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1. 证明:

a> $K_\lambda = \lambda K_1 + (1-\lambda)K_2 > \lambda K_2 + (1-\lambda)K_2 = K_2$

且 $K_\lambda = \lambda K_1 + (1-\lambda)K_2 < \lambda K_1 + (1-\lambda)K_1 = K_1$.

记 Φ_1: $P_t(K_\lambda)$ Φ_2: $\lambda P_t(K_1) + (1-\lambda)P_t(K_2)$.

若 $\tau_1 = T$. 则有

$V_{\tau_1}(\Phi_1) = P_T(K_\lambda) = (K_\lambda - S_T)^+$

$V_{\tau_1}(\Phi_2) = \lambda P_T(K_1) + (1-\lambda)P_T(K_2) = \lambda(K_1 - S_T)^+ + (1-\lambda)(K_2 - S_T)^+$

$$= \begin{cases} K_\lambda - S_T, & S_T < K_2 \\ \lambda(K_1 - S_T), & K_2 < S_T < K_1 \\ 0, & S_T > K_1 \end{cases}$$

$$= \begin{cases} K_\lambda - S_T, & S_T < K_2 \\ K_\lambda - S_T + (1-\lambda)(S_T - K_2), & K_2 < S_T < K_1 \\ 0, & S_T > K_1 \end{cases}$$

$\geq V_{\tau_1}(\Phi_1)$

若 $\tau_1 < T$, 由美式看跌期性质有

$V_{\tau_1}(\Phi_1) = P_{\tau_1}(K_\lambda) = (K_\lambda - S_{\tau_1})^+$

$V_{\tau_1}(\Phi_1) = \lambda P_{\tau_1}(K_1) + (1-\lambda)P_{\tau_1}(K_2)$

$= \lambda(K_1 - S_{\tau_1}) + (1-\lambda)(K_2 - S_{\tau_1})^+$

$$= \begin{cases} K_\lambda - S_{\tau_1}, & S_{\tau_1} < K_2 \\ \lambda(K_1 - S_{\tau_1}), & K_2 < S_{\tau_1} < K_1 \end{cases}$$

$$= \begin{cases} K_\lambda - S_{\tau_1}, & S_{\tau_1} < K_2 \\ K_\lambda - S_{\tau_1} + (1-\lambda)(S_{\tau_1} - K_2), & K_2 < S_{\tau_1} < K_1 \end{cases}$$

$\geq V_{\tau_1}(\Phi_1)$

综上所述, 始终有 $V_{\tau_1}(\Phi_2) \geq V_{\tau_1}(\Phi_1)$, 由无套利原理

应有 $V_t(\Phi_2) \geq V_t(\Phi_1)$

$\Rightarrow P_t(K_\lambda) \leq \lambda P_t(K_1) + (1-\lambda)P_t(K_2)$, $t < \tau_1 < T$. a.s.

b>

① 先证右边式子

若 $\tau_1 < T$，记 $\Phi_1: P_t(k_1) + k_2$ $\quad \Phi_2: P_t(k_2) + k_1$

则 $$V_{\tau_1}(\Phi_1) = k_1 - S_{\tau_1} + k_2 e^{r(\tau_1 - t)}$$

$$V_{\tau_1}(\Phi_2) = (k_2 - S_{\tau_1})^+ + k_1 e^{r(\tau_1 - t)}$$

$$= \begin{cases} k_2 - S_{\tau_1} + k_1 e^{r(\tau_1 - t)}, & S_{\tau_1} < k_2 \\ k_1 e^{r(\tau_1 - t)}, & S_{\tau_1} > k_2 \end{cases}$$

$$\geq V_{\tau_1}(\Phi_1).$$

若 $\tau_1 = T$. 则有

$$V_T(\Phi_1) = (k_1 - S_T)^+ + k_2 e^{r(T-t)}$$

$$V_T(\Phi_2) = (k_2 - S_T)^+ + k_1 e^{r(T-t)}$$

则 $$V_T(\Phi_2) = (k_1 - k_2) e^{r(T-t)} + (k_2 - S_T)^+ - (k_1 - S_T)^+$$

$$= \begin{cases} (k_1 - k_2) e^{r(T-t)} - (k_1 - k_2), & S_T < k_2 < k_1 \\ (k_1 - k_2) e^{r(T-t)} - k_1 + S_T, & k_2 < S_T < k_1 \\ (k_1 - k_2) e^{r(T-t)}, & S_T > k_1 > k_2 \end{cases}$$

$$> 0$$

综上所述由无套利原理 $V_t(\Phi_1) \leq V_t(\Phi_2)$

② 证左边式子.

记 $\Phi: P_t(k_1) - P_t(k_2)$.

若 $\tau_1 = T$. 则 $$V_T(\Phi) = (k_1 - S_T)^+ - (k_2 - S_T)^+$$

$$= \begin{cases} k_1 - k_2, & S_T < k_2 < k_1 \\ k_1 - S_T, & k_2 < S_T < k_1 \\ 0, & k_2 < k_1 < S_T \end{cases}$$

$$\geq 0$$

若 $\tau_1 < T$. 则 $$V_{\tau_1}(\Phi) = k_1 - S_{\tau_1} - (k_2 - S_{\tau_2})^+$$

$$= \begin{cases} k_1 - k_2, & S_{\tau_1} < k_2 < k_1 \\ k_1 - S_{\tau_1}, & k_2 < S_{\tau_1} < k_1 \end{cases}$$

$$\geq 0$$

故由无套利 $V(\Phi) \geq 0$

$$\Rightarrow 0 < P_t(k_1) - P_t(k_2) < k_1 - k_2, \quad t \leq \tau_1 \leq T \text{ a.s.}$$

2. 解:

a> t 时刻持有 $\bar{P}_t$，找 Δ 份 S_t 对冲风险. 即 t 时刻组合为 $\bar{P}_t - \Delta S_t$，在 $t+\Delta t$ 时刻则有

$$\bar{P}_t^u - \Delta S_t u = \bar{P}_t^d - \Delta \cdot S_t d = (\bar{P}_t - \Delta S_t)\rho$$

其中 $S_t u$ 为原生资产 S_t 的上升价格，$S_t d$ 为下降价格，$\rho = 1 + r\Delta t$

则

$$0 - \Delta \cdot S_t u = 1 - \Delta \cdot S_t d = (\bar{P}_t - \Delta S_t)\cdot \rho$$

$$\Rightarrow \quad \Delta = \frac{0-1}{(u-d)S_t} = -\frac{1}{(u-d)S_t}$$

$$\bar{P}_t = \frac{1}{\rho} \times \frac{u-\rho}{u-d}$$

b> $\bar{C}_t + \bar{P}_t$ 是一个无风险投资组合，$\bar{C}_{t+\Delta t} + \bar{P}_{t+\Delta t} = 1 \quad \Rightarrow \quad \bar{C}_t + \bar{P}_t = \frac{1}{\rho}$

$$\bar{C}_t = \frac{1}{\rho} - \bar{P}_t = \frac{1}{\rho}\cdot\frac{\rho - d}{u-d}$$

所以 $S_t = S_t u * \bar{C}_t + S_t d * \bar{P}_t$

3. 解:

$$\cancel{dV = \left\{\frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right]\right\} dt}$$

$$\cancel{+ \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2}$$

$$dV = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 S_i^2 \frac{\partial^2 V}{\partial S_i^2}$$

$$+ \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j}$$

4. 解:

PDE 模型:

$$\begin{cases} \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0, & k_1<S<k_2, 0<t<T \\ V(S,T)=0, & k_1<S<k_2 \\ V(k_1,t)=V_p(k,t), & 0<t<T \\ V(k_2,t)=V_c(k,t), & 0<t<T \end{cases}$$

其中 V_c 与 V_p 是标准期权价格

概率模型

$$V(S,t)=e^{-rT}E^{Q}[f(S_T)\mid S_t=S]$$

$$f(S_T)=\begin{cases}(S_T-k)^+, & S_T>k_2 \\ 0 & k_1<S_T<k_2 \\ (k-S_T)^+, & k_2<S_T\end{cases}$$

5. 解:

$$\begin{cases}\min\{-LV, V-(S-k)^+\}=0, & 0\le S\le L, 0\le t\le T \\ V(L,t)=L-k, & 0\le t\le T \\ V(S,T)=(S-k)^+, & 0\le t\le T\end{cases}$$

其中 $-LV=-[\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV]$

引入惩罚函数 $F_\varepsilon(V^\varepsilon-g)=-\frac{1}{\varepsilon}(V^\varepsilon-g)$，其中 $g=(S-k)^+$，$\varepsilon>0$

划分网格区域为 $Q=\{(n\Delta t, j\Delta x)\mid 0\le n\le N, 0\le j\le M, N=\frac{T}{\Delta t}, M=\frac{L}{\Delta x}\}$

则 $t^{n+1}\to t^n$ 有

$$-\delta_t^- V_i^{n+1}-\frac{\sigma^2}{2}\delta_x^2V_i^{n+1}-(r-q-\frac{\sigma^2}{2})\delta_x V_i^{n+1}+rV_i^n+F_\varepsilon(V_i^{n+1}-g)=0$$

有

$$V_j^n=\max\{\frac{1}{1+r\Delta t}[\alpha V_{j-1}^{n+1}+\beta V_j^{n+1}+\gamma V_{j+1}^{n+1}], g_i\}$$

$$V_j^N=g_j, \quad V_M^n=L-k$$

其中 $\alpha=\frac{\sigma^2\Delta t}{2(\Delta x)^2}-(r-q-\frac{\sigma^2}{2})\frac{\Delta t}{2\Delta x}$　　$\beta=1-\sigma^2\frac{\Delta t}{(\Delta x)^2}$

$$\gamma=\frac{\sigma^2\Delta t}{2(\Delta x)^2}+(r-q-\frac{\sigma^2}{2})\frac{\Delta t}{2\Delta x}$$

6. 解:

在 $\{-M<x<M,\ -M<y<M,\ 0\le t\le T\}$ 上建立网格

$t_n = n\Delta t, \quad 0\le n\le N, \quad N=\frac{T}{\Delta t}$

$x_i = i\Delta x, \quad -\frac{M}{\Delta x} < i < \frac{M}{\Delta x}$

$y_j = j\Delta y, \quad -\frac{M}{\Delta y} < j < \frac{M}{\Delta y}$

$V|_{x=-M} = e^{y} \qquad V|_{y=-M} = e^{x} \qquad V|_{x=M} = e^{M} \qquad V|_{y=M} = e^{M}$

记 $V_{ij}^{n} = V(x_i, y_j, t_n)$.

作差分有 $\frac{\partial V}{\partial t} = \frac{V_{ij}^{n+1} - V_{ij}^{n}}{\Delta t} \qquad \frac{\partial V}{\partial x} = \frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2\Delta x} \qquad \frac{\partial V}{\partial t} = \frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2\Delta y}$

$$\frac{\partial^2 V}{\partial x^2} = \frac{V_{i+1,j}^{n+1} - 2V_{ij}^{n+1} + V_{i+1,j}^{n+1}}{(\Delta x)^2} \qquad \frac{\partial^2 V}{\partial y^2} = \frac{V_{i,j+1}^{n+1} - 2V_{ij}^{n+1} + V_{i,j-1}^{n+1}}{(\Delta y)^2}$$

当 $\rho<0$ 时, 作九点差分有

$$\frac{\partial^2 V}{\partial x\partial y} = \frac{V_{i+1,j}^{n+1} + V_{i-1,j}^{n+1} + V_{i,j+1}^{n+1} + V_{i,j-1}^{n+1} - 2V_{ij}^{n} - V_{i-1,j+1}^{n+1} - V_{i+1,j-1}^{n+1}}{2\Delta x\Delta y}$$

代入可求得

$$\begin{aligned}
V_{ij}^{n} = \frac{1}{1+r\Delta t}\Big[&\Big(1 - \frac{\sigma_1^2\Delta t}{(\Delta x)^2} - \frac{\sigma_2^2\Delta t}{(\Delta y)^2} - \rho\sigma_1\sigma_2\cdot\frac{\Delta t}{\Delta x\Delta y}\Big)V_{ij}^{n+1} \\
&+ \Big(\frac{\sigma_1^2\Delta t}{2(\Delta x)^2} + \frac{\Delta t}{2\Delta x}\big(r - \frac{\sigma_1^2}{2}\big) + \frac{\rho\sigma_1\sigma_2\Delta t}{2\Delta x\Delta y}\Big)V_{i+1,j}^{n+1} \\
&+ \Big(\frac{\sigma_1^2\Delta t}{2(\Delta x)^2} - \frac{\Delta t}{2\Delta x}\big(r - \frac{\sigma_1^2}{2}\big) + \frac{\rho\sigma_1\sigma_2\Delta t}{2\Delta x\Delta y}\Big)V_{i-1,j}^{n+1} \\
&+ \Big(\frac{\sigma_2^2\Delta t}{2(\Delta y)^2} + \frac{\Delta t}{2\Delta y}\big(r - \frac{\sigma_2^2}{2}\big) + \frac{\rho\sigma_1\sigma_2\Delta t}{2\Delta x\Delta y}\Big)V_{i,j+1} \\
&+ \Big(\frac{\sigma_2^2\Delta t}{2(\Delta y)^2} - \frac{\Delta t}{2\Delta y}\big(r - \frac{\sigma_2^2}{2}\big) + \frac{\rho\sigma_1\sigma_2\Delta t}{2\Delta x\Delta y}\Big)V_{i,j-1}^{n+1}\Big]
\end{aligned}$$

$$= \frac{1}{1+r\Delta t}\left[\alpha V_{ij}^{n+1} + \beta_+ V_{i+1,j}^{n+1} + \beta_- V_{i-1,j}^{n+1} + \gamma_+ V_{i,j+1}^{n+1} + \gamma_- V_{i,j-1}^{n+1} + \mu_+ V_{i-1,j+1}^{n+1} + \mu_- V_{i+1,j-1}^{n+1}\right]$$

记 $\alpha_1 = \frac{\sigma^2\Delta t}{(\Delta x)^2} > 0 \qquad \alpha_2 = \frac{\sigma_2^2\Delta t}{(\Delta y)^2} > 0. \qquad \mu = \mu_+ = \mu_-$

该格式保持稳定性要求 $\alpha, \beta_\pm, \gamma_\pm, \mu_\pm \ge 0$

则需要满足以下不等式组

$$
\begin{cases}
\alpha = 1-\alpha_1-\alpha_2+2\mu \geq 0 & ① \\
\beta_+ = \frac{\alpha_1}{2}\left[1+\frac{1}{\sigma_1^2}\left(r-\frac{\sigma_1^2}{2}\right)\Delta x\right]-\mu \geq 0 & ② \\
\beta_- = \frac{\alpha_1}{2}\left[1-\frac{1}{\sigma_1^2}\left(r-\frac{\sigma_1^2}{2}\right)\Delta x\right]-\mu \geq 0 & ③ \\
\gamma_+ = \frac{\alpha_1}{2}\left[1+\frac{1}{\sigma_2^2}\left(r-\frac{\sigma_2^2}{2}\right)\Delta x\right]-\mu \geq 0 & ④ \\
\gamma_- = \frac{\alpha_2}{2}\left[1-\frac{1}{\sigma_2^2}\left(r-\frac{\sigma_2^2}{2}\right)\Delta x\right]-\mu \geq 0 & ⑤ \\
\mu_+ = \mu_- \geq 0 & ⑥
\end{cases}
$$

因为 $\rho<0$，$\sigma_1, \sigma_2, \Delta t, \Delta x, \Delta y \geq 0$ 故 $\mu_+ = \mu_- \geq 0$ 恒成立

由①可得 $\alpha_1+\alpha_2 \leq 1+2\mu$

由②+③可得 $\alpha_1 \geq 2\mu$. 由④+⑤可得 $\alpha_2 \geq 2\mu$

故有 $2\mu \leq \alpha_1 \leq 1$, $2\mu \leq \alpha_2 \leq 1$

使②③④⑤成立还要有

$$\frac{\alpha_1}{2}\left[1-\frac{1}{\sigma_1^2}\left|r-\frac{\sigma_1^2}{2}\right|\Delta x\right]-\mu \geq 0$$

$$\frac{\alpha_2}{2}\left[1-\frac{1}{\sigma_2^2}\left|r-\frac{\sigma_2^2}{2}\right|\Delta y\right]-\mu \geq 0$$

$$\Rightarrow \quad \frac{1}{\sigma_1^2}\left(\left|r-\frac{\sigma_1^2}{2}\right|-\frac{\rho\sigma_1}{\Delta y}\right)\Delta x \leq 1$$

$$\frac{1}{\sigma_2^2}\left(\left|r-\frac{\sigma_2^2}{2}\right|-\frac{\rho\sigma_2}{\Delta x}\right)\Delta y \leq 1$$

故使 $\alpha, \beta_\pm, \gamma_\pm, \mu_\pm \geq 0$ 的条件为

$$
\begin{cases}
-\frac{\rho\sigma_2}{\sigma_1} \leq \frac{\Delta y}{\Delta x}, \quad -\frac{\rho\sigma_1}{\sigma_2} \leq \frac{\Delta x}{\Delta y} \\
\frac{\sigma_1^2\Delta t}{(\Delta x)^2} \leq 1, \quad \frac{\sigma_2^2\Delta t}{(\Delta y)^2} \leq 1 \\
\frac{1}{\sigma_1^2}\left(\left|r-\frac{\sigma_1^2}{2}\right|-\frac{\rho\sigma_2}{\Delta y}\right)\Delta x \leq 1 \\
\frac{1}{\sigma_2^2}\left(\left|r-\frac{\sigma_2^2}{2}\right|-\frac{\rho\sigma_1}{\Delta x}\right)\Delta y \leq 1
\end{cases}
$$

故需要调整网格 $\Delta t, \Delta x, \Delta y$ 来控制 $\alpha, \beta_\pm, \gamma_\pm, \mu_\pm \geq 0$ 从而保正稳定性.