3 习题

习题 3.1. 设参数化单位球面 $\mathbb{S}^2 : \mathbf{r}(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$.

(1) 求球面上的一组正交活动标架 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 以及对应的 1 形式 ω_i, ω_{ij}.

(2) 利用正交活动标架求球面的第一、二基本形式.

(3) 利用正交活动标架验证球面的 *Gauss-Codazzi* 方程.

解. (1) 直接计算有

$$\begin{aligned}
\mathbf{r}_u =&(-\sin u \cos v, -\sin u \sin v, \cos u),\\
\mathbf{r}_v =&(-\cos u \sin v, \cos u \cos v, 0),\\
\mathbf{n} =&-(\cos u \cos v, \cos u \sin v, \sin u),
\end{aligned}$$

故而正交标架为

$$\begin{aligned}
\mathbf{e}_1 =&(-\sin u \cos v, -\sin u \sin v, \cos u),\\
\mathbf{e}_2 =&(-\sin v, \cos v, 0),\\
\mathbf{e}_3 =&-(\cos u \cos v, \cos u \sin v, \sin u).
\end{aligned}$$

由于 $d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv = \mathbf{e}_1 du + \cos u dv \mathbf{e}_2$, 故而

$$\omega_1 = \langle d\mathbf{r}, \mathbf{e}_1 \rangle = du \quad \text{以及} \quad \omega_2 = \langle d\mathbf{r}, \mathbf{e}_2 \rangle = \cos u dv.$$

又直接计算得

$$d\mathbf{e}_1 = (-\cos u \cos v, -\cos u \sin v, -\sin u)du + (\sin u \sin v, -\sin u \cos v, 0)dv$$

以及

$$d\mathbf{e}_2 = (-\cos v, -\sin v, 0)dv,$$

所以

$$\begin{cases}
\omega_{12} = \langle d\mathbf{e}_1, \mathbf{e}_2 \rangle = -\sin u dv = -\omega_{21},\\
\omega_{13} = \langle d\mathbf{e}_1, \mathbf{e}_3 \rangle = du = -\omega_{31},\\
\omega_{23} = \langle d\mathbf{e}_2, \mathbf{e}_3 \rangle = \cos u dv = -\omega_{32}.
\end{cases}$$

(2) 由 (1) 中的结果有曲面 $\mathbb{S}^2$ 的第一基本形式为

$$\mathbf{I} = \omega_1^2 + \omega_2^2 = du^2 + \cos^2 u dv^2,$$

第二基本形式为

$$\mathbf{II} = \omega_1\omega_{13} + \omega_2\omega_{23} = du^2 + \cos^2 u dv^2.$$

(3) 直接计算有

$$\begin{cases} d\omega_{12} = -\cos u du \wedge dv, \\ d\omega_{13} = 0, \\ d\omega_{23} = -\sin u du \wedge dv, \end{cases}$$

故而

$$\omega_{12} = -\cos u du \wedge dv = \omega_{13} \wedge \omega_{32},$$

即 Gauss 方程成立. 同理

$$\begin{cases} d\omega_{13} = 0 = \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} = -\sin u du \wedge dv = \omega_{21} \wedge \omega_{13}, \end{cases}$$

即 Codazzi 方程成立. □

习题 3.2. *设 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 为曲面 $\mathbf{S}$ 的正交活动标架, $\mathbf{e}_1, \mathbf{e}_2$ 是曲面的主方向, κ_1, κ_2 为相应的主曲率. 证明*

$$\begin{cases} d\kappa_1 \wedge \omega_1 = (\kappa_2 - \kappa_1)\omega_{12} \wedge \omega_2, \\ d\kappa_2 \wedge \omega_2 = (\kappa_1 - \kappa_2)\omega_{21} \wedge \omega_1. \end{cases}$$

证明. 由于 $\mathbf{e}_1, \mathbf{e}_2$ 为曲面的主方向, 所以 $-\mathcal{W}$ 在 $\{\mathbf{e}_1, \mathbf{e}_2\}$ 下的坐标变换矩阵

$$B = \begin{bmatrix} \kappa_1 & 0 \\ 0, & \kappa_2 \end{bmatrix}.$$

故而由命题 10.3 有

$$\omega_{13} = \kappa_1\omega_1, \quad \omega_{23} = \kappa_2\omega_2.$$

利用运动方程直接计算有

$$d\omega_{13} = d\kappa_1 \wedge \omega_1 + \kappa_1 d\omega_1 = d\kappa_1 \wedge \omega_1 + \kappa_1\omega_{12} \wedge \omega_2,$$

以及

$$d\omega_{23} = d\kappa_2 \wedge \omega_2 + \kappa_2 d\omega_2 = d\kappa_2 \wedge \omega_2 + \kappa_2 \omega_{21} \wedge \omega_1,$$

进而 Codazzi 方程 $\begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{cases}$ 等价于

$$\begin{cases} d\kappa_1 \wedge \omega_1 = (\kappa_2 - \kappa_1)\omega_{12} \wedge \omega_2, \\ d\kappa_2 \wedge \omega_2 = (\kappa_1 - \kappa_2)\omega_{21} \wedge \omega_1. \end{cases}$$

□

习题 3.3. 设 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 和 $\{\mathbf{r}; \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \mathbf{e}_3\}$ 是曲面 $\mathbf{S}$ 的两组正交活动标架满足

$$\begin{bmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}.$$

记 ω_i, ω_{ij} 及 $\bar{\omega}_i$, $\bar{\omega}_{ij}$ 分别为标架 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 和 $\{\mathbf{r}; \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \mathbf{e}_3\}$ 下对应的 1 形式. 证明:

$$\bar{\omega}_{12} = \omega_{12} + d\theta,$$

以及

$$\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = \frac{d\bar{\omega}_{12}}{\bar{\omega}_1 \wedge \bar{\omega}_2}.$$

证明. 记

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} := R(\theta),$$

则直接计算有

$$\bar{\omega}_{12} = \langle d\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2 \rangle = \langle d(R\mathbf{e}_1), R\mathbf{e}_2 \rangle = \langle dR\mathbf{e}_1, R\mathbf{e}_2 \rangle + \langle d\mathbf{e}_1, \mathbf{e}_2 \rangle.$$

又

$$dR = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} d\theta = R(\theta + \frac{\pi}{2}) d\theta,$$

故而

$$\langle dR\mathbf{e}_1, R\mathbf{e}_2 \rangle = \langle R(\theta + \frac{\pi}{2})\mathbf{e}_1 d\theta, R\mathbf{e}_2 \rangle = \langle R(\theta)\mathbf{e}_2 d\theta, R\mathbf{e}_2 \rangle = d\theta,$$

进而

$$\bar{\omega}_{12} = \omega_{12} + d\theta.$$

利用 $d^2 = 0$, 我们得到

$$d\bar{\omega}_{12} = d(\omega_{12} + d\theta) = d\omega_{12}.$$

由命题 10.4 有 $\omega_1 \wedge \omega_2 = \bar{\omega}_1 \wedge \bar{\omega}_2$, 故而

$$\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = \frac{d\bar{\omega}_{12}}{\bar{\omega}_1 \wedge \bar{\omega}_2}.$$

□

参考文献

[dC16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Dover Publications, Inc., Mineola, NY, 2016. Revised & updated second edition of [MR0394451].