Portfolio Theory
Solutions to Tutorial 7

1. (a) By writing out $Ay = b$ as

$$\begin{pmatrix} 2C & -\mu & -\vec{1} \\ \mu^T & 0 & 0 \\ \vec{1}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ E_P \\ 1 \end{pmatrix}$$

Equation (1) in the question is the result of multiplying out the first n rows of this matrix representation.

Equations (2) and (3) in the question are the result of multiplying out the last two rows.

(b)

$$\begin{aligned} 2C\pi - \lambda_1\mu - \lambda_2\vec{1} &= 0 \\ \Rightarrow 2C\pi &= \lambda_1\mu_1 + \lambda_2\vec{1} \\ \Rightarrow \pi &= \frac{1}{2}C^{-1}\left(\lambda_1\mu_1 + \lambda_2\vec{1}\right) \end{aligned}$$

(c) We still have to find the values of λ_1 and λ_2. But from equation (4) in the question we can see that the optimal portfolio is always going to be a linear combination of two vectors $C^{-1}\mu$ and $C^{-1}\vec{1}$. These vectors are not proper portfolios but we can define

$$\pi_A = C^{-1}\mu/(\vec{1}^T C^{-1}\mu), \quad \text{and} \quad \pi_B = C^{-1}\vec{1}/(\vec{1}^T C^{-1}\vec{1})$$

so that $\pi_A^T\vec{1} = 1$ and $\pi_B^T\vec{1} = 1$.

So any optimal (i.e.efficient) portfolio will be a linear combination of portfolios A and B: that is, the two-fund theorem.

(d) This problem can be solved using the Lagrangian function with only one constraint, $\pi^T\vec{1} = 1$.

$$\begin{aligned} \mu &= (E[X_1], E[X_2], \ldots, E[X_n])^T \ , \\ \vec{1} &= (1, 1, \ldots, 1)^T \ , \\ \pi &= (\pi_1, \pi_2, \ldots, \pi_n)^T \ , \\ y &= (\pi_1, \pi_2, \ldots, \pi_n, \lambda_2)^T \ , \\ b &= (0, 0, \ldots, 0, 1)^T \end{aligned}$$

$$A = \begin{pmatrix} 2C & -\vec{1} \\ \vec{1}^T & 0 \end{pmatrix}$$

where C is the covariance matrix of the random vector X of returns. Then, the system of $n+1$ linear equations is:

$$Ay = b$$

$\Rightarrow$ Solution is given by

$$y = A^{-1}b$$

Following the solution of the more general constrained problem we have:

$$\begin{aligned}
2C\pi - \lambda_2\vec{1} &= 0 && (1)\\
\pi^T\vec{1} &= 1 && (2)\\
\Rightarrow \pi &= \frac{\lambda_2}{2}C^{-1}\vec{1} && (3)\\
\text{and} \quad \pi^T\vec{1} = \frac{\lambda_2}{2}\vec{1}^T C^{-1}\vec{1} &= 1 && (4)\\
\Rightarrow \lambda_2 &= \frac{2}{\vec{1}^T C^{-1}\vec{1}} && (5)\\
\Rightarrow \pi &= \frac{C^{-1}\vec{1}}{\vec{1}^T C^{-1}\vec{1}} && (6)
\end{aligned}$$

(e) One efficient portfolio is the minimum variance portfolio

$$\pi_B = \frac{C^{-1}\vec{1}}{\vec{1}^T C^{-1}\vec{1}}$$

while the other will be

$$\pi_A = \frac{C^{-1}\mu}{\vec{1}^T C^{-1}\mu}.$$

2. Consider the portfolio:

$$P = \pi_A R_A + \pi_B R_B + \pi_C R_C$$

The variance of the return on this portfolio is σ_P^2, where:

$$\sigma_P^2 = 36\pi_A^2 + 9\pi_B^2 + 225\pi_C^2 + 18\pi_A\pi_B + 36\pi_A\pi_C + 36\pi_B\pi_C$$

We have to minimise σ_P^2 subject to the single constraint:

$$\pi_A + \pi_B + \pi_C = 1$$

Hence the Lagrangian function is $L(\underline{\pi}, \alpha)$, where:

$$\begin{aligned} L(\underline{\pi}, \alpha) &= 36\pi_A^2 + 9\pi_B^2 + 225\pi_C^2 + 18\pi_A\pi_B + 36\pi_A\pi_C + 36\pi_B\pi_C \\ &\quad -\alpha(\pi_A + \pi_B + \pi_C - 1) \end{aligned}$$

Differentiating with respect to π_A, π_B, π_C and α and setting the partial derivatives equal to zero, we get the four simultaneous linear equations:

$$\begin{aligned} 0 &= 72\pi_A + 18\pi_B + 36\pi_C - \alpha \\ 0 &= 18\pi_B + 18\pi_A + 36\pi_C - \alpha \\ 0 &= 450\pi_C + 36\pi_A + 36\pi_B - \alpha \\ 0 &= -(\pi_A + \pi_B + \pi_C - 1) \end{aligned}$$

These equations are not that difficult to solve by hand. They can also be expressed as $Ay = b$, where:

$$\begin{pmatrix} 72 & 18 & 36 & -1 \\ 18 & 18 & 36 & -1 \\ 36 & 36 & 450 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$y^T = (\pi_A, \pi_B, \pi_C, \alpha); \qquad b^T = (0, 0, 0, 1)$$

The inverse of A is:

$$A^{-1} = \begin{pmatrix} 0.01852 & -0.01852 & 0 & 0 \\ -0.01852 & 0.02104 & -0.00253 & 1.04545 \\ 0 & -0.00253 & 0.00253 & -0.04545 \\ 0 & -1.04545 & 0.04545 & 17.1818 \end{pmatrix}$$

Hence the portfolio with minimum variance is:

$$\pi_A = 0; \quad \pi_B = 1.04545; \quad \pi_C = -0.04545$$

(For this portfolio, $\sigma_P = 2.93\%$ and $\mathrm{E}[R_P] = 7.45\%$.)

3. (a) We calculate the expected rates of return as:

$$\begin{aligned} \mu_i &= \alpha_i + \beta_i \mu_M \\ \mu_1 &= \alpha_1 + \beta_1 \mu_M = 2 + 0.5 \times 9 = 6.5\% \\ \mu_2 &= \alpha_2 + \beta_2 \mu_M = -1 + 1.5 \times 9 = 12.5\% \\ \mu_3 &= \alpha_3 + \beta_3 \mu_M = 1 + 1.2 \times 9 = 11.8\% \end{aligned}$$

The variances are given by:

$$\begin{aligned}
\sigma_i^2 &= \beta_i^2\sigma_M^2 + \sigma_{\xi_i}^2 \\
\sigma_1^2 &= \beta_1^2\sigma_M^2 + \sigma_{\xi_1}^2 = 0.5^2 \times 20 + 5 = 10\%\% \\
\sigma_2^2 &= \beta_2^2\sigma_M^2 + \sigma_{\xi_2}^2 = 1.5^2 \times 20 + 8 = 53\%\% \\
\sigma_3^2 &= \beta_3^2\sigma_M^2 + \sigma_{\xi_3}^2 = 1.2^2 \times 20 + 4 = 32.8\%\%
\end{aligned}$$

The covariances are given by:

$$\begin{aligned}
\sigma_{ij} &= \beta_i\beta_j\sigma_M^2 \\
\sigma_{12} &= \beta_1\beta_2\sigma_M^2 = 0.5 \times 1.5 \times 20 = 15\%\% \\
\sigma_{13} &= \beta_1\beta_3\sigma_M^2 = 0.5 \times 1.2 \times 20 = 12\%\% \\
\sigma_{23} &= \beta_2\beta_3\sigma_M^2 = 1.2 \times 1.5 \times 20 = 36\%\%
\end{aligned}$$

(b) We express the problem in matrix form as $Ay = b$, where

$$y = (\pi_1, \pi_2, \pi_3, \alpha, \beta)^T$$
$$b = (0, 0, 0, E_P, 1)^T$$

where $E_P = 10\%$ and A is given by:

$$\begin{pmatrix} 2\sigma_1^2 & 2\sigma_{12} & 2\sigma_{13} & -\mu_1 & -1 \\ 2\sigma_{12} & 2\sigma_2^2 & 2\sigma_{23} & -\mu_2 & -1 \\ 2\sigma_{13} & 2\sigma_{23} & 2\sigma_3^2 & -\mu_3 & -1 \\ \mu_1 & \mu_2 & \mu_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 30 & 24 & -6.5 & -1 \\ 30 & 106 & 72 & -12.5 & -1 \\ 24 & 72 & 65.6 & -11.8 & -1 \\ 6.5 & 12.5 & 11.8 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Hence:

$$A^{-1} = \begin{pmatrix} 0.00062 & 0.004692 & -0.00531 & -0.18464 & 2.174469 \\ 0.004692 & 0.035524 & -0.04022 & 0.030604 & -0.3933 \\ -0.00531 & -0.04022 & 0.045527 & 0.154033 & -0.78117 \\ -0.18464 & 0.030604 & 0.154033 & -1.31219 & 7.607052 \\ 2.174469 & -0.3933 & -0.78117 & 7.607052 & -62.3881 \end{pmatrix}$$

and $\pi_1 = 0.3281, \pi_2 = -0.0873, \pi_3 = 0.7592.$

4. By the definition of the estimators,

$$\begin{aligned}
\mathrm{E}(\hat{\mu}) &= \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(R_i) = \mu, \\
\mathrm{E}(\hat{\sigma}^2) &= \frac{1}{n-1}\mathrm{E}\left(\sum_{i=1}^{n}(R_i - \hat{\mu})^2\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\mathrm{E}(R_i^2) - \frac{2}{n}\sum_{j=1}^{n}\mathrm{E}(R_iR_j) + \frac{1}{n^2}\sum_{j,k=1}^{n}\mathrm{E}(R_jR_k)\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\left(\mu^2 + \sigma^2\right) - \frac{2}{n}\left(n\mu^2 + \sigma^2\right) + \frac{1}{n^2}\left(n^2\mu^2 + n\sigma^2\right)\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\sigma^2 - \frac{1}{n}\sigma^2\right) = \sigma^2.
\end{aligned}$$

5. (a) We calculate the expected rates of return as:

$$\begin{aligned}
\mu_i &= a_i + b_{iM}E(I_M) + b_{iB}E(I_B) + b_{iO}E(I_O) \\
\mu_A &= a_A + b_{AM}E(I_M) + b_{AB}E(I_B) + b_{AO}E(I_O) \\
&= 1 + 1.2 \times 10 + 0.9 \times (-2) + 0 = 11.2\% \\
\mu_B &= 2 + 0.9 \times 10 + 1.1 \times (-2) + 0 = 8.8\% \\
\mu_C &= 3 + 0.5 \times 10 + 0 + 0.8 \times 5 = 12\% \\
\mu_D &= 3 + 0.4 \times 10 + 0 + 1.3 \times 5 = 13.5\%
\end{aligned}$$

The variances are given by:

$$\begin{aligned}
\sigma_i^2 &= b_{iM}^2\sigma_{IM}^2 + b_{iB}^2\sigma_{IB}^2 + b_{iO}^2\sigma_{IO}^2 + \sigma_{\xi_i}^2 \\
\sigma_A^2 &= b_{AM}^2\sigma_{IM}^2 + b_{AB}^2\sigma_{IB}^2 + b_{AO}^2\sigma_{IO}^2 + \sigma_{\xi_A}^2 \\
&= 1.2^2 \times 15 + 0.9^2 \times 3 + 1 = 25.03\%\% \\
\sigma_B^2 &= b_{BM}^2\sigma_{IM}^2 + b_{BB}^2\sigma_{IB}^2 + b_{BO}^2\sigma_{IO}^2 + \sigma_{\xi_B}^2 \\
&= 0.9^2 \times 15 + 1.1^2 \times 3 + 1 = 16.78\%\% \\
\sigma_C^2 &= b_{CM}^2\sigma_{IM}^2 + b_{CB}^2\sigma_{IB}^2 + b_{CO}^2\sigma_{IO}^2 + \sigma_{\xi_C}^2 \\
&= 0.5^2 \times 15 + 0.8^2 \times 14 + 8 = 20.71\%\% \\
\sigma_D^2 &= b_{DM}^2\sigma_{IM}^2 + b_{DB}^2\sigma_{IB}^2 + b_{DO}^2\sigma_{IO}^2 + \sigma_{\xi_D}^2 \\
&= 0.4^2 \times 15 + 1.3^2 \times 14 + 3 = 29.06\%\%
\end{aligned}$$

The covariances are given by:

$$\begin{aligned}
\sigma_{ij} &= b_{iM}b_{jM}\sigma_{IM}^2 + b_{iB}b_{jB}\sigma_{IB}^2 + b_{iO}b_{jO}\sigma_{IO}^2 \\
\sigma_{AB} &= 1.2 \times 0.9 \times 15 + 0.9 \times 1.1 \times 3 + 0 = 19.17\%\% \\
\sigma_{AC} &= 1.2 \times 0.5 \times 15 = 9\%\% \\
\sigma_{AD} &= 1.2 \times 0.4 \times 15 = 7.2\%\% \\
\sigma_{BC} &= 0.9 \times 0.5 \times 15 = 6.75\%\% \\
\sigma_{BD} &= 0.9 \times 0.4 \times 15 = 5.4\%\% \\
\sigma_{CD} &= 0.5 \times 0.4 \times 15 + 0 + 0.8 \times 1.3 \times 14 = 17.56\%\%
\end{aligned}$$

(b) We have portfolio P with weights:

$$\pi_A = \pi_B = 0.3; \quad \pi_C = \pi_D = 0.2$$

The mean and variance of a portfolio are calculated in the usual way as:

$$\mu_P = \pi_A\mu_A + \pi_B\mu_B + \pi_C\mu_C + \pi_D\mu_D$$

$$\begin{aligned}
\sigma_P^2 &= \pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + \pi_C^2\sigma_C^2 + \pi_D^2\sigma_D^2 + 2\pi_A\pi_B\mathrm{Cov}(A,B) \\
&\quad +2\pi_A\pi_C\mathrm{Cov}(A,C) + 2\pi_A\pi_D\mathrm{Cov}(A,D) + 2\pi_B\pi_C\mathrm{Cov}(B,C) \\
&\quad +2\pi_B\pi_D\mathrm{Cov}(B,D) + 2\pi_C\pi_D\mathrm{Cov}(C,D)
\end{aligned}$$

We obtain the required means, variances and covariances from part (a).

$$\mu_P = 0.3 \times 11.2 + 0.3 \times 8.8 + 0.2 \times 12 + 0.2 \times 13.5 = 11.1\%$$

$$\begin{aligned}
\sigma_P^2 &= 0.3^2 \times 25.03 + 0.3^2 \times 16.78 + 0.2^2 \times 20.71 + 0.2^2 \times 29.06 \\
&\quad +2 \times 0.3 \times 0.3 \times 19.17 + 2 \times 0.3 \times 0.2 \times 9 + 2 \times 0.3 \times 0.2 \times 7.2 \\
&\quad +2 \times 0.3 \times 0.2 \times 6.75 + 2 \times 0.3 \times 0.2 \times 5.4 + 2 \times 0.2 \times 0.2 \times 17.56 \\
&= 14.01\%\%
\end{aligned}$$