Graph Shortest Path

Notes by Yan Yan

Contents

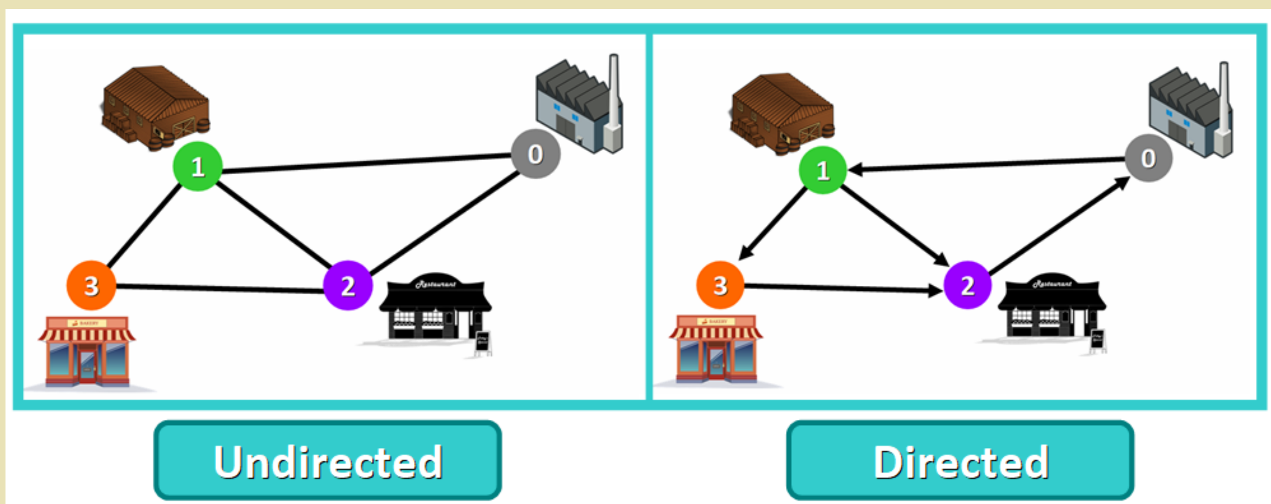
- Dijkstra’s Algorithm
- Bellman-Ford's Algorithm

Learning Objectives

- Understand the process of Dijkstra's Algorithm and Bellman-Ford's Algorithm
- Implement the two algorithms to find the shortest path on a given graph
- Describe the differences of the two algorithms
- Remember the time complexities of the two algorithms

Dijkstra’s algorithm

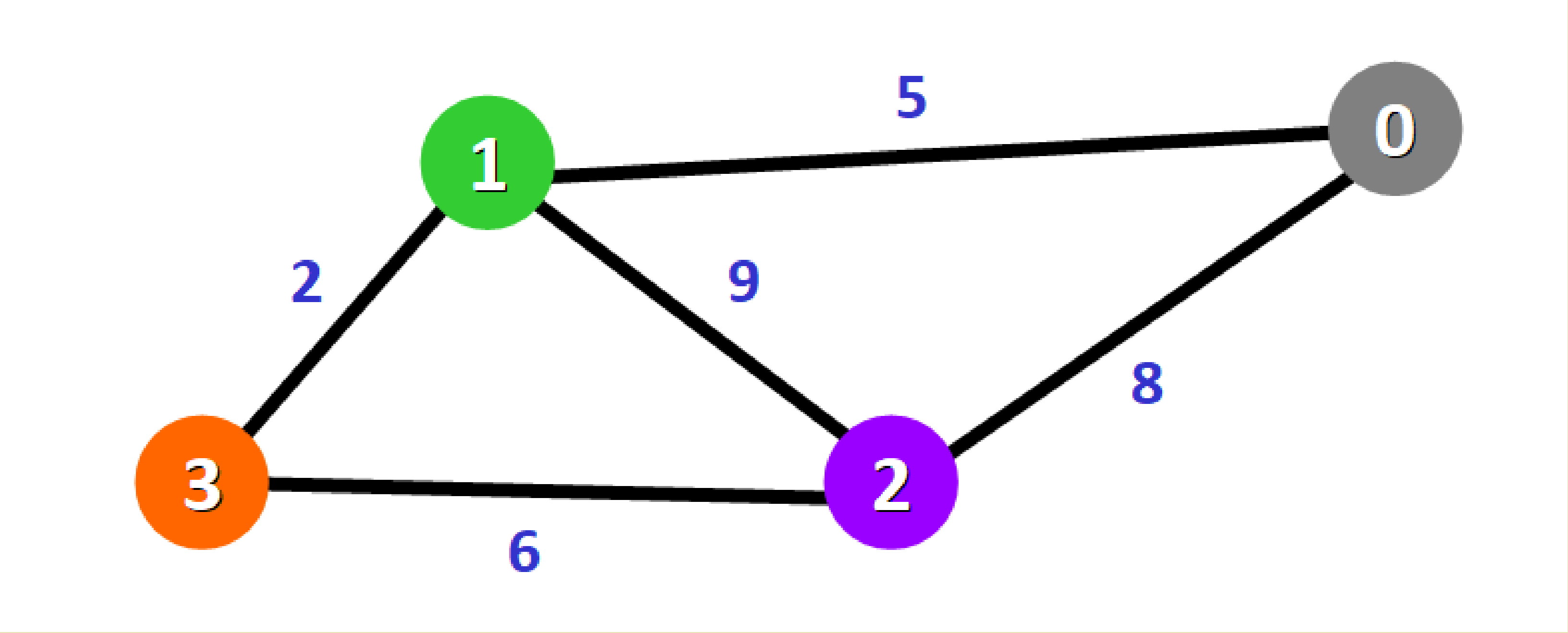
- Dijkstra's shortest path algorithm, created by Edsger Dijkstra
  - Dutch computer scientist and software engineer
- Can be directed or undirected graph with positive weights



https://www.freecodecamp.org/news/dijkstras-shortest-path-algorithm-visual-introduction/#:~:text=Dijkstra's%20Algorithm%20finds%20the%20shortest,node%20and%20all%20other%20nodes.

Dijkstra’s algorithm

- Example on a weighted graph
- What is the shortest path between 1 and 2?



https://www.freecodecamp.org/news/dijkstras-shortest-path-algorithm-visual-introduction/#:~:text=Dijkstra's%20Algorithm%20finds%20the%20shortest,node%20and%20all%20other%20nodes.

Dijkstra’s algorithm

- Dijkstra's shortest path algorithm determines the shortest path from a start vertex (source) to each vertex in a graph.
  - The algorithm keeps track of the currently known shortest distance from each vertex to the source vertex and it updates these values if it finds a shorter path.
  - Once the algorithm has found the shortest path between the source vertex and another vertex, that vertex is marked as "visited" and added to the path.
  - The process continues until all the vertices in the graph have been added to the path.

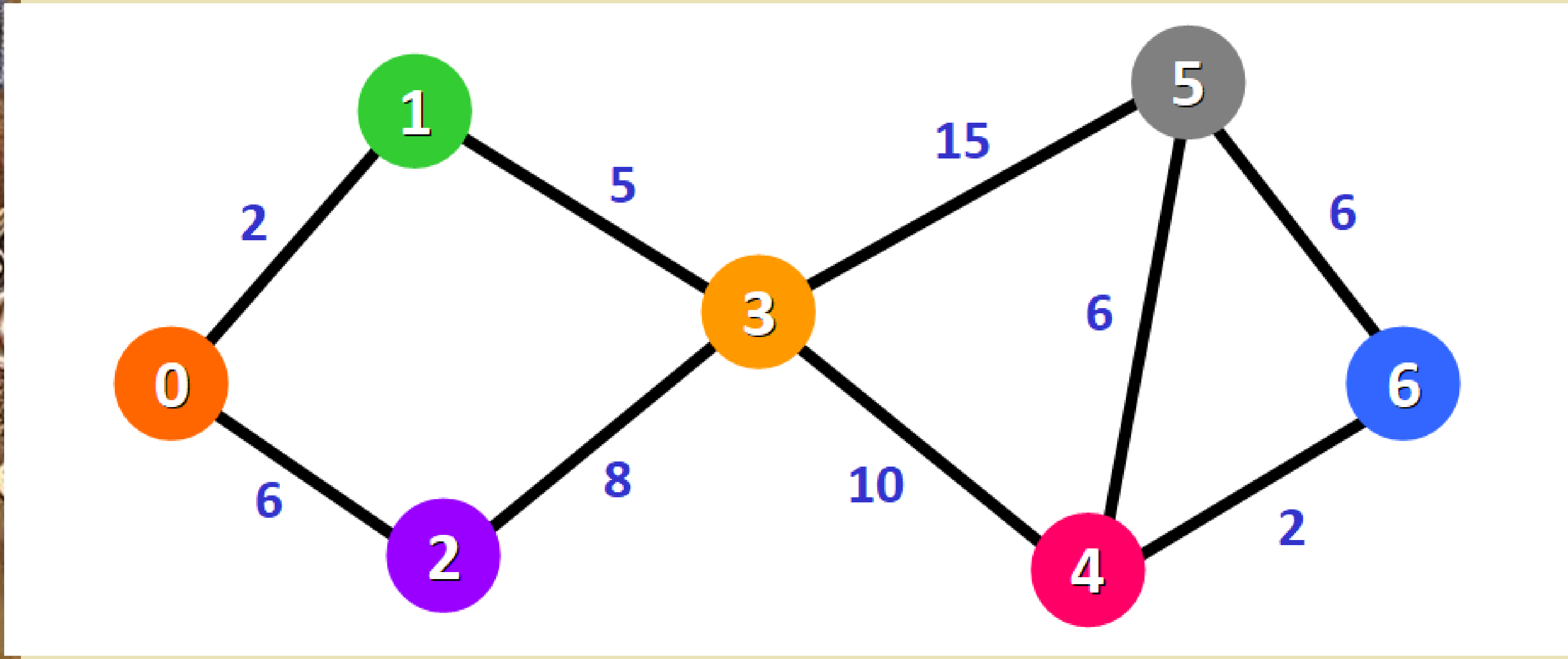
Dijkstra’s algorithm

- For each vertex, Dijkstra's algorithm determines the vertex's distance and predecessor pointer.
- A vertex's **distance** is the shortest path distance from the start vertex.
- A vertex's **predecessor** pointer points to the previous vertex along the shortest path from the start vertex.

Zybook 11.9 Algorithm: Dijkstra's shortest path

Dijkstra's algorithm

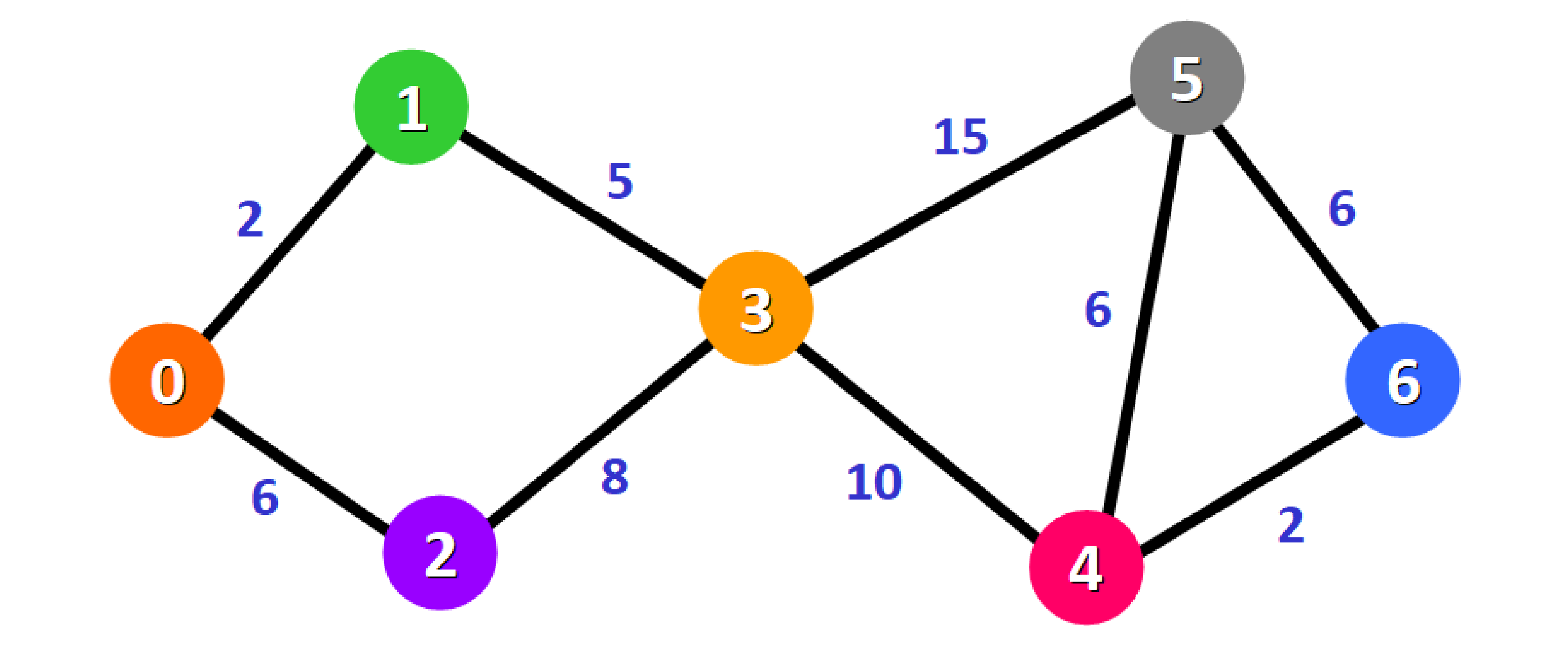
- Example procedure



https://www.freecodecamp.org/news/dijkstras-shortest-path-algorithm-visual-introduction/#:~:text=Dijkstra's%20Algorithm%20finds%20the%20shortest,node%20and%20all%20other%20nodes.

Dijkstra’s algorithm

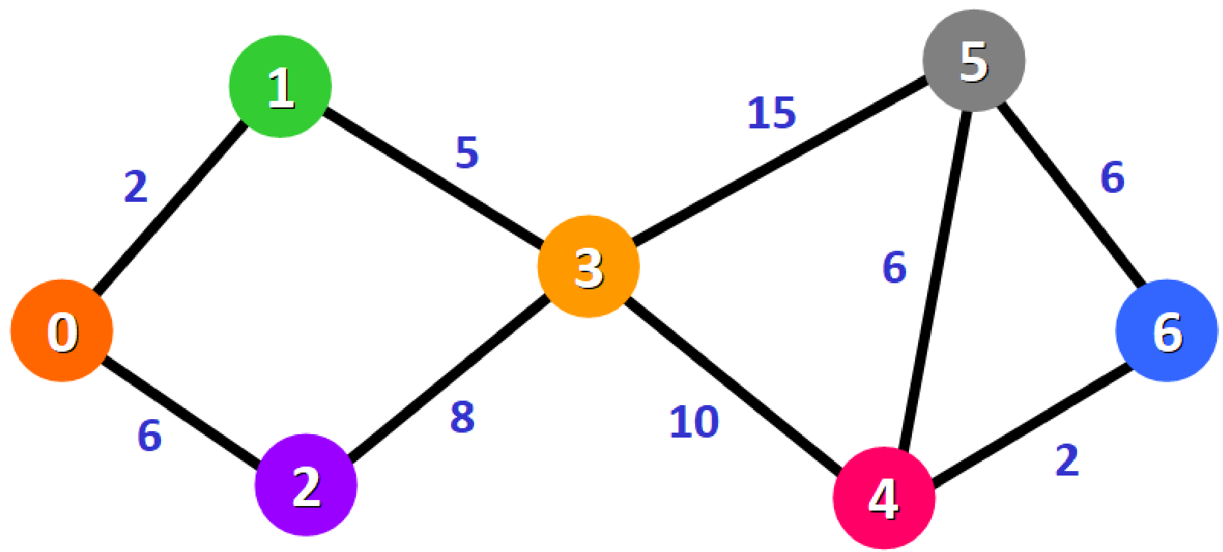
- Example procedure sources 0



  – We will find the shortest path from node 0 to node 1, from node 0 to node 2, from node 0 to node 3, and so on for every node in the graph.

Dijkstra’s algorithm

Step 1: The distance from the source node to itself is 0.



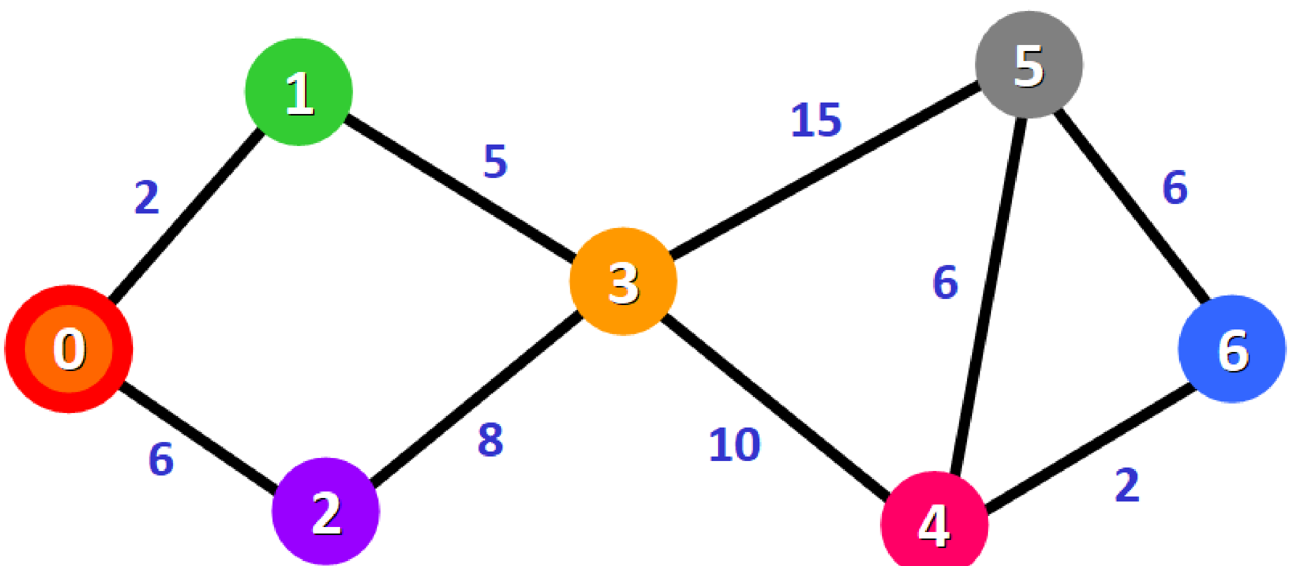
Unvisited Nodes: {0, 1, 2, 3, 4, 5, 6}

Distance:

0: 0
1: ∞
2: ∞
3: ∞
4: ∞
5: ∞
6: ∞

Dijkstra’s algorithm

Step 1: Visit node 0. Add to path. Current: {0}.



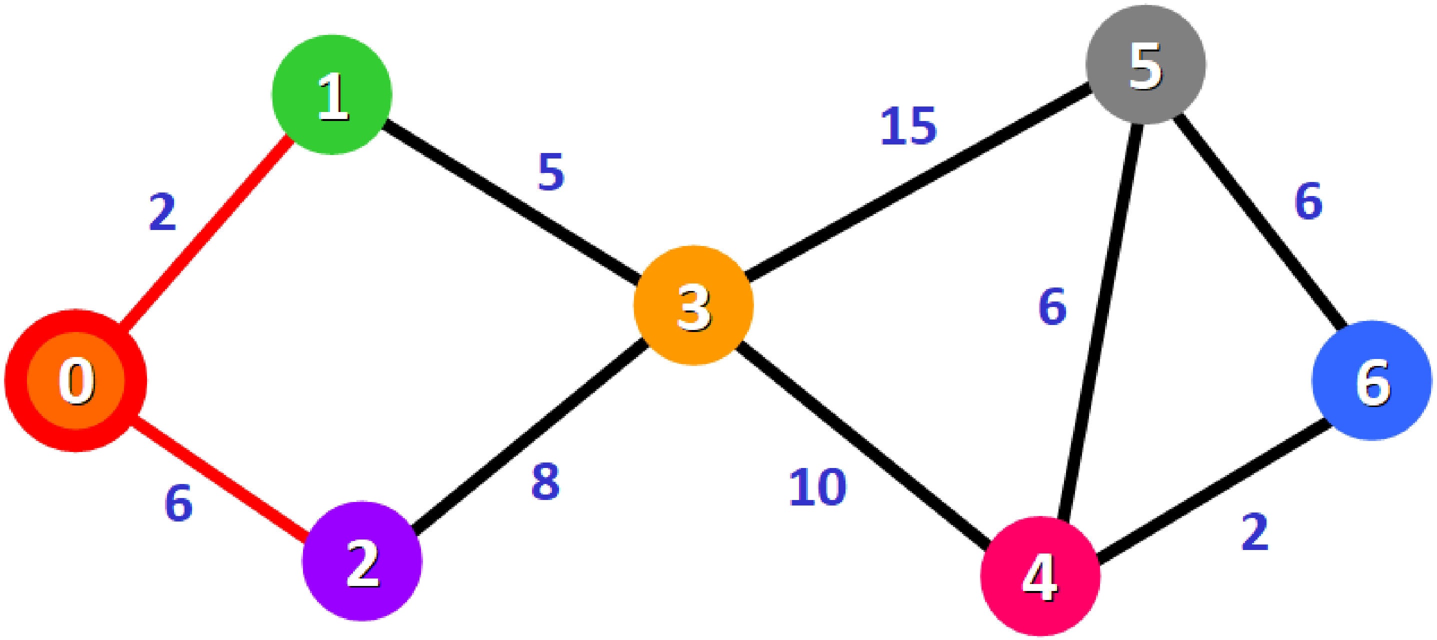
Unvisited Nodes: {~~0~~, 1, 2, 3, 4, 5, 6}

Distance:

0: 0
1: ∞
2: ∞
3: ∞
4: ∞
5: ∞
6: ∞

Dijkstra’s algorithm

Step 2: Start checking the distance from node 0 to its adjacent nodes (1 and 2), and update the distances from node 0 to them.



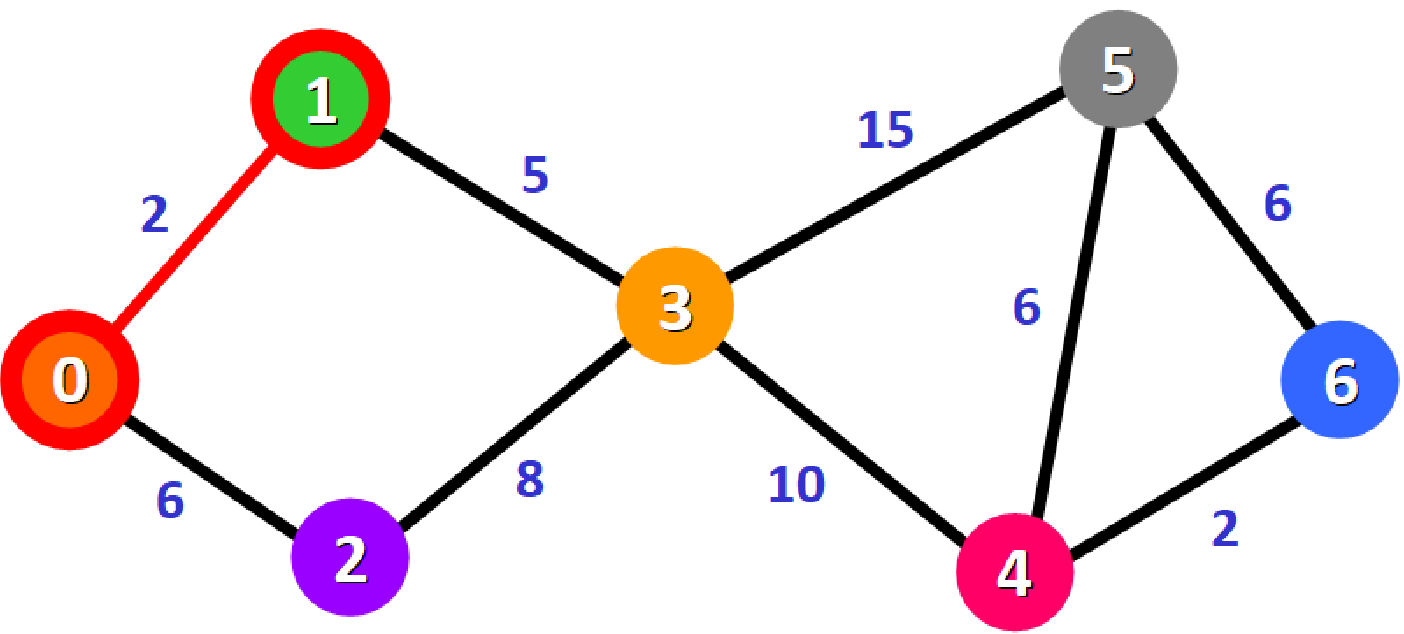
Distance:

0: 0
1: $\cancel{\infty}$ 2
2: $\cancel{\infty}$ 6
3: ∞
4: ∞
5: ∞
6: ∞

Unvisited Nodes: {~~0~~, 1, 2, 3, 4, 5, 6}

Dijkstra’s algorithm

Step 2: Select the node that is closest to the source node based on the current known distances. Mark it as visited. Add it to the path. Current: {0, 1}.



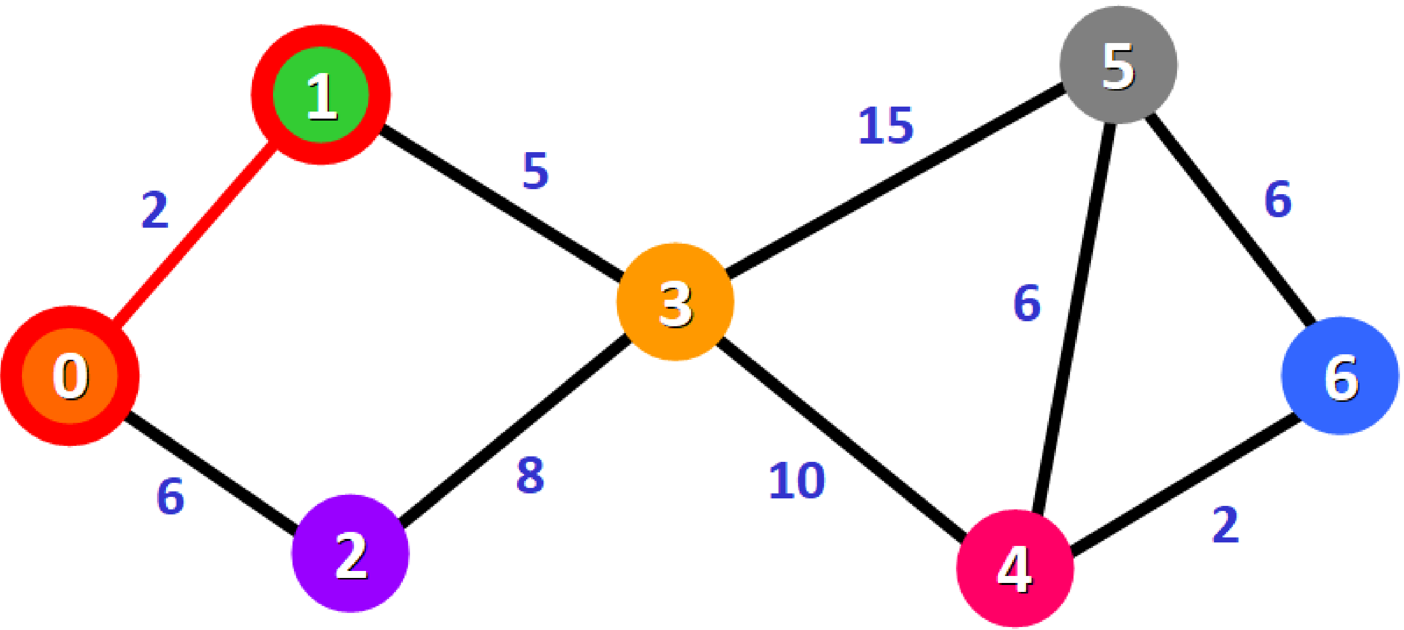
Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ∞
4: ∞
5: ∞
6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, 2, 3, 4, 5, 6}

Dijkstra’s algorithm

Step 3: Analyze nodes adjacent to nodes in the path. Here they are 2, 3. Only need to update distance for 3.



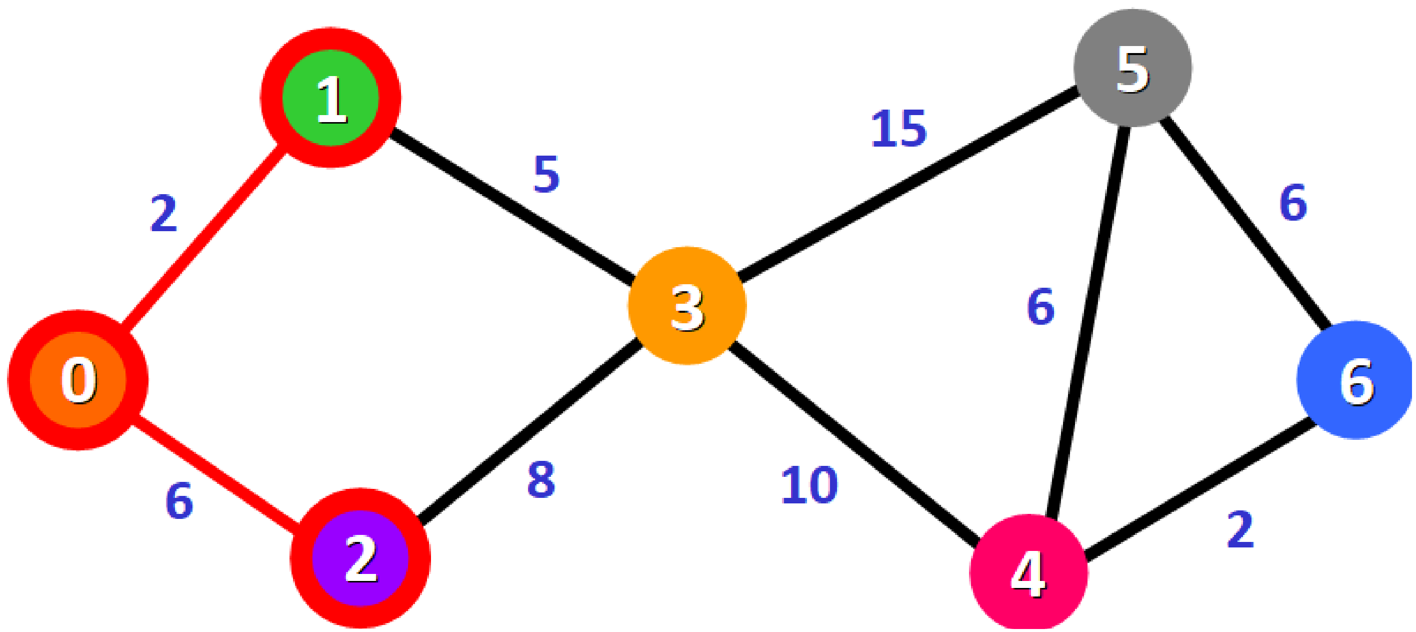
Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ~~∞~~ 7
4: ∞
5: ∞
6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, 2, 3, 4, 5, 6}

Dijkstra’s algorithm

Step 3: select the **unvisited** node with the (currently known) shortest distance to the source node. Add to path. Current: {0, 1, 2}.



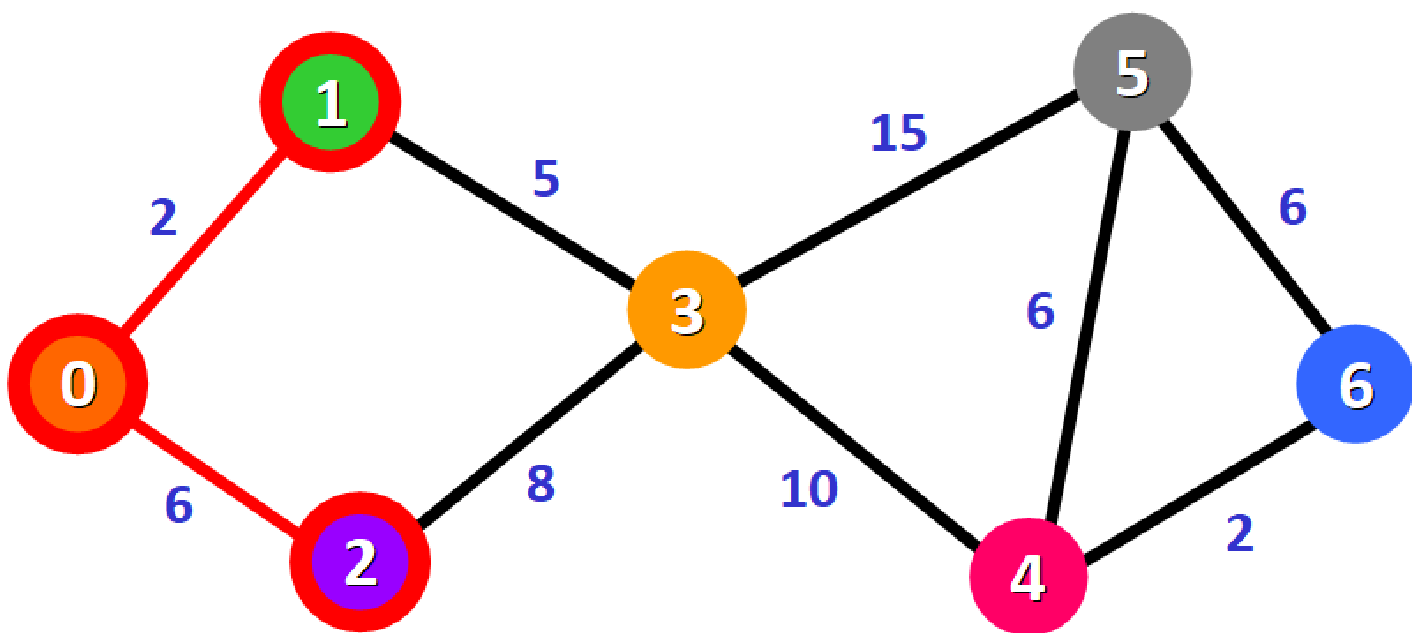
Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ~~∞~~ 7
4: ∞
5: ∞
6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, ~~2~~, 3, 4, 5, 6}

Dijkstra’s algorithm

Step 4: Repeat previous step. Analyze nodes adjacent to nodes in the path. Only 3. Check if updates needed. Here, 2+5=7<(6+8). No updates.



Distance:

0: 0

1: $\cancel{\infty}$ 2

2: $\cancel{\infty}$ 6

3: $\cancel{\infty}$ 7

4: ∞

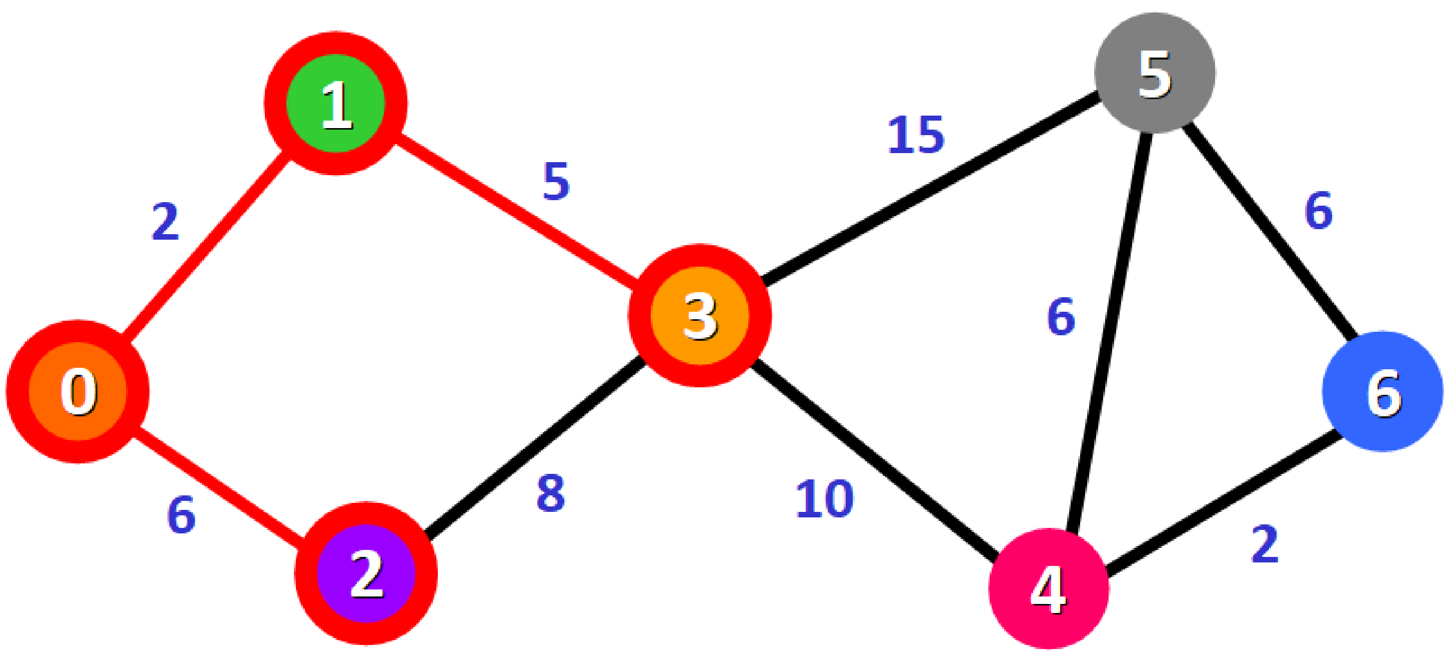
5: ∞

6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, ~~2~~, 3, 4, 5, 6}

Dijkstra’s algorithm

Step 4: Add **unvisited** nodes with the shortest known distance to the source. Add to path. Current path: {0,1,2,3}.



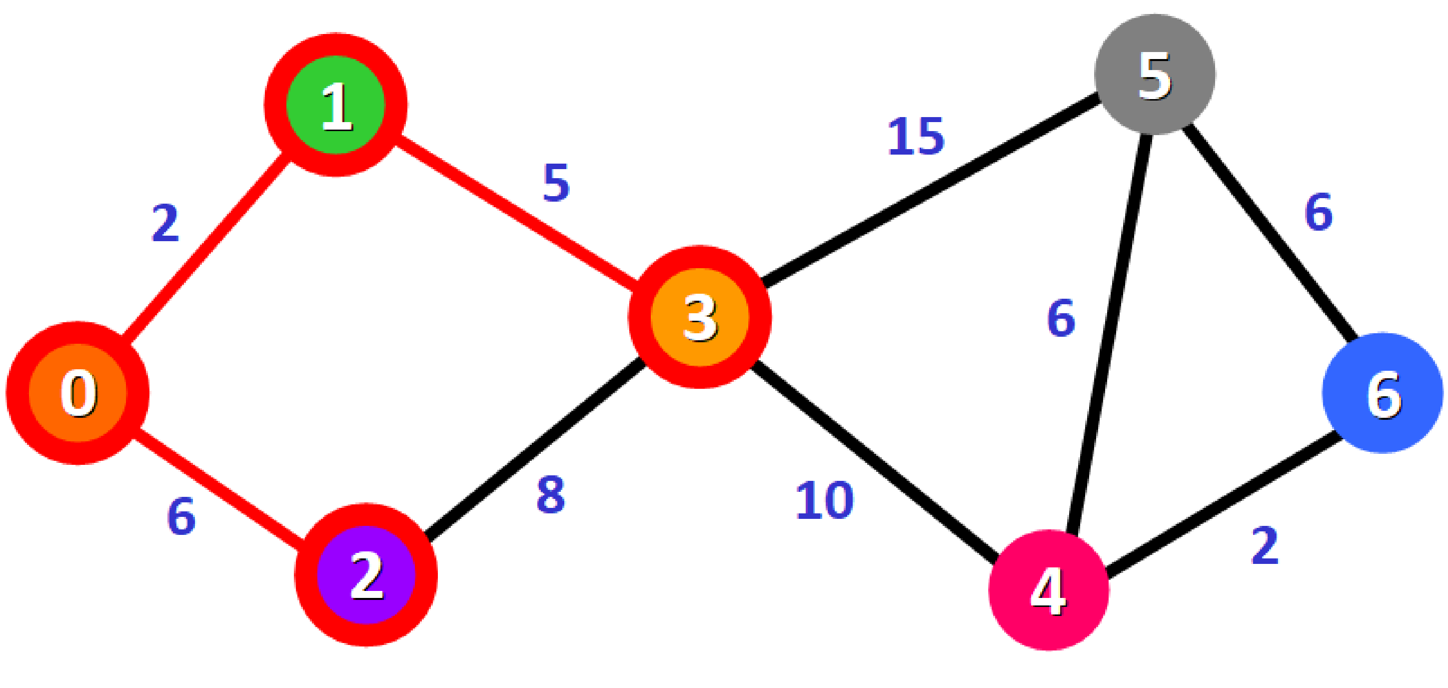
Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ~~∞~~ 7
4: ∞
5: ∞
6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, ~~2~~, ~~3~~, 4, 5, 6}

Dijkstra’s algorithm

Step 5: check new adjacent nodes to nodes in the path. Update distance (4,5). Choose the unvisited node with shortest distance to add to path.



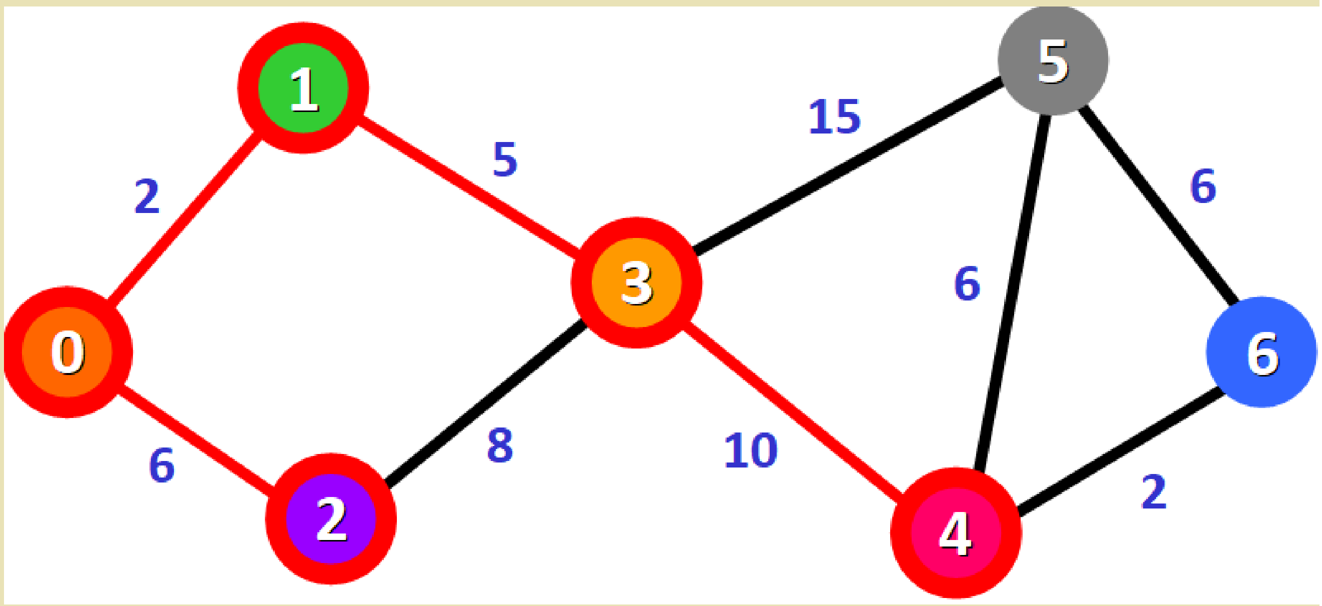
Distance:

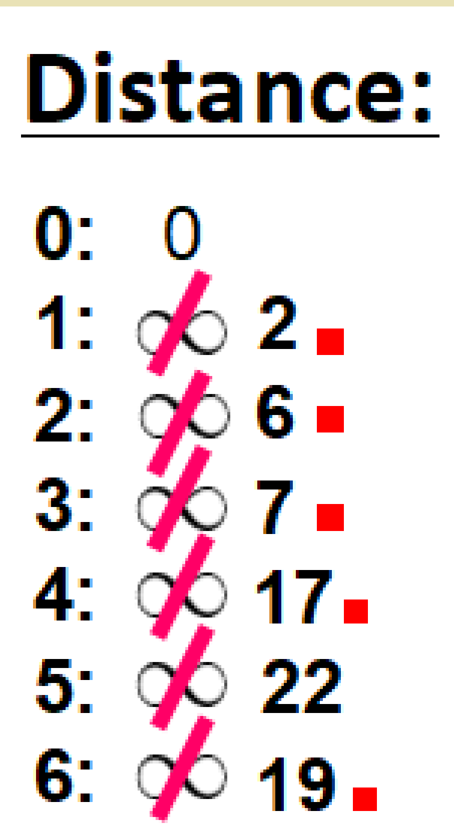
0: 0
1: $\cancel{\infty}$ 2
2: $\cancel{\infty}$ 6
3: $\cancel{\infty}$ 7
4: $\cancel{\infty}$ 17
5: $\cancel{\infty}$ 22
6: ∞

Unvisited Nodes: {~~0~~, ~~1~~, ~~2~~, ~~3~~, ~~4~~, 5, 6}

Dijkstra's algorithm

Step 6: check new adjacent nodes to nodes in the path. Update distance (not need for 5, needed for 6).





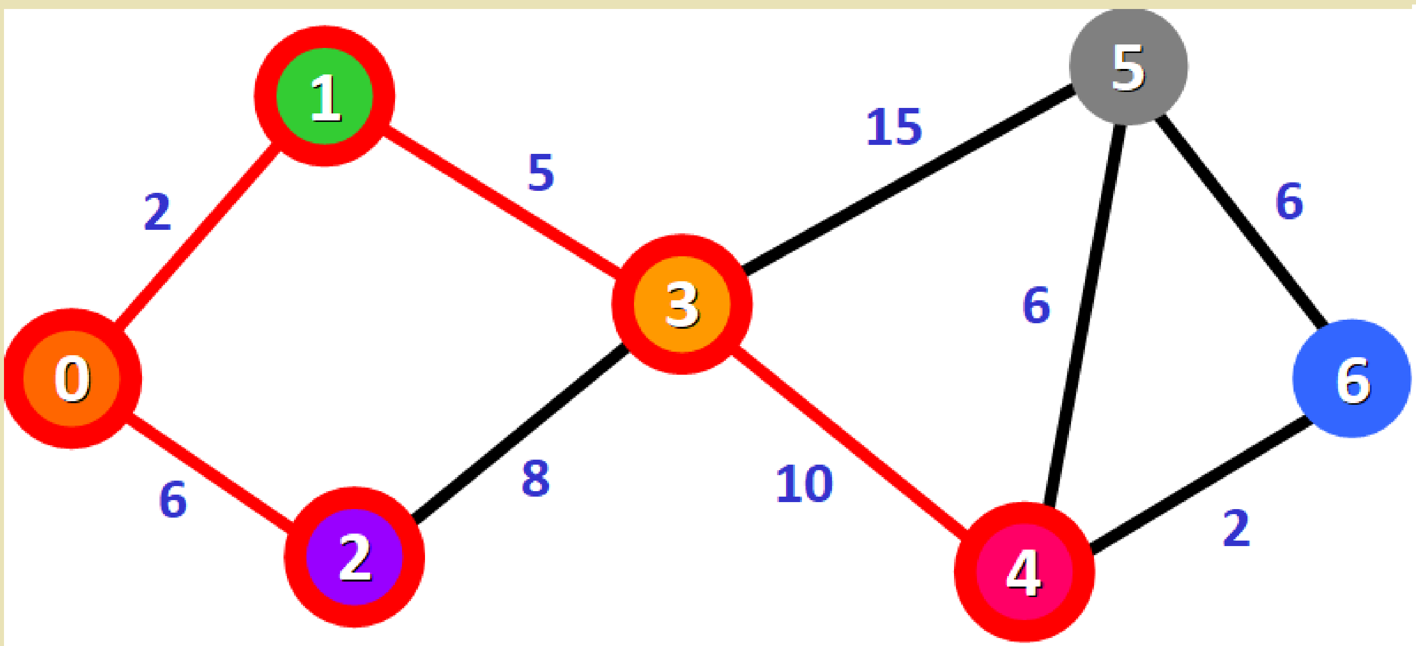
Two Options for reaching 5:

3 -> 5 (current) **22**

4 -> 5 **23** (17+ 6).

Dijkstra's algorithm

Step 6: Choose the unvisited node with shortest distance to the source(6). Add to the path.



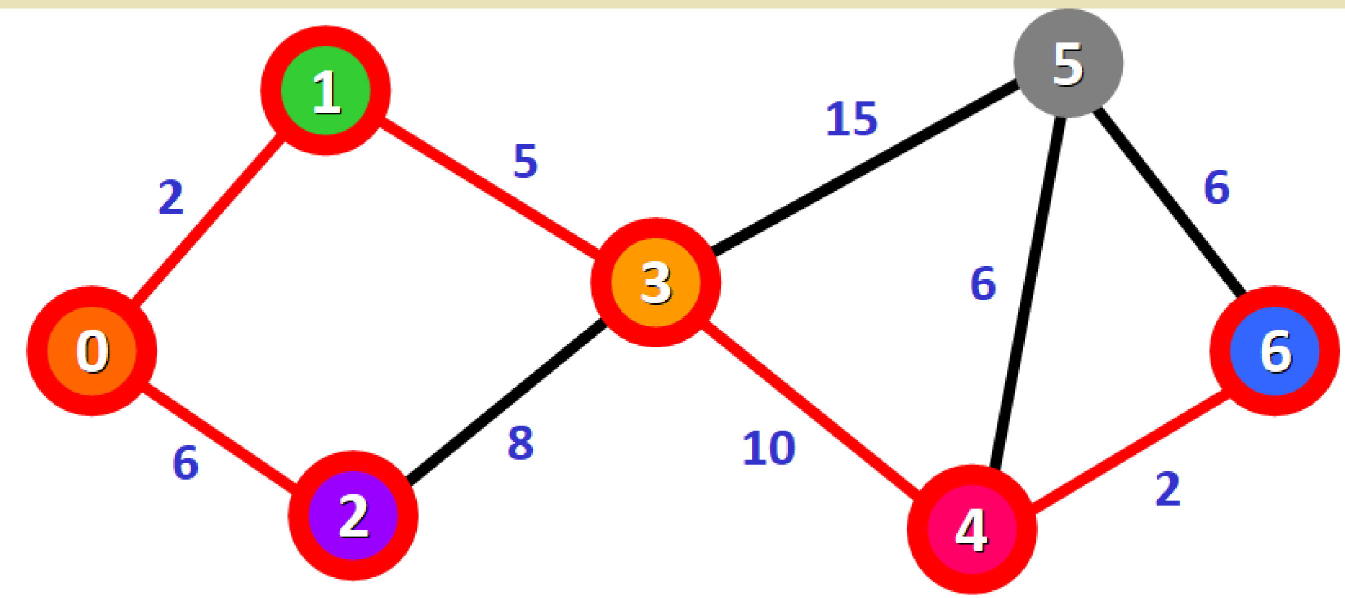
Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ~~∞~~ 7
4: ~~∞~~ 17
5: ~~∞~~ 22
6: ~~∞~~ 19

Unvisited Nodes: {~~0~~, ~~1~~, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~}

Dijkstra's algorithm

Step 7: check new adjacent nodes to nodes in the path. Update distance
Choose the unvisited node with shortest distance to add to path.



Distance:

0: 0
1: ~~∞~~ 2
2: ~~∞~~ 6
3: ~~∞~~ 7
4: ~~∞~~ 17
5: ~~∞~~ 22
6: ~~∞~~ 19

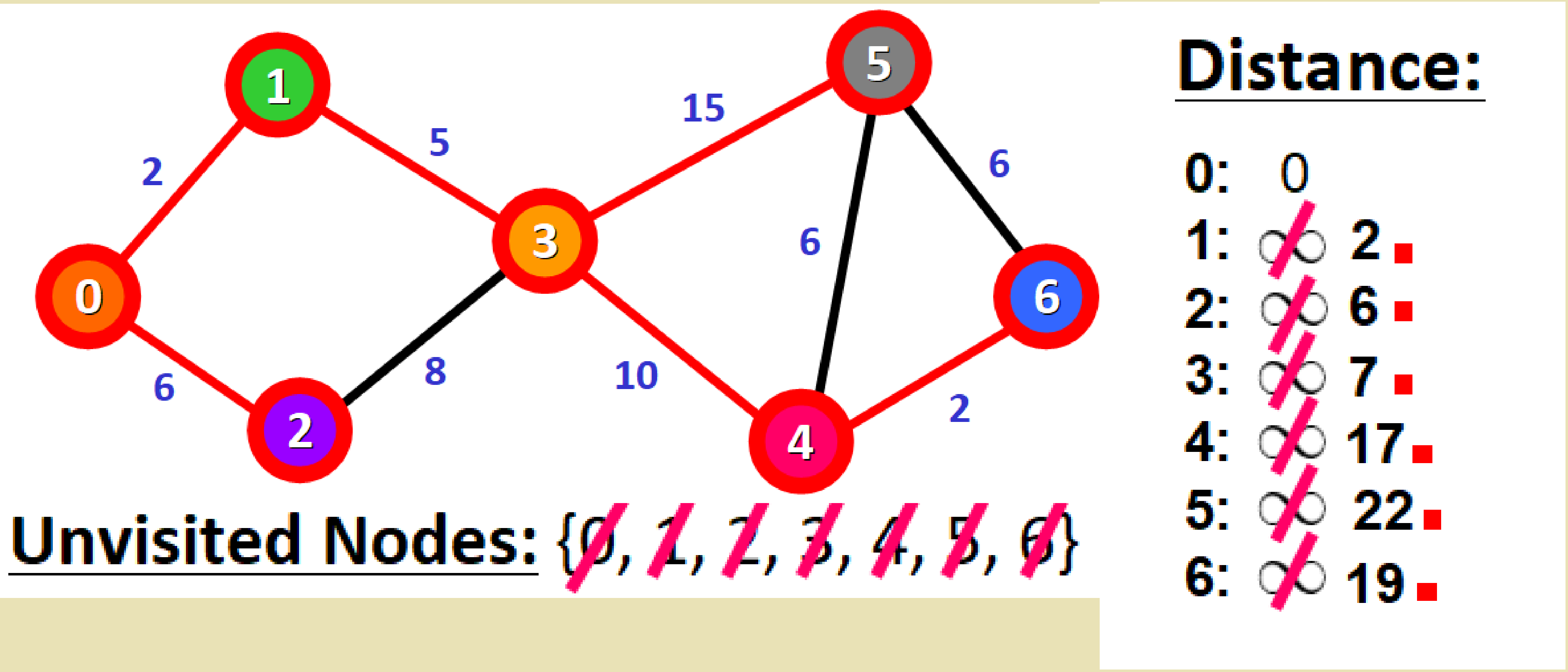
Two Options for reaching 5:

3 -> 5 (current) **22**

6 -> 5 **25** (19+ 6)

Dijkstra's algorithm

Step 7: All nodes visited. The red lines mark the edges that belong to the shortest path.

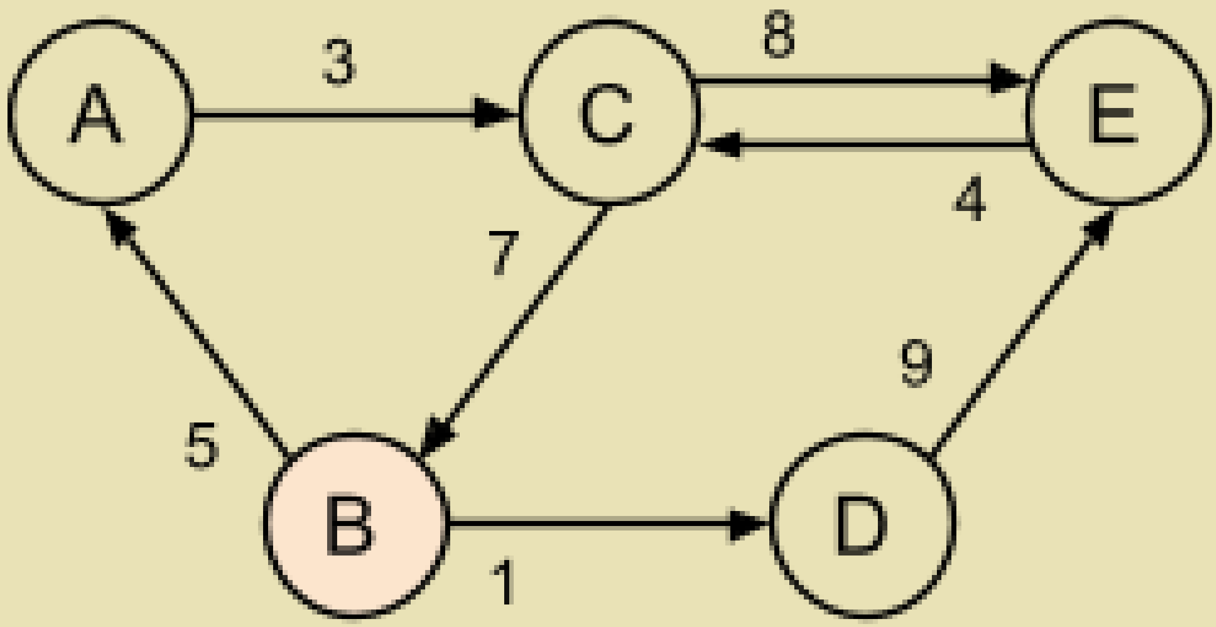


Dijkstra's algorithm

```
Dijkstra(Graph, source):
 Initialize:
  Q ← set of all vertices in Graph
  distance[v] ← ∞ for each vertex v in Graph
  previous[v] ← UNDEFINED for each vertex v in Graph
  distance[source] ← 0

 While Q is not empty:
  u ← vertex in Q with smallest distance[u]
  Remove u from Q

  For each neighbor v of u:
   alt ← distance[u] + length(u, v)
   If alt < distance[v]:
    distance[v] ← alt
    previous[v] ← u

 Return distance[], previous[]
```

Dijkstra's Algorithm -- Example

- Perform Dijkstra's shortest path algorithm on the graph below with starting vertex B.



- Which vertex is visited after B?

Zybook 11.9 Algorithm: Dijkstra's shortest path

Algorithm efficiency

- If the unvisited vertex queue is implemented using a list, the runtime for Dijkstra's shortest path algorithm is $O(V^2)$.
- The outer loop executes V times to visit all vertices. In each outer loop execution, dequeuing the vertex from the queue requires searching all vertices in the list, which has a runtime of O(V).

Algorithm efficiency

- For each vertex, the algorithm follows the subset of edges to adjacent vertices; following a total of E edges across all loop executions. Given $E < V^2$, the runtime is $O(V*V + E) = O(V^2 + E) = O(V^2)$.
- Implementing the unvisited vertex queue using a standard binary heap reduces the runtime to $O((E + V) \log V)$, and using a Fibonacci heap data structure (not discussed in this material) reduces the runtime to $O(E + V \log V)$.

Bellman-Ford's Algorithm

- The Bellman-Ford shortest path algorithm, created by Richard Bellman and Lester Ford, Jr., determines the shortest path from a start vertex to each vertex in a graph.
  - Can work on graphs with negative weights, but no negative weight cycles.
- A vertex's **distance** is the shortest path distance from the start vertex.
- A vertex's **predecessor pointer** points to the previous vertex along the shortest path from the start vertex.

Zybook 11.10 Algorithm: Bellman-Ford's shortest path

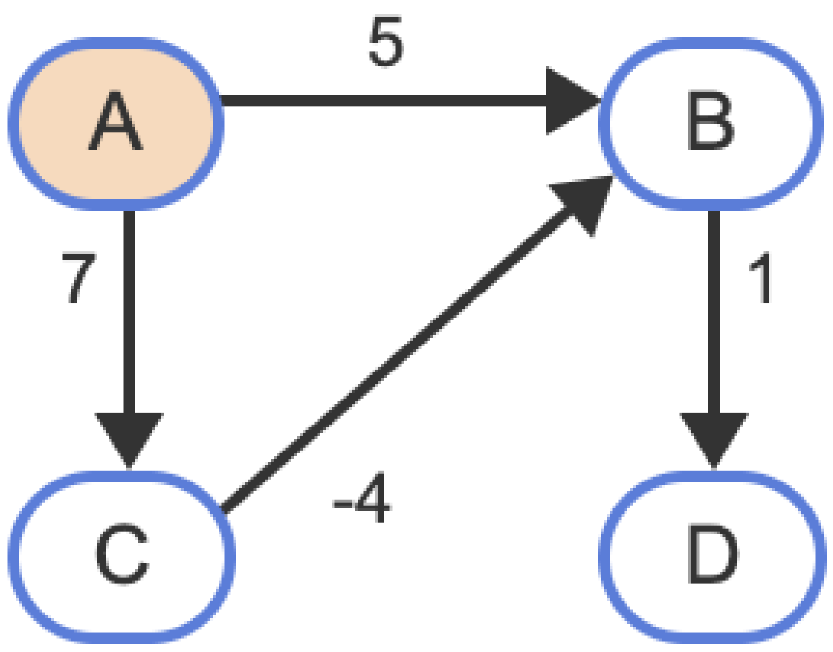
Bellman-Ford's Algorithm

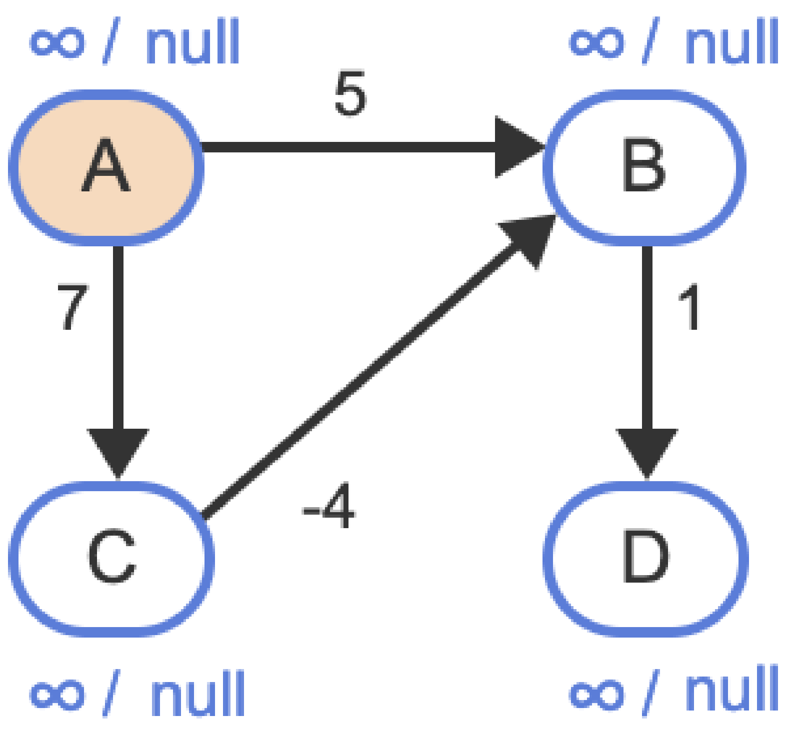
- Initialize all vertices' current distances to infinity (∞) and predecessors to null, and assigns the start vertex with a distance of 0.
- Perform V-1 main iterations, visiting all vertices in the graph in each iteration by checking all **edges**.
  – For each edge (u, v) with weight w: If going through u gives a shorter path to v from the source (i.e., distance[v] > distance[u] + w), we update the distance[v] as distance[u] + w.
  – This is also called "Relaxation of Edges".

Zybook 11.10 Algorithm: Bellman-Ford's shortest path

Bellman-Ford's Algorithm

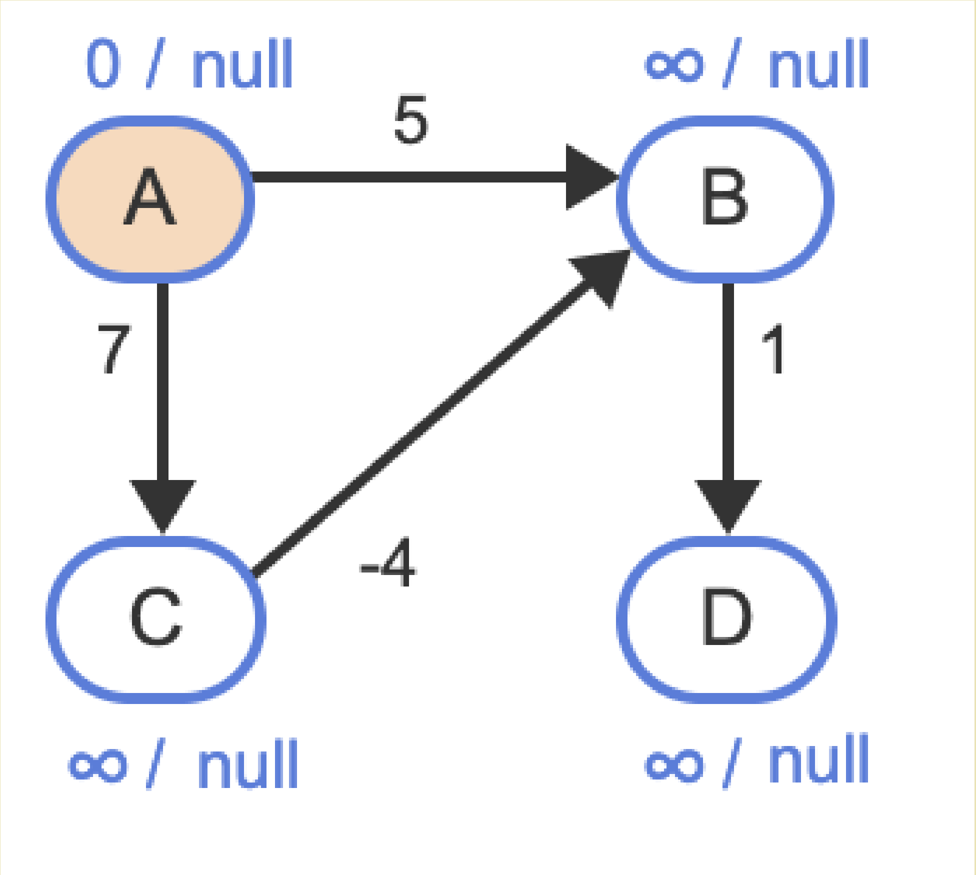
Initialization





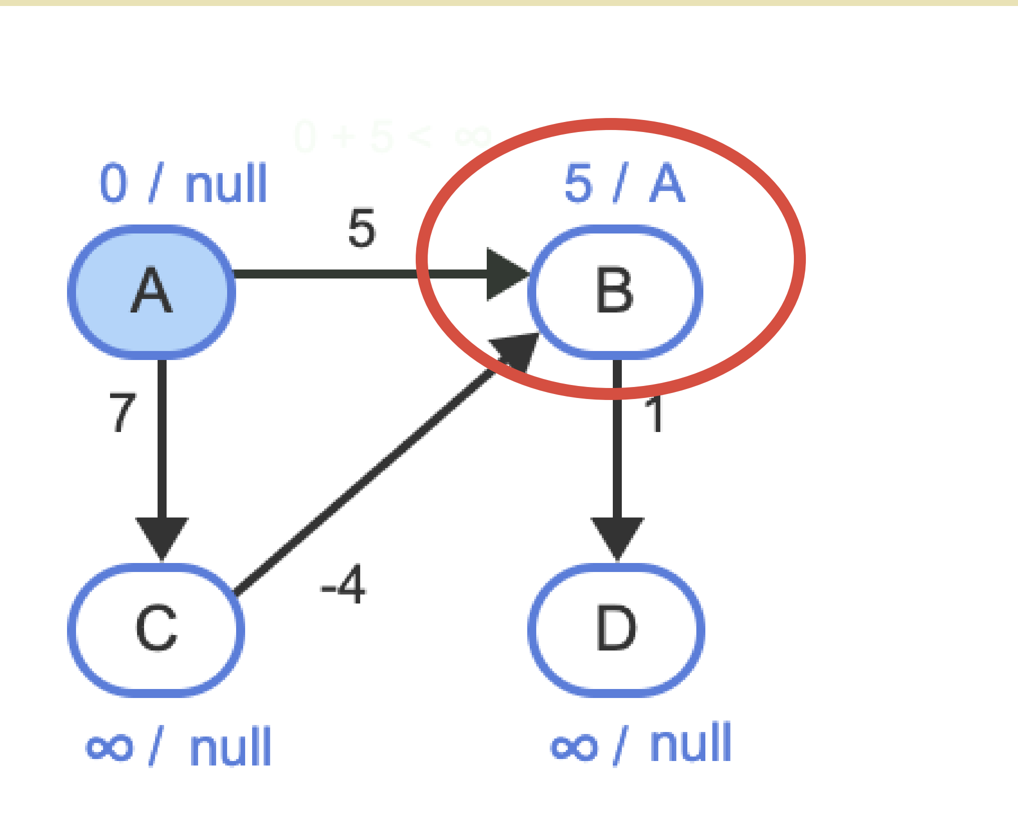
Bellman-Ford's Algorithm

Initialization



Bellman-Ford's Algorithm

Iteration 1



Edges

A→ B

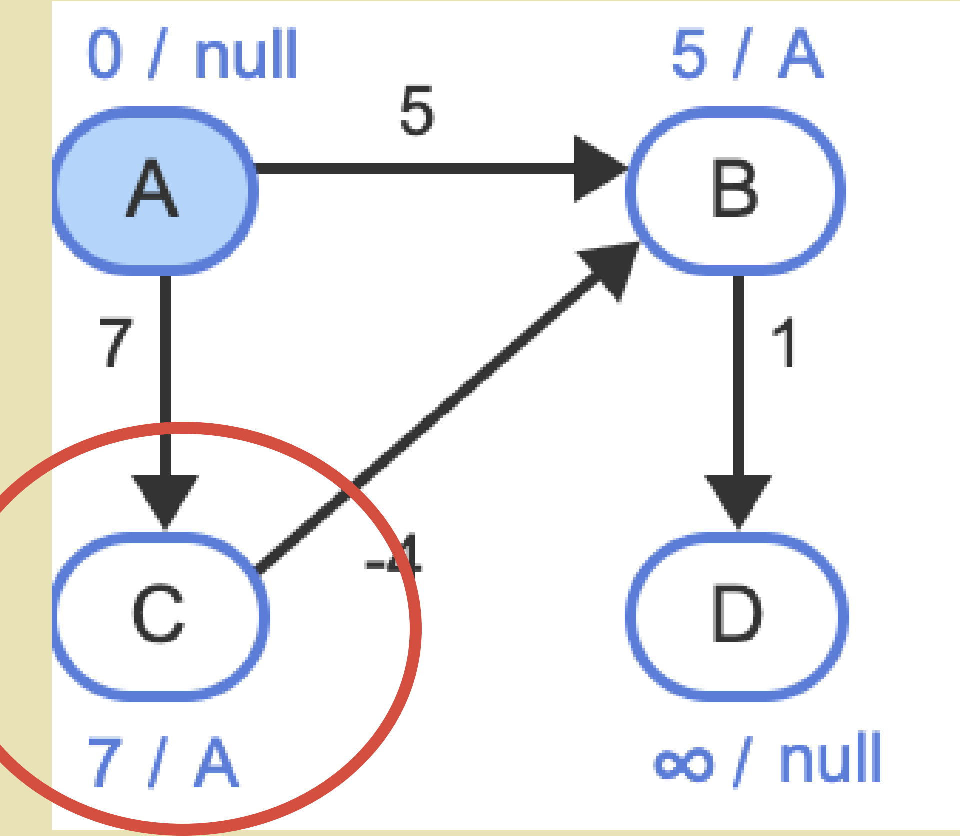
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 1



Edges

A→ B

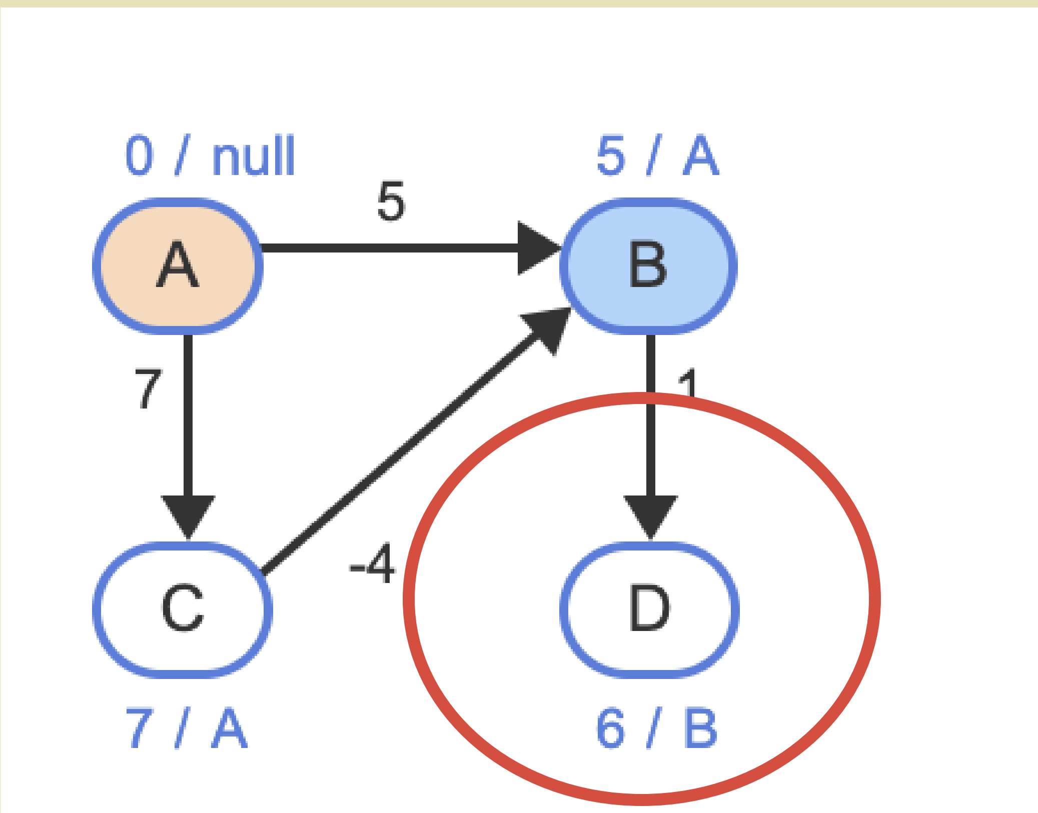
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 1



Edges

A→ B

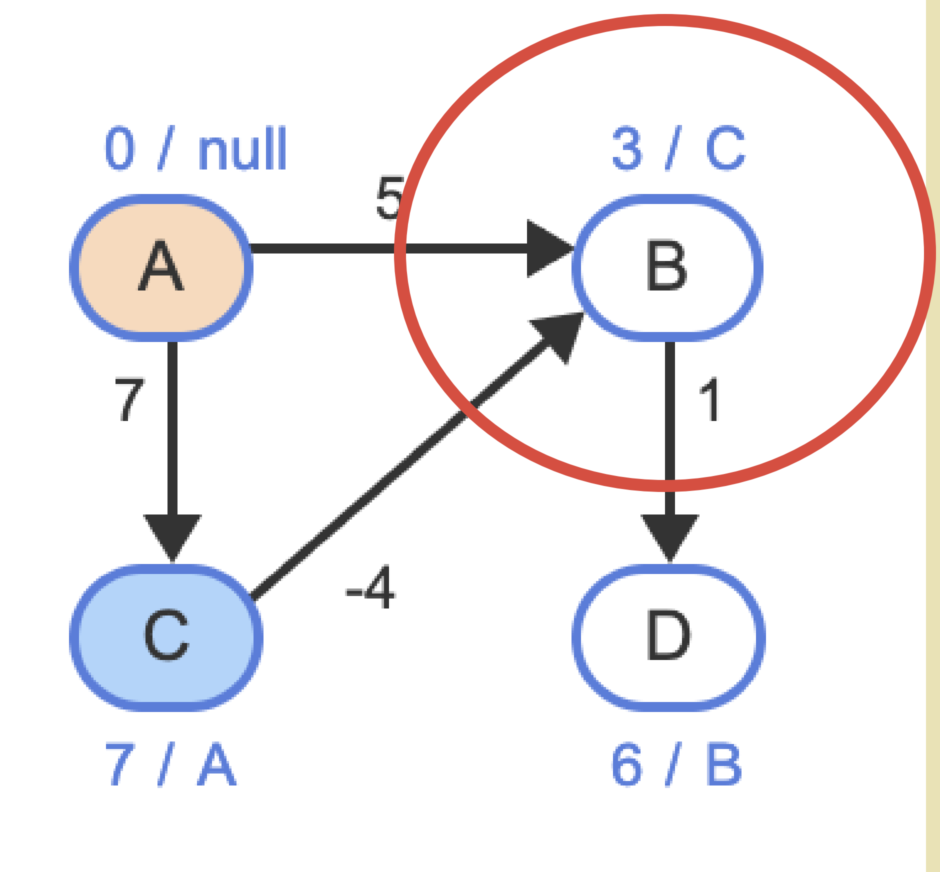
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 1



Edges

A→ B

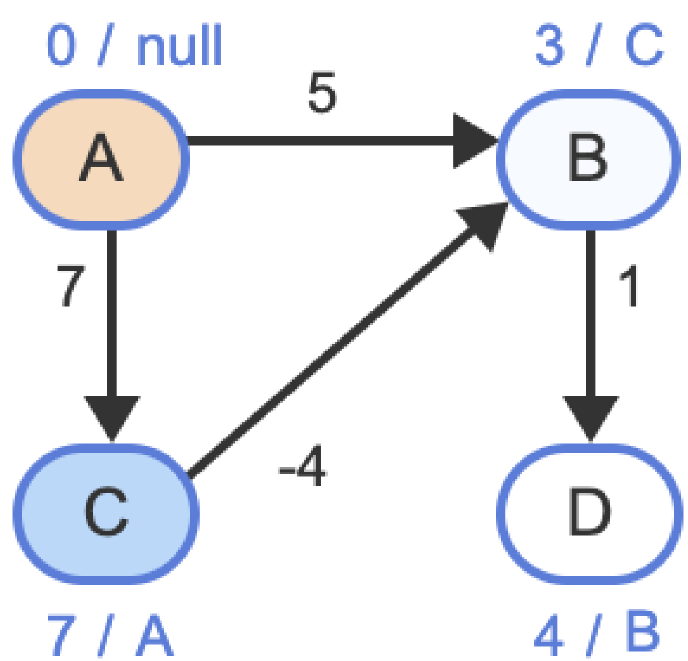
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 2



Edges

A→ B

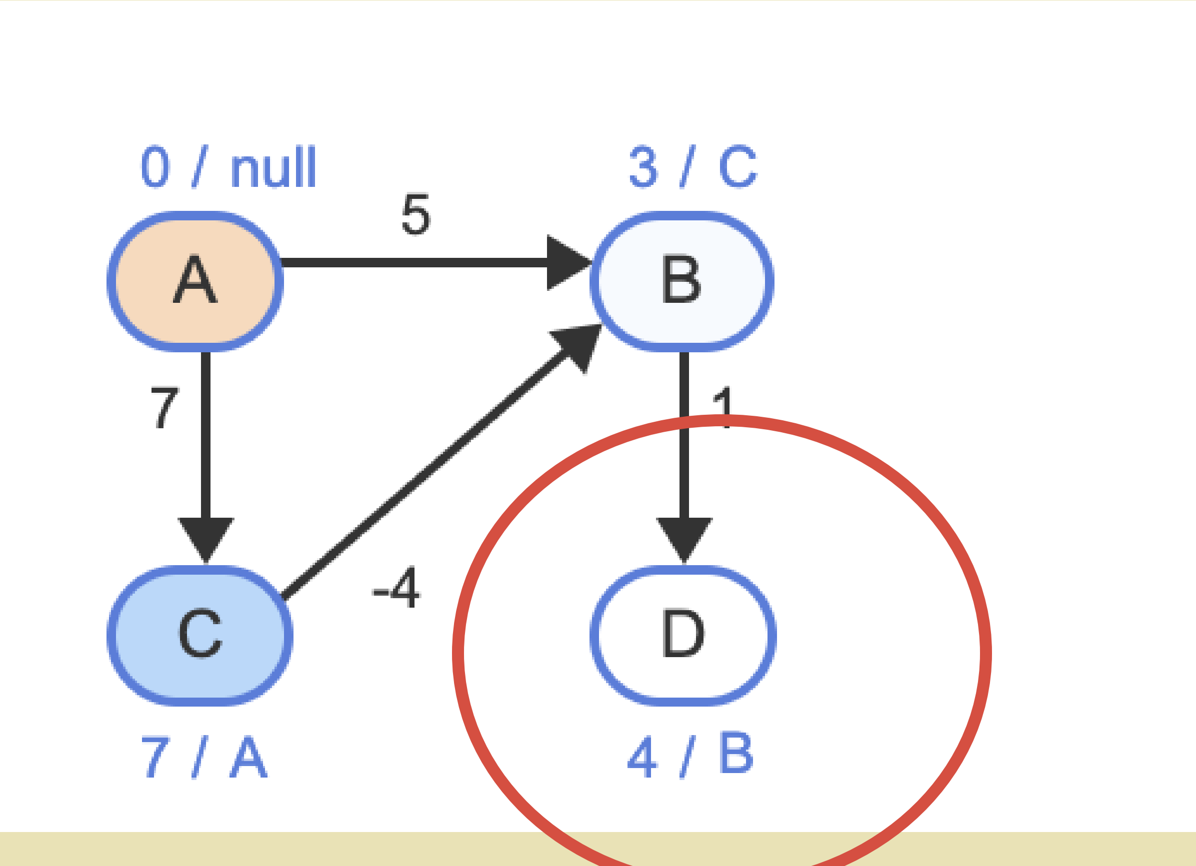
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 2



Edges

A→ B

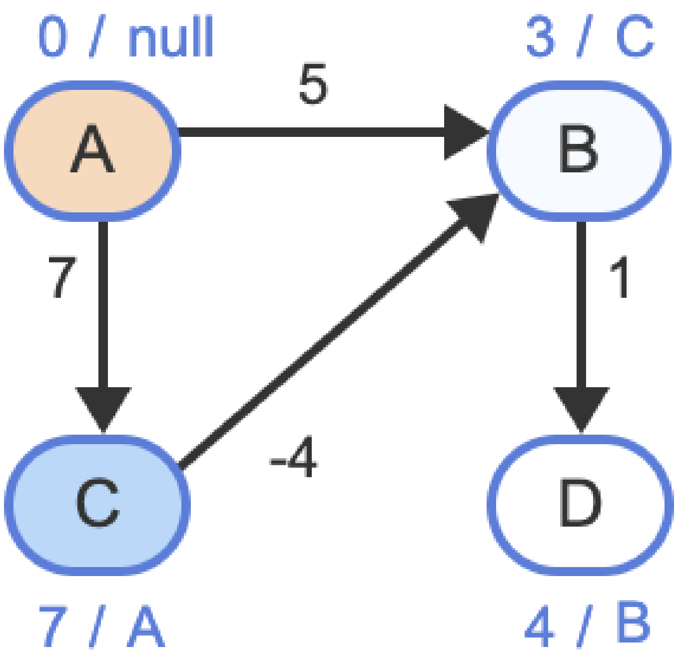
A→ C

B→D

C→B

Bellman-Ford's Algorithm

Iteration 3 (last one)



Edges

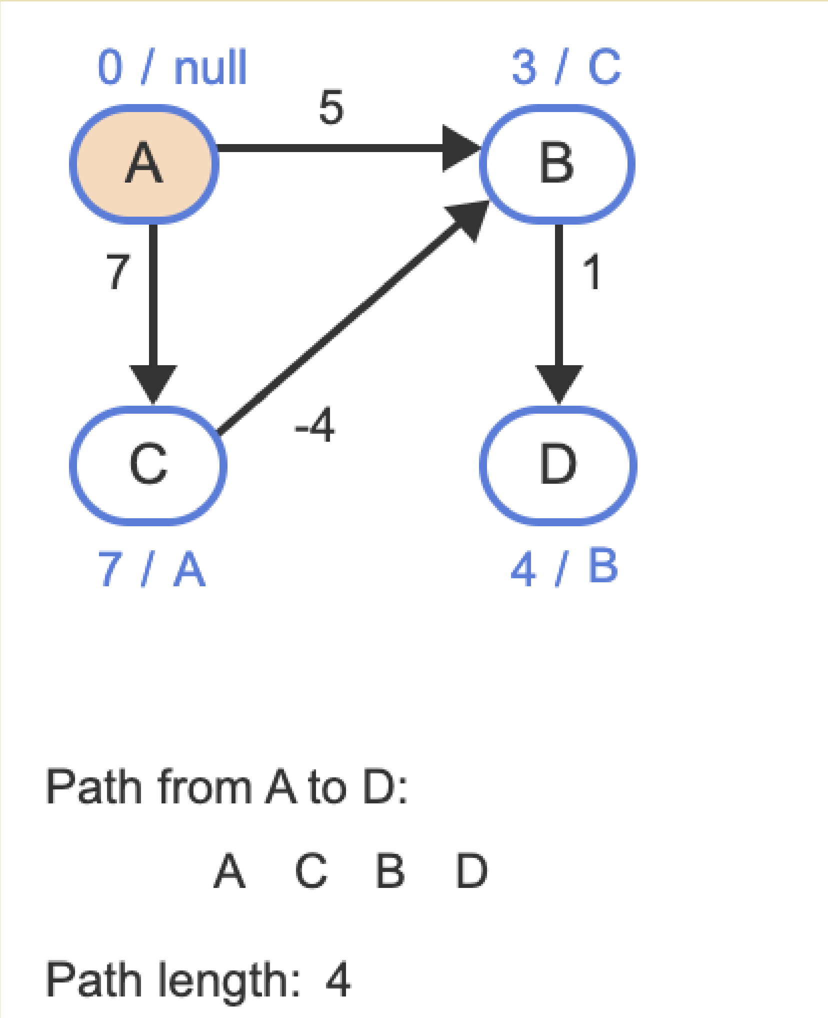
A→ B

A→ C

B→D

C→B

Bellman-Ford's Algorithm



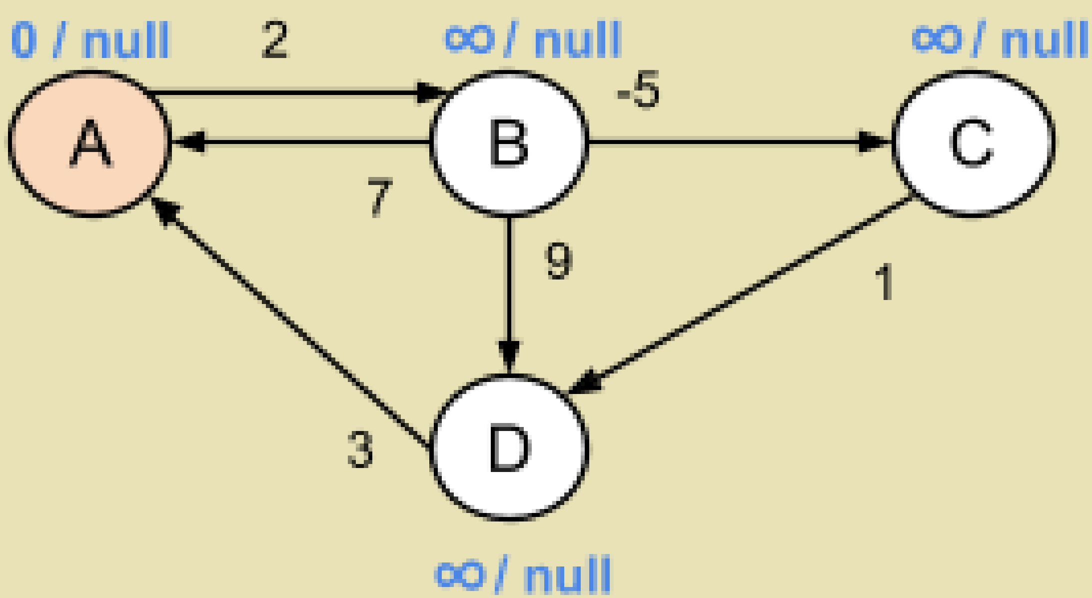
Shortest path and length

Bellman-Ford's algorithm

```
BellmanFord(startVertex):
# Initialize all vertices with infinite distance and null predecessor
    for each vertex vertex in graph:
        vertex.distance = Infinity
        vertex.predecessor = null
# Set the starting vertex distance to 0
    startVertex.distance = 0

# Relax all edges for (number of vertices - 1) times
    for iteration = 1 to (number of vertices - 1):
        for each vertex currentVertex in graph:
            for each adjacent vertex adjacentVertex of currentVertex:
                edgeWeight = weight of edge(currentVertex,
adjacentVertex)
                newDistance = currentVertex.distance + edgeWeight

                # If a shorter path is found, update distance and
predecessor
                if newDistance < adjacentVertex.distance:
                    adjacentVertex.distance = newDistance
                    adjacentVertex.predecessor = currentVertex
# Check for negative weight cycle (later slides)
```

Bellman-Ford's Algorithm -- Example

- Perform Bellman-Ford's shortest path algorithm on the graph below with starting vertex A. Order of edge list is given.

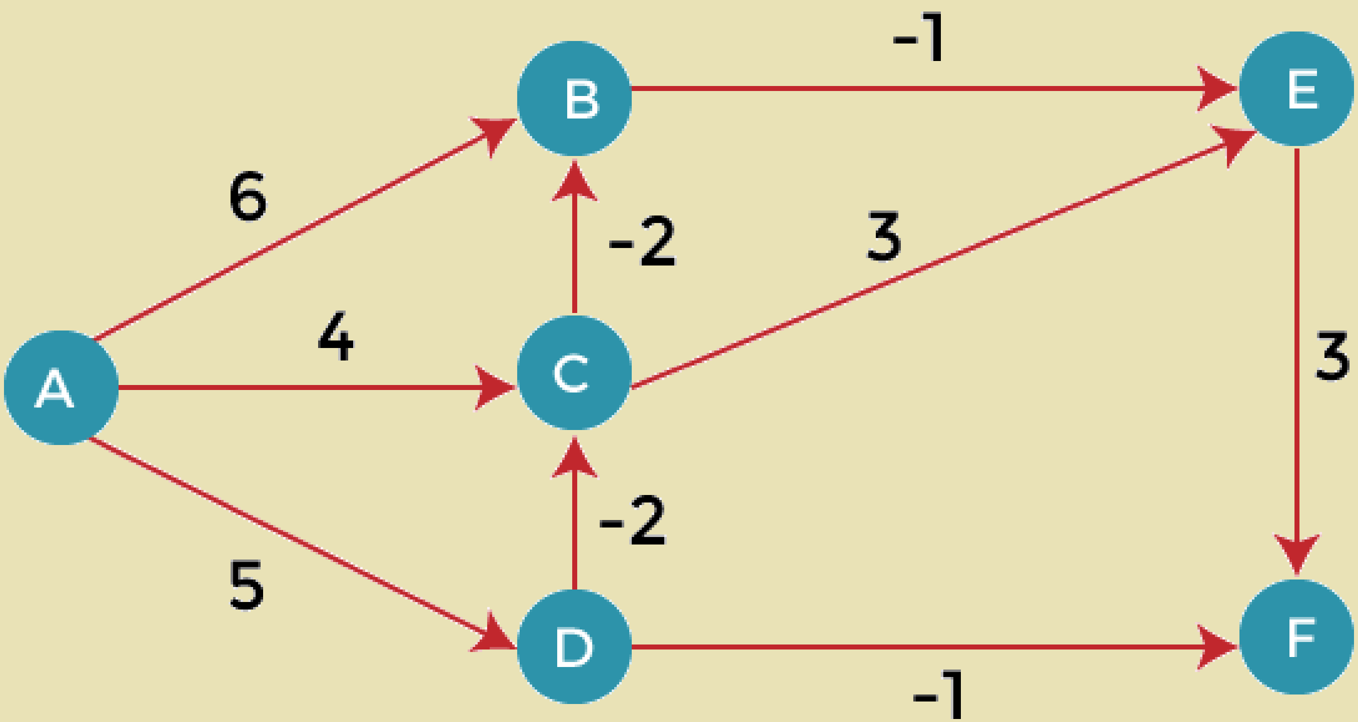


Edges
A→ B
B→ A
B→ C
B→D
C→D
D→A

Zybook 11.10 Algorithm: Bellman-Ford's shortest path

Bellman-Ford's Algorithm -- Example

- Another good example https://www.javatpoint.com/bellman-ford-algorithm

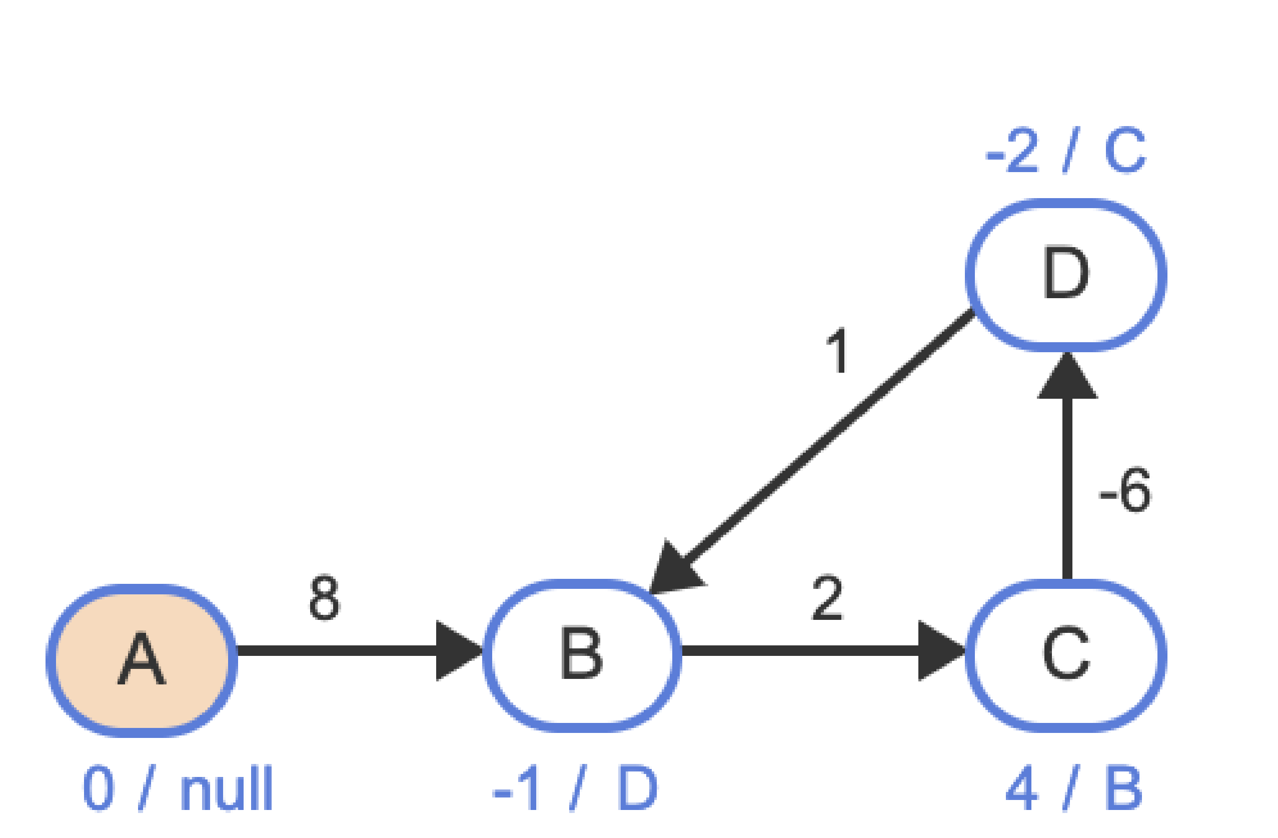


Bellman-Ford's Algorithm

- Negative weight cycle check (after V-1 iterations)
  - For each vertex in the graph, adjacent vertices are checked for a shorter path.
  - If shorter path from startV to adjV is still found, a negative edge weight cycle exists

Bellman-Ford's Algorithm

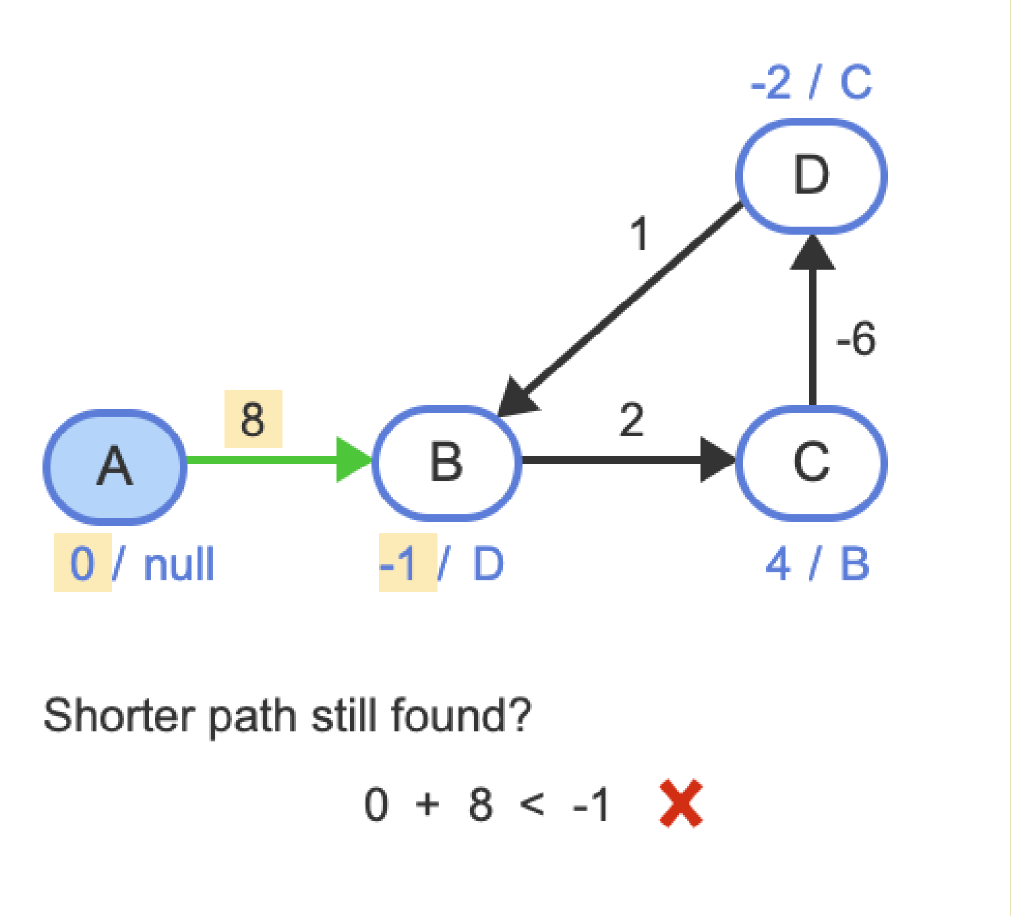
- Negative weight cycle check (after V-1 iterations)



Edges
A→ B
B→ C
C→D
D→B

Bellman-Ford's Algorithm

- Negative weight cycle check (after V-1 iterations)



Edges

A→ B

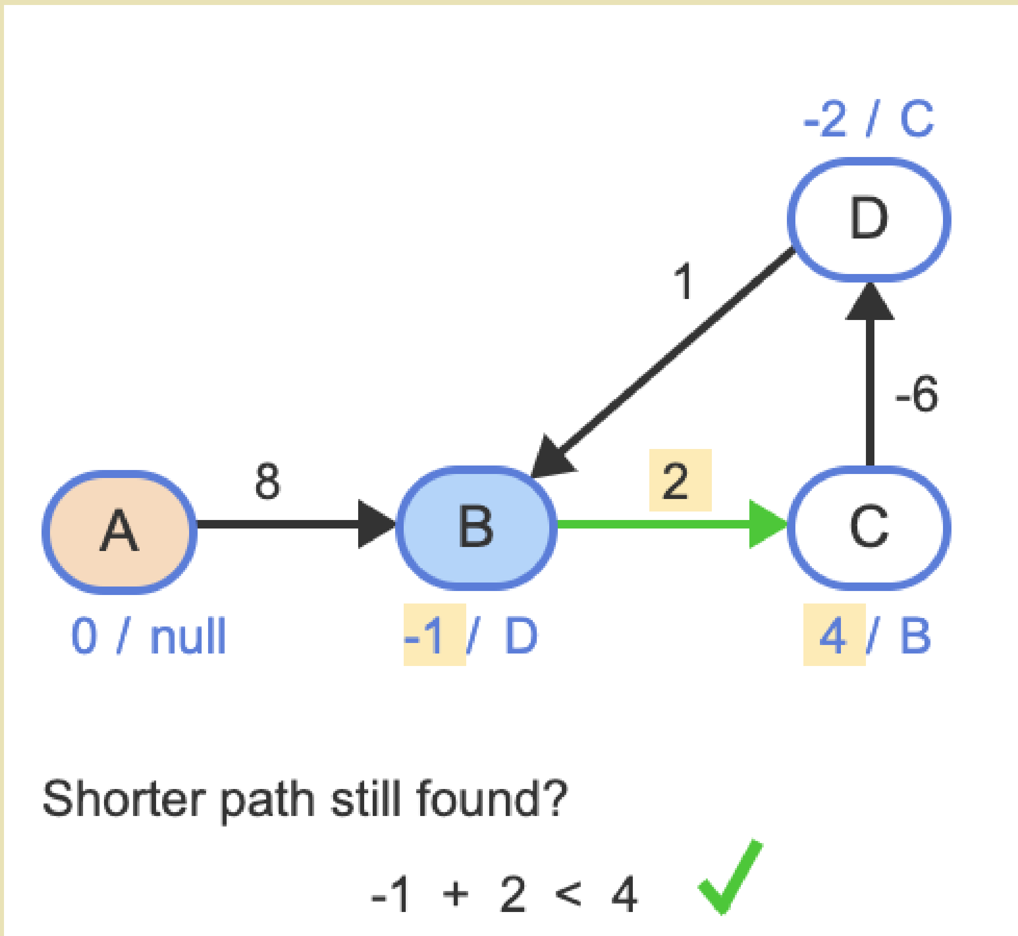
B→ C

C→D

D→B

Bellman-Ford's Algorithm

- Negative weight cycle check (after V-1 iterations)



Edges

A→ B

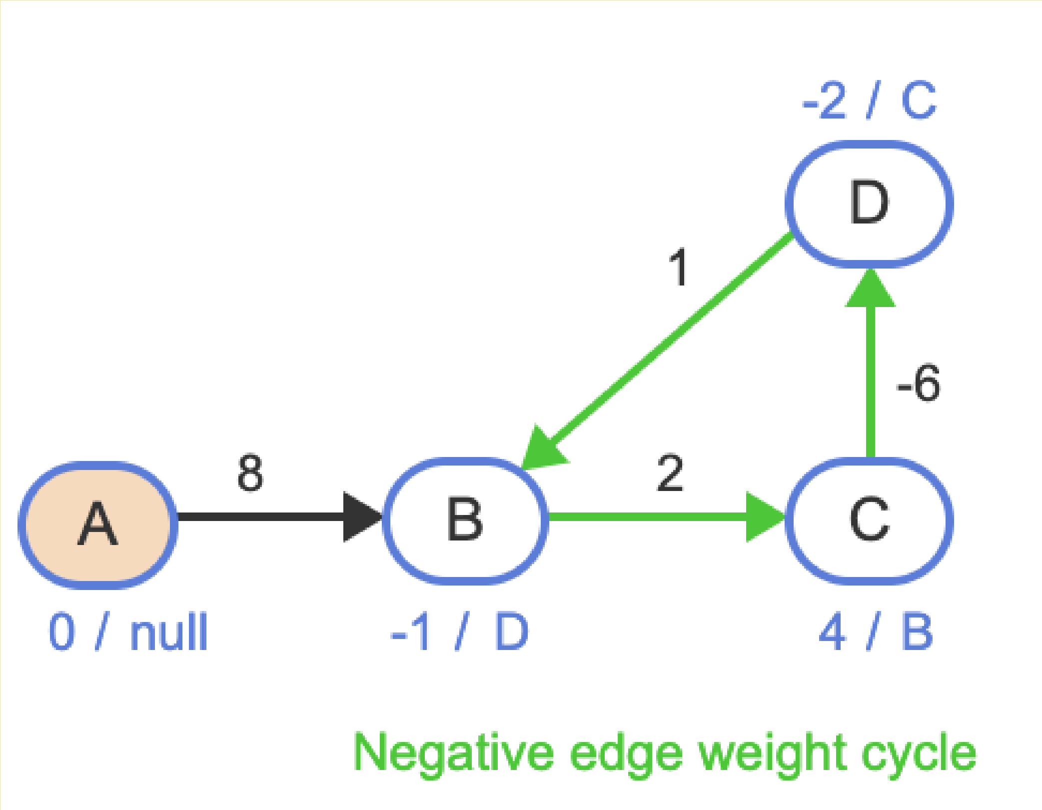
B→ C

C→D

D→B

Bellman-Ford's Algorithm

- Negative weight cycle check (after V-1 iterations)



Edges

A→ B

B→ C

C→D

D→B

Bellman-Ford's algorithm negative weight cycle

```
# Check for negative weight cycle
    for each vertex currentVertex in graph:
        for each adjacent vertex adjacentVertex of currentVertex:
            edgeWeight = weight of edge(currentVertex, adjacentVertex)
            if currentVertex.distance + edgeWeight < adjacentVertex.distance:
                print("Graph contains a negative weight cycle")
                return
```

Bellman-Ford's Algorithm

- Time complexity
  - Best Case: O(E), when distance array after 1st and 2nd relaxation are same , we can simply stop further processing.
  - Worst Case: O(V*E)

References and Useful Resources

- Dijkstra's Shortest Path Algorithm - A Detailed and Visual Introduction https://www.freecodecamp.org/news/dijkstras-shortest-path-algorithm-visual-introduction/#:~:text=Dijkstra's%20Algorithm%20finds%20the%20shortest,node%20and%20all%20other%20nodes.
- Bellman-Ford's Algorithm https://www.geeksforgeeks.org/bellman-ford-algorithm-dp-23/
- Dijkstra's vs Bellman-Ford Algorithm https://medium.com/@brianpatrao1996/dijkstras-vs-bellman-ford-algorithm-383e4771c2cb#:~:text=One%20of%20the%20main%20benefits,is%20the%20number%20of%20vertices.

That’s about this lecture!