Portfolio Theory
Solutions to Tutorial 6

1. (a) $x_2 = 1 - x_1$. Hence

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + 2(1-x_1)^2 = 3x_1^2 - 4x_1 + 2 \\ \frac{\partial f}{\partial x_1} &= 6x_1 - 4 = 0 \\ \Rightarrow x_1 &= 2/3 \\ \Rightarrow x_2 &= 1/3 \end{aligned}$$

(b)

$$\begin{aligned} L(x_1, x_2, \lambda) &= x_1^2 + 2x_2^2 - \lambda(x_1 + x_2 - 1) \\ \frac{\partial L}{\partial x_1} &= 2x_1 - \lambda = 0 && (1) \\ \frac{\partial L}{\partial x_2} &= 4x_2 - \lambda = 0 && (2) \\ \frac{\partial L}{\partial \lambda} &= -x_1 - x_2 + 1 = 0 && (3) \\ (1) \Rightarrow x_1 &= \lambda/2 \\ (2) \Rightarrow x_2 &= \lambda/4 = x_1/2 \\ (3) \Rightarrow x_1 &= 2/3 \quad \text{and} \quad x_2 = 1/3 \end{aligned}$$

2. (a)

$$\begin{aligned} \mathrm{E}[R_{P_1^*}] &= 1.427 \times 10 + 0.147 \times 15 - 0.574 \times 20 = 4.995\% \text{ (say 5\%)} \\ \mathrm{E}[R_{P_2^*}] &= 0.742 \times 10 + 0.517 \times 15 - 0.259 \times 20 = 9.995\% \text{ (say 10\%)} \end{aligned}$$

(b) The required portfolio is $P_3^* = \lambda P_1^* + (1-\lambda)P_2^*$, where:

$$\mathrm{E}(P_3^*) = 8 = \lambda \mathrm{E}(R_{P_1^*}) + (1-\lambda)\mathrm{E}(R_{P_2^*}) = 5\lambda + 10(1-\lambda)$$

which gives $\lambda = 0.4$ and so $P_3^* = (\pi_A, \pi_B, \pi_C) = (1.016, 0.369, -0.385)$.

(c) Let P_4^* be the solution to $(*)$ for $E_P = 15\%$. Then:

$$P_4^* = 2P_2^* - P_1^* = (0.057, 0.887, 0.056)$$

Since P_4^* solves $(*)$ and all its weights are in the range $[0, 1]$, it must also be the solution to $(**)$.

3. If we look at the covariance matrix, we can recognize that assets 1 and 3 are equivalent in terms of variance and covariances. Therefore, they should have the same weight in the minimum variance portfolio, let the weight be π. The variance of the portfolio is

$$\begin{aligned} \sigma^2 &= 4\pi^2 + 9(1-2\pi)^2 + 4\pi^2 + 2(3)\pi^2 + 2(3)\pi(1-2\pi) + 2(3)\pi(1-2\pi) \\ &= (4 + 36 + 4 + 6 - 12 - 12)\pi^2 + (-36 + 6 + 6)\pi + 9 \\ &= 26\pi^2 - 24\pi + 9. \end{aligned}$$

Differerntate the variance with respect to π, we have

$$\frac{d\sigma^2}{d\pi} = 52\pi - 24 = 0 \quad \Rightarrow \quad \pi = \frac{6}{13}.$$

The minimum variance portfolio is $(6/13, 1/13, 6/13)$. The expected return is 5%.

4. (a) Let the composition of the portfolio be $\pi = (\pi_A, \pi_B, \pi_C)$.
   Using the result of Tutorial 5 Question 4, for a portfolio with uncorrelated assets, the weights of the assets in the minimum variance portfolio are inversely proportional to the variance of the assets. That is,

   $$\pi_A : \pi_B : \pi_C = 1 : 1/2 : 1/3 = 6 : 3 : 2.$$

   Therefore, the minimum variance portfolio is (6/11, 3/11, 2/11) and the variance of the return of the portfolio is 6/11.

   (b) The Lagrangian is

   $$\begin{aligned} L(\pi, \alpha, \beta) &= 1\pi_A^2 + 2\pi_B^2 + 3\pi_C^2 - \alpha(4\pi_A + 5\pi_B + 6\pi_C - 6) \\ &\quad -\beta(\pi_A + \pi_B + \pi_C - 1) \end{aligned}$$

   From the Lagrangian, we have

   $$\begin{aligned} \frac{\partial L}{\partial \pi_A} &= 2\pi_A - 4\alpha - \beta = 0 \quad (1) \\ \frac{\partial L}{\partial \pi_B} &= 4\pi_B - 5\alpha - \beta = 0 \quad (2) \\ \frac{\partial L}{\partial \pi_C} &= 6\pi_C - 6\alpha - \beta = 0 \quad (3) \\ \frac{\partial L}{\partial \alpha} &= -4\pi_A - 5\pi_B - 6\pi_C + 6 = 0 \quad (4) \\ \frac{\partial L}{\partial \beta} &= -\pi_A - \pi_B - \pi_C + 1 = 0 \quad (5). \end{aligned}$$

   Substituting (1), (2) and (3) into (4) and (5) we get

   $$\begin{aligned} \frac{4(4\alpha+\beta)}{2} + \frac{5(5\alpha+\beta)}{4} + \frac{6(6\alpha+\beta)}{6} &= 6 \quad (4') \qquad \text{and} \\ \frac{4\alpha+\beta}{2} + \frac{5\alpha+\beta}{4} + \frac{6\alpha+\beta}{6} &= 1 \quad (5'). \end{aligned}$$

   Simplify the equations, they become

   $$81\alpha + 17\beta = 24 \qquad \text{and} \qquad 51\alpha + 11\beta = 12.$$

   We obtain $\alpha = 5/2$, $\beta = -21/2$, and the efficient portfolio is (-1/4, 1/2, 3/4). The variance of the return of the portfolio is 9/4.

5. (a) We express the problem in matrix form as $Ay = b$, where

   $$y = (\pi_1, \pi_2, \pi_3, \pi_4, \alpha, \beta)^T \qquad b = (0, 0, 0, 0, E_P, 1)^T$$

and A is given by

$$\begin{pmatrix} 20 & 16 & 0 & 30 & -6 & -1 \\ 16 & 30 & 0 & 32 & -7 & -1 \\ 0 & 0 & 80 & 0 & -8 & -1 \\ 30 & 32 & 0 & 70 & -10 & -1 \\ 6 & 7 & 8 & 10 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

(b) For A as above, A^{-1} is (use a computer package such as R!):

$$\begin{pmatrix} 0.034 & -0.044 & -0.002 & 0.012 & -0.276 & 2.333 \\ -0.044 & 0.061 & -0.004 & -0.013 & -0.001 & 0.306 \\ -0.002 & -0.004 & 0.009 & -0.003 & 0.053 & -0.125 \\ 0.012 & -0.013 & -0.003 & 0.005 & 0.224 & -1.514 \\ 0.276 & 0.001 & -0.053 & -0.224 & 1.543 & -8.082 \\ -2.333 & -0.306 & 0.125 & 1.514 & -8.082 & 54.635 \end{pmatrix}.$$

(c) From the equation $y = A^{-1}b$, we have:

$$\pi_1(E_P) = -0.276 \times E_P + 2.333$$

for E_P expressed as a percentage rate of return.

6. We know that the efficient frontier is a straight line in the case of borrowing and lending at the risk-free rate. The line passes through two points; firstly the risk-free asset, and secondly a risky market portfolio, M. We know that regardless of the investor's attitude to risk, the investor will invest in the same risky portfolio M. The question asks us to find M.

   The variance-covariance matrix for the three assets, A, B and F, is:

$$C = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   The minimum variance portfolio for an expected rate of return E_P is:

$$\pi_A A + \pi_B B + \pi_F F$$

   where:

$$\begin{pmatrix} 8 & 6 & 0 & -6 & -1 \\ 6 & 20 & 0 & -9 & -1 \\ 0 & 0 & 0 & -4 & -1 \\ 6 & 9 & 4 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ E_P \\ 1 \end{pmatrix} = \begin{pmatrix} \pi_A \\ \pi_B \\ \pi_F \\ \alpha \\ \beta \end{pmatrix}$$

   Inverting the matrix gives:

$$\begin{pmatrix} 8 & 6 & 0 & -6 & -1 \\ 6 & 20 & 0 & -9 & -1 \\ 0 & 0 & 0 & -4 & -1 \\ 6 & 9 & 4 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0.15625 & -0.0625 & -0.09375 & 0.0625 & -0.25 \\ -0.0625 & 0.025 & 0.0375 & 0.175 & -0.7 \\ -0.09375 & 0.0375 & 0.05625 & -0.2375 & 1.95 \\ -0.0625 & -0.175 & 0.2375 & 0.775 & -3.1 \\ 0.25 & 0.7 & -1.95 & -3.1 & 12.4 \end{pmatrix}$$

We can choose any value for E_P greater than the risk-free rate, 4%. For $E_P = 8\%$, we find that:

$$\pi_A = 0.25; \quad \pi_B = 0.7; \quad \pi_F = 0.05$$

Hence, the investor should hold the risky assets in the proportions:

$$A = 0.25/0.95 = 0.263; \qquad B = 0.7/0.95 = 0.737$$