☆ 变分问题

Step 1 问题转化． eg. $M_\varphi=\{w\in C^1(\bar{\Omega})\mid w(x,y)=\varphi(x,y),(x,y)\in\partial\Omega\}$

设 u 为问题的解，取集合 M_0 满足 $\forall v\in M_0$，$\forall \varepsilon\in\mathbb{R}$，$u+\varepsilon v\in M_\varphi$

对于 $v\in M_0$，作函数 $j(\varepsilon)\triangleq J(u+\varepsilon v)$，$\varepsilon\in\mathbb{R}$.

Step 2．计算 $j'(\varepsilon)$，$j'(0)=0$

Step 3．导出 Euler 方程，干掉 v

Step 4．充分性，反推回代得 $j'(0)=0$，再证 $j''(\varepsilon)\geq 0$.

考虑变分问题
$$\begin{cases} u\in M_0,\quad J(u)=\min\limits_{v\in M_0} J(v) \\ M_0=\{v(x)\in C^1([0,1])\mid v(0)=v(1)=0\} \\ J(v)=\frac{1}{2}\int_0^1[(v')^2-v^2+2v]\,dx. \end{cases}$$
设 $u\in C^2([0,1])$ 为问题的解，导出 u 满足的 ODE.

Step 1. 设 u 为问题的解，$\forall v\in M_0$，$\forall\varepsilon\in\mathbb{R}$，有 $u+\varepsilon\cdot v\in M_0$

$$j(\varepsilon)\triangleq J(u+\varepsilon v)=\frac{1}{2}\int_0^1\left[((u+\varepsilon v)')^2-(u+\varepsilon\cdot v)^2+2(u+\varepsilon v)\right]dx$$
$$=\frac{1}{2}\int_0^1\left[(u'+\varepsilon v')^2-(u^2+\varepsilon^2v^2+2uv\varepsilon)+2u+2\varepsilon v\right]dx$$

Step 2. $j(\varepsilon)$ 在 $\varepsilon=0$ 时达到最小值

$$j'(\varepsilon)=\frac{1}{2}\int_0^1\left[2(u'+\varepsilon v')\cdot v'-2\varepsilon v^2-2uv+2v\right]dx$$
$$=\int_0^1 u'v'\,dx+\int_0^1\varepsilon\cdot[(v')^2-v^2]\,dx+\int_0^1(1-u)\cdot v\,dx$$

则 $j'(0)=\int_0^1 u'v'dx+\int_0^1(1-u)v\,dx=0$

Step 3 转化 $v\longrightarrow u$

$$\int_0^1 u'v'dx=\int_0^1 u'dv=u'\cdot v\Big|_0^1-\int_0^1 v\cdot u''dx=-\int_0^1 v\cdot u''dx$$

$$\Rightarrow\ j'(0)=\int_0^1 v\cdot(-u''-u+1)\,dx=0,\quad\forall v\in M_0$$

由于 $[0,1]$ 为有界区域，$u''-u+1$ 连续，故由定理知 u 满足

$$\begin{cases}-u''-u+1=0\\ u(0)=u(1)=0\end{cases}$$

Step 4. 充分性，反推即可.

☆ 特征线法求解一阶线性方程的 Cauchy ~~方程~~ 问题　只有初始条件，没有边界条件

$$\begin{cases} u_t + a(x,t)\cdot u_x + b(x,t)\cdot u = f(x,t), & x\in\mathbb{R},\ t>0 \\ u(x,0) = \phi(x), & x\in\mathbb{R} \end{cases}$$

①. 求特征线 $x = x(t)$，令 $U = u(x(t),t)$

$$\frac{dU(t)}{dt} = u_t + u_x\cdot\frac{dx}{dt} = u_t + a(x,t)\cdot u_x \quad (比较)$$

$$\begin{cases} \frac{dx}{dt} = a(x,t), & t>0 \\ x(0) = c \end{cases}$$

②. 沿特征线化简方程并求解

原方程化为 $\begin{cases} \frac{dU}{dt} + b(x,t)\cdot U = f(x(t),t) \\ U(0) = u(x(0),0) = \phi(c) \end{cases}$，解出 $U(t)$.

③. 变回原变量，得解

eg1. $\begin{cases} u_t + 2u_x + u = xt, & x\in\mathbb{R},\ t>0 \\ u(x,0) = 2-x, & x\in\mathbb{R} \end{cases}$

① 求特征线 $x = x(t)$，令 $U = u(x(t),t)$

$$\frac{dU(t)}{dt} = u_t + u_x\cdot\frac{dx}{dt} = u_t + 2u_x \Rightarrow \frac{dx}{dt} = 2 \Rightarrow 特征线为\ x = 2t+c$$

②. ~~$\frac{dU}{dt}+$~~ 沿特征线，原方程化为

$$\begin{cases} \frac{dU(t)}{dt} = xt - U(t) = (2t+c)t - U(t) \\ U(0) = u(x(0),0) = 2 - x(0) = 2-c \end{cases} \Rightarrow U(t) = de^{-t} + 2t^2 + ct - 4t - c + 4$$

$$\Rightarrow d - c + 4 = 2 - c \Rightarrow d = -2$$

故 $U(t) = -2e^{-t} + 2t^2 + (c-4)t + 4 - c$

③. 由 $x = 2t+c \Rightarrow c = x-2t \Rightarrow u(x,t) = -2e^{-t} + 2t^2 + (x-2t-4)(t-1)$

eg2. $\begin{cases} u_t + x\cdot u_x = 0, & 0<x<+\infty,\ -\infty<t<+\infty \\ u(x,-\ln x) = x^2-1, & 0<x<1 \\ u(x,0) = 0, & 1\le x<+\infty \end{cases}$

1°. 当 $1\le x<+\infty$ 时，用特征线法，$x = x(t)$

① $\frac{dU(t)}{dt} = u_t + u_x\cdot\frac{dx}{dt} = u_t + x\cdot u_x$

$\Rightarrow x(t) = c\cdot e^t$，$c>0$

② $\begin{cases} \frac{dU(t)}{dt} = 0 \\ U(0) = u(x(0),0) = 0 \end{cases} \Rightarrow U(t) = C_1 \Rightarrow U(t) = 0$

因此 $u(x,t) = 0$，$1\le x<+\infty$

2°. 当 $0<x<1$ 时，用分离变量法

令 $u(x,t) = X(x)T(t)$，则有

$$\frac{T'(t)}{T(t)} = -x\cdot\frac{X'(x)}{X(x)} \triangleq -\lambda$$

$\Rightarrow T'(t) + \lambda T(t) = 0$，$x\cdot X'(x) - \lambda\cdot X(x) = 0$

于是 $\frac{X'(x)}{X(x)} = -\frac{\lambda}{x}$

① $\lambda = 0$，$X(x) = C_1\ (C_1\neq 0)$，$T(t) = C_2\ (C_2\neq 0)$

☆ 弦振动方程的初值问题

$$\begin{cases} \Box u = u_{tt} - a^2 u_{xx} = f(x,t), & x\in\mathbb{R},\ t>0 \\ u(x,0) = \varphi(x), & x\in\mathbb{R} \\ u_t(x,0) = \psi(x), & x\in\mathbb{R}. \end{cases} \xrightarrow{\text{拆分为三}} \begin{cases} \Box u = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = 0 \end{cases} \quad \begin{cases} \Box u = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x) \end{cases} \quad \begin{cases} \Box u = f \\ u(x,0) = 0 \\ u_t(x,0) = 0 , \end{cases}$$

D'Alembert Formula: $u(x,t) = \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(s)\,ds + \dfrac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(s,\tau)\,ds\right]d\tau$

当 $f=0$ 时. $u(x,t) = \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(s)\,ds$.

eg. $\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0) = x, & x\in\mathbb{R} \\ u_t(x,0) = x^2 & x\in\mathbb{R}. \end{cases}$

将问题拆分为二:

① $\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0) = x, & x\in\mathbb{R} \\ u_t(x,0) = 0, & x\in\mathbb{R} \end{cases}$ ② $\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R}.\ t>0 \\ u(x,0) = 0, & x\in\mathbb{R} \\ u_t(x,0) = x^2, & x\in\mathbb{R} \end{cases}$

由 D'Alembert Formula:

① 的解: $u_1 = \dfrac{(x+t)+(x-t)}{2} = x$

② 的解: $u_2 = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} s^2\,ds = \dfrac{1}{6}\left[(x+t)^3-(x-t)^3\right] = x^2t + \dfrac{1}{3}t^3$

从而原方程的解为 $u = u_1 + u_2 = x + x^2t + \frac{1}{3}t^3$.

☆ 对称延拓法 —— 半无界问题（限制 $0\le x<\infty$）.

⇒. 弦振动方程的半无界问题 / 热传导方程的半无界问题

$$\begin{cases} \Box u = u_{tt} - a^2\cdot u_{xx} = f(x,t), & 0\le x<\infty,\ 0<t<\infty \\ u|_{t=0} = \varphi(x), & 0\le x<\infty \\ u_t|_{t=0} = \psi(x), & 0\le x<\infty \\ u|_{x=0} = g(t), & t>0 \end{cases}$$ (第一边值)

Step 1. 边界条件齐次化

做变换 $v(x,t)=u(x,t)-xg$

Step 2. 对 φ. ψ. f 对 x 作奇延拓

$\bar\varphi = \begin{cases} \varphi(x), & x\ge 0 \\ -\varphi(-x), & x<0 \end{cases}$ $\bar\psi=\{\cdots$ $\bar f=\cdots$

Step 3. 求解 Cauchy 问题

Step 4. $u=\bar u|_{\Omega}$

第二边值： $u_x|_{x=0}=g(t)$ ⟶ 偶延拓.

$$\begin{cases} u_t - a^2\cdot u_{xx} = f(x,t), & 0\le x<\infty,\ 0<t<\infty \\ u|_{t=0} = \varphi(x) & 0\le x<\infty \\ u|_{x=0} = g(t), & t\ge 0 \end{cases}$$

同上.

eg. $$\begin{cases} u_t - a^2\cdot u_{xx} = f(x,t), & 0<x<+\infty,\ t>0 \\ u_x(0,t)=1, & t\ge 0 \\ u(x,0)=\varphi(x), & 0\le x<\infty. \end{cases}$$

①. 边界条件齐次化.

令 $v(x,t)=u(x,t)+w(x,t)$，则 $v_x(0,t)=u_x(0,t)+w_x(0,t)$

$=1+w_x(0,t)$

$=0$

则 $w_x(0,t)=-1$

设 $w(x,t)=-x$，则 $v(x,t)=u(x,t)-x$

原方程化为 $$\begin{cases} v_t - a^2\cdot v_{xx} = f(x,t), & 0<x<+\infty,\ t>0 \\ v_x|_{x=0}=0, & t\ge 0 \\ v|_{t=0}=\varphi(x)-x, & 0<x<+\infty \end{cases}$$

②. 作偶延拓.

令 $\bar f(x,t)=\begin{cases} f(x,t), & x\ge 0,\ t\ge 0 \\ f(-x,t), & x<0,\ t\ge 0 \end{cases}$，$\bar\varphi(x)=\begin{cases} \varphi(x)-x & x\ge 0 \\ \varphi(-x)+x & x<0 \end{cases}$

则 $\bar f$、$\bar\varphi$ 均为关于 x 的偶函数

③. 求解初值问题

$$\begin{cases} \bar v_t - a^2\cdot \bar v_{xx} = \bar f(x,t), & -\infty<x<+\infty,\ t>0 \\ \bar v(x,0)=\bar\varphi(x), & -\infty<x<+\infty. \end{cases}$$

☆ 分离变量法 —— 混合问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & \underbrace{0 < x < l}, \ t > 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \\ u_t(x,0) = \psi(x), & 0 \le x \le l \\ u(0,t) = u(l,t) = 0, & t > 0 \end{cases}$$

边界条件齐次

Step 1. 导出特征问题

设 $u(x,t) = X(x)\cdot T(t)$ 为方程的解，代入原方程得：$X(x)\cdot T''(t) - a^2 T(t)\cdot X''(x) = 0$

$$\Rightarrow \frac{T''(t)}{a^2\cdot T(t)} = \frac{X''(x)}{X(x)} \triangleq -\lambda$$

找到所有具有变量分离形式的非零特解

$$\Rightarrow \begin{cases} X''(x) + \lambda\cdot X(x) = 0 \\ T''(t) + a^2\lambda\cdot T(t) = 0 \end{cases}$$

$X(0)\cdot T(t) = X(l)\cdot T(t) = 0$

而 $u(x,t) \not\equiv 0$，故 $T(t) \not\equiv 0$，从而 $X(0) = X(l) = 0$.

$$\Rightarrow \begin{cases} X''(x) + \lambda\cdot X(x) = 0, & 0 < x < l \\ X(0) = X(l) = 0 \end{cases} \quad \text{(特征问题)}$$

Step 2 求解特征问题

Thm. 对于齐次ODE定解问题 $\begin{cases} X''(x) + \lambda\cdot X(x) = 0, & 0 < x < l \\ -\alpha_1 X'(0) + \beta_1 X(0) = 0 \\ \alpha_2 X'(l) + \beta_2 X(l) = 0 \end{cases}$，$\alpha_i \ge 0,\ \beta_i \ge 0$，$\alpha_i + \beta_i > 0$

其所有的特征值均非负，当 $\beta_1 + \beta_2 > 0$ 时，所有特征值为正数

eg: $\begin{cases} X''(x) + \lambda\cdot X(x) = 0 \\ X'(0) = 0 \\ X'(1) = 0 \end{cases}$

① $\lambda < 0$，由定理，此情况不存在

② $\lambda = 0$，$X''(x) = 0 \Rightarrow X(x) = ax + b$，$X'(0) = X'(1) = 0 \Rightarrow a = 0$

取 $X(x) = 1$

③ $\lambda > 0$，特征方程为 $x^2 + \lambda = 0 \Rightarrow x_1 = -\sqrt{\lambda}\cdot i$，$x_2 = \sqrt{\lambda}\cdot i$

$X(x) = C_1\cdot\cos(\sqrt{\lambda}\cdot x) + C_2\cdot\sin(\sqrt{\lambda}\cdot x)$

$X'(x) = -C_1\cdot\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}\cdot x) + C_2\cdot\sqrt{\lambda}\cdot\cos(\sqrt{\lambda}\, x)$

$$\begin{cases} X'(0) = C_2\cdot\sqrt{\lambda} = 0 \\ X'(1) = -C_1\cdot\sqrt{\lambda}\cdot\sin\sqrt{\lambda} + C_2\cdot\sqrt{\lambda}\cdot\cos\sqrt{\lambda} = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1\cdot\sin\sqrt{\lambda} = 0 \end{cases}$$

为使 $X(x)$ 为非零函数，$\sin\sqrt{\lambda} = 0 \Rightarrow \lambda = (n\pi)^2$，$n = 1, 2, \ldots$

$\Rightarrow X_n(x) = C_n\cdot\cos(n\pi x)$，$n = 1, 2, \ldots$，$C_n \ne 0$

对同一个特征值，我们取一个特征函数，i.e.，$C_n = 1$

综上 特征值 $\lambda_n = (n\pi)^2$，$n = 0, 1, 2, \ldots$，对应特征函数 $X_n = \cos(n\pi x)$，$n = 0, 1, 2, \ldots$

☆ Fourier 变换

Def $\forall f(x)\in L(-\infty,+\infty)$，

$$F:\ L(-\infty,+\infty)\longrightarrow L^{\infty}(-\infty,+\infty)$$
$$f(x)\longmapsto \hat{f}(\lambda)\triangleq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)\cdot e^{-i\lambda x}\,dx$$

$$G:\ L^{\infty}(-\infty,+\infty)\longrightarrow L(-\infty,+\infty)$$
$$f(x)\longmapsto f^{\vee}(\lambda)\triangleq\lim_{N\to+\infty}\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}f\cdot e^{i\lambda x}\,d\lambda$$

F 称为 Fourier 变换，G 称为 Fourier 逆变换，记为 $[f(x)]^{\wedge}=\hat{f}(\lambda)$

$$\begin{tikzcd} L(-\infty,+\infty)\ni f(x) \arrow[r, bend left, "F"] & \hat{f}(\lambda)\in L^{\infty}(-\infty,+\infty) \arrow[l, bend left, "f\in C^1(-\infty,+\infty)" description, "G"] \end{tikzcd}$$

性质

① 线性. $(a_1 f_1(x)+a_2 f_2(x))^{\wedge}=a_1\hat{f}_1(\lambda)+a_2\hat{f}_2(\lambda)$.

② 微商. $\left(\dfrac{d f(x)}{dx}\right)^{\wedge}=i\lambda\cdot\hat{f}(\lambda)$

③ 积分 $\left(\int_0^x f(t)\,dt\right)^{\wedge}=\dfrac{1}{i\lambda}\hat{f}(\lambda)$

④ 乘多项式. $[x f(x)]^{\wedge}=i\cdot\dfrac{d}{d\lambda}\cdot\hat{f}(\lambda)$

eg1. $\begin{cases} u_t-a^2u_{xx}=f(x,t), & x\in\mathbb{R},\ t>0\\ u(x,0)=\varphi(x), & x\in\mathbb{R}\end{cases}$

$$\left[\frac{\partial u(x,t)}{\partial t}\right]^{\wedge}=\frac{\partial\hat{u}(\lambda,t)}{\partial t}$$ (求导与积分交换次序)

(关于 x 作 Fourier 变换)
(t 为参数)

$$\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]^{\wedge}=(i\lambda)^2\cdot\hat{u}(\lambda,t)=-\lambda^2\cdot\hat{u}(\lambda,t)$$

$[u(x,0)]^{\wedge}=u(\lambda,0)$.　$[\varphi(x)]^{\wedge}=\varphi(\lambda)$.

eg2. $\begin{cases} u_{xxt}-u=f(x,t), & -\infty<x<+\infty,\ 0<t<+\infty\\ u(x,0)=\varphi(x), & -\infty<x<+\infty.\end{cases}$

$$\left[\frac{\partial^3 u}{\partial x^2\partial t}\right]^{\wedge}=(i\lambda)^2\cdot\frac{\partial\hat{u}(\lambda,t)}{\partial t}=-\lambda\cdot\frac{\partial\hat{u}(\lambda,t)}{\partial t}$$

☆ 广义函数

Def 若 $f: \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ 为连续线性泛函，则称 f 是一个广义函数

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\cdot\varphi(x)\,dx, \quad \forall \varphi(x) \in \mathscr{D}(\mathbb{R}).$$

性质. ① 线性: $\langle \alpha f+\beta g, \varphi(x) \rangle = \alpha\langle f, \varphi(x)\rangle + \beta\cdot\langle g, \varphi(x)\rangle$

② 广义函数与 C^∞ 函数之积: $\forall f \in \mathscr{D}'(\mathbb{R}),\ g \in C^\infty(\mathbb{R})$

$$\langle fg, \varphi(x)\rangle = \langle f, g(x)\varphi(x)\rangle \quad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

③ $\langle f', \varphi(x)\rangle = -\langle f, \varphi'(x)\rangle$. $\langle f^{(k)}, \varphi(x)\rangle = (-1)^k\cdot\langle f, \varphi^{(k)}(x)\rangle$

δ函数

$\langle \delta, \varphi(x)\rangle = \varphi(0), \ \forall \varphi \in \mathscr{D}(\mathbb{R})$. $H' = \delta$

Leibniz公式

eg1. ① 在 $\mathscr{D}'(\mathbb{R})$ 意义下. fg 之广义导数为 $f'g + fg'$. $f \in \mathscr{D}'(\mathbb{R})$, $g \in C^\infty(\mathbb{R})$

$$\begin{aligned}\langle (fg)', \varphi\rangle &= -\langle fg, \varphi'(x)\rangle = -\langle f, g\varphi'(x)\rangle \\ &= -\langle f, (g\varphi)' - g'\varphi\rangle \\ &= -\langle f, (g\varphi)'\rangle + \langle f, g'\varphi\rangle \\ &= \langle f', g\varphi\rangle + \langle f, g'\varphi\rangle \\ &= \langle f'g, \varphi\rangle + \langle fg', \varphi\rangle \\ &= \langle f'g + fg', \varphi\rangle\end{aligned}$$

② $\varphi(x)\cdot\delta(x) = \varphi(0)\cdot\delta(x)$

$$\begin{aligned}\langle \varphi(x)\cdot\delta(x), \psi(x)\rangle &= \langle \delta(x), \varphi(x)\psi(x)\rangle \\ &= \varphi(0)\cdot\psi(0) \\ &= \varphi(0)\cdot\langle \delta, \psi\rangle\end{aligned}$$

eg2. 设 $u(x) = \begin{cases} \sin x, & x>0 \\ 0, & x \le 0 \end{cases}$. 证明: $u(x)$ 在广义函数意义下满足 $u''+u = \delta(x)$

先证①

要记住δ函数. 只用看δ定义

$u(x) = \sin x\cdot H(x)$

$u'(x) = \sin x\cdot H'(x) + \cos x\cdot H(x)$

$u''(x) = \cos x\cdot H'(x) + \sin x\cdot H''(x) + \cos x\cdot H'(x) - \sin x\cdot H(x)$

$\Rightarrow u''(x) + u(x) = 2\cos x\cdot\delta(x) + \sin x\cdot\delta'(x) = 2\varphi(0) + \sin x\cdot\delta'(x)$

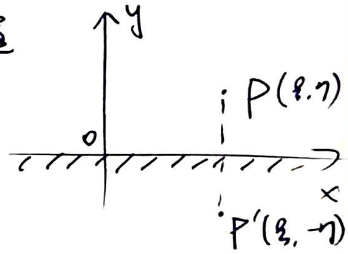
$$\begin{aligned}\langle u''+u, \varphi(x)\rangle &= \langle 2\cos x\cdot\delta(x) + \sin x\cdot\delta'(x), \varphi(x)\rangle \\ &= 2\langle \cos x\cdot\delta(x), \varphi(x)\rangle + \langle \sin x\cdot\delta'(x), \varphi(x)\rangle \\ &= 2\langle \delta(x), \cos x\cdot\varphi(x)\rangle + \langle \delta'(x), \sin x\cdot\varphi(x)\rangle \\ &= 2\cos 0\cdot\varphi(0) - \langle \delta(x), \cos x\cdot\varphi(x) + \sin x\cdot\varphi'(x)\rangle \\ &= 2\varphi(0) - \varphi(0) \\ &= \varphi(0).\end{aligned}$$

☆ Green 函数 —— 位势方程的 Green 函数（镜像法）

Def　求解位势方程第一边值问题 $\begin{cases} -\Delta u = f, & (x,y)\in\Omega \\ u=\varphi, & (x,y)\in\partial\Omega \end{cases}$

设函数 $g=g(x,y;\xi,\eta)$ 对任意的 $(\xi,\eta)\in\Omega$ 关于 (x,y) 在 $\bar{\Omega}$ 上有任意二阶连续偏导数且满足 $\begin{cases} -\Delta g = 0, & (x,y)\in\Omega \\ g(x,y) = -T(x,y;\xi,\eta), & (x,y)\in\partial\Omega \end{cases}$，则称函数 $G(x,y;\xi,\eta)=T+g$ 为边值问题的 Green 函数。$\mathbb{R}^2 \longrightarrow T(x,y;\xi,\eta) = -\frac{1}{2\pi}\cdot\ln\sqrt{(x-\xi)^2+(y-\eta)^2}$

eg1. 求解上半平面位势方程的第一边值问题的 Green 函数 $\begin{cases} -\Delta u = f, & (x,y)\in\mathbb{R}^2_+ \\ u(x,0)=\varphi(x), & -\infty<x<\infty \end{cases}$



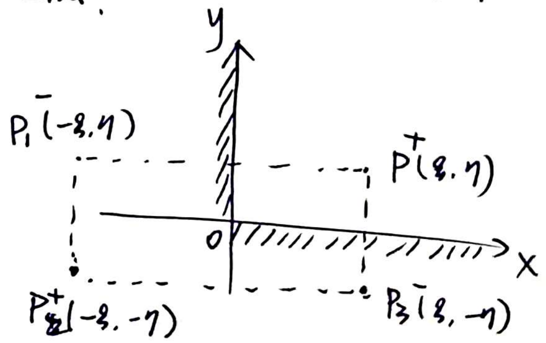
$\forall P(\xi,\eta)\in\mathbb{R}^2_+$，在 P 点放置一个单位正电荷，在 $P'(\xi,-\eta)$ 放置一个单位负电荷。

$\forall M\in\partial\Omega$，i.e. $M(x,0)$，$|MP|=|MP'|$，这两个电位势在M点大小相等，方向相反，因此相互抵消：$\left[T(x,y;\xi,\eta)-T(x,y;\xi,-\eta)\right]\Big|_{y=0}=0$

$$\Rightarrow G(x,y;\xi,\eta)=T(x,y;\xi,\eta)-T(x,y;\xi,-\eta) = -\frac{1}{4\pi}\cdot\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}$$

eg2. 求解第一象限位势方程的第一边值问题的 Green 函数。

$$\begin{cases} u_{xx}+u_{yy} = f(x,y), & 0<x<+\infty,\ 0<y<+\infty \\ u(x,0)=\varphi(x), & 0\le x<+\infty \\ u(0,y)=\psi(x), & 0\le y<+\infty \end{cases}$$



$\forall P(\xi,\eta)\in\Omega:=\{(x,y): 0<x<+\infty, 0<y<+\infty\}$

在 $P(\xi,\eta)$ 放置一个单位正电荷

在 $P_1(-\xi,\eta)$ ……负

在 $P_2(-\xi,-\eta)$ ……正

在 $P_3(\xi,-\eta)$ ……负

$\forall M\in\partial\Omega$，① 若 $M(0,y)$，$y\ge0$，$|MP_1|=|MP|$，$|MP_2|=|MP_3|$

P点产生的电位势与 P_1 点产生的电位势大小相等，方向相反，因而相互抵消

P_2 ―― ―― P_3 ―― ―― ……

② 若 $M(x,0)$，$x>0$，$|MP|=|MP_3|$，$|MP_1|=|MP_2|$

……

因此在 $\partial\Omega$ 上，M点的电势为 0。

$$\Rightarrow G(x,y;\xi,\eta)=T(x,y;\xi,\eta)-T(x,y;-\xi,\eta)+T(x,y;-\xi,-\eta)-T(x,y;\xi,-\eta)$$

$$=-\frac{1}{4\pi}\cdot\ln\frac{\left[(x-\xi)^2+(y-\eta)^2\right]\cdot\left[(x+\xi)^2+(y+\eta)^2\right]}{\left[(x+\xi)^2+(y-\eta)^2\right]\cdot\left[(x-\xi)^2+(y+\eta)^2\right]}$$

☆ 能量方法 — L_2 模估计.

用能量方法证明下述混合问题
$$\begin{cases} u_t - u_{xx} = f(x,t), & 0<x<1,\ 0<t\le 1 \\ u_x(0,t)=0,\ u(1,t)=0, & 0\le t\le 1 \\ u(x,0)=\varphi(x), & 0\le x\le 1 \end{cases}$$
在 $C^2(\bar{Q})$ 中的解是唯一的，其中 $Q=\{(x,t): 0<x<1,\ 0<t\le 1\}$

设 u_1. u_2 均为 $C^2(\bar{Q})$ 中的不同解.

记 $w(x,t)=u_1(x,t)-u_2(x,t)$，则 $w\in C^2(\bar{Q})$，且满足
$$\begin{cases} w_t - w_{xx} = 0, & 0<x<1,\ 0<t\le 1 \\ w_x(0,t)=w(1,t)=0, & 0\le t\le 1 \\ w(x,0)=0, & 0\le x\le 1 \end{cases}$$

$$\Longrightarrow \quad w\cdot w_t - w\cdot w_{xx} = 0$$

$$\Rightarrow \quad \frac{1}{2}(w^2)_t = w\cdot w_{xx}$$ 在 $[0,1]\times[0,t]$ 上积分:

$$= \int w\, dw_x = w\cdot w_x - \int w_x\, dx$$

$$\frac{1}{2}\int_0^1\int_0^t (w^2)_t\, dx\, dt = \int_0^1 \underbrace{\int_0^t w\cdot w_{xx}\, dx}\, dt$$

$$左边 = \frac{1}{2}\left[\int_0^1 w^2(x,t)dx - \int_0^1 w^2(x,0)\,dx\right] = \frac{1}{2}\int_0^1 w^2(x,t)\,dx$$

$$右边 = \int_0^1 (w\cdot w_x)\Big|_{x=0}^{x=1} dt - \int_0^1\int_0^1 w_x^2\, dx\, dt = -\int_0^1\int_0^1 w_x^2\, dx dt \le 0$$

$$\Longrightarrow \quad \int_0^1 w^2(x,t)\,dx \le 0$$

假设 $\exists\, x_0\in(0,1)$, s.t. $w^2(x_0,t)>0$，则 $\exists\,\delta>0$, s.t.
$$w^2(x,t) \ge \frac{w^2(x_0,t)}{2} > 0, \quad \forall\, x\in(x_0-\delta, x_0+\delta)$$

从而
$$\int_0^1 w^2(x,t)\,dx = \int_0^{x_0-\delta} w^2(x,t)\,dx + \int_{x_0-\delta}^{x_0+\delta} w^2(x,t)\,dx + \int_{x_0+\delta}^1 w(x,t)\,dt$$
$$\ge 0 + \int_{x_0-\delta}^{x_0+\delta} \frac{w^2(x_0,t)}{2}\,dx + 0$$
$$= w^2(x_0,t)\cdot\delta > 0$$

因此 $w^2(x,t)\equiv 0$，从而 $w(x,t)\equiv 0$，$u_1\equiv u_2$.

\#

★ 极值原理与最大模估计

Thm.（弱极值原理）若 $u\in C^{2,1}(Q)\cap C(\bar{Q})$ 满足 $u_t-a^2\cdot\Delta u\le 0$ $(\ge)$ 于 Q，则 u 在 $\bar{Q}$ 上的最大值在 $\partial_p Q$ 上取到，i.e. $\max\limits_{\bar{Q}}u=\max\limits_{\partial_pQ}u$ $\left(\min\limits_{\bar{Q}}u=\min\limits_{\partial_pQ}u\right)$.

eg1. 初边值问题的唯一性. $\begin{cases}u_t-a^2\cdot\Delta u=f(x,t), & (x,t)\in Q\\ u=\varphi(x,t), & (x,t)\in\partial_pQ\end{cases}$ $(*)$

Pf. u_1. u_2 均为解. 令 $u=u_1-u_2$，则 $u=u_1-u_2\in C^{2,1}(Q)\cap C(\bar{Q})$. 且满足 $(*)$

应用极值原理即可

eg2. 比较原理

设 $u,v\in C(\bar{Q})\cap C^{2,1}(Q)$，且 $\begin{cases}u_t-a^2u_{xx}\le v_t-a^2v_{xx} & 于 Q\\ u\le v & 于 \partial_pQ\end{cases}$，则 $u\le v$ 于 $\bar{Q}$

Pf. 令 $w=u-v$，则 $Lu-Lv\le 0$ 于 Q，$w\le 0$ 于 ∂_pQ. 由极值原理，$\max\limits_{\bar{Q}}w=\max\limits_{T}w\le 0$

$\Rightarrow$ $w\le 0$ 于 $\bar{Q}$，即 $u\le v$ 于 $\bar{Q}$.

eg3. 设 $\varphi\in C^0(\partial_pQ)$，$c\in\mathbb{R}$，$u\in C^{2,1}(Q)\cap C(\bar{Q})$ 是问题 $\begin{cases}u_t-\Delta u-cu=0, & (x,t)\in Q\\ u=\varphi(x,t). & (x,t)\in\partial_pQ\end{cases}$

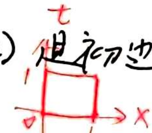
的解. 证明在 $\bar{Q}$ 上，$|u(x,t)|\le e^{ct}\cdot\max\limits_{\partial_pQ}|\varphi|$

Pf. 令 $u_1=e^{-ct}\cdot u$，则 $u=e^{ct}\cdot u_1$，$u_1\in C^{2,1}(Q)\cap C(\bar{Q})$

$$u_t-\Delta u-cu=c\cdot e^{ct}\cdot u_1+e^{ct}\cdot u_{1t}-e^{ct}\cdot\Delta u_1-ce^{ct}u_1=e^{ct}\cdot(u_{1t}-\Delta u_1)$$

从而 u_1 是问题 $\begin{cases}u_{1t}-\Delta u_1=0, & (x,t)\in Q\\ u_1=e^{-ct}\cdot\varphi(x,t), & (x,t)\in\partial_pQ\end{cases}$ 的解，由 Thm，$\max\limits_{\bar{Q}}|u_1|=\max\limits_{\partial_pQ}|e^{-ct}\varphi|\le\ldots$ #

eg4. 设 $Q=(0,1)\times(0,1]$，$u(x,t)\in C(\bar{Q})\cap C^{2,1}(Q)$ 是初边值问题



$\begin{cases}u_t-a\cdot u_{xx}+b\cdot u_x+cu=0 \quad (f(x,t)\le 0), & 0<x<1,\ 0<t\le 1\\ u(0,t)=0,\ u(1,t)=0, & 0\le t\le 1\\ u(x,0)=\phi(x). & 0\le x\le 1\end{cases}$ 的古典解，$a>0$，$c>0$，$\phi(x)>0$.

证明：$0\le u(x,t)\le\max\limits_{[0,1]}\phi(x)$. $(x,t)\in\bar{Q}$

Pf. 令 $v=u\cdot e^{-\frac{b}{2a}x-\left(\frac{b^2}{4a}-b-c\right)t}$.

$\Rightarrow$ $u_t-a\cdot u_{xx}+bu_x+cu=(v_t-av_{xx})\cdot e^{\frac{b}{2a}x+\left(\frac{b^2}{4a}-b-c\right)t}$

$\Rightarrow$ v 是问题 $\begin{cases}v_t-a\cdot v_{xx}=0, & 0<x<1,\ 0<t\le 1\\ v(0,t)=v(1,t)=0, & 0\le t\le 1\\ v(x,0)=\phi(x)\cdot e^{-\frac{b}{2a}x}, & 0\le x\le 1\end{cases}$ 的解.

$\Rightarrow$ $\max\limits_{\bar{Q}}v=\max\limits_{\partial_pQ}v=\max\limits_{0\le x\le 1}e^{-\frac{b}{2a}x}\cdot\phi(x)$，$(x,t)\in\bar{Q}$

$\min\limits_{\bar{Q}}v=\min\limits_{\partial_pQ}v=0$，$(x,t)\in\bar{Q}$.

$\Rightarrow$ $u=v\cdot e^{\frac{b}{2a}+\left(\frac{b^2}{4a}-b-c\right)t}\ge 0$

$u=v\cdot e^{\frac{b}{2a}+\left(\frac{b^2}{4a}-b-c\right)t}\le e^{\frac{b}{2a}+\left(\frac{b^2}{4a}-b-c\right)t}\cdot\max\limits_{[0,1]}e^{-\frac{b}{2a}x}\cdot\phi(x)$

① 设 $f<0$，则 u 不能在 $(0,1)\times(0,1]$ 内达到最大值

若不然，设 $\exists P_0(x_0,t_0)\in Q$. s.t. $u(x_0,t_0)=\max\limits_{\bar{Q}}u(x,t)$

则 $f(x_0,t_0)=(\underbrace{u_t}_{\ge 0}-a\underbrace{u_{xx}}_{\le 0}+\cancel{bu_x}+cu)|_{P_0}\ge 0$

这与 $f<0$ 矛盾

② 考虑 $f\le 0$. $\forall\varepsilon>0$，考虑 $v=u-\varepsilon t$，则

$Lv=Lu-\varepsilon-c\varepsilon t$

$=f-\varepsilon-c\varepsilon t<0$

由①知，v 一定不能在 Q 内达到最大值，故 $\max\limits_{\bar{Q}}v=\max\limits_{T}v$

$\Rightarrow$ $\max\limits_{\bar{Q}}u(x,t)=\max\limits_{\bar{Q}}(v+\varepsilon t)\le\max\limits_{\bar{Q}}v+\varepsilon\le\max\limits_{T}u+\varepsilon$ …… $\varepsilon\to 0$