第七次作业 6.38，6.39

6.38　设$\xi_1,\xi_2,\cdots,\xi_n$ 是取自正态母体$N\left(\mu,\sigma^2\right)$ 的一个子样, 其中μ 已知. 证明:

i) $S_n^2=\frac{1}{n}\sum_{i=1}^n(\xi_i-\mu)^2$ 是σ^2 的相合估计.（例题）

ii) $\hat{\sigma}=\frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^n|\xi_i-\mu|$ 是σ 的无偏估计, 并求其有效率.

解: i) 因为$\frac{nS_n^2}{\sigma^2}=\sum_{i=1}^n\left(\frac{\xi_i-\mu}{\sigma}\right)^2\sim\chi^2(n)$, 所以

$$E\left(\frac{nS_n^2}{\sigma^2}\right)=n,\quad D\left(\frac{nS_n^2}{\sigma^2}\right)=2n.$$

即

$$ES_n^2=\sigma^2,\quad DS_n^2=\frac{2\sigma^4}{n}.$$

母体的概率函数为: $f\left(x;\sigma^2\right)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$,

$$\ln f\left(x;\sigma^2\right)=-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln\sigma^2-\frac{1}{2\sigma^2}(x-\mu)^2,$$
$$\frac{\partial\ln f\left(x;\sigma^2\right)}{\partial\sigma^2}=-\frac{1}{2\sigma^2}+\frac{1}{2\sigma^4}(x-\mu)^2,$$
$$\frac{\partial^2\ln f\left(x;\sigma^2\right)}{\partial\left(\sigma^2\right)^2}=\frac{1}{2\sigma^4}-\frac{1}{\sigma^6}(x-\mu)^2,$$
$$I\left(\sigma^2\right)=-E\left(\frac{\partial^2\ln f\left(\xi;\sigma^2\right)}{\partial\left(\sigma^2\right)^2}\right)=-\frac{1}{2\sigma^4}+\frac{1}{\sigma^6}E(\xi-\mu)^2=-\frac{1}{2\sigma^4}+\frac{1}{\sigma^4}=\frac{1}{2\sigma^4}.$$

因为$\frac{1}{nI(\sigma^2)}=\frac{2\sigma^4}{n}=D\left(S_n^2\right)$, 所以$S_n^2$ 是σ^2 的有效估计.

ii) 因为$\frac{\xi_i-\mu}{\sigma}\sim N(0,1)$, 所以

$$\begin{aligned}E\frac{|\xi_i-\mu|}{\sigma}&=\int_{-\infty}^{\infty}|x|\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,\mathrm{d}x\\&=2\int_0^{\infty}x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,\mathrm{d}x\\&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-\frac{1}{2}x^2}\,\mathrm{d}\frac{x^2}{2}\\&=\sqrt{\frac{2}{\pi}}.\end{aligned}$$

所以

$$E|\xi_i-\mu|=\sqrt{\frac{2}{\pi}}\sigma,i=1,2,\cdots,n.$$

$$E\hat{\sigma} = \frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n} E\left|\xi_i - \mu\right| = \frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n}\sqrt{\frac{2}{\pi}}\sigma = \sigma,$$

即$\hat{\sigma}$ 是σ 的无偏估计.

下面计算$D(|\xi_i - \mu|)$.

$$D\left(|\xi_i - \mu|\right) = E\left(|\xi_i - \mu|\right)^2 - \left(E\left|\xi_i - \mu\right|\right)^2 = E\left(\xi_i - \mu\right)^2 - \left(\sqrt{\frac{2}{\pi}}\sigma\right)^2$$
$$= D\xi_i - 2\sigma^2/\pi = \sigma^2 - 2\sigma^2/\pi = \frac{\pi - 2}{\pi}\sigma^2.$$

所以

$$D\hat{\sigma} = \frac{\pi}{2n^2}\sum_{i=1}^{n} D\left|\xi_i - \mu\right| = \frac{\pi - 2}{2n}\sigma^2.$$

由于$\sigma = \sqrt{\sigma^2} \triangleq g\left(\sigma^2\right)$, 故由上可知$\hat{\sigma}$是$g\left(\sigma^2\right)$的无偏估计. 而$g'\left(\sigma^2\right) = \frac{1}{2\sqrt{\sigma^2}} = \frac{1}{2\sigma}$. 因此

$$\frac{\left[g'\left(\sigma^2\right)\right]^2}{nI\left(\sigma^2\right)} = \frac{\left[1/(2\sigma)\right]^2}{n\frac{1}{2\sigma^4}} = \frac{\sigma^2}{2n} < \frac{\pi - 2}{2n}\sigma^2 = D\hat{\sigma}.$$

所以$\hat{\sigma}$ 不是σ 的有效估计. $\hat{\sigma}$ 的有效率为

$$\frac{\sigma^2}{2n}\Big/\frac{(\pi - 2)\sigma^2}{2n} - \frac{1}{\pi - 2}.$$

6.39 设$\xi_1, \xi_2, \cdots, \xi_n$ 是取自具有下列指数分布的一个子样.

$$f(x) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & \text{else.} \end{cases}$$

证明: $\bar{\xi} = \frac{1}{n}\sum_{i=1}^{n}\xi_i$ 是θ 的无偏、相合、有效估计.

证明：用ξ 表示母体, 那么

$$E\xi = \theta, D\xi = \theta^2.$$

由此可知(定理5.1):

$$E\bar{\xi} = E\xi = \theta.$$
$$D\bar{\xi} = D\xi/n = \theta^2/n.$$

所以$\bar{\xi}$ 是θ 的无偏估计.

因为$E\xi_i = E\xi = \theta$, 且$\xi_1, \xi_2, \cdots, \xi_n, \cdots$ 是独立同分布的随机变量序列, 故由辛钦大数定律可知:

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \xrightarrow{P} \theta.$$

即$\bar{\xi} = \sum_{i=1}^{n}\xi_i$ 是θ 的相合估计.

又

$$\ln f(x;\theta) = -\ln\theta - \frac{x}{\theta},$$
$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{x}{\theta^2},$$
$$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}.$$

因为

$$I(\theta) = -E\left(\frac{\partial^2 \ln f(\xi;\theta)}{\partial\theta^2}\right) = -\frac{1}{\theta^2} + \frac{2}{\theta^3}E\xi = -\frac{1}{\theta^2} + \frac{2}{\theta^3}\theta = \frac{1}{\theta^2}.$$

故

$$\frac{1}{nI(\theta)} = \frac{\theta^2}{n} = D\bar{\xi},$$

所以$\bar{\xi}$ 是θ 的有效估计.