Portfolio Theory
Tutorial 7

1. Suppose that you have n risky assets available, $X_1, \ldots, X_n$.

   Recall from Chapter 4, the use of the Lagrangian function to solve the minimum variance problem can be solved by solving a system of $n+2$ linear equations.

   Define

$$\begin{aligned}
\mu &= (E[X_1], E[X_2], \ldots, E[X_n])^T \ , \\
\vec{1} &= (1, 1, \ldots, 1)^T \ , \\
\pi &= (\pi_1, \pi_2, \ldots, \pi_n)^T \ , \\
y &= (\pi_1, \pi_2, \ldots, \pi_n, \lambda_1, \lambda_2)^T \ , \\
b &= (0, 0, \ldots, 0, E_P, 1)^T
\end{aligned}$$

$$A = \begin{pmatrix} 2C & -\mu & -\vec{1} \\ \\ \mu^T & 0 & 0 \\ \vec{1}^T & 0 & 0 \end{pmatrix}$$

   where C is the covariance matrix of the random vector X of returns. Then, the $n+2$ equations are:

$$Ay = b$$

   $\Rightarrow$ Solution is given by

$$y = A^{-1}b$$

   (a) Show that the system of linear equations can be written as

$$\begin{aligned}
2C\pi - \lambda_1\mu - \lambda_2\vec{1} &= 0 && (1) \\
\pi^T\mu &= E_P && (2) \\
\pi^T\vec{1} &= 1. && (3)
\end{aligned}$$

   where (1) is the matrix representation of the first n linear equations.

   (b) Hence show that the optimal vector π is of the form

$$\pi = \frac{1}{2}C^{-1}\left(\lambda_1\mu + \lambda_2\vec{1}\right). \qquad (4)$$

   (c) Discuss how this provides an alternative proof of the two-fund theorem.

   (d) Consider the problem: minimise $Var(X_\pi)$ subject to $\pi^T\vec{1} = 1$. (That is, find the minimum variance portfolio.)

   Show that the minimum variance portfolio is

$$\pi_{min} = \frac{C^{-1}\vec{1}}{\vec{1}^T C^{-1}\vec{1}}.$$

   (e) Use equation (4) to identify two efficient portfolios.

For Questions 2 and 3, you will need to use a suitable computer package to invert a large matrix.

2. The returns on three assets A, B and C are such that:

$$E[R_A] = 14\%; \quad E[R_B] = 8\%; \quad E[R_C] = 20\%$$

$$\sigma_A = 6\%; \quad \sigma_B = 3\%; \quad \sigma_C = 15\%$$

$$\rho_{AB} = 0.5; \quad \rho_{AC} = 0.2; \quad \rho_{BC} = 0.4$$

where ρ denotes the correlation between 2 assets. Assume short selling is allowed. Find the portfolio with the minimum variance which can be constructed from these three assets.
Hint: Write down the Langrangian function, noting that there is no constraint on the expected return. Write down the four partial derivatives of the Langrangian. Derive the four simultaneous linear equations to be satisfied by the weights in the minimum variance portfolio. Express these equations in matrix form and solve using your favourite computer programme.

3. The rates of return on three assets 1, 2 and 3 can be modelled by a single index model using the following parameters.

	1	2	3
α	2	-1	1
β	0.5	1.5	1.2
$\sigma^2_{\xi_i}$	5	8	4

Note - α has units %, and $\sigma^2_{\xi_i}$ has units %%.

(a) Given that the market has an expected rate of return of 9% and variance of 20%%, calculate the mean and variance of the rate of return on each asset and the covariance of the rates of return between each pair of assets.

(b) An investor can invest only in assets 1, 2 and 3. Determine the minimum variance portfolio for an expected rate of return of 10%. Short sales are allowed.

4. Let R_i, for $i = 1, 2, \ldots, n$, be independent samples of a return R of mean μ and variance σ^2. Define the estimators

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} R_i,$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \hat{\mu})^2.$$

Show that $E(\hat{\mu}) = \mu$ and $E(\hat{\sigma}^2) = \sigma^2$. (That means, they are unbiased estimators of μ and σ^2, respectively.)

5. The following industry index model has been suggested to model the rates of return on banks and oil companies. You may assume that the indices are orthogonal.

$$R_i = a_i + b_{iM} I_M + b_{iB} I_B + b_{iO} I_O + \xi_i$$

The banks A and B, and the oil companies C and D have the following parameters in this model:

	A	B	C	D
a_i	1	2	3	3
b_{iM}	1.2	0.9	0.5	0.4
b_{iB}	0.9	1.1	0	0
b_{iO}	0	0	0.8	1.3
$\sigma^2_{\xi_i}$	1	1	8	3

Note - a_i has units %, and $\sigma^2_{\xi_i}$ has units %%.

(a) You are given that the market has an expected rate of return of 10% and variance of 15%%. The banking index I_B has mean -2% and variance 3%%. The oil index I_O has mean 5% and variance 14%%. Calculate the mean and variance of the rate of return on each of these four assets and the covariance of the rates of return between pairs of assets.

(b) A portfolio has 30% invested in each bank, and 20% invested in each oil company. What is the mean and variance of the rate of return on this portfolio?