check that $\sum_{i=1}^{n} \pi_i = 1$),

$$\pi_i = p\sigma_i^{-2} = \sigma_i^{-2}\Big(\sum_{j=1}^{n} \sigma_j^{-2}\Big)^{-1},$$

and the variance of the portfolio is

$$\sum_{i=1}^{n} (p\sigma_i^{-2})^2 \sigma_i^2 \quad = \quad p^2 \sum_{i=1}^{n} \sigma_i^{-2} \quad = \quad \left(\sum_{i=1}^{n} \sigma_i^{-2}\right)^{-1}.$$