苏州大学 泛函分析（双语）课程试卷 (A) 卷 共 1 页

(考试形式 开卷 2022 年 6 月)

1. (15 marks) Let $p > 1$. Prove that the standard norm on space ℓ^p can be induced by an inner product if and only if $p = 2$.

2. (15 marks) Let Y be a linear subspace of an inner product space X over $\mathbb{F}$. Show that $x \in Y^\perp$ if and only if $d(x, Y) = \|x\|$.

3. (10 marks) Recall that the sequence $\mathcal{F} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt : n \in \mathbb{N} \right\}$ is an orthonormal basis in the Hilbert space $L^2[-\pi, \pi]$. Prove that $\{\sqrt{\frac{2}{\pi}} \sin nt : n \in \mathbb{N}\}$ is an orthonormal basis in the Hilbert space $L^2[0, \pi]$.

4. (10 marks) Let X be a normed linear space and let f be a linear functional on X. Show that f is bounded (or continuous) on X if and only if the kernel of f, $\mathrm{Ker}(f) = \{x \in X : f(x) = 0\}$, is closed in X.

5. (15 marks) Let $\{a_n\}$ be a sequence of real or complex numbers. Define a linear operator T on ℓ^2 by $Tx = (a_1 x_1, a_2 x_2, \cdots)$ for $x = (x_1, x_2, \cdots) \in \ell^2$. Prove that $T : \ell^2 \to \ell^2$ is bounded if and only if $\{a_n\}$ is bounded and in this case $\|T\| = \sup\limits_{n \geqslant 1} |a_n|$.

6. (15 marks)

   (a) Write down the definitions of weak convergence and strong convergence for a sequence $\{x_n\}$ in a normed linear space X.

   (b) Let $\{T_n\} \subset \mathscr{B}(X, Y)$, where X is a Banach space and Y is a normed linear space. Prove that if for each $x \in X$ the sequence $\{T_n x\}$ is weakly converges in Y, then $\{\|T_n\|\}$ is bounded.

7. (10 marks) Let X be a normed linear space. Show that

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}.$$

8. (10 marks) Let $\mathcal{H}$ be a complex Hilbert and let $T \in \mathscr{B}(\mathcal{H})$. Prove that T is self-adjoint (i.e., $T = T^*$) if and only if for each $x \in \mathcal{H}$, $\langle Tx, x \rangle$ is a real number.