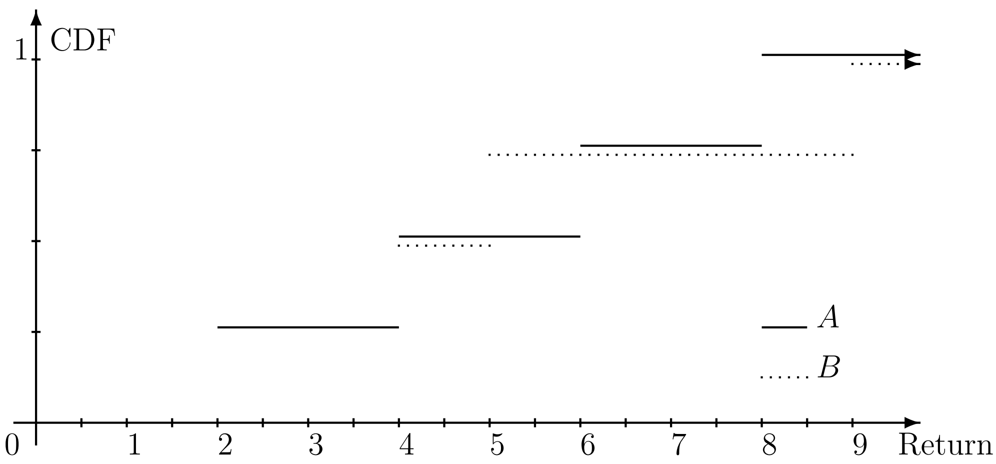
**Portfolio Theory**
**Solutions to Tutorial 2**

1. The following figure shows the distribution functions for investments $A$ and $B$.




(a)

$$F_A(2\%) = 0.25 \quad > \quad 0 = F_B(2\%)$$

$$F_A(5\%) = 0.5 \quad < \quad 0.75 = F_B(5\%)$$

Hence, neither investment has first order stochastic dominance over the other.

(b) For $x < 5\%$, $F_A(x) \geq F_B(x)$ and so,

$$\int_{-\infty}^{x} F_A(y)\,dy \geq \int_{-\infty}^{x} F_B(y)\,dy \quad (x \leq 5\%)$$

$$\begin{aligned} \int_{-\infty}^{6\%} F_A(y)\,dy &= 0.25 \times (4-2) + 0.5 \times (6-4) = 1.5\% \\ \int_{-\infty}^{6\%} F_B(y)\,dy &= 0.5 \times (5-4) + 0.75 \times (6-5) = 1.25\% < \int_{-\infty}^{6\%} F_A(y)\,dy \end{aligned}$$

Hence,

$$\int_{-\infty}^{x} F_A(y)\,dy \geq \int_{-\infty}^{x} F_B(y)\,dy$$

for all $x$ since $F_A(y) \geq F_B(y)$ for $y \geq 6\%$ and the inequality is strict at $x = 6\%$. Hence, $B$ has second order stochastic dominance over $A$.

2. (i) $E(A) = E(B) = 110$, $E(C) = 95$ and $E(D) = 115$.

(ii) (a) Since $D \geq C$ almost surely (with at least 1 strict inequality), $D$ has absolute dominance over $C$. No other portfolios have absolute dominance over any other.

(b) Since absolute dominance implies 1st- and 2nd-order dominance, we have $D \geq_{\text{sd}} C$. By plotting the graphs of the 4 distribution function, it is easy to see that no other portfolios have 1st-order dominance over any other.

(c) We need only check the values of $\int_{-\infty}^{x} F(y)dy$ and show that $\int_{-\infty}^{x} F_i(y)dy < \int_{-\infty}^{x} F_j(y)dy$ for the values of $x$ where one of the distribution functions changes value (i.e. makes a jump), since for all other values of $x$ in between, no distribution function will have changed value, and so they will be accumulating areas under their graphs at constant rates which implies that the same inequality continues to hold at those intermediate values of $x$. The following table of values is easily obtained from plots of the distribution functions.

| | $\int_{-\infty}^{x} F_A(y)dy$ | $\int_{-\infty}^{x} F_B(y)dy$ | $\int_{-\infty}^{x} F_C(y)dy$ | $\int_{-\infty}^{x} F_D(y)dy$ |
|---|---|---|---|---|
| $x = 40$ | 0 | 0 | 0 | 0 |
| $x = 50$ | 0 | 2.5 | 0 | 0 |
| $x = 60$ | 0 | 5 | 2.5 | 0 |
| $x = 80$ | 0 | 10 | 12.5 | 0 |
| $x = 90$ | 2.5 | 15 | 17.5 | 0 |
| $x = 100$ | 5 | 20 | 22.5 | 5 |
| $x = 120$ | 15 | 30 | 32.5 | 15 |
| $x = 130$ | 22.5 | 37.5 | 37.5 | 20 |
| $x = 140$ | 30 | 45 | 45 | 25 |
| $x = 200$ | 90 | 90 | 105 | 85 |

From (a), we already have $D \geq_{\text{ssd}} C$. In addition, we have from the table above

$$\int_{-\infty}^{x} F_D(y)dy \leq \int_{-\infty}^{x} F_A(y)dy \leq \int_{-\infty}^{x} F_B(y)dy \quad \Rightarrow \quad D \geq_{\text{ssd}} A \geq_{\text{ssd}} B$$
$$\int_{-\infty}^{x} F_D(y)dy \leq \int_{-\infty}^{x} F_A(y)dy \leq \int_{-\infty}^{x} F_C(y)dy \quad \Rightarrow \quad D \geq_{\text{ssd}} A \geq_{\text{ssd}} C$$

There is no 2nd-order dominance between $B$ and $C$: e.g.
$\int_{-\infty}^{50} F_B(y)dy = 2.5 > \int_{-\infty}^{50} F_C(y)dy = 0$
$\int_{-\infty}^{80} F_B(y)dy = 10 < \int_{-\infty}^{80} F_C(y)dy = 12.5$

3. For $Y = A, B$ we have

$$\begin{aligned} F_Y(x) &= \mathrm{P}\{Y \leq x\} = \mathrm{P}\left\{\frac{Y-\mu_Y}{\sigma} \leq \frac{x-\mu_Y}{\sigma}\right\} \\ &= \Phi\left(\frac{x-\mu_Y}{\sigma}\right), \end{aligned}$$

where $\Phi$ denotes the standard normal distribution function. Since

$$\mu_A > \mu_B,$$

we have for all $x$

$$\frac{x-\mu_A}{\sigma} < \frac{x-\mu_B}{\sigma},$$

hence

$$\Phi\left(\frac{x-\mu_A}{\sigma}\right) < \Phi\left(\frac{x-\mu_B}{\sigma}\right).$$

Thus

$$F_A(x) < F_B(x) \quad \forall x,$$

*i.e.* $A$ has first order stochastic dominance over $B$.

4. (a)

$$F_Y(x) = \mathrm{P}\{Y \le x\} = \Phi\left(\frac{x-\mu}{\sigma_Y}\right) \text{ for } Y = A, B.$$

For $x < \mu$ we have

$$\frac{x-\mu}{\sigma_A} > \frac{x-\mu}{\sigma_B}$$

and for $x > \mu$ we have

$$\frac{x-\mu}{\sigma_A} < \frac{x-\mu}{\sigma_B}.$$

Thus

$$\begin{aligned} F_A(x) &> F_B(x) \text{ for } x < \mu \\ \text{and } F_A(x) &< F_B(x) \text{ for } x > \mu \end{aligned}$$

*i.e.* neither $A$ nor $B$ has first order stochastic dominance over the other portfolio.

(b) We want to show that

$$\int_{-\infty}^{x} (F_A(y) - F_B(y))\, dy \ge 0 \quad \forall x$$

with strict inequality for some $x$. From part (a) above we know that

$$\begin{aligned} F_A(y) &> F_B(y) \quad \forall y < \mu \\ F_A(y) &< F_B(y) \quad \forall y > \mu \end{aligned}$$

and thus

$$\int_{-\infty}^{x} (F_A(y) - F_B(y))\, dy > 0 \quad \forall x \le \mu.$$

For $x > \mu$ we have

$$\begin{aligned} \int_{-\infty}^{x} (F_A(y) - F_B(y))\, dy &= \int_{-\infty}^{\infty} (F_A(y) - F_B(y))\, dy - \\ &\qquad \int_{x}^{\infty} (F_A(y) - F_B(y))\, dy \\ &> \int_{-\infty}^{\infty} (F_A(y) - F_B(y))\, dy \end{aligned}$$

(since the second integral on the RHS is $< 0$)

Putting $x = y - \mu$ in the integral below gives

$$\int_{-\infty}^{\infty} (F_A(y) - F_B(y))\, dy = \int_{-\infty}^{\infty} (\Phi(x/\sigma_A) - \Phi(x/\sigma_B))\, dx.$$

By symmetry, $\Phi(-x) = 1 - \Phi(x)$, so $\Phi(-x/\sigma_A) - \Phi(-x/\sigma_B) = \Phi(x/\sigma_B) - \Phi(x/\sigma_A)$ (i.e. the integrand above is an odd function). Hence

$$\int_{-\infty}^{\infty} (\Phi(x/\sigma_A) - \Phi(x/\sigma_B))\, dx = 0.$$

This concludes the proof.