第四次作业 5.6，5.9，5.24，5.28

5.6 设母体$\xi \sim N(\mu, 4)$, $(\xi_1, \xi_2, \cdots, \xi_n)$ 是取自此母体的一个子样, $\bar{\xi}$ 为子样均值. 试问: 子样容量多大, 才能使

(1) $E\left(|\bar{\xi}-\mu|^2\right) \leq 0.1$;

(2) $E(|\bar{\xi}-\mu|) \leq 0.1$;

(3) $P(|\bar{\xi}-\mu| \leq 0.1) \geq 0.95$.

解. (1)

$$E\left(|\bar{\xi}-\mu|^2\right) = D\bar{\xi} = \tfrac{4}{n} \leqslant 0.1,$$
$$n \geqslant \tfrac{4}{0.1} = 40.$$

所以当n 取40 时, 可以使得$E\left(|\bar{\xi}-\mu|^2\right) \leq 0.1$.

(2) 由题意可知$\eta = \frac{\sqrt{n}}{2}(\bar{\xi}-\mu) \sim N(0,1)$, 且

$$E|\eta| = \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}}|x|e^{-\frac{1}{2}x^2}dx = 2\int_{0}^{\infty} \tfrac{1}{\sqrt{2\pi}}xe^{-\frac{1}{2}x^2}dx = -\tfrac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\Big|_0^{\infty} = \sqrt{\tfrac{2}{\pi}}$$

故

$$E(|\bar{\xi}-\mu|) = \frac{2}{\sqrt{n}}E\left|\frac{\sqrt{n}}{2}(\bar{\xi}-\mu)\right| = \frac{2}{\sqrt{n}}E|\eta| = \frac{2}{\sqrt{n}}\sqrt{\frac{2}{\pi}} \leq 0.1 \Rightarrow n \geq \frac{800}{\pi} \approx 254.7,$$

所以当n 取255 时, 可以使得$E(|\bar{\xi}-\mu|) \leq 0.1$.

(3).

$$P(|\bar{\xi}-\mu| \leq 0.1) = P\left(\left|\frac{\sqrt{n}}{2}(\bar{\xi}-\mu)\right| \leq \frac{0.1\sqrt{n}}{2}\right) \geq 0.95$$
$$\Rightarrow 2\Phi\left(\frac{0.1\sqrt{n}}{2}\right) - 1 \geq 0.95 \Rightarrow \Phi\left(\frac{0.1\sqrt{n}}{2}\right) \geq 0.975$$
$$\Rightarrow \frac{0.1\sqrt{n}}{2} \geq 1.96 \Rightarrow n \geq 39.2^2 = 1536.6.$$

即当$n \geq 1537$ 时, 才能使$P(|\bar{\xi}-\mu| \leq 0.1) \geq 0.95$. □

5.9. 设母体$\xi \sim N\left(\mu, \sigma^2\right)$, 子样方差$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\xi_i - \bar{\xi}\right)^2$. 求$ES_n^2, DS_n^2$, 并证明当$n$ 增大时, 他们分别为$\sigma^2 + o\left(\frac{1}{n}\right)$ 和$\frac{2\sigma^4}{n} + o\left(\frac{1}{n}\right)$.

解: $ES_n^2 = \frac{(n-1)\sigma^2}{n} = \sigma^2 - \frac{1}{n}\sigma^2 = \sigma^2 + o(1)$. (注: 习题中有错误, 不是$o\left(\frac{1}{n}\right)$, 而是$o(1)$, 即无穷小.)

对于后一问, 只需利用P_{239} 的定理5.1, 我们在这里需计算μ_2, μ_4.

$$\mu_2 = D\xi = \sigma^2,$$

$$\begin{aligned}
\mu_4 &= E(\xi-\mu)^4 = \int_{-\infty}^{\infty}(x-\mu)^4 p_\xi(x)dx \\
&= \int_{-\infty}^{\infty} x^4 \tfrac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\tfrac{1}{2}\tfrac{x^2}{\sigma^2}\right\}dx = \int_{-\infty}^{\infty} x^3 \tfrac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\tfrac{1}{2}\tfrac{x^2}{\sigma^2}\right\}d\tfrac{x^2}{2} \\
&= -x^3\tfrac{\sigma}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\tfrac{x^2}{\sigma^2}\right\}\Big|_{-\infty}^{\infty} + 3\sigma^2\int_{-\infty}^{\infty} x^2 \tfrac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\tfrac{1}{2}\tfrac{x^2}{\sigma^2}\right\}dx \\
&= 3\sigma^4.
\end{aligned}$$

把μ_2, μ_4 的结果代入定理5.1, 可知:

$$DS_n^2 = \sigma^4\left[\frac{2}{n} - \frac{2}{n^2}\right] = \frac{2\sigma^4}{n} + o\left(\frac{1}{n}\right).$$

实际上, 我们也可以这样计算: 令随机变量$\eta \sim \chi^2(n)$, 那么

$$\begin{aligned}
E\eta &= \int_0^{\infty} x\tfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{1}{2}x}dx = \tfrac{2^{\frac{n+2}{2}}\Gamma(\frac{n+2}{2})}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} = n \\
E\eta^2 &= \int_0^{\infty} x^2\tfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{1}{2}x}dx = n(n+2).
\end{aligned}$$

因此$E\eta = n, D\eta = 2n$. 从以上可知:

$$D\left(S_n^2\right) = \frac{\sigma^4}{n^2}D\left(\frac{nS_n^2}{\sigma^2}\right) = 2(n-1)\frac{\sigma^4}{n^2} = \frac{2\sigma^2}{n} + o\left(\frac{1}{n}\right).$$

5.24 设母体ξ 以等概率取四个值$0, 1, 2, 3$, 现从中获得一个容量为3 的子样,试分别求$\xi_{(1)}$ 与$\xi_{(3)}$ 的分布。

解.

$$\begin{aligned}
P\left(\xi_{(1)} \geqslant k\right) &= P\left(\xi_i \geqslant k, i = 1,2,3\right) \\
&= \left(\frac{4-k}{4}\right)^3, k = 0,1,2,3 \\
P\left(\xi_{(3)} \leqslant k\right) &= P\left(\xi_i \leqslant k, i = 1,2,3\right) \\
&= \left(\frac{k+1}{4}\right)^3, k = 0,1,2,3
\end{aligned}$$

从上两式,易得:

$\xi_{(1)}$	0	1	2	3
P	$\frac{37}{64}$	$\frac{19}{64}$	$\frac{7}{64}$	$\frac{1}{64}$

$\xi_{(3)}$	0	1	2	3
P	$\frac{1}{64}$	$\frac{7}{64}$	$\frac{19}{64}$	$\frac{37}{64}$

$\square$

5.28 设母体ξ 的密度函数

$$f(x)=\begin{cases}6x(1-x), & 0<x<1\\ 0, & otherwise\end{cases}$$

由此母体抽取一个字样$(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$ 又$\xi_{(1)}<\xi_{(2)}<\xi_{(3)}<\xi_{(4)}<\xi_{(5)}$ 是子样的次序统计量,求$\xi_{(3)}$的密度函数.

解. 易得分布函数为

$$F(x)=\begin{cases}1, & x\geqslant 1\\ 3x^2-2x^3 & 0<x<1\\ 0, & x\leq 0\end{cases}$$

所以

$$f_{\xi_{(3)}}(x)=\begin{cases}6\frac{5!}{2!2!}(3x^2-2x^3)^2(1-3x^2+2x^3)^2x(1-x), & 0<x<1\\ 0, & otherwise\end{cases}$$

即

$$f_{\xi_{(3)}}(x)=\begin{cases}180x^5(1-x)(3-2x)^2\left(1-3x^2+2x^3\right)^2, & 0<x<1\\ 0, & otherwise\end{cases}$$

$\square$