Chapter 5. 解线性方程组的直接方法

Ex] . 用列主元消去法解线性方程组

$$\begin{cases} 12x_1 - 3x_2 + 3x_3 = 15 \\ -18x_1 + 3x_2 - x_3 = -15 \\ x_1 + x_2 + x_3 = 6 \end{cases}$$

并求出系数矩阵 A 的行列式 $\det A$ 的值.

Solve. $Ax = b$. 其中 $A = \begin{bmatrix} 12 & -3 & 3 \\ -18 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 15 \\ -15 \\ 6 \end{bmatrix}$

则.

$$(A \mid b) \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} -18 & 3 & -1 & -15 \\ 12 & -3 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{array}\right] \quad l_{21} = \frac{a_{21}}{a_{11}} = -\frac{2}{3} \quad l_{31} = \frac{a_{31}}{a_{11}} = -\frac{1}{18}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 18 & 3 & -1 & -15 \\ 0 & -1 & \frac{7}{3} & 5 \\ 0 & \frac{7}{6} & \frac{17}{18} & \frac{31}{6} \end{array}\right] \quad l_{32} = -\frac{7}{6}$$

$$\longrightarrow \left[\begin{array}{ccc|c} -18 & 3 & -1 & -15 \\ 0 & -1 & \frac{7}{3} & 5 \\ 0 & 0 & \frac{11}{3} & 11 \end{array}\right].$$

因此解为 $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. $\det A = -66$.

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Ex 9. 用追赶法求解三对角方程组 $Ax=b$，其中

$$A=\begin{bmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&0\\0&0&-1&2&-1\\0&0&0&-1&2\end{bmatrix}\quad b=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

Solve. 设 A 有分解如下：

$$\begin{bmatrix}2&-1&&&\\-1&2&-1&&\\&-1&2&-1&\\&&-1&2&-1\\&&&-1&2\end{bmatrix}=\begin{bmatrix}\alpha_1&&&&\\-1&\alpha_2&&&\\&-1&\alpha_3&&\\&&-1&\alpha_4&\\&&&-1&\alpha_5\end{bmatrix}\begin{bmatrix}1&\beta_1&&&\\&1&\beta_2&&\\&&1&\beta_3&\\&&&1&\beta_4\\&&&&1\end{bmatrix}$$

由公式(3.14)，
$$\begin{cases}b_1=\alpha_1\\ c_1=\alpha_i\beta_1\\ b_i=\alpha_i\beta_{i-1}+\alpha_i, & i=2,3,4,5.\\ c_i=\alpha_i\beta_i & i=2,3,4.\end{cases}$$

其中 b_i，c_i 分别为 A 的主对角下边和上边的元素。

计算可得：$\alpha_1=2$，$\alpha_2=\frac{3}{2}$，$\alpha_3=\frac{4}{3}$，$\alpha_4=\frac{5}{4}$，$\alpha_5=\frac{6}{5}$

$\beta_1=-\frac{1}{2}$，$\beta_2=-\frac{2}{3}$，$\beta_3=-\frac{3}{4}$，$\beta_4=-\frac{4}{5}$

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$$\begin{bmatrix} 2 & & & & \\ -1 & \frac{3}{2} & & & \\ & -1 & \frac{4}{3} & & \\ & & -1 & \frac{5}{4} & \\ & & & -1 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

可得： $y_1 = \frac{1}{2}$. $y_2 = \frac{1}{3}$, $y_3 = \frac{1}{4}$. $y_4 = \frac{1}{5}$. $y_5 = \frac{1}{6}$.

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$$\begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{bmatrix}$$

可得： $x_5 = \frac{1}{6}$. $x_4 = \frac{1}{3}$. $x_3 = \frac{1}{2}$. $x_2 = \frac{2}{3}$. $x_1 = \frac{5}{6}$.

即： $x = \begin{bmatrix} \frac{5}{6} \\ \frac{2}{3} \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$ 为 $Ax = b$ 的解. #.

Remark. 编程求解过程参考读书笔记中的 chase.m 程序.

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Ex10. 用改进的平方根法解线性方程组

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solve. 设

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & l_{21} & l_{31} \\ & 1 & l_{32} \\ & & 1 \end{bmatrix}$$

$\Rightarrow$ $d_1 = 2$, $l_{21} = -\frac{1}{2}$, $l_{31} = \frac{1}{2}$

$d_2 = -\frac{5}{2}$, $l_{32} = -\frac{7}{5}$.

$d_3 = \frac{27}{5}$.

解 $\begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ \frac{1}{2} & -\frac{7}{5} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$\Rightarrow$ $y_1 = 4$, $y_2 = 7$, $y_3 = \frac{69}{5}$.

再由

$$\begin{bmatrix} 2 & & \\ & -\frac{5}{2} & \\ & & \frac{27}{5} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ & 1 & -\frac{7}{5} \\ & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ \frac{69}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ & 1 & -\frac{7}{5} \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & & \\ & -\frac{5}{2} & \\ & & \frac{27}{5} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 4 \\ 7 \\ \frac{69}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -\frac{14}{5} \\ \frac{23}{9} \end{bmatrix}.$$

因此. $x_3 = \frac{23}{9}$, $x_2 = \frac{7}{9}$, $x_1 = \frac{10}{9}$.

即. 方程组之解为 $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{7}{9} \\ \frac{23}{9} \end{bmatrix}$. #