Portfolio Theory
Tutorial 3

1. X and Y are random variables with strictly increasing, continuous distribution functions. If $X \geq_{sd} Y$, show that there exists random variables X^* and Y^* such that

$$X \stackrel{d}{=} X^*, \quad Y \stackrel{d}{=} Y^* \quad \text{and} \quad X^* \geq Y^* \quad a.s..$$

(Note: That is, X and Y have the same distributions as X^* and Y^*, respectively, and $X^* \geq Y^*$ almost surely.)
Hints: Let F_X and F_Y be the distribution function of X and Y, respectively. Consider $X^* = X$ and $Y^* = F_Y^{-1}(F_X(X^*))$.

2. There are three independent random variables, X, Y and Z. Show that if $a > 0$, we have

$$X \geq_{sd} Y \Rightarrow (aX + Z) \geq_{sd} (aY + Z).$$

3. Suppose that $f(x)$ is a strictly increasing, continuous function (it is not a density function!).

   (a) Show that, if $X \geq_{sd} Y$, then $f(X) \geq_{sd} f(Y)$.

   (b) It is known that $X \geq_{sd} Y \Rightarrow \mathrm{E}(X) > \mathrm{E}(Y)$. Using the result in part (a), prove

$$X \geq_{sd} Y \Rightarrow \mathrm{E}(u(X)) > \mathrm{E}(u(Y))$$

   for all strictly increasing continuous utility functions $u(x)$.

4. The random variable X represents the total return on a portfolio from an investment of 90. X has a triangular distribution with the following density function:

$$\begin{aligned} f_X(x) &= (x - 20)/80^2 \qquad \text{for } 20 \leq x \leq 100 \\ f_X(x) &= (180 - x)/80^2 \qquad \text{for } 100 \leq x \leq 180 \end{aligned}$$

   (a) Calculate the semi-variance of X.

   (b) Calculate the variance of X.

   (c) Calculate the probability of a shortfall below 30 for X.

   (d) Calculate the expected shortfall below 30 for X.

   (e) Calculate the 95% VaR and Conditional Tail Expectation for this portfolio.

5. An investor can choose between two investments, A and B.

   Investment A - the rate of return is distributed uniformly between 4% and 10%.

   Investment B - the rate of return can take only discrete values with the following probabilities:

$Prob$	B
$\frac{1}{5}$	12%
$\frac{1}{5}$	10%
$\frac{1}{5}$	8%
$\frac{2}{5}$	3%

   For a unit investment in each of A and B, calculate the following:

   (a) mean return,

   (b) variance of return,

   (c) semi-variance of return,

   (d) the shortfall probability with a level of 1.06,

   (e) the mean shortfall with a level of 1.05.