Solutions to Marked Exercises of Functional Analysis-An Introduction (2nd Ed.)

S.1 Solutions to Exercises Marked with (✍) of Chapter 1

1.1 (✍) **First solution:** $|a+b| = |a| + |b| \Leftrightarrow \mathbf{Re}\,(\overline{a}b) = |a||b|$. Let $k = a/b$. It gives that $|k| = |a|/|b|$, then $k = \mathbf{Re}\,(k) = |a|/|b| \geqslant 0$.

Second solution: $b \neq 0 \Rightarrow a = kb$ for some $k \in \mathbb{C}$. Denote $k := x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, then $|a+b| = |a| + |b| \Leftrightarrow x = \sqrt{x^2 + y^2} \Leftrightarrow k = x \geqslant 0$.

1.2 (✍) The case of $p = 1$ is trivial. If $p > 1$, then by the Hölder inequality (1.6) we get the result.

1.3 (✍)

(i) By assumptions and the Hölder inequality we have for all $x \in (0,1)$,

$$\begin{aligned} |f(x)| &= \Big| \int_0^x f'(t)\,\mathrm{d}t \Big| \leqslant \int_0^x |f'(t)|\,\mathrm{d}t \\ &\leqslant \Big(\int_0^x \mathrm{d}t \Big)^{1/q} \Big(\int_0^x |f'(t)|^p\,\mathrm{d}t \Big)^{1/p} \leqslant x^{1/q} \Big(\int_0^1 |f'(x)|^p\,\mathrm{d}x \Big)^{1/p}. \end{aligned}$$

(ii) By the above (i) we see that for all $x \in (0,1)$ and $r > 0$,

$$\int_0^1 |f(x)|^r \, dx \leqslant \Big(\int_0^1 x^{r/q} \, dx \Big) \Big(\int_0^1 |f'(x)|^p \, dx \Big)^{r/p}$$
$$= \frac{q}{q+r} \Big(\int_0^1 |f'(x)|^p \, dx \Big)^{r/p},$$

then we get the result.

1.5 (✍) By the triangle inequality and the symmetry for a metric we see that $d(x,y) \leqslant d(x,x') + d(x',y') + d(y,y')$ and $d(x',y') \leqslant d(x,x') + d(x,y) + d(y,y')$, then the result.

1.6 (✍)

(i) The function ρ given in (i) is not a metric in $\mathbb{R}$ since the triangle inequality (M3) may not hold. For example, let $x = 1, y = 0$ and $z = -1$, then $\rho(x,y) + \rho(y,z) = 2 < 4 = \rho(x,z)$.

(ii) The function ρ given in (ii) is a metric. Indeed, this ρ clearly satisfies the axioms (M1) and (M2) for a metric. To check the triangle inequality (M3), let $x, y, z \in X$ be arbitrary, then we have $\sqrt{|x-y|} + \sqrt{|y-z|} \geqslant \sqrt{|x-y| + |y-z|} \geqslant \sqrt{|x-z|}$, that is, $\rho(x,y) = \sqrt{|x-y|}$ satisfies (M3).

1.7 (✍)

(i) The function ρ given in (i) obviously satisfies the axioms (M1) and (M2) for a metric since d is a metric. The triangle inequality (M3) follows from the inequality $(\mathbf{T}_2)$ (i.e. (1.1)). Hence ρ is a metric on X.

(ii) Clearly, the function ρ given in (ii) satisfies the axioms (M1) and (M2) for a metric. To check the triangle inequality (M3), let $x, y, z \in X$ be arbitrary.

Case I. Either $d(x,y) \geqslant 1$ or $d(y,z) \geqslant 1$.

If $d(x,z) \geqslant 1$, then

$$\rho(x,y) + \rho(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$$
$$\geqslant 1 = \min\{1, d(x,z)\} = \rho(x,z).$$

If $d(x,z) < 1$, then

$$\rho(x,y) + \rho(y,z) \geqslant 1 > d(x,z) = \min\{1, d(x,z)\} = \rho(x,z).$$

Case II. $d(x, y) < 1$ and $d(y, z) < 1$.

If $d(x, z) < 1$, then

$$\begin{aligned}\rho(x,z) &= \min\{1, d(x,z)\} = d(x,z) \leqslant d(x,y) + d(y,z)\\ &= \min\{1, d(x,y)\} + \min\{1, d(y,z)\} = \rho(x,y) + \rho(y,z).\end{aligned}$$

If $d(x, z) \geqslant 1$, then

$$\begin{aligned}\rho(x,z) &= \min\{1, d(x,z)\} = 1 \leqslant d(x,z) \leqslant d(x,y) + d(y,z)\\ &= \min\{1, d(x,y)\} + \min\{1, d(y,z)\} = \rho(x,y) + \rho(y,z).\end{aligned}$$

1.8 (✍) By Example 1.2.5 s is a metric space. Let $x = \{\xi_k\}, y = \{\eta_k\} \in s$ be arbitrary, then we have

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|} \leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

since

$$\frac{|\xi_k - \eta_k|}{(1 + |\xi_k - \eta_k|} < 1 \quad \text{for all } k \in \mathbb{N},$$

which infers that $\sup\limits_{x,y \in s} d(x, y) \leqslant 1$. For each $n \in \mathbb{N}$ we choose an $x_n = \{\xi_{n,k}\} \in s$ and another $y_0 = \{\eta_{0,k}\} \in s$, with $\xi_{n,k} = n$ for all $k \in \mathbb{N}$ and $\eta_{0,k} = 0$ for all $k \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(x_n, y_0) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{n}{1+n} = \lim_{n\to\infty} \frac{n}{1+n} = 1,$$

it follows that for each $\varepsilon : 0 < \varepsilon < 1$ we may choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1-\varepsilon}{\varepsilon}$, so that $d(x_{n_0}, y_0) > 1 - \varepsilon$, i.e., $\sup\limits_{x,y\in s} d(x, y) = 1$.

1.9 (✍) The function ρ obviously satisfies the axioms (M2) and (M3) for a metric. For arbitrary continuous functions $x(t)$ and $y(t)$ on $[0, 1]$, that is, $x, y \in X$, we trivially have $\rho(x, y) \geqslant 0$. Obverse that $\rho(x, y) = 0$ if and only if $|x(t) - y(t)| = 0$ for all t in $[a, b] \Leftrightarrow x(t) \equiv y(t)$ in $[a, b]$, i.e., $x = y$. Hence ρ satisfies (M1), and so ρ is a metric on X.

1.13 (✍) The conclusion follows immediately from Exercise 1.5. Explicitly, obverse that

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y) \ \to 0 \text{ as } n \to \infty$$

by Exercise 1.5 since both $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ as $n \to \infty$ by the assumptions.

1.14 (✍) Let $\{x_n\}$ be an arbitrary Cauchy sequence in X, then for each $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m > N_1$. Now suppose that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, say, $x_{n_k} \to x$ for some $x_0 \in X$ as $k \to \infty$. It follows that there exists a positive integer K such that $n_K > N_1$ and $d(x_{n_K}, x_0) < \varepsilon/2$. Taking $N = n_K$, we obtain from the above that for all $n > N$

$$d(x_n, x_0) \leqslant d(x_n, x_N) + d(x_N, x_0) < \varepsilon.$$

Thus the whole sequence $\{x_{n_k}\}$ is convergent.

1.15 (✍) *Necessity.* Let $\{x_n\}$ be a sequence in (X, d) with $x_n \to x \in X$ as $n \to \infty$, that is, for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ and choose $K \in \mathbb{N}$ such that $n_K > N$, then for the above $\varepsilon > 0$ we have $d(x_{n_k}, x) < \varepsilon$ for all $k > K$, which means that $x_{n_k} \to x$ as $k \to \infty$.

Sufficiency. Suppose that every subsequence of $\{x_n\}$ converges. Since $\{x_n\}$ itself is a subsequence of $\{x_n\}$, it follows that $\{x_n\}$ converges.

1.16 (✍) It is easy to check. In fact, for each $k \in \mathbb{N}$ let $x_k = (\xi_1^{(k)}, \cdots, \xi_n^{(k)}) \in \mathbb{F}^n$ and $x = (\xi_1, \cdots, \xi_n) \in \mathbb{F}^n$, then

$$d(x_k, x) = \sqrt{\sum_{i=1}^{n} \left|\xi_i^{(k)} - \xi_i\right|} \to 0 \text{ as } k \to \infty$$

if and only if for each $i = 1, \cdots, n$, $\left|\xi_i^{(k)} - \xi_i\right| \to 0$ as $k \to \infty$, that is, $\xi_i^{(k)} \to \xi_i$ as $k \to \infty$.

Also, for each $k \in \mathbb{N}$ let $x_n(t) \in C[a, b]$ and $x(t) \in C[a, b]$, then

$$d(x_n, x) = \max_{a \leqslant t \leqslant b} |x_n(t) - x(t)| \to 0 \text{ as } n \to \infty$$

if and only the sequence $\{|x_n(t) - x(t)|\}$ uniformly converges to 0 on $[a, b]$ as $n \to \infty$, that is, $\{x_n(t)\}$ uniformly converges to $x(t)$ as $n \to \infty$.

1.17 (✍) *Necessity.* Let $\{x_n\}$ be a Cauchy sequence in X, i.e., $d(x_n, x_m) \to 0$ as $n, m \to \infty$, in particular, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Sufficiency. Let $p \in \mathbb{N}$ be arbitrary and let $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Since

$$\begin{aligned} d(x_n, x_{n+p}) &\leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+p})\} \\ &\leqslant \max\{d(x_n, x_{n+1}), \max\{d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+p})\}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+p})\} \\ &\leqslant \cdots \\ &\leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \cdots, d(x_{n+p-1}, x_{n+p})\}, \end{aligned}$$

which yields that $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$ by the assumption $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence.

1.20 (✍) The smallest $r = \sqrt{2}$. Note that the condition $y \in S(x, r)$ means that $d(x, y) \max\limits_{t \in [0, 2\pi]} |\sin t - \cos t| \leqslant r$, hence the smallest $r = \sqrt{2}$ since $\max\limits_{t \in [0, 2\pi]} |\sin t - \cos t| = \sqrt{2}$.

1.21 (✍) Let $\{x_n\}$ be a sequence in a metric space (X, d) with $x_n \to x \in X$ as $n \to \infty$, then there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n > N$. Choose $M = \max\{d(x_1, x), \cdots, d(x_N, x), 1\}$, we have $d(x_n, x) < M + 1$ for all $n \in \mathbb{N}$, that is, $\{x_n\} \subset B(x, M + 1)$, hence $\{x_n\}$ is bounded.

1.22 (✍)

(i) Let $x \in A'$ be arbitrary, then x is an accumulation point of A, which means that $B(x, \varepsilon) \backslash \{x\} \cap A \neq \varnothing$ for every $\varepsilon > 0$. Clearly $B(x, \varepsilon) \backslash \{x\} \cap B \neq \varnothing$ for every $\varepsilon > 0$ since $A \subset B$, it gives that x is also an accumulation point of B, hence $x \in B'$, so that $A' \subset B'$.
Let $x \in A^\circ$ be arbitrary, then x is an interior point of A, which means that there exists an open ball $B(x, r)$ such that $B(x, r) \subset A$. Also $B(x, r) \subset B$ since $A \subset B$. It follows that x is also an interior point of B. Hence $x \in B^\circ$ and so $A^\circ \subset B^\circ$.
By the above, $\overline{A} \subset \overline{B}$ is obvious since $\overline{A} = A' \cup A$, $\overline{B} = B' \cup B$ and $A \subset B$.

(ii) Note that $A \subset A \cup B$ and $B \subset A \cup B$, we see that $A' \cup B' \subset (A \cup B)'$ follows from (i). To prove the reverse direction, we let $x \in (A \cup B)'$

be arbitrary, then $B(x,\varepsilon)\backslash\{x\}\cap(A\cup B)\neq\varnothing$ for all $\varepsilon>0$, it follows that either $B(x,\varepsilon)\backslash\{x\}\cap A\neq\varnothing$ or $B(x,\varepsilon)\backslash\{x\}\cap B\neq\varnothing$ must hold for all $\varepsilon>0$ since otherwise $B(x,\varepsilon)\backslash\{x\}\cap(A\cup B)=\varnothing$ for some $\varepsilon>0$, a contradiction. Hence $x\in A'$ or $x\in B'$, so that $x\in A'\cup B'$, thus $(A\cup B)'\subset A'\cup B'$ and then $(A\cup B)'=A'\cup B'$.

(iii) If A is open, then every point of A is an interior point of A by the definition, meaning $A\subset A^\circ$. Since $A^\circ\subset A$ holds trivially, it follows that $A=A^\circ$. On the other hand, if $A=A^\circ$, then A is open since A° is open by noting $(A^\circ)^\circ=A^\circ$.

(iv) If A is closed, then A contains all its accumulation points by the definition, meaning $A'\subset A$. It implies $\overline{A}\subset A$ since $\overline{A}=A\cup A'$, so that $A=\overline{A}$ since $A\subset\overline{A}$. On the other hand, if $A=\overline{A}$, then $\overline{A}$ is closed since $\overline{A}$ is closed by noting $(\overline{A})'=A'\cup(A')'$ and $(A')'\subset A'$ (cf. Remark 1.4.4).

1.23 (✍) Note that X is a subspace of $\mathbb{R}$, by Theorem 1.4.7, we have

(i) $[0,3)$ is closed and open in X, e.g. $[0,3)=[0,3]\cap X=(-1,3)\cap X$, where $[0,3]$ is a closed subset in $\mathbb{R}$ and $(-1,3)$ is an open subset in $\mathbb{R}$.

(ii) $[4,5)$ is open but not closed in X, e.g. $[4,5)=(3,5)\cap X$, where $(3,5)$ is an open subset in $\mathbb{R}$. But for all closed subset F in $\mathbb{R}$, $[4,5)\neq F\cap X$.

(iii) $(6,7)$ is closed and open in X, e.g. $(6,7)=[6,7]\cap X=(6,7)\cap X$, where $[6,7]$ is a closed subset in $\mathbb{R}$ and $(6,7)$ is an open subset in $\mathbb{R}$.

(iv) $\{8\}$ is closed and open in X, e.g. $\{8\}=[7,8]\cap X=(7,9)\cap X$, where $[7,8]$ is a closed subset in $\mathbb{R}$ and $(7,9)$ is an open subset in $\mathbb{R}$.

(v) $[0,3)\cup[4,5)$ is open but not closed in X, which follows from (i) and (ii).

(vi) $[0,3)\cup(6,7)$ is closed and open in X, which follows from (i) and (iii).

(vii) $(6,7)\cup\{8\}$ is closed and open in X, which follows from (i) and (iv).

(viii) $[1,2)$ is not close nor open in X since $[1,2)\neq F\cap X$ for all closed subset F in $\mathbb{R}$ and $[1,2)\neq G\cap X$ for all open subset G in $\mathbb{R}$.

(ix) $(1,2)$ is open but not closed in X, e.g. $(1,2) = (1,2) \cap X$, where $(1,2)$ is an open subset in $\mathbb{R}$. But $(1,2) \neq F \cap X$ for all closed subset F in $\mathbb{R}$.

(x) $[1,2]$ is closed but not open in X, e.g. $[1,2] = [1,2] \cap X$, where $[1,2]$ is a closed subset in $\mathbb{R}$. But $[1,2] \neq G \cap X$ for all open subset G in $\mathbb{R}$.

1.24 (✍) Since each open ball is an open set in X, it follows by Theorem 1.4.2 that if A is a union of some open balls, then A is open. On the other hand, if A is open, then for each $a \in A$ there exists an open ball $B(a, r_a) \subset A$ of radius $r_a > 0$ centered at a, which implies that $\bigcup\limits_{a\in A} B(a, r_a) \subset A$. Clearly, $A \subset \bigcup\limits_{a\in A} B(a, r_a)$, so that $A = \bigcup\limits_{a\in A} B(a, r_a)$.

1.25 (✍) Since A° is open, it follows that $A^\circ \subset \bigcup_{\substack{G\subset A \\ G \text{ open}}} G$. On the other hand, If G is open such that $G \subset A$, then $G = G^\circ$ and so $G^\circ \subset A^\circ$ by Exercises 1.22. Hence $G \subset A^\circ$, so that $A^\circ = \bigcup_{\substack{G\subset A \\ G \text{ open}}} G$.

Since $\overline{A}$ is closed and $\overline{A} \supset A$, it gives that $\overline{A} \supset \bigcap_{\substack{F\supset A \\ F \text{ closed}}} F$. If F is closed such that $F \supset A$, then $F = \overline{F}$ and $\overline{F} \supset \overline{A}$ by Exercise 1.22. Hence $F \supset \overline{F}$, so that $\overline{A} = \bigcap_{\substack{F\supset A \\ F \text{ closed}}} F$.

1.29 (✍) Suppose that $\{x_n\}$ is an arbitrary Cauchy sequence in a metric space (X, d). So there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m > N$. Let $M = 1 + \max\limits_{1\leqslant i\leqslant N} d(x_{N+1}, x_i)$. Clearly, $\{x_n\} \subset B(x_{N+1}, M)$. Thus $\{x_n\}$ is bounded in X.

1.30 (✍) Let $\{x_n\}$ be an arbitrary Cauchy sequence in (X, ρ), then for each $\varepsilon > 0$ with $\varepsilon < 1/2$ there exists an $N \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon < 1/2$ for all $m, n > N$. Noting that

$$d(x_m, x_n) = \frac{\rho(x_m, x_n)}{1 - \rho(x_m, x_n)} < 2\rho(x_m, x_n) < \varepsilon \quad \text{for all } m, n > N,$$

we see that $\{x_n\}$ is also a Cauchy sequence in (X, d) and then there exists an $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ since (X, d) is complete. It follows that $\rho(x_n, x) \to 0$ as $n \to \infty$ since $\rho(x_n, x) < d(x_n, x)$, that is, $\{x_n\}$ converges to x in (X, ρ) as $n \to \infty$. Hence (X, ρ) is complete.

1.31 (✍) We first consider $\mathbb{F}^n$. Let $\{x_m\}$ be an arbitrary Cauchy sequence in $\mathbb{F}^n$, writing $x_m = \left(\xi_1^{(m)}, \cdots, \xi_n^{(m)}\right)$. Then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$d\left(x_m, x_r\right) = \Big(\sum_{j=1}^{n} \big|\xi_j^{(m)} - \xi_j^{(r)}\big|^2\Big)^{1/2} < \varepsilon \quad \text{for all } m, r > N. \tag{E1-1}$$

Squaring, we have for $m, r > N$ and $j = 1, \cdots, n$, $\big|\xi_i^{(m)} - \xi_i^{(n)}\big| < \varepsilon$. This shows that for each fixed j, $(1 \leqslant j \leqslant n)$, the sequence $\left(\xi_j^{(1)}, \xi_j^{(2)}, \cdots\right)$ is a Cauchy sequence of real or complex numbers, so it converges by the Cauchy convergence criterion for sequences of numbers, say, $\xi_j^{(m)} \to \xi_j \in \mathbb{F}$ as $m \to \infty$. Using these n limits, we define $x = \left(\xi_1, \cdots, \xi_n\right)$. Clearly, $x \in \mathbb{F}^n$. Letting $n \to \infty$ in (E1-1), we have $d(x_m, x) \leqslant \varepsilon$ for all $m > N$. Which infers that x is the limit of $\{x_m\}$ and proves completeness of $\mathbb{F}^n$ because $\{x_m\}$ was an arbitrary Cauchy sequence. Completeness of $\mathbb{F}$ follows from that of $\mathbb{F}^n$ with $n = 1$.

To prove the completeness of $C[a, b]$, we consider any Cauchy sequence $\{x_m\}$ in $C[a, b]$. Then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$d\left(x_m, x_n\right) = \max_{t \in [a,b]} \big|x_m(t) - x_n(t)\big| < \varepsilon \quad \text{for all } m, n > N.$$

It follows by the Cauchy uniform convergence criterion for sequences of functions that there exists an $x \in C[a, b]$ such that $x_n(t)$ converges to $x(t)$ uniformly on $[a, b]$ as $n \to \infty$. This shows that $d(x_n, x) = \max_{t \in [a,b]} \big|x_n(t) - x(t)\big| \to 0$ as $n \to \infty$. Hence $C[a, b]$ is complete.

1.32 (✍) By Example 1.5.3, we knew that ℓ^∞ is complete. Clearly c, c_0 are the subspaces of ℓ^∞. If we show that c and c_0 are closed in ℓ^∞ then c and c_0 are complete as a consequence of Theorem 1.5.1. We now prove that c, c_0 are closed in ℓ^∞.

In fact, suppose that $x = \{\xi_i\}$ is an arbitrary point of $\overline{c}$, the closure of c in ℓ^∞. Then there exists a sequence $\{x_n\}$ in c, writing $x_n = \{\xi_{n,i}\}$, such that $x_n \to x$ as $n \to \infty$. Hence, for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for each $i \in \mathbb{N}$, $|\xi_{N,i} - \xi_i| \leqslant \sup_{i \in \mathbb{N}} |\xi_{N,i} - \xi_i| = d(x_N, x) < \varepsilon$. Since $x_N = \{\xi_{N,i}\}$ is a convergent sequence of real or complex numbers, there exists an $M \in \mathbb{N}$ such that $|\xi_{N,i} - \xi_{N,j}| < \varepsilon$ for all $i, j > M$. So $|\xi_i - \xi_j| \leqslant |\xi_{N,i} - \xi_i| + |\xi_{N,j} - \xi_j| + |\xi_{N,i} - \xi_{N,j}| \leqslant 3\varepsilon$ for all $i, j > M$. This shows that $\{\xi_i\}$ is a convergent sequence of real or complex numbers. Hence $x \in c$, which proves closedness of c in ℓ^∞.

To prove the closedness of c_0 in ℓ^∞, we consider an arbitrary $x = \{\xi_i\} \in \overline{c_0}$. In the same way like above, given any $\varepsilon > 0$, there exists $N > 0$ such that for each $i \in \mathbb{N}$, $|\xi_{N,i} - \xi_i| < \varepsilon$. Since $x_N = \{\xi_{N,i}\}$ is a sequence of real or complex numbers which converges to 0 as $n \to \infty$, there exists $M > 0$ such that $|\xi_{N,i}| < \varepsilon$ for all $i > M$. So $|\xi_i| \leqslant |\xi_{N,i}| + |\xi_i - \xi_{N,i}| < 2\varepsilon$ for all $i > M$. This shows that $\{\xi_i\}$ is a null sequence of real or complex numbers, so that $x \in c_0$. Hence c_0 is closed in ℓ^∞.

1.33 (✍) Let $\{x_n\}$ be an arbitrary Cauchy sequence in a discrete metric space $\mathcal{D}$. Then there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m > N$, which means that $x_m = x_n = x_{N+1}$ for all $n, m > N$ by the definition of the discrete metric. Hence for each $\varepsilon > 0$ we have $d(x_n, x_{N+1}) = 0 < \varepsilon$ for all $n > N$, this shows that $\{x_n\}$ converges to x_{N+1} as $n \to \infty$ and proves the completeness of $\mathcal{D}$.

1.34 (✍) Let any $x \in \overline{Y}$, then there exists $\{x_n\} \subset Y$ such that $x_n \to x$ as $n \to \infty$, which gives that $x_n(a) \to x(a)$ as $n \to \infty$ and $x_n(b) \to x(b)$ as $n \to \infty$. Since $x_n(a) = x_n(b)$, we have $x(a) = x(b)$, thus $x \in Y$. Hence Y is closed in $C[a, b]$, so Y is complete by Theorem 1.5.1.

1.35 (✍) *Necessity.* We let $d_n = \sup\{d(x, y) : x, y \in A_n\}$ and let $x_n \in A_n$ for each $n \in \mathbb{N}$. Similarly as in the proof of Theorem 1.5.2, we can show that $\{x_n\}$ is a Cauchy sequence in X and then it converges to $x \in X$ since (X, d) is complete. It follows that $x \in A_n$ for all $n \in \mathbb{N}$ since A_n is closed for each $n \in \mathbb{N}$. Hence $x \in \bigcap_{n=1}^{\infty} A_n$. If $y \in \bigcap_{n=1}^{\infty} A_n$ then $d(x, y) \leqslant d_n$ for all $n \in \mathbb{N}$, which gives that $d(x, y) = 0$ and then $x = y$. Therefore $\bigcap_{n=1}^{\infty} A_n = \{x\}$.

Sufficiency. Suppose that $\{x_n\}$ is an arbitrary Cauchy sequence in (X, d). Then for $\varepsilon = 1/2^{k+1}$ we choose $n_k \in \mathbb{N}$ such that $d(x_{n_k}, x_m) < 1/2^{k+1}$ for all $m > n_k$. We may assume that $\{n_k\}$ is increasing and we set $A_k = \{x \in X : d(x_{n_k}, x) < 1/2^k\}$, then $A_{k+1} \subset A_k$ since for each $y \in A_{k+1}$,

$$d(x_{n_k}, x) \leqslant d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x) < 1/2^k$$

and $\sup\{d(x,y) : x, y \in A_n\} \to 0$ as $n \to \infty$. Hence there exists $x \in \bigcap_{n=1}^{\infty} A_n$ such that $d(x_{n_k}, x) \leqslant 1/2^k$, which means that $x_{n_k} \to x$ as $k \to \infty$. By Exercise 1.14 we see that $x_n \to x$ as $n \to \infty$.

1.36 (✍) For each $n \in \mathbb{N}$ let $x_n = n$, then $\{x_n\}$ is a Cauchy sequence in the space since $d(x_n, x_m) = |n^{-1} - m^{-1}| \to 0$ as $m, n \to \infty$. But $\{x_n\}$ does not converge in that space since for any positive integer k, $d(x_n, k) = |n^{-1} - k^{-1}| \to k^{-1} \neq 0$ as $n \to \infty$, so that the space is incomplete (not complete).

1.39 (✍) We claim that the sequence $\{x_n\}$ of (X, ρ) in which each x_n is defined by

$$x_n(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < 1/2, \\ nt - n/2, & \text{if } 1/2 \leqslant t \leqslant 1/2 + 1/n, \\ 1, & \text{if } 1/2 + 1/n < t \leqslant 1 \end{cases}$$

is Cauchy but it is not convergent in (X, ρ).

In fact, $\rho(x_n, x_m) = \int_0^1 |x_n(t) - x_m(t)|\, \mathrm{d}t \leqslant |1/m - 1/n|$, which goes to 0 as $m, n \to \infty$, so that $\{x_n\}$ is Cauchy. Let

$$x(t) = \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant 1/2, \\ 1, & \text{if } 1/2 < t \leqslant 1, \end{cases}$$

then this function $x(t)$ is the limits of the sequence of functions $\{x_n(t)\}$ in (X, ρ) as $n \to \infty$ since

$$\rho(x_n, x) = \int_{1/2}^{1/2+1/n} |x_n(t) - x(t)|\, \mathrm{d}t = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty.$$

But the $x \notin X$, which shows that (X, ρ) (not $C[a, b]$) is incomplete.

1.40 (✍)

Necessity. If S is nowhere dense in X, i.e., $\overline{S}$ has no interior point, then for any $x \in X$ and any $\varepsilon > 0$ satisfying $B(x, \varepsilon) \cap \overline{S}^c \neq \varnothing$, so that $x \in \overline{\overline{S}^c}$. It follows that $X = \overline{\overline{S}^c}$, thus $\overline{S}^c$ is dense in X.

Sufficiency. Assume that $\overline{S}^c$ is dense in X, i.e., $X = \overline{\overline{S}^c}$. If $\overline{S}$ would have an interior point x_0, then there could exist $\delta > 0$ such that $B(x_0,\delta) \subset \overline{S}$, so that $B(x_0,\delta) \cap \overline{S}^c = \varnothing$, that is, $x_0 \notin \overline{\overline{S}^c}$, a contradiction. Hence $\overline{S}^\circ = \varnothing$ and S is nowhere dense in X.

1.42 (✍) Since $[0,1]$ is closed in $\mathbb{R}$, we know that $[0,1]$ is complete. If $[0,1]$ were countable, say, $[0,1] = \{x_1, x_2 \cdots\}$, then $[0,1] = \bigcup\limits_{i=1}^{\infty}\{x_i\}$, which is contrary to the Baire category theorem since every single-point set $\{x_i\}$ is nowhere dense in $\mathbb{R}$.

1.44 (✍)

Necessity. By Theorem 1.6.2, we see that for each $r \in \mathbb{R}$ the set $\{x \in X : f(x) < r\} = f^{-1}((-\infty, r))$ is open in X since f is continuous and $(-\infty, r)$ in open in $\mathbb{R}$, so that the set $\{x \in X : f(x) \geqslant r\}$ is closed in X. Similarly the set $\{x \in X : f(x) \leqslant r\}$ is closed in X.

Sufficiency. Clearly, for given $a, b \in \mathbb{R}$ with $a < b$ the set $\{x \in X : a < f(x) < b\}$ is open in X since sets $\{x \in X : f(x) \leqslant a\}$ and $\{x \in X : f(x) \geqslant b\}$ are closed in X by assumption. For each open set G in $\mathbb{R}$, we have $G = \bigcup\limits_{n=1}^{\infty}(\alpha_n, \beta_n)$, where (α_n, β_n) is an open interval for each $n \in \mathbb{N}$, thus

$$f^{-1}(G) = \bigcup_{n=1}^{\infty}\{x \in X : \alpha_n < f(x) < \beta_n\}$$

is open in X, so that f is continuous by Theorem 1.6.2.

1.45 (✍)

(i) If $x \in A$, then

$$0 \leqslant d(x, A) = \inf_{y \in A} d(x, y) \leqslant d(x, x) = 0,$$

so $d(x, A) = 0$.

(ii) The converse of the above (i) is not true in general, for example, let $A = (0, 1]$ in $\mathbb{R}$ and $x = 0$, we have

$$d(x, A) = \inf_{y \in (0,1]} |0 - y| = \inf_{y \in (0,1]} |y| = 0,$$

but $0 \notin (0, 1]$.

(iii) Clearly, $d(x, \overline{A}) \leqslant d(x, A)$ since $A \subset \overline{A}$. Let $b \in \overline{A}$ be arbitrary, then there exists $\{y_n\} \subset A$ such that $y_n \to b$ as $n \to \infty$, which implies from Exercise 1.13 that $d(x, y_n) \to d(x, b)$ as $n \to \infty$. But $d(x, y_n) \geqslant d(x, A)$ for all $n \in \mathbb{N}$, thus we get $d(x, b) \geqslant d(x, A)$ for all $b \in \overline{A}$. Therefore $d(x, \overline{A}) = \inf_{b \in \overline{A}} d(x, b) \geqslant d(x, A)$, so that $d(x, \overline{A}) = d(x, A)$.

Suppose that $x \in \overline{A}$, then there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$ as $n \to \infty$. Hence

$$0 \leqslant d(x, A) = \inf_{a \in A} d(x, a) \leqslant d(x, x_n) \to 0 \text{ as } n \to \infty,$$

so that $d(x, A) = 0$. On the other hand, if

$$d(x, A) = \inf_{a \in A} d(x, a) = 0,$$

then there exists $\{a_n\} \subset A$ such that $d(x, a_n) \to d(x, A) = 0$ as $n \to \infty$. Thus $a_n \to x$ as $n \to \infty$, meaning $x \in \overline{A}$.

(iv) Since

$$\begin{aligned} d(x, A) &= \inf_{a \in A} d(x, a) \leqslant \inf_{a \in A} (d(x, y) + d(y, a)) \\ &= d(x, y) + \inf_{a \in A} d(y, a) = d(x, y) + d(y, A), \end{aligned}$$

it follows that $d(x, A) - d(y, A) \leqslant d(x, y)$. Changing x with y, we have $d(y, A) - d(x, A) \leqslant d(x, y)$, then the conclusion follows.

1.46 (✍) Both the continuity and the uniform continuity of f follow immediately from (iv) of Exercise 1.45. In the following, without recourse to (iv) of Exercise 1.45, we will show that f is uniformly continuous on X, and then f is continuous on X.

Indeed, for every $\varepsilon > 0$, let $\delta = \varepsilon$. By the definition of the infimum there exists $y \in A$ such that $0 \leqslant d(x, y) - f(x) < \varepsilon$ for all $x \in X$. Hence $f(x') - f(x) \leqslant d(x', y) - f(x) \leqslant d(x', x) + d(x, y) - f(x) < 2\varepsilon$ for all $x, x' \in X$ with $d(x, x') < \delta$. Similarly, $f(x) - f(x') < 2\varepsilon$ whenever $d(x, x') < \delta$. Thus $|f(x) - f(x')| < 2\varepsilon$ whenever $d(x, x') < \delta$, so f is uniformly continuous on X.

1.47 (✍) This is a consequence of Exercisers 1.44, 1.46 and Theorem 1.6.2.

1.48 (✍) *Necessity.* Suppose F is a closed subset of a metric space (X, ρ), then

$$F = \{x \in X : \rho(x, F) = 0\}$$

by (iii) of Exercise 1.45 since $\overline{F} = F$. Clearly,

$$\{x \in X : \rho(x, F) = 0\} = \bigcap_{n=1}^{\infty} \{x \in X : \rho(x, F) < 1/n\}$$

and each set $\{x \in X : \rho(x, F) < 1/n\}$ is open in X by Exercise 1.47, hence F is an intersection of a countable number of open sets.

Sufficiency. Let G be an open subset of X, then G^{c} is closed in X, so that $G^{\mathrm{c}} = \bigcap_{n=1}^{\infty} G_n$ with open subsets G_n of X by the above, hence

$$G = \left(\bigcap_{n=1}^{\infty} G_n \right)^{\mathrm{c}} = \bigcup_{n=1}^{\infty} G_n^{\mathrm{c}},$$

where each G_n^c is closed in X. Therefore G is a union of a countable number of closed sets.

1.49 (✍) For each $x \in X$ let

$$f(x) = \frac{d(x, F_1)}{d(x, F_1) + d(x, F_2)},$$

then $f(x)$ is continuous on X by Exercise 1.46,. Since F_1 and F_2 are closed in X, it follows from (i) and (iii) of Exercise 1.45 that $d(x, F_1) = 0$ if $x \in F_1$ and $d(x, F_2) = 0$ if $x \in F_2$. Hence $f(x) = 0$ if $x \in F_1$ and $f(x) = 1$ if $x \in F_2$, where we have used the facts that $d(x, F_1) > 0$ if $x \in F_2$ and $d(x, F_2) > 0$ if $x \in F_1$, since if $x \in F_2$ and $x \in F_1$ respectively, then $x \notin F_1$ and $x \notin F_2$ respectively by assumption $F_1 \cap F_2 = \varnothing$.

1.56 (✍) The necessity is clear since every compact subset in a metric space must be bounded and closed by Theorem 1.7.2.

Sufficiency. Suppose that A is a bounded and closed subset in $\mathbb{R}^n$, with the Euclidean metric. Let $\{x_k\}$ be an arbitrary sequence in A. By the Bolzano-Weierstrass theorem, we see that $\{x_k\}$ has a convergent subsequence $\{x_{n_j}\}$ since $\{x_k\}$ is bounded, say, $x_{n_j} \to x_0$ in $\mathbb{R}^n$ as $j \to \infty$, for some $x_0 \in \mathbb{R}^n$. It is clear that $x_0 \in A$ since A is closed, so that A is compact by the arbitrary choice of $\{x_n\}$.

For readers familiar with the Bolzano-Weierstrass theorem in $\mathbb{R}$ *we will* *give a detailed proof for the sufficiency in the following.*

Let $\{x_k\}$ be an arbitrary sequence in A. Each x_k has the form $x_k = (\xi_{k,1}, \cdots, \xi_{k,n})$ with $\xi_{k,j} \in \mathbb{R}$ for each $k \in \mathbb{N}$, $j = 1, \cdots, n$. Clearly, the sequence of numbers $\{\xi_{k,j}\}$ (j fixed) is bounded by the boundedness of $\{x_k\}$. By the Bolzano-Weierstrass theorem in $\mathbb{R}$, we see that $\{\xi_{k,1}\}$ has a subsequence $\{\xi_{k_1,1}\}$ such that $\xi_{k_1,1} \to \xi_1$ in $\mathbb{R}$ as $k_1 \to \infty$, for some $\xi_1 \in \mathbb{R}$. Also, $\{\xi_{k_1,2}\}$ has a convergent subsequence $\{\xi_{k_2,2}\}$ such that $\xi_{k_2,2} \to \xi_2$ in $\mathbb{R}$ as $k_2 \to \infty$, for some $\xi_2 \in \mathbb{R}$. Continuing in this way, after n steps we obtain a subsequence of numbers $\{\xi_{k_n,n}\}$ such that $\xi_{k_n,n} \to \xi_n$ in $\mathbb{R}$ as $k_n \to \infty$, for some $\xi_n \in \mathbb{R}$. Using these n limits, we define an $x = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$, then by Example 1.3.3 we get that $\{x_k\}$ has a subsequence $\{x_{k_n}\}$ which converges to x as $k_n \to \infty$. It is clear that $x \in A$ since A is closed, so that A is compact by the arbitrary choice of $\{x_n\}$.

1.57 (✍)

(i) X is complete since $[0, 1]$ and $\{2, 3, \cdots\}$ are two closed subspaces in $\mathbb{R}$ so that X is closed in $\mathbb{R}$ by the completeness of $\mathbb{R}$.

(ii) X is separable since the countable set of all rational numbers in $[0, 1]$ and all numbers of $\{2, 3, \cdots\}$ is dense in X.

(iii) X is noncompact since X is not bounded in $\mathbb{R}$.

1.58 (✍) The sufficiency is clear.

Necessity. If A contains infinitely many elements, then there exists a countable subset $\{x_1, x_2, \cdots\}$ such that $x_n \neq x_m$ if $n \neq m$. Hence $d(x_n, x_m) = 1$ if $n \neq m$, so that $\{x_n\}$ cannot have a convergent subsequence, which contradicts the compactness of A. Therefore A is a finite set.

1.60 (✍) For each $n \in \mathbb{N}$, we take $a_n \in A_n$, then $\{a_n\} \subset A_1$ so that there exists subsequence $\{a_{n_k}\} \subset A_1$ such that $a_{n_k} \to a \in A_1$ as $k \to \infty$ since X is compact and A_1 is closed. For each $n \in \mathbb{N}$ there exists $K_n \in \mathbb{N}$ such that $n_k > n$ for all $k > K$. It follows that $a_{n_k} \in A_{n_k} \subset A_n$, so that $a \in A_n$ since each A_n is closed. Hence $a \in \bigcap_{n=1}^{\infty} A_n$, i.e., $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.

1.61 (✍) By the definition of the supremum, we know that there exist two sequences $\{u_n\}$ and $\{v_n\}$ in E such that

$$d(u_n, v_n) \to \sup_{u,v \in E} d(u, v) \text{ as } n \to \infty.$$

Since E is a compact subset of X, it follows that $\{u_n\}$ has a convergent subsequence $\{u_{n_i}\}$ such that $u_{n_i} \to x \in E$ as $i \to \infty$. Again there exist a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ and some $y \in E$ such that $v_{n_{i_j}} \to y$ as $j \to \infty$ since $\{v_{n_i}\}$ is a sequence in E. By the continuity of a metric (cf. Exercise 1.13), we see that

$$d(x, y) = \lim_{j \to \infty} d(u_{n_{i_j}}, v_{n_{i_j}}) = \sup_{u,v \in E} d(u, v).$$

1.62 (✍) By the definition of the infimum, we see that there exist two sequences $\{x_n\} \subset F_1$ and $\{y_n\} \subset F_2$ such that

$$d(x_n, y_n) \to \inf\{d(x, y) : x \in F_1, y \in F_2\} = d(F_1, F_2) \text{ as } n \to \infty.$$

Since F_1 is compact, we have a subsequence x_{n_i} of $\{x_n\}$ and a point $x_0 \in F_1$ such that $x_{n_i} \to x_0$ as $i \to \infty$. Also, by the compactness of F_2, we have a subsequence $y_{n_{i_j}}$ of $\{y_n\}$ and a point $y_0 \in F_2$ such that $y_{n_{i_j}} \to y_0$ as $j \to \infty$. Applying Exercise 1.13, we obtain that

$$d(x_0, y_0) = \lim_{j \to \infty} d(x_{n_{i_j}}, y_{n_{i_j}}) = d(F_1, F_2).$$

1.63 (✍) Since

$$\begin{aligned} d(F_1, F_2) &= \inf\{d(x, y) : x \in F_1, y \in F_2\} \\ &= \inf_{x \in F_1} \inf_{y \in F_2} d(x, y) \\ &= \inf_{x \in F_1} d(x, F_2), \end{aligned}$$

it follows from the definition of the infimum that there exists a sequence $\{x_n\} \subset F_1$ such that

$$d(x_n, F_2) \to \inf_{x \in F_1} d(x, F_2) = d(F_1, F_2) \text{ as } n \to \infty.$$

Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ as $k \to \infty$ by the compaction of F_1, where x is a point in F_1. Suppose that $d(F_1, F_2) = 0$. Clearly,

$$d(x, F_2) = \lim_{k\to\infty} d(x_{n_k}, F_2) = d(F_1, F_2) = 0.$$

By Exercises 1.45 and the closeness of F_2, we see that $x \in F_2$, so that $F_1 \cap F_2 \supset \{x\} \neq \varnothing$.

1.64 (✍) By the assumption that $|f(x) - f(y)| \leqslant L|x - y|$ holds for all $f \in M$ and all $x, y \in [a, b]$, we know that M is equicontinuous. Since also $|f(x_0)| \leqslant m$ for all $f \in M$, we obtain

$$|f(x)| \leqslant |f(x) - f(x_0)| + |f(x_0)| \leqslant L|x - x_0| + m \leqslant L(b - a) + m$$

for all $f \in M$ and all $x \in [a, b]$. i.e., M is uniformly bounded. The Arzelà-Ascoli theorem gives that M is relatively compact in $C[a, b]$.

1.65 (✍) Since

$$\int_a^b |f'(x)|^2\, dx \leqslant \int_a^b \left(|f(x)|^2 + |f'(x)|^2\right)\, dx \leqslant k \text{ for all } f \in M,$$

as in the proof of Example 1.7.3, we have

$$|f(x) - f(a)| \leqslant \sqrt{k(b-a)} = k_1 \text{ for all } f \in M \text{ and } x \in [a, b].$$

Hence $|f(a)| \leqslant |f(x) - f(a)| + |f(x)| \leqslant k_1 + |f(x)|$ for all $f \in M$. Therefore, for all $f \in M$,

$$\begin{aligned}
(b-a)|f(a)| &= \int_a^b |f(a)|\, dx \leqslant \int_a^b k_1\, dx + \int_a^b |f(x)|\, dx \\
&\leqslant k_1(b-a) + \left(\int_a^b |f(x)|^2\, dx\right)^{1/2} \left(\int_a^b 1\, dx\right)^{1/2} \\
&\leqslant k_1(b-a) + (b-a)^{1/2} \left(\int_a^b \left(|f(x)|^2 + |f'(x)|^2\right)\, dx\right)^{1/2} \\
&\leqslant k_1(b-a) + \sqrt{k(b-a)},
\end{aligned}$$

then the conclusion follows from Example 1.7.3.

1.66 (✍) By Definition 1.4.2 we see that there exists an open ball $B(x_0, r) \subset C[a, b]$ such that $M \subset B(x_0, r)$ since M is bounded in $C[a, b]$, which means that $d_{C[a,b]}(x, x_0) = \max_{t\in[a,b]} |x(t) - x_0(t)| < r$ for all $x \in M$. In particular, $\max_{t\in[a,b]} |x(t)| < r + \max_{t\in[a,b]} |x_0(t)|$ for all $x \in M$. Denote $L := r + \max_{t\in[a,b]} |x_0(t)|$. Let $t_1, t_2 \in [a, b]$ with $t_1 \leqslant t_2$, then for all $y \in A$, $x \in M$ we have

$$\begin{aligned}|y(t_2) - y(t_1)| = \left|\int_{t_1}^{t_2} x(s)\, ds\right| &\leqslant \left(\max_{t\in[a,b]} |x(t)|\right)(t_2 - t_1)\\ &\leqslant L(t_2 - t_1),\end{aligned}$$

which yields that $|y(t_1) - y(t_2)| \leqslant L|t_2 - t_1|$ for all $y \in A$ and $t_1, t_2 \in [a, b]$. Note that $y(0) = 0$ for all $y \in A$, so the conclusion follows by Exercise 1.64.

1.67 (✍) Applying the Taylor formula for each function g_n ($n \in \mathbb{N}$), we get that for each $x \in [0, 1]$,

$$g_n(x) = g_n(0) + g_n'(0)x + \frac{g''(\xi_1)}{2}x^2,$$

where $\xi_1 \in (0, x) \subset [0, 1]$. Thus,

$$|g_n(x)| = \frac{|g''(\xi_1)|x^2}{2} \leqslant \frac{1}{2} \text{ for all } x \in [0, 1] \text{ and } n \in \mathbb{N}$$

by the assumptions, i.e., $\{g_n\}$ is uniformly bounded. Now, applying the mean value theorem of differential calculus (the Lagrange formula) twice, we obtain that for all $x, x_0 \in [0, 1]$ and all $n \in \mathbb{N}$,

$$\begin{aligned}|g_n(x) - g_n(x_0)| &= |g'(\xi_2)|\, |x - x_0|\\ &= |g'(\xi_2) - g'(0)|\, |x - x_0|\\ &= |g''(\xi_3)||\xi_2|\, |x - x_0|\\ &\leqslant |\xi_2|\, |x - x_0| \leqslant |x - x_0|,\end{aligned}$$

where ξ_2 lies between x and x_0, $\xi_3 \in (0, \xi_2) \subset [0, 1]$. Hence, $\{g_n\}$ is equicontinuous. The conclusion follows by the Arzelà-Ascoli theorem.

1.68 (✍) Applying the mean value theorem of differential calculus, we have

$$|f(x_1) - f(x_2)| = |f'(\xi)|\, |x_1 - x_2| \leqslant L|x_1 -_2|$$

for all $f \in M$ and $x_{1,2} \in [a, b]$, where ξ lies between x_1 and x_2. Therefore M is equicontinuous. Let $f \in M$ and $x \in [a,b]$, then there exists $x_f \in [a,b]$ such that $f(x_f) = 0$ by the assumption (ii), where x_f depends on the function $f \in M$. Therefore,

$$\begin{aligned}|f(x)| &= |f(x) - f(x_f)| = |x - x_f||f'(\xi_1)| \\ &\leqslant L|x - x_0| \leqslant L(|x| + |x_0|) \leqslant 2L\max\{b, -a\}\end{aligned}$$

for all $f \in M$ and $x \in [a,b]$, where ξ_1 lies between x and x_f, thus, M is uniformly bounded. It follows from the Arzelà-Ascoli theorem that M is relatively compact in $C[a,b]$.

1.69 (✍)

(i) The answer is no, since the family $\{\sin nx : n \in \mathbb{N}\}$ is not equicontinuous by Example 1.7.4 and $\{\sin nx : n \in \mathbb{N}\} \subset \{\sin \alpha x : \alpha \in \mathbb{R}\}$.

(ii) The answer is yes. Noting that both

$$|f_\alpha(x)| = |\sin(\alpha + x)| \leqslant 1 \quad \text{and} \quad |f'_\alpha(x)| = |\cos(x + \alpha)| \leqslant 1$$

hold for all $\alpha \in \mathbb{R}$ and all $x \in [0,1]$. Applying the mean value theorem of differential calculus, we have

$$|f_\alpha(x) - f_\alpha(y)| = |x - y||f'_\alpha(\xi)| \leqslant |x - y|$$

for all $\alpha \in \mathbb{R}$ and all $x, y \in [0,1]$, where ξ lies between x and x. It follows from Exercise 1.64 that $\{f_\alpha : \alpha \in \mathbb{R}\}$ is relatively compact in $C[0,1]$.

(iii) The answer is no, since the considered set $M = \{\arctan(\alpha x) : \alpha \in \mathbb{R}\}$ is not equicontinuous. Indeed, if M were equicontinuous, then for each $\varepsilon > 0$ there would exist $\delta_\varepsilon > 0$ such that for all $x, y \in [0,1]$, with $|x - y| < \delta_\varepsilon$, we could have $|\arctan(\alpha x) - \arctan(\alpha y)| < \varepsilon$. Choosing an $n_\varepsilon \in \mathbb{N}$ such that $1/n < \delta_\varepsilon$ for all $n > n_\varepsilon$, and taking $x = 1/n$, $y = 0$, $\alpha = n$, we get that $|x - y| = 1/n < \delta_\varepsilon$ for all $n > n_\varepsilon$, so, for all $n > n_\varepsilon$, it holds that

$$|\arctan(1) - \arctan(0)| = \frac{\pi}{4} < \varepsilon,$$

which is a contradiction since $\varepsilon > 0$ is arbitrary.

(iv) The answer is yes. Since $|f_\alpha(0)| = |e^{-\alpha}| = e^{-\alpha} \leqslant 1$ for all $\alpha \geqslant 0$, and

$$\int_0^1 |f'_\alpha(x)|^2 \, \mathrm{d}x = \int_0^1 e^{2x-2\alpha} \, \mathrm{d}x = \frac{e^2-1}{2} e^{-2\alpha} \leqslant \frac{e^2-1}{2}$$

for all $\alpha \geqslant 0$, by Example 1.7.3, we see that the considered set $\{f_\alpha : \alpha \geqslant 0\}$ is relatively compact in $C[0,1]$.

1.72 (✍) For example, let $f(x) = \sin x$ for all $x \in (0,1]$, then f is continuous with $f((0,1]) \subset (0,1]$, but f has no fixed point on $(0,1]$ since the only possible solution of the equation $\sin x = x$ is $x = 0 \notin (0,1]$.

1.73 (✍) Since

$$\begin{aligned} d(Tx,Ty) = |Tx - Ty| &= \left| \frac{x-y}{2} + \frac{y-x}{xy} \right| \\ &= \left| \frac{1}{2} - \frac{1}{xy} \right| |x-y| \leqslant \frac{1}{2}|x-y| = \frac{1}{2} d(x,y) \end{aligned}$$

for all $x, y \in [1, +\infty)$, we know that T is a contraction with a contractivity factor $c = 1/2$. The smallest contractivity factor c is also $1/2$ since for any $n \geq 2$ and let $c_n = 1/2 - 1/n < 1/2$, then we can find $x_n = 2$ and $y_n = n$ such that $d(Tx_n, Ty_n) = \left(\frac{1}{2} - \frac{1}{2n}\right) d(x_n, y_n) > c_n d(x_n, y_n)$.

1.74 (✍) Clearly, by Example 1.2.3 we know that $(\mathbb{R}^n, \rho)$ is a metric space.

Sufficiency. Denote $\widehat{c} := \max\limits_{0 \leqslant i \leqslant n} \sum\limits_{j=1}^n |a_{ij}|$ and suppose that $\widehat{c} < 1$. Since for every $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we have

$$\begin{aligned} \rho(Tx,Ty) &= \max_{0 \leqslant i \leqslant n} \left| \sum_{j=1}^n a_{ij} x_j + b_i - \sum_{j=1}^n a_{ij} y_j - b_i \right| \\ &= \max_{0 \leqslant i \leqslant n} \left| \sum_{j=1}^n a_{ij}(x_j - y_j) \right| \\ &\leqslant \max_{0 \leqslant i \leqslant n} \sum_{j=1}^n |a_{ij}||x_j - y_j| \\ &\leqslant \left(\max_{0 \leqslant i \leqslant n} \sum_{j=1}^n |a_{ij}| \right) \max_{0 \leqslant j \leqslant n} |x_j - y_j| \\ &= \widehat{c}\, \rho(x,y), \end{aligned}$$

which gives that T is a contraction mapping on $(\mathbb{R}^n, \rho)$, with a contractivity factor $\widehat{c}$.

Necessity. Suppose that T is a contraction mapping on $(\mathbb{R}^n, \rho)$. Then there exists a $c \in [0, 1)$ such that $\rho(Tx, Ty) \leqslant c\,\rho(x, y)$ for every $x, y \in (\mathbb{R}^n, \rho)$. In particular, this inequality holds for $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^n$ and and $x_k = (x_{k,1}, \cdots, x_{k,n}) \in \mathbb{R}^n$, $k = 1, \cdots, n$, where each $x_{i,j} = \operatorname{sgn} a_{k,j}$ (k fixed), that is,

$$x_{k,j} = \begin{cases} 1, & \text{if } a_{kj} > 0, \\ 0, & \text{if } a_{kj} = 0, \\ -1, & \text{if } a_{kj} < 0. \end{cases}$$

So, $\rho(Tx_k, T\mathbf{0}) \leqslant c\,\rho(x_k, \mathbf{0})$ for all $k = 1, \cdots, n$, which yields that

$$\begin{aligned} \max_{1\leqslant k\leqslant n} \sum_{j=1}^{n} |a_{kj}| &= \max_{1\leqslant k\leqslant n} \sum_{j=1}^{n} a_{kj} \operatorname{sgn} a_{k,j} = \max_{1\leqslant k\leqslant n} \left| \sum_{j=1}^{n} a_{kj} \operatorname{sgn} a_{k,j} \right| \\ &= \max_{1\leqslant k\leqslant n} \left| \sum_{j=1}^{n} a_{kj} x_{k,j} \right| = \max_{1\leqslant k\leqslant n} \left| \sum_{j=1}^{n} (a_{kj} x_{k,j} + b_k) - b_k \right| \\ &= \rho(Tx_k, \mathbf{0}) \leqslant c\,\rho(x_k, \mathbf{0}) = c < 1. \end{aligned}$$

1.75 (✍) By the mean value theorem of differential calculus, we see that for all $x, y \in \mathbb{R}$ with $x \neq y$ there exists some ξ which lies between x and y such that

$$|Tx - Ty| = \left| 1 - \frac{1}{1+\xi^2} \right| |x - y| = \left(\frac{\xi^2}{1+\xi^2} \right) |x - y| < |x - y|,$$

this gives that $d(Tx, Ty) < d(x, y)$ for all $x, y \in \mathbb{R}$ with $x \neq y$. But T has no fixed point in $\mathbb{R}$ since the equation $Tx = x$, i.e., $\arctan x = \pi/2$, has no solution in $\mathbb{R}$.

1.76 (✍) Since for all $x, y \in C[0, 1]$ we have

$$d(Tx, Ty) = \max_{t\in[0,1]} |tx(t) - ty(t)| \leqslant \max_{t\in[0,1]} |x(t) - y(t)| = d(x, y),$$

which shows that T is a non-expansive mapping on $C[0, 1]$.

For each $x \in K$, note that $(Tx)(t) = tx(t)$ for all $t \in [0, 1]$. It is clear that $0 \leqslant (Tx)(t) \leqslant 1$ for all $t \in [0,1]$, $(Tx)(0) = 0x(0) = 0$ and $(Tx)(1) = x(1) = 1$, so that $Tx \in K$, and then $T(K) \subset K$.

If there would exist $\tilde{x} \in K$ such that $T\tilde{x} = \tilde{x}$, then $t\tilde{x}(t) = \tilde{x}(t)$ may hold for all $t \in [0,1]$, so that $\tilde{x}(t) \equiv 0$ for all $t \in [0,1]$, which is inconsistent with the assumption $\tilde{x} \in K$, hence T has no fixed point in K.

1.77 (✍) Let $\varepsilon = (1-\alpha_0)/2$, then by the definition of the infimum there exists $N \in \mathbb{N}$ such that

$$\sup_{x \neq y} \frac{d(T^N x, T^N y)}{d(x,y)} < \alpha_0 + \varepsilon = \frac{1+\alpha_0}{2}.$$

Denote $\bar{c} := (1+\alpha_0)/2$, then we see that $\bar{c} < 1$ and so that

$$d(T^N x, T^N y) \leqslant \sup_{x \neq y} \frac{d(T^N x, T^N y)}{d(x,y)} d(x,y) \leqslant \bar{c}\, d(x,y) \quad \text{for all } x, y \in X,$$

which means that T^N is a contraction on X. By the Banach fixed point theorem we get that T^N has a unique fixed point on X, so does T by Theorem 1.8.2.

1.78 (✍) By Theorem 1.5.1 we see that $S(x_0, r)$ is a complete subspace of (X,d) since (X,d) is complete and $S(x_0,r)$ is closed in (X,d). Note that for each $y \in S(x_0,r)$, that is, $d(y,x_0) < r$, we have

$$\begin{aligned} d(Ty, x_0) &\leqslant d(Ty, Tx_0) + d(Tx_0, x_0) \\ &\leqslant \theta d(y, x_0) + (1-\theta) r \\ &< \theta r + (1-\theta) r = r \end{aligned}$$

by the assumptions that $d(x_0, Tx_0) < (1-\theta)r$ and $d(Tx, Ty) \leqslant \theta d(x,y)$ for all $x, y \in X$, which means that $T(S(x_0,r)) \subset S(x_0,r)$. So T is a contraction mapping on $S(x_0,r)$. By Theorem 1.8.1, we get that T has a unique fixed point on $S(x_0,r)$.

1.79 (✍)

a) The answer is no. For example, let $X = (0,1) \subset \mathbb{R}$, with the Euclidean matric, and $f(x) = x/2$. Clearly, $f(X) \subset X$, f is a contraction on X since $|f(x) - f(y)| = (1/2)|x-y|$ for all $x, y \in X$ and X is an incomplete metric space by Theorem 1.5.1 since X is not closed in $\mathbb{R}$ and $\mathbb{R}$ is complete. But f dose not have any fixed point in X since the only possible solution in $\mathbb{R}$ for the equation $f(x) = x$, i.e., $x = x/2$, is $x = 0 \notin X$.

b) The answer is no. For example, one may choose the metric space $\mathbb{R}$, with the Euclidean matric, and the function f given in Exercise 1.75. Here, we will present another example.

Let $X = [2, \infty) \subset \mathbb{R}$, with the Euclidean matric, and $f(x) = x + 1/x$. Then X is a complete metric space by Theorem 1.5.1 since X is closed in $\mathbb{R}$ and $\mathbb{R}$ is complete. Moreover, f satisfies the following conditions: $f(X) \subset X$ and

$$|f(x) - f(y)| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| 1 - \frac{1}{xy} \right| \, |x - y| < |x - y|$$

for every $x, y \in X$, with $x \neq y$. But f dose not have any fixed point in $\mathbb{R}$.

1.80 (✍) **First solution:** Denote

$$\tilde{c} := \sup_{x \neq y} \frac{d(Tx, Ty)}{d(x, y)},$$

then $0 < \tilde{c} \leqslant 1$ since $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. By the definition of the supremum there exist sequences $\{x_n\}, \{y_n\} \subset X$ such that $d(Tx_n, Ty_n)/d(x_n, y_n) \to \tilde{c}$ as $n \to \infty$. Using the argument in the proof of Exercise 1.61 we get two subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $\{y_{n_k}\} \subset \{y_n\}$ such that both $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$ as $k \to \infty$ since X is compact, where $x_0, y_0 \in X$. If $x_0 \neq y_0$, then $c = d(Tx_0, Ty_0)/d(x_0, y_0) < 1$, so that $d(Tx, Ty) \leqslant \tilde{c}d(x, y)$ for all $x, y \in X$ with $x \neq y$. This inequality clearly holds for all $x, y \in X$. Hence T is a contraction on X. Note that X is complete by Theorem 1.7.1 since X is compact. Now the Banach fixed point theorem infers that T has a unique fixed point on X.

Second solution: Clearly, T is continuous on X. Let $f(x) = d(Tx, x)$ for all $x \in X$, then f is also continuous on X since T and d are continuous, and $f(x) \geqslant 0$ for all $x \in X$. By Theorem 1.7.4, we see that there exists an $x_0 \in X$ such that $f(x_0) = \min\limits_{x \in X} f(x)$ since X is compact. We claim that $f(x_0) = 0$. Otherwise, let $x_1 = Tx_0$, then we have

$$f(x_1) = d(Tx_1, x_1) = d(Tx_1, Tx_0) < d(x_1, x_0) = d(Tx_0, x_0) = f(x_0),$$

which contradicts that $f(x_0)$ is the minimum of $f(x)$ on X. Hence $f(x_0) = 0$, so that $Tx_0 = x_0$, i.e., x_0 is a fixed point of T. The uniqueness of the fixed point for T can be deduced by assumption that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$.