Segment 3: Causal Inference with Regression (in randomized studies)

Section 06: Causal Inference as Model-Based Imputation

Model-Based Imputation for Estimation of Causal Effects

- Build a model for all potential outcomes (depending on unknown parameters)
- Use observed data to learn these parameters
- View model as an *imputation model*
  - To *predict* the unobserved potential outcomes
- Predict the "unobserved" potential outcome for each unit
- Evaluate the causal effect $\tau_{SATE} = \frac{1}{n}\sum\left(Y_i^1 - Y_i^0\right)$
- Repeat...most naturally motivated from a Bayesian perspective

Motivating Toy Example with 6 Units

Adapted from textbook by Imbens and Rubin

Randomized evaluation of a job training program:

- The National Supported Work Program
  - Originally analyzed by Lalonde (1986), now standard dataset for illustration
- Treatment (Z): Random assignment to a job training program or a control group
- Units: Men substantially disadvantaged in the labor market
- Outcome (Y): Post-program earnings

Motivating Toy Example with 6 Units

Table 8.2. ***First Six Observations from NSW Program Data***

Unit	Potential Outcomes		(z_i) Treatment	Observed Outcome
	Y_i^0 $Y_i(0)$	Y_i^1 $Y_i(1)$	W_i	Y_i^{obs}
1	0	?	0	0
2	?	9.9	1	9.9
3	12.4	?	0	12.4
4	?	3.6	1	3.6
5	0	?	0	0
6	?	24.9	1	24.9

Note: Question marks represent missing potential outcomes.

Regression: Causal Effect is a Parameter

We know in this case that an estimate of τ_{SATE} can come from a linear regression model:

$$Y_i = \alpha + \underset{\substack{\uparrow \\ \text{ATE}}}{\tau} W_i + \varepsilon_i$$

- Estimation of regression parameters $\Leftrightarrow$ estimation of $\tau_{SATE} = \frac{1}{6}\sum_{i=1}^{6}(Y_i^t - Y_i^c)$
- Consider an alternative perspective where estimatin of τ will derive from imputations from the model
- This can ground a more general perspective on causal inference as a solution to a missing data problem
  - In general, the *causal estimand* may not correspond to a *model parameter*

Example of Simple Imputation Procedure

Estimand of interest: Average Treatment Effect

$$\tau_{SATE} = \frac{1}{6}\sum_{i=1}^{6}(Y_i^t - Y_i^c)$$

$$= \frac{1}{6}\sum_{i=1}^{6}\left(W_i(Y_i^{obs} - Y_i^{mis}) + (1 - W_i)(Y_i^{mis} - Y_i^{obs})\right)$$

...impute a predicted value for each $Y_i^{mis} \rightarrow \hat{Y}_i^{mis}$

$$\hat{\tau} = \frac{1}{6}\sum_{i=1}^{6}\left(W_i(Y_i^{obs} - \hat{Y}_i^{mis}) + (1 - W_i)(\hat{Y}_i^{mis} - Y_i^{obs})\right)$$

But how to impute $\hat{Y}_i^{mis}$?

Simple, naive: Impute each missing outcome with the average of the observed potential outcome with that treatment

Table 8.3. ***The Average Treatment Effect Using Imputation of Average Observed Outcome Values within Treatment and Control Groups for the NSW Program Data***

Unit	Potential Outcomes		Treatment	Observed Outcome
	$Y_i(0)$	$Y_i(1)$	W_i	Y_i^{obs}
1	0	(12.8)	0	0
2	(4.13)	9.9	1	9.9
3	12.4	(12.8)	0	12.4
4	(4.13)	3.6	1	3.6
5	0	(12.8)	0	0
6	(4.13)	24.9	1	24.9
Average	4.13	12.8		
Diff (ATE):	8.67			

Not Surprising but....

- Estimated effect (8.7) is the same as the simple $\bar{Y}_t^{obs} - \bar{Y}_c^{obs}$
  - Not surprising given the imputation method
- But no uncertainty in imputed values, $\hat{Y}_i^{mis}$
  - Looks particularly suspect given the observed data...
- Randomization allows us to deduce the *mean* of Y_i^{mis}, but not the exact values
- I.e., Pretty easy to get the imputations right "on average"

Another Simple Approach

Instead of just set $\hat{Y}_i^{mis}$ equal to $\bar{Y}^{obs}$ in the "other" treatment group...

Draw Y_i^{mis} from the random distribution of Y_j^{obs} in the "other" treatment group.

E.g., for unit with $W_i = 0$, draw $Y_i^{mis} = Y_i^t$ from the empirical distribution of $Y_i^{obs} | W_i = t$.

In the $N = 6$ example, for unit 1 with $W_i = 0$, draw Y_i^{mis} to be 9.9, 3.6, or 24.9 with probability 1/3 each.

Repeat for each unit and calculate the ATE
Repeat multiple times...

Multiple Imputation

Multiple Imputations of $\mathbf{Y}^{mis}$

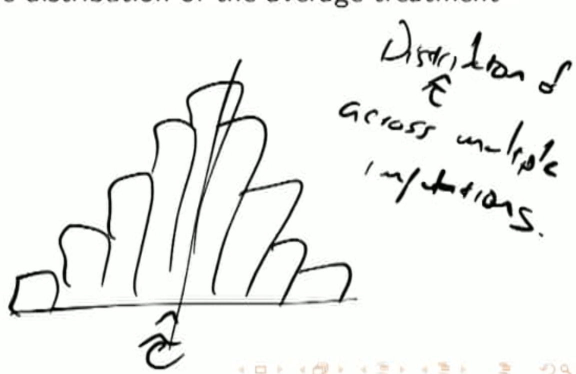
2 imputations

Table 8.4. ***The Average Treatment Effect Using Imputed Draws from the Empirical Distributions within Treatment and Control Groups for the First Six Units from the NSW Program Data***

Unit	Potential Outcomes $Y_i(0)$	Potential Outcomes $Y_i(1)$	Treatment W_i	Observed Outcome Y_i^{obs}
Panel A: First draw				
1	0	(3.6)	0	0
2	(12.4)	9.9	1	9.9
3	12.4	(9.9)	0	12.4
4	(12.4)	3.6	1	3.6
5	0	(9.9)	0	0
6	(0)	24.9	1	24.9
Average	6.2	10.3		
Diff (ATE):	4.1			
Panel B: Second draw				
1	0	(9.9)	0	0
2	(0)	9.9	1	9.9
3	12.4	(24.9)	0	12.4
4	(0)	3.6	1	3.6
5	0	(3.6)	0	0
6	(0)	24.9	1	24.9
Average	2.1	12.8		
Diff (ATE):	10.7			

Result: A Distribution of Effect Estimates

- In this simple example, there are $3^6 = 729$ different imputations, all equally likely
- Calculate the corresponding ATE for each
  - "Overall" point estimate is the mean of the 729 estimates
  - Standard deviation of the 729 estimates
- Result: $\hat{\tau} = 8.7$ with standard deviation 3.1
- Now we have an entire distribution of the average treatment effect



General Motivation

- We have already seen that causal inference problems can be motivated as missing data problems
- Reliable model for $\mathbf{Y}^{mis} \Rightarrow$ reliable model for causal effects
- General stochastic simulation
  1. Sample (or "impute") values for $\mathbf{Y}^{mis}$
  2. Calculate causal effects
  3. Repeat...

Predicting Unobserved Potential Outcomes with Regression

We have seen how to generate (posterior) predictions from linear regression models:

- `posterior_predict(model, newdata)`: Predictive distribution of y
  - Useful for making predictions for a particular unit
  - $\rightarrow \tau_{SATE}$

- `posterior_linpred(model, newdata)`: Posterior distribution of *average* response
  - Useful for making predictions for what would happen on average across the population
  - $\rightarrow \tau_{PATE}$

Predicting Potential Outcomes with Regression

Use the mechanics of Bayesian prediction to estimate the causal effect:

- Fit the regression model
  - `stan_glm(model, data)`
- Then repeat for $k = 1, 2, \ldots, K$
  1. Predict unobserved potential outcomes from the model
     - `posterior_predict(model, newdata)` for τ_{SATE}
     - `posterior_linpred(model, newdata)` for τ_{PATE}
  2. Calculate the causal estimand
     - E.g., $\hat{\tau}^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\left(W_i(Y_i^{obs} - \hat{Y}_i^{mis(k)}) + (1 - W_i)(\hat{Y}_i^{mis(k)} - Y_i^{obs})\right)$
- Use simulations $\hat{\tau}^{(1)}, \hat{\tau}^{(2)}, \ldots, \hat{\tau}^{(K)}$ to calculate estimates/uncertainty for the causal effect

- (Will be equivalent to parameter from *linear regression model*, but not necessarily true for other models)

Predicting Unobserved Potential Outcomes

Linear regression with interactions between ~~D~~ Z and all X has the following features:

- Essentially imputes the missing potential outcomes
- Does so separately for the treated and control units
- Imputing Y_i^0 for $Z_i = 1$ units, only uses Y_i^{obs} for $Z_i = 0$ units, without dependence on Y_i^1
  - And *vice versa*
- Entails some robustness properties by clearly separating imputation of control and treated outcomes

Regression with Covariates and Interactions

(still with complete randomization)

$\hat{Y}_i^0$

$N(0, \sigma^2)$

$$Y_i^{obs} = \alpha + \theta Z_i + X_i\beta + Z_i X_i \gamma + \epsilon_i$$

For $Z_i = 1$:

- $\hat{Y}_i(0) = \hat{\alpha} + X_i\hat{\beta} + \epsilon_i \Rightarrow \sim N(\hat{\alpha} + X_i\hat{\beta}, \hat{\sigma}^2)$
- $\hat{\tau}_i = Y_i^{obs} - \hat{Y}_i(0)$

For $Z_i = 0$

- $\hat{Y}_i(1) = \hat{\alpha} + \hat{\theta} + X_i\hat{\beta} + X_i\hat{\gamma} + \varepsilon \Rightarrow N(\,\cdot\,, \hat{\sigma}^2)$
- $\hat{\tau}_i = \hat{Y}_i(1) - Y_i^{obs}$

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\tau}_i = \sum_{i=1}^{N}\left(Z_i(Y_i^1 - \hat{Y}_i^0) + (1 - Z_i)(\hat{Y}_i^1 - Y_i^0)\right) = \hat{\tau}^{\text{ols}}$$