# MA2034 Worksheet 2 Fall 2024

Friday, October 11, 2024 8:16 AM

MA2034 Worksheet...



**The answers to the questions are to be uploaded to Gradescope on the Worksheet 2 Answer Form, which is different than this question form.**

## I. Mixing: A Polluted Pond[1]

Consider a pond that has an initial volume of 12,000 cubic ~~feet~~ meters, and that at $t = 0$ the water in the pond is clean and unpolluted. There are two streams flowing into the pond, stream $A$ and stream $B$, and one stream flowing out, stream $C$. Suppose 600 cubic meters per day of water flow from stream $A$ into the pond and 850 cubic meters per day of water flow from stream $B$ into the pond. 1450 cubic meters flow out of the pond each day into stream $C$.

At $t = 0$, the water flowing into the pond from stream A becomes contaminated with a potent carcinogen, $Q$ at a concentration of 2 kilograms per 1000 cubic meters. Suppose the water in the pond is well mixed so the concentration of the carcinogen at any given time is constant. As a way to address this pollution, it is decided by the local government to fill in the pond by dumping sand into it. At $t = 0$ the authorities dump 50 cubic meters of sand into the pond each day. To adjust for the incoming sand, the rate that the water flows out increases to 1500 cubic meters per day so that the water does not overflow the banks of the pond.

Let $Q(t)$ represent the amount of the carcinogen in the pond at time, $t$.

1. Find a differential equation that models this situation. Write the equation in standard form. Explain your reasoning.

2. If you treat the differential equation that you found in (a) as a linear first order DE. What is the integrating factor?

3. Find the solution to the initial value problem for the differential equation that you found in (a). You can use a numerical solver, you do not need to show your work.

[1] Adapted from P. Blancard et al



4. Graph your solution to (a) with the initial value $Q(0) = 0$.

5. Does your graph of the solution make sense to you in terms of the interval of existence of the problem? Explain your answer.

## II. Second Order Linear Differential Equations: Resonance, Beats and Damping[2]

Every object has a resonant frequency – the natural frequency at which it vibrates. For example, if you rub the rim of a wine glass with a damp finger, you might hear a faint hum – that is the resonant frequency of the glass. Or you can simply tap the glass and hear the same frequency. Resonance is a phenomenon that occurs when the matching vibrations of another object increase the amplitude of an object's oscillations.

Any structure is susceptible to damage by the force of resonance. In this project, you will study the phenomena of mechanical resonance and understand its role using second-order linear ordinary differential equations.

Some common examples of resonance:

### Pushing a child on a swing

The first push or pump sets the swing in motion. Each subsequent push or pump, if it is delivered at just the right time, increases the amplitude of swing.

### Shattering wine glass with human voice

You can watch a glass being shattered with a human voice on Outrageous Acts of Science https://www.youtube.com/watch?v=z6oqPB07X3o

### Marching soldiers cause suspension bridge collapse

Built in 1826, the Broughton Suspension Bridge was an iron chain suspension bridge in England and one of Europe's first suspension bridges. It collapsed five years later due to mechanical resonance induced by troops marching over the bridge in step, throwing 74 soldiers into the river. As a result

[2] Adapted from Simiode: Jue Wang, Department of Mathematics, Union College, Schenectady NY USA



of the incident, the British Army issued an order that troops should "break step" when crossing a bridge.

More recently, in 2000 right after opening the London Millenium Bridge pedestrian bridge developed a wobble that was related, in part, to the resonant effect of the pedestrians synchronized walking. The bridge needed to be shut down for several days to fix the problem.

### Crash of Aircrafts

On September 29, 1959, a commercial Braniff airline, a Lockheed L-188 Electra, crashed after 23 minutes into the flight. The officials determined that the left wing had failed, but they could not determine the cause of the catastrophe. Six months later a second Electra aircraft crashed on March 17, 1960, with its right wing found 5 miles from the crash site. NASA, Boeing, and Lockheed engineers determined that violent flutter had torn the wings off the two planes. They were able to duplicate the flutter in wind tunnel simulations. The weakened engine mounts would develop the propeller-whirl flutter. The fatal resonance could build up and tear the plane apart in 30 seconds.

### How Did These Happen and Why?

When the frequency of a periodic external force matches the natural frequency of the mechanical system, resonance may occur which builds up the oscillation to such tremendous magnitudes that the system may fall apart. Because of potential damages, mechanical resonance is in general to be avoided, especially in designing infrastructures and vibrating systems.

To illustrate the effect of resonance, let us model the system with an oscillator driven by a periodic external force,

$$mx'' + bx' + kx = F(t). \tag{1}$$

For simplicity, take $F(t) = F_0 \sin(\omega t)$ or $F_0 \cos(\omega t)$, where $\omega$ is the angular frequency.

**The natural frequency of a mechanical system depends only on the physical paramaters. In the case of a simple harmonic oscillator the natural frequency $\omega_0$ depends only the mass and spring constant**

$$\omega_0 = \sqrt{\frac{k}{m}}.$$



## The Definition of Mechanical Resonance

**Resonance happens when the natural frequency $\omega_0$ and the forcing frequency $\omega$ of an undamped ($b = 0$) harmonic oscillator are the same. For example,**

$$x'' + \omega_0^2 x = \cos(\omega_0 t)$$

In the vibration of a wine glass, $x(t)$ is the amplitude of vibration at time $t$. In the torsional dynamics of a suspension bridge, $x(t)$ represents the angle deflection from the horizontal position . In the motion of the aircraft wing, $x(t)$ is the vertical position of the center of mass of the wing.

## Questions to be uploaded to Gradescope on the Worksheet 2 Answer Form.

### Pure Resonance $\omega = \omega_0$

1. Suppose a suspension bridge is modeled as a simple harmonic oscillator supported by cables which act as the springs and have a spring constant of 16,000 newtons/meter and that the road of the bridge has a mass of 1,000 kg. Further, suppose that the wind or marching soldiers create an external force to the bridge, $F(t) = 6,000\cos(\omega t)$ newtons.

   Let's first examine **pure resonance** by neglecting damping. Equation(1) becomes

$$1000x'' + 16000x = 6000\cos(\omega t),$$

$$x'' + 16x = 6\cos(\omega t). \tag{2}$$

   (a) First, look at the associated homogeneous solution. Use a numerical solver to plot $x(t)$, the solution of

$$x'' + 16x = 0$$

   for $t$ in $[0, 20]$. **You must use computer plotting software to generate your graph.** The Slopes app has an **Oscillations** activity which makes this very simple.

   (b) Now, look at the nonhomogeneous case.

      i. Solve equation (2) by using the Method of Undetermined Coefficients for $\omega = 4$ and $x(0) = x'(0) = 0$. What will be the form of the trial solution and why do you need to use this form?



      ii. Find the solution of this IVP using the Method of Undetermined Coefficients. Show all of your work.

      iii. Now Plot $x(t)$ for $t$ in $[0, 20]$ for your solution to (2), the resonance situation. Again you must have a computer generated graph.

   (c) Upload both graphs and explain why the graphs are different. Describe how your second the graph illustrates the phenomena of resonance.

### Beats $\omega \neq \omega_0$

2. The phenomena of BEATS occurs when the frequency of the driving force oscillation is close to the natural frequency of the system.

   (a) Plot the solution $x(t)$ of equation (2) for both $\omega = 4.5$ and $\omega = 6$, $x(0) = x'(0) = 0$ on $[0, 20]$ . You must use software generate your graphs.

   (b) How are these graphs different from the graph of equation (2) for resonance?

   (c) Based on your graphs of $\omega = 4.4$ and $\omega = 6$, how does the period of the beat frequency appear to change as $\omega$ gets farther from the natural resonant frequency?

### Damping

3. In the physical world, **damping** is always present. We modify equation (2) and add a damping coefficient, $b$. In the equation below which models mechanical vibrations, $m$ is the mass and $k$ is the spring constant denoting the stiffness of the spring. We can take $m = 1$ for simplicity. $m, b, k \geq 0$. If we look at the general form of the equation with no applied force:

$$mx'' + bx' + kx = 0. \tag{3}$$

   In this section we will explore the effects of increasing the damping coefficient on the solutions of the above DE.

   (a) The Undamped Harmonic Oscillator. NO DAMPING $b = 0$ You have already graphed this in II.1.b. You do not need to graph it again, but you will be comparing the damped motion solutions to this undamped case.

$$x'' + 16x = 0 \qquad x(0) = 5,\ x'(0) = 0$$



   (b) Now suppose we have "light" damping. Let $b = 1$

$$x'' + x' + 16x = 0$$

      i. Graph the solutions to the above equation on $[0, 20]$ for $x(0) = 5, \quad x'(0) = 0$. Upload your file.
      ii. Use software and write the solution to this IVP. You do not need to show your work.
      iii. If we define transient terms as terms that go to zero as time goes to infinity, are there any transient terms? If so, identify them.

   (c) Now suppose we add more damping. In each of the following cases, look at the graph of the DE with increased damping and also, if you are using the slopes app, play the simulation. How does the motion of the oscillator change?

      i. Look at the graph of Let $b = 3$

$$x'' + 3x' + 16x = 0$$

      ii. Look at the graph of $b = 6$

$$x'' + 6x' + 16x = 0$$

      iii. Now suppose we add even more damping. Let $b = 8$ and look at the graph of

$$x'' + 8x' + 16x = 0$$

      iv. Upload the graph of $b = 8$ and describe how the motion of oscillator changes from undamped, to light damping then finally to critical damping



**Multiple Choice Questions on SHM, Damping, Resonance, and Beats**

(a) Which of the following best describes simple harmonic motion (SHM)?
   (A) Motion that repeats in space but not in time
   (B) Motion that is always accelerated and never returns to its initial position
   (C) Motion that is periodic and oscillates about an equilibrium position with a restoring force proportional to the displacement
   (D) Motion under a constant velocity

(b) An underdamped harmonic oscillator differs from an undamped oscillator in that it:
   (A) Does not oscillate
   (B) Oscillates with an amplitude that decreases over time
   (C) Oscillates at a higher frequency than the undamped oscillator
   (D) Maintains constant amplitude over time

(c) Critical damping is characterized by:
   (A) The system returning to equilibrium as quickly as possible without oscillating
   (B) The system oscillating with a constant amplitude indefinitely
   (C) The system experiencing maximum amplitude oscillations
   (D) The system taking the longest time to return to equilibrium

(d) The natural frequency of a system is defined as:
   (A) The frequency at which the system dissipates energy
   (B) The frequency at which the system oscillates when not subjected to any damping or external forces
   (C) The highest frequency at which the system can oscillate
   (D) The frequency at which the system experiences resonance with external forces

(e) Resonance in a mechanical system occurs when:
   (A) The frequency of external forces matches the system's natural frequency, leading to maximum amplitude oscillations
   (B) The system is critically damped
   (C) The damping force becomes zero
   (D) The amplitude of oscillations decreases to zero

(f) Which statement about beats is true?
   (A) Beats occur when two waves of the same frequency interfere with each other
   (B) Beats are a result of critical damping in a system
   (C) Beats are produced by the superposition of two waves of slightly different frequencies



   (D) The frequency of beats is determined by the average of the two combining frequencies

(g) In the context of damping, the damping coefficient:
   (A) Has no effect on the period of the oscillation
   (B) Increases the natural frequency of the system
   (C) Determines whether the system is underdamped, critically damped, or overdamped
   (D) Is only relevant in systems with resonance

(h) Which of the following best describes the phenomenon of resonance?
   (A) The gradual decrease in amplitude of an oscillating system
   (B) The rapid increase in amplitude when an oscillating system's natural frequency is matched by an external frequency
   (C) The disappearance of oscillatory motion in a damped system
   (D) The splitting of a single frequency into multiple frequencies