Portfolio Theory
Solutions to Tutorial 4

1. (a) Let R_X denote the rate of return on X as a percentage, so that $R_X \sim U(-5, 25)$ and the pdf of R_X is f_X, where:

$$f_X(x) = \frac{1}{30} \qquad \text{for } -5 < x < 25$$

Hence,

$$\mathrm{E}[X] = \int_{-5}^{25} x f_X(x)\, dx = \left[\frac{x^2}{60}\right]_{-5}^{25} = 10\%$$
$$\mathrm{Var}[X] = \int_{-5}^{25} (x-10)^2 f_X(x)\, dx = \left[\frac{(x-10)^3}{90}\right]_{-5}^{25} = 75\%\%$$

Hence X has the same mean and variance of return as Y.

(b) The total return from investing 1,000,000 in X, T_X, is uniformly distributed on (950,000; 1,250,000). Hence the 95% value at risk is $x_{95\%}$, where:

$$\mathrm{P}[10^6 - T_X > x_{95\%}] = 0.05$$

so that:

$$10^6 - x_{95\%} = 950,000 + 0.05 \times (1,250,000 - 950,000)$$

and hence $x_{95\%} = £35,000$ Using similar notation, the 95% value at risk for Y, denoted $y_{95\%}$, satisfies:

$$\mathrm{P}[1 - T_Y > y_{95\%}] = 0.05$$

where $T_Y \sim N(1.1,\ 0.0075)$ in millions of pounds. Hence:

$$\frac{1 - y_{95\%} - 1.1}{\sqrt{0.0075}} = -1.645$$

and hence:

$$y_{95\%} = 0.042461 = £42,461$$

(c) Investments X and Y have the same mean and variance for their returns. However, the normally distributed return on Y is more likely to have a very low value. Hence the investor concerned about value at risk would choose investment X. This is an example where using the variance alone does not tell us all the information we require about the distribution of returns.

2. Let X_1, X_2 and X_3 be the results of the three dices. They are independent and identically distributed. We know that

$$\mathrm{E}(X_1) = 3.5, \quad \mathrm{E}(X_1^2) = (1+4+9+16+25+36)/6 = \frac{91}{6}$$

Therefore,

$$\begin{aligned} \text{Expected value of the prize} &= \mathrm{E}(X_1X_2X_3) = \mathrm{E}^3(X_1) = 42.875. \\ \text{Variance of the prize} &= \mathrm{Var}(X_1X_2X_3) = \mathrm{E}[(X_1X_2X_3)^2] - \mathrm{E}^2(X_1X_2X_3) \\ &= \mathrm{E}^3(X_1^2) - \mathrm{E}^6(X_1) = 1650.4890. \end{aligned}$$

3. (a)

$$\begin{aligned}\mathrm{Var}\,(\pi_1 X_1 + \pi_2 X_2) &= \mathrm{E}((\pi_1 X_1 + \pi_2 X_2)^2) - \mathrm{E}^2(\pi_1 X_1 + \pi_2 X_2) \\ &= \mathrm{E}(\pi_1^2 X_1^2 + 2\pi_1\pi_2 X_1 X_2 + \pi_2^2 X_2^2) - (\pi_1^2 \mathrm{E}^2(X_1) + 2\pi_1\pi_2 \mathrm{E}(X_1)\mathrm{E}(X_2) + \pi_2^2 \mathrm{E}^2(X_2)) \\ &= \pi_1^2\sigma_1^2 + 2\pi_1\pi_2\sigma_{12} + \pi_2^2\sigma_2^2.\end{aligned}$$

(b) For the case $n = 1$, we have

$$\mathrm{Var}(\pi_1 X_1) = \mathrm{E}((\pi_1 X_1)^2) - \mathrm{E}^2(\pi_1 X_1) = \pi_1^2 \sigma_{11}.$$

Assume the statement is true when $n = k$. When $n = k + 1$,

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{k+1} \pi_i X_i\right) &= \mathrm{Var}\left(\sum_{i=1}^{k} \pi_i X_i + \pi_{k+1} X_{k+1}\right) \\ &= \mathrm{Var}\left(\sum_{i=1}^{k} \pi_i X_i\right) + 2\mathrm{Cov}\left(\sum_{i=1}^{k} \pi_i X_i, \pi_{k+1} X_{k+1}\right) + \mathrm{Var}\,(\pi_{k+1} X_{k+1}) \\ &= \sum_{i,j=1}^{k} \pi_i\pi_j\sigma_{ij} + 2\sum_{i=1}^{k} \pi_i\pi_{k+1}\sigma_{i,k+1} + \pi_{k+1}^2\sigma_{k+1,k+1} \;=\; \sum_{i,j=1}^{k+1} \pi_i\pi_j\sigma_{ij}.\end{aligned}$$

Therefore, the equation holds for all positive integer n.

4. (a) Using the result from the lecture note, we know that it is possible to form a portfolio with no risk using two assets which are perfectly negatively correlated. For a portfolio, $\pi_X + \pi_Y = 1$. Moreover,

$$\begin{aligned}\mathrm{Var}\,(\pi_X X + \pi_Y Y) &= \pi_X^2\sigma_X^2 - 2\pi_X\pi_Y\sigma_X\sigma_Y + \pi_Y^2\sigma_Y^2 \\ &= (\pi_X\sigma_X - \pi_Y\sigma_Y)^2 = 0 \\ \Rightarrow \quad 20\pi_X - 30\pi_Y = 0 \quad &\Rightarrow \quad \pi_X = 3/5, \quad \pi_Y = 2/5.\end{aligned}$$

(b) There is no change as the result does not use the normal distribution information.

5. (a) $E[A] = 65$, $E[B] = 60$ and $E[C] = 60$.

$$\begin{aligned}\mathrm{Var}[A] &= (35^2 + 10^2 + 25^2)/3 = 650 \\ \mathrm{Var}[B] &= (15^2 + 0^2 + 15^2)/3 = 150 \\ \mathrm{Var}[C] &= (30^2 + 0^2 + 30^2)/3 = 600\end{aligned}$$

B is clearly better than C (lower variance and equal mean). But we cannot order A (it has the highest mean and highest variance).

Note - A clearly has first order stochastic dominance over C, but we cannot use this fact. We normally will use estimates of mean and variance without knowing the true underlying distribution.

(b) C is independent of A and B, so we know that

$$\mathrm{Cov}(A, C) = \mathrm{Cov}(B, C) = 0$$

We can now calculate

$$\mathrm{Cov}(A, B) = (-35 * -15 + 10 * 0 + 25 * 15)/3 = 300$$

$E[(A+B)/2] = 62.5$, $E[(A+C)/2] = 62.5$, $E[(C+B)/2] = 60$ and $E[(A+B+C)/3] = 185/3$.

$$\text{Var}[P] = \sum \pi_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} \pi_i \pi_j \sigma_{ij}$$

$$\text{Var}[(A+B)/2] = \frac{1}{4}650 + \frac{1}{4}150 + 2\frac{1}{4}300 = 350$$

$$\text{Var}[(A+C)/2] = \frac{1}{4}650 + \frac{1}{4}600 + 2\frac{1}{4}0 = 312.5$$

$$\text{Var}[(C+B)/2] = \frac{1}{4}600 + \frac{1}{4}150 + 2\frac{1}{4}0 = 187.5$$

$$\text{Var}[(A+B+C)/3] = \frac{1}{9}650 + \frac{1}{9}150 + \frac{1}{9}600 + 2\frac{1}{9}300 = 222.2$$

So using mean and variance we get the following order of preference:

$$B > (B+C)/2 > C$$

$$(A+B+C)/3 > C$$

$$(A+C)/2 > (A+B)/2 > C$$

We cannot choose for other portfolios.

(c) For any two assets X and Y we can calculate the variance as

$$\sigma_P^2 = \pi_X^2 \sigma_X^2 + 2\pi_X(1-\pi_X)\sigma_{XY} + (1-\pi_X)^2\sigma_Y^2$$

If the covariance is not equal to $\pm\sigma_X\sigma_Y$, then by differentiation with respect to π_X we show that the minimum variance portfolio occurs when

$$\pi_X = \frac{-\text{Cov}(X,Y) + \sigma_Y^2}{\sigma_X^2 - 2\text{Cov}(X,Y) + \sigma_Y^2}$$

In Question 1 we calculated the mean, variance and covariance to be

$$\text{Var}(A) = 650 \quad \text{Var}(B) = 150 \quad \text{Cov}(A,B) = 300$$

Hence

$$\pi_A = \frac{-300 + 150}{650 - 2 \times 300 + 150} = -0.75$$

So the lowest variance portfolio is $P = -0.75A + 1.75B$, where

$$E[P] = -0.75 \times 65 + 1.75 \times 60 = 56.25$$

$$\text{Var}[P] = (-0.75)^2 \times 650 + 2(-0.75)(1.75) \times 300 + (1.75)^2 \times 150 = 37.5$$

(d) If we are not allowed a negative investment in A for the portfolio A & B, we choose the allowable portfolio nearest to $P = -0.75A + 1.75B$, i.e. the lowest risk is to invest entirely in B. Below is a plot of the variance
$650\pi_A^2 + 2\pi_A(1-\pi_A) \times 300 + 150(1-\pi_A)^2$ for $-1.5 < \pi_A < 1.5$.
We see that the minimum occurs at $\pi_A = -0.75$ when there are no restrictions on π_A, but when $0 \leq \pi_A \leq 1$, the minimum occurs at $\pi_A = 0$.