二、洛朗展式

[16] $f(z)=\dfrac{1}{z^2(z-i)}$ 以 $z=i$ 为中心展成洛朗级数

$f(z)$ 奇点为 $0, i$

① 当 $0<|z-i|<1$，$f(z)=\dfrac{1}{z-i}\times\left(\dfrac{1}{i+z-i}\right)^2$

$$\frac{1}{i+z-i}=\frac{1}{i}\frac{1}{1-\left(-\frac{z-i}{i}\right)}=\frac{1}{i}\sum_{n=0}^{+\infty}(i)^n(z-i)^n \quad 0<\left|-\frac{z-i}{i}\right|<1$$

$$\left(\frac{1}{i+z-i}\right)^2=-\left(\frac{1}{i+z-i}\right)'=i\sum_{n=1}^{+\infty}(i)^n\, n\,(z-i)^{n-1}=\sum_{n=0}^{+\infty}(i)^{n+2}(n+1)(z-i)^n$$

故 $f(z)=\sum_{n=0}^{+\infty}(i)^{n+2}(n+1)(z-i)^{n-1}$

② 当 $1<|z-i|<+\infty$，$f(z)=\dfrac{1}{z-i}\times\dfrac{1}{(z-i)^2}\times\left(\dfrac{1}{\frac{i}{z-i}+1}\right)^2$

$$\frac{1}{1-\left(\frac{-i}{z-i}\right)}=\sum_{n=0}^{+\infty}(-i)^n\frac{1}{(z-i)^n} \quad 0<\left|\frac{-i}{z-i}\right|<1$$

$$\left(\frac{1}{1+\frac{i}{z-i}}\right)^2=-\frac{(z-i)^2}{-i}\left(\frac{1}{1+\frac{i}{z-i}}\right)'=-\sum_{n=1}^{+\infty}(-i)^{n-1}(-n)\frac{1}{(z-i)^{n-1}}$$

$$=\sum_{n=0}^{+\infty}(-i)^n(n+1)\frac{1}{(z-i)^n}$$

故 $f(z)=\sum_{n=0}^{+\infty}(-i)^n(n+1)\dfrac{1}{(z-i)^{n+3}}$

[7,19] 求 $f(z)=\frac{1}{(z^2+1)^2}$ 在 $z=i$ 去心邻域洛朗展式，并求其收敛范围

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$$f(z)=\frac{1}{(z+i)^2(z-i)^2}=\frac{1}{(z-i)^2}\times\frac{1}{(2i+z-i)^2}=-\frac{1}{4(z-i)^2}\left(\frac{1}{1-\left(-\frac{z-i}{2i}\right)}\right)^2$$

在 $0<|z-i|<2$ 上，$0<\left|\frac{z-i}{2i}\right|<1$

$$\frac{1}{1-\left(-\frac{z-i}{2i}\right)}=\sum_{n=0}^{+\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n,\quad 0<\left|\frac{z-i}{2i}\right|<1$$

$$\left(\frac{1}{1+\frac{z-i}{2i}}\right)^2=-2i\left(\frac{1}{1+\frac{z-i}{2i}}\right)'=-\sum_{n=1}^{+\infty}(-1)^n\,n\left(\frac{z-i}{2i}\right)^{n-1}$$

$$=\sum_{n=0}^{+\infty}(-1)^n(n+1)\left(\frac{z-i}{2i}\right)^n$$

故 $f(z)=-\frac{1}{4(z-i)^2}\sum_{n=0}^{+\infty}(-1)^n(n+1)\left(\frac{z-i}{2i}\right)^n,\quad 0<|z-i|<2$