Portfolio Theory
Solutions to Tutorial 8

1. (a) All efficient portfolios lie on a straight line of the form

$$E[R_e] = a + b\sigma_e$$

The risk-free asset is an efficient portfolio with expected return 1.04 and variance 0%%, hence

$$a = 1.04$$

Asset A must also lie on the capital market line since it is efficient. Asset A has expected return 1.13 and *standard deviation* of 10%. Hence

$$1.13 = 1.04 + b \times 0.10 \qquad \Rightarrow \qquad b = 0.9$$

(b) The market portfolio is also an efficient portfolio and so must satisfy the capital market line. We are given that the expected return on the market is 1.09, hence

$$1.09 = 1.04 + 0.9\sigma_M \qquad \Rightarrow \qquad \sigma_M = 5.56\%$$

Hence the variance of the market is 30.9 %%.

2. The security market line for any portfolio, P, is given by

$$E[R_P] = R_F + \beta_P(\mu_M - R_F).$$

All assets should lie on this line. Therefore by considering assets A and B we can obtain two simultaneous equations:

$$1.07 = R_F + 0.8(E[R_M] - R_F)$$
$$1.10 = R_F + 1.2(E[R_M] - R_F).$$

If we multiply the first equation by 1.5 and subtract the second equation we obtain

$$0.505 = 0.5R_F \qquad \Rightarrow \qquad R_F = 1.01$$

Substituting into the first equation yields

$$1.07 = 1.01 + 0.8(E[R_M] - 1.01) \qquad \Rightarrow \qquad E[R_M] = \mu_M = 1.085$$

3. All the assets should have a positive market capitalization. For a market portfolio, the weight of all the component assets should be non-negative. As the market portfolio is efficient, it must be on the efficient frontier and can be formed by the two portfolios X and Y. Let M be the market portfolio and $M = \pi X + (1 - \pi)Y$.

(a) We have

$$\begin{aligned} 0.6\pi + 0.8(1-\pi) &\geq 0 \quad \text{and} \\ 0.2\pi - 0.2(1-\pi) &\geq 0 \quad \text{and} \\ 0.2\pi + 0.4(1-\pi) &\geq 0. \end{aligned}$$

Solve the system of inequalities and obtain

$$\begin{cases} 4 & \geq & \pi \\ \pi & \geq & 1/2 \\ 2 & \geq & \pi \end{cases} \qquad \Rightarrow \qquad 2 \geq \pi \geq 1/2.$$

$\mathrm{E}(X) = 11\% > \mathrm{E}(Y) = 9\%$, hence,
the minimum possible expected return of $M = 1/2(11) + 1/2(9) = 10(\%)$,
the maximum possible expected return of $M = 2(11) - 1(9) = 13(\%)$.

(b) If X is the minimum variance portfolio, it should also the efficient portfolio with the lowest expected return. Therefore, the minimum possible expected return of M should be 11%. The maximum value does not change.

4. (a) Let the returns of asset i and the market portfolio M be R_i and R_M, respectively. We have

$$R_M = \sum_{i=1}^{n} w_i R_i.$$

Using the fact that the return of different assets are mutually independent, we have

$$\begin{aligned} \mathrm{Var}(R_M) &= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 \\ \mathrm{Cov}(R_j, R_M) &= \sum_{i=1}^{n} \mathrm{Cov}(R_j, w_i R_i) &= w_j \sigma_j^2. \end{aligned}$$

By the definition of β_j, we have $\beta_j = \mathrm{Cov}(R_j, R_M)/\mathrm{Var}(R_M) = w_j \sigma_j^2 / \sum_{i=1}^n w_i^2 \sigma_i^2$.

(b) We know that β of a portfolio equals to the weighted average $\beta_i' s$ of the component assets. For the market portfolio,

$$\beta_M = \sum_{j=1}^{n} w_j \beta_j = \sum_{j=1}^{n} w_j \frac{w_j \sigma_j^2}{\sum_{i=1}^n w_i^2 \sigma_i^2} = 1.$$

5. (a) $X_i(T) = R(i,1) \times \ldots \times R(i,T)$

(b) Choose the strategy $i = 1$ or 2 that maximises $E[\log R(i,t)]$
Strategy 1: $E[\log R(1,t)] = (\log 0.9 + \log 1.1 + \log 1.3)/3 = 0.0841$
Strategy 2: $E[\log R(2,t)] = (\log 0.85 + \log 1.05 + \log 1.25 + \log 1.45)/4 = 0.1202$
Strategy 2 is higher, so choose strategy 2 under the Kelly criterion.

(c)

$$\begin{aligned} \text{maximise over } i \;\; E[u(X_i(T))] &= E[\frac{1}{\gamma} \prod_{t=1}^{T} R(i,t)^\gamma] \\ &= \frac{1}{\gamma} \prod_{t=1}^{T} E[R(i,t)^\gamma] \\ &= \frac{1}{\gamma} E[R(i,t)^\gamma]^T \end{aligned}$$

If $\gamma > 0$ this means choose the i that maximises $E[R(i,t)^\gamma]$.

If $\gamma < 0$ this means choose the i that minimises $E[R(i,t)^\gamma]$.

Strategy 1: $E[R(i,t)^\gamma] = 1.046 \ \ (\gamma = 0.5) \ \ 0.8613 \ \ (\gamma = -5)$

Strategy 2: $E[R(i,t)^\gamma] = 1.067 \ \ (\gamma = 0.5) \ \ 0.8802 \ \ (\gamma = -5)$

So Strategy 2 is best for $\gamma = 0.5$ ($\gamma > 0$ so choose the maximum) and Strategy 1 is best for $\gamma = -5$ ($\gamma < 0$ so choose the minimum).

Intuition: There is no first or second order dominance. Strategy 2 has a higher mean but also a higher variance. The higher means that strategy 2 will be preferred by investors with low risk aversion. But as risk aversion increases (γ decreases) the higher variance (and the lower, worst outcome of 0.85) will cause very risk averse investors to select Strategy 1.