Portfolio Theory
Tutorial 4

1. An investor can choose between two investments, X and Y.

   Investment X: the rate of return is distributed uniformly between -5% and 25%.

   Investment Y: the rate of return is distributed normally with mean 10% and variance $75\%\%$.

   (a) Show that the rate of return on X has the same mean and variance as the rate of return on Y.

   (b) The investor can invest his entire wealth of £1 million in X or Y. Calculate the 95% value at risk for these two investments.

   (c) Comment on your results from parts (a) and (b).

2. There is a game using three dices to determine the value of the prize. Three dices are rolled and the prize equals to the product of the three resulting values. Find the expected value and variance of the prize.

3. Let σ_i and σ_{ij} denote the standard deviation of X_i and the covariance between X_i and X_j, respectively. Hence, $\sigma_{ii} = \sigma_i^2$.

   (a) By using the fact that $\mathrm{Cov}(X_1, X_2) = \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2)$, show
   $$\mathrm{Var}\left(\pi_1X_1 + \pi_2X_2\right) = \pi_1^2\sigma_1^2 + 2\pi_1\pi_2\sigma_{12} + \pi_2^2\sigma_2^2.$$

   (b) Prove
   $$\mathrm{Var}\left(\sum_{i=1}^{n} \pi_i X_i\right) = \sum_{i,j=1}^{n} \pi_i\pi_j\sigma_{ij},$$
   for all positive integer n by mathematical induction.

4. There are two investment vehicles with return following normal distribution. We use X and Y to denote their returns. Given

   $\mathrm{E}(X) = 10\%, \mathrm{Var}(X) = (20\%)^2, \mathrm{E}(Y) = 12\%, \mathrm{Var}(Y) = (30\%)^2, \rho_{XY} = -1.$

   (a) Find the composition of the portfolio formed by X and Y which has no risk.

   (b) If the returns are following other distributions rather than normal, would the result in part (a) change?

5. Investors can choose between 3 investment assets, A, B and C. The returns from these portfolios depend on which of three scenarios occurs; the relevant information is shown in the following table.

Scenario	*Prob*	A	B	C
1	1/3	30	45	30
2	1/3	75	60	60
3	1/3	90	75	90

(a) By considering only the mean and variance of each asset, state which assets are preferred to which others. (Note that you may not be able to distinguish between assets.)

(b) Suppose now that C is independent of A and B (and takes the values in the table above with the probabilities indicated). Suppose also that A and B still take the same values according to the scenario with the same probabilities. Calculate the covariance between each pair of assets. Hence determine any preferences between the following portfolios, using only mean and variance: A, B, C, $(A+B)/2$, $(A+C)/2$, $(B+C)/2$ and $(A+B+C)/3$.

(c) Investors can construct portfolios consisting of weight π_A in A and $1-\pi_A$ in B. Find the minimum variance portfolio with no restrictions on short selling. Find the mean and variance of this portfolio.

(d) Would your answer to (c) be different if short selling is not allowed?