苏州大学 泛函分析（双语） 课程期中试卷

(考试形式　开卷　　2022 年 4 月 22 日)

1. (5 marks) Let $X = [-1, 0) \cup \{3\}$ be a metric space with the Euclidean metric of $\mathbb{R}$. Indicate whether $\{3\}$ is open or closed or nowhere dense in X, and state the reasons.

2. (10 marks) Let (X, d) be complete and let $\rho = d/(1 + d)$. Show that (X, ρ) is complete.

3. (15 marks) Let $\{g_n(x)\}$ be a sequence of continuously differentiable real-valued functions on $[0, 1]$. Suppose that there exists $M > 0$ such that $\max\{|g_n(0)|, |g_n'(x)|\} \leqslant M$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$. Show that $\{g_n(x)\}$ has a subsequence which is uniformly convergent on $[0, 1]$, and derive from this that $\{\cos(x + n)\}$ has a subsequence which uniformly converges on $[0, 1]$.

4. (20 marks)

   (a) Let (X, d) be a compact metric space and let a mapping $T : X \to X$ satisfy that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ and $x \neq y$. Prove that T has a unique fixed point on X.

   (b) If (X, d) is a noncompact metric space, is the statement of the above (a) true? Give a proof or a counterexample.

5. (5 marks) Recall that c_0 is a normed linear space of all sequence of numbers which converge to 0 with the ℓ^∞ norm $\|x\|_{c_0} = \sup\limits_{n \geqslant 1} |\xi_n|$ for all $x = \{\xi_n\} \in c_0$. Prove that the above norm can be written as $\|x\|_{c_0} = \max\limits_{n \geqslant 1} |\xi_n|$ for all $x = \{\xi_n\} \in c_0$.

6. (20 marks)

   (a) Let $1 < p < q < \infty$ and let E be a Lebesgue measurable set of $\mathbb{R}$ or $\mathbb{R}^n$. If $\operatorname{meas}(E) < \infty$, show that

   $$L^\infty(E) \subsetneqq L^q(E) \subsetneqq L^p(E).$$

   If $\operatorname{meas}(E) = \infty$, does the set inclusion $L^q(E) \subset L^p(E)$ hold? Give a proof or a counterexample.

   (b) Let $1 < p < q < \infty$. Show that $\ell^p \subsetneqq \ell^q \subsetneqq \ell^\infty$ and for $x \in \ell^p$,

   $$\lim_{p \to \infty} \|x\|_{\ell^p} = \|x\|_{\ell^\infty}.$$

7. (15 marks) Recall that Riesz's Lemma states as in the following:

   Suppose that X is a normed linear space, Y is a closed linear subspace of X such that $Y \neq X$. Then for every real number α with $0 < \alpha < 1$ there exists $x_\alpha \in X$ satisfying $\|x_\alpha\| = 1$ and $\|x_\alpha - y\| \geqslant \alpha \ \forall\, y \in Y$.

   (a) Prove Riesz's Lemma.

   (b) If the subspace Y is not closed in X, is Riesz's Lemma still true? Give a proof or a counterexample.

8. (10 marks) Let $p(x)$ be a semi-norm defined on a linear space X, that is, $p(x)$ satisfies all axioms for norm except that $p(x)$ may be zero for non-zero elements. Let $N = \{x \in X : p(x) = 0\}$. Show that N is a linear subspace of X and

$$p_0(\pi(x)) = p(x) \quad \text{for all } \pi(x) \in X/N$$

   defines a norm on the quotient space X/N.