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Research Article

The Representation of Circular Arc by Using Rational Cubic Timmer Curve

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In CAD/CAM systems, rational polynomials, in particular the Bézier or NURBS forms, are useful to approximate the circular arcs. In this paper, a new representation method by means of rational cubic Timmer (RCT) curves is proposed to effectively represent a circular arc. The turning angle of a rational cubic Bézier and rational cubic Ball circular arcs without negative weight is still not more than $4\pi/3$ and π, respectively. The turning angle of proposed approach is more than Bézier and Ball circular arcs with easier calculation and determination of control points. The proposed method also provides the easier modification in the shape of circular arc showing in several numerical examples.

1. Introduction

The study of curves plays a significant role in Computer Aided Geometric Design (CAGD) and Computer Graphics (CG) in particular parametric forms because it is easy to model curves interactively [1]. CAGD copes with the representation of free form curves. In parametric form, it is important which basis functions are used to represent the circular arcs. Circular arcs are extensively used in the fields of CAGD and CAD/CAM systems, since circular arcs can be represented by parametric (rational) polynomials instead of polynomials in explicit form. Faux and Pratt [2] represented only an elliptic segment whose turning angle is less than π by a rational quadratic Bézier curve.

Generally, a rational cubic Bézier is used to extend the turning angle for conics. The maximum turning angle of a rational cubic circular arc is not more than $4\pi/3$ [3]. Only the negative weight conditions can extend its expressing range to 2π (not equal to 2π) and such conditions are not cooperative in CAD systems because they lose the convex hull property [4]. A rational quartic Bézier curve can express any circular arc whose central angle is less than 2π, and it requires at least rational Bézier curve of degree five to represent a full circle without resorting to negative weights [5]. Fang [6] presented a special representation for conic sections by a rational quartic Bézier curve which has the same weight for all control points but the middle one. G.-J. Wang and G.-Z. Wang [7] presented the necessary and sufficient conditions for the rational cubic Bézier representation of conics by applying coordinate transformation and parameter transformation. Hu and Wang [8] derived the necessary and sufficient conditions on control points and weights for the rational quartic Bézier representation of conics by using two specials kinds, degree reducible and improperly parameterization. Usually rational cubic and rational quartic Bézier curves are used for representation of conic sections or circular arcs representation which can also be found in several papers [9–12]. Hu and Wang [13] constructed necessary and sufficient conditions for conic representation in rational low degree Bézier form and the transformation formula from Bernstein basis to Said-Ball basis.

In this work, we present a rational cubic representation of a circular arc using Timmer basis functions. It is able to approximate a circular arc up to 2π (but not including 2π) without resorting to negative weights in contrast of rational cubic Bézier and rational cubic Ball circular arcs representation methods. Necessary and sufficient conditions are derived for a representation of RCT circular arc.

The remainder of this paper is as follows. The cubic alternative basis functions with geometric properties are given in Section 2. The RCT curve with some geometric curve properties is presented in Section 3 and the weight formulation for Timmer circular arc is also part of this section. The RCT curve as conic segment is represented in Section 4. The Timmer circular arc representation method is discussed in Section 5 and some numerical examples of the proposed method are presented in Section 6. Section 7 is devoted to conclusions.

2. Bézier-Like Cubic Basis Functions

In this section, the Bézier-like cubic basis functions $B_i(u)$, $i = 0, 1, 2, 3$ are defined in [14, 15] as

$$\begin{aligned} B_0(u) &= (1-u)^2(1+(2-m)u), \\ B_1(u) &= m(1-u)^2 u, \\ B_2(u) &= m(1-u)u^2, \\ B_3(u) &= u^2(1+(2-m)(1-u)) \quad u \in [0,1], \end{aligned} \tag{1}$$

where m is a shape parameter. For $m = 2$, the cubic basis function are the Ball basis functions [14], $m = 3$, the basis functions are Bernstein Bézier basis functions [1], and if $m = 4$, then (1) represents a Timmer cubic basis functions [15, 16]; see Figure 1.

Theorem 1 (see [15]). *The Bézier-like cubic basis functions $B_i(u)$, $i = 0, 1, 2, 3$ satisfy the following properties.*

(a) Positivity. For $0 \le m \le 3$, the basis function (1) *are nonnegative on the interval $u \in [0, 1]$, but this property is no more satisfied when $m = 4$.*

(b) Partition of Unity. The sum of the Bézier-like cubic basis functions is one on the interval $u \in [0, 1]$. That is,

$$\sum_{i=0}^{3} B_i(u) = 1. \tag{2}$$

(c) Monotonicity. For the given value of the parameter m, $B_0(u)$ is monotonically decreasing and $B_3(u)$ is monotonically increasing.

(d) Symmetry

$$B_i(u; m) = B_{3-i}(1-u; m), \quad i = 0, 1, 2, 3. \tag{3}$$

3. A Rational Cubic Bézier-Like Curve

The rational cubic Bézier-like curve with single shape parameter m can be defined as follows.

Definition 2. Given the control points P_i $(i = 0, 1, 2, 3)$ in $\mathbb{R}^2$, the rational cubic Bézier-like curve is

$$r(u) = \frac{\sum_{i=0}^{3} w_i P_i B_i(u)}{\sum_{i=0}^{3} w_i B_i(u)}, \quad u \in [0,1]. \tag{4}$$

In standard representation the weights $w_0 = w_3 = 1$ and the middle weights are positive and in this paper the middle weights are considered to be equal to $w_1 = w_2 = w$. Figure 2 illustrates cubic curves with $m = 2, 3$, and 4 with $w = 1$ to represent the ordinary cubic curves.

There are some geometric properties of rational cubic Bézier-like curve as follows.

Theorem 3. *The rational cubic Bézier-like curve satisfies the following properties.*

(a) End Point Properties. Consider

$$\begin{aligned} & r(0) = P_0, \qquad r(1) = P_3, \\ & r'(0) = m(P_1 - P_0)w, \qquad r'(1) = m(P_3 - P_2)w, \\ & r''(0) = 2\Big((3-m)P_3 + mwP_2 + mw(-2+m-mw)P_1 \\ & \qquad + \big(-3 + m^2(-1+w)w + m(1+w)\big)P_0\Big), \\ & r''(1) = 2\Big((3-m)P_0 + mwP_1 + mw(-2+m-mw)P_2 \\ & \qquad + \big(-3 + m^2(-1+w)w + m(1+w)\big)P_3\Big). \end{aligned} \tag{5}$$

(b) Symmetry. The control points P_i and P_{3-i} define the same rational cubic Bézier-like curve in different parameterization; that is, $r(u; m, P_i) = r(1-u; m, P_{3-i})$

(c) Geometric Invariance. The partition of unity property of Bézier-like cubic basis functions assures the invariance of the shape of the rational cubic Bézier-like curve under translation and rotation of its control points.

(d) Convex Hull Property. In general, rational cubic curve does not satisfy the convex hull property for $m \in \mathbb{R}$. But if $m \in [0, 3]$, it is always confined to the convex hull of its defining control points. For $m = 4$ and $w > 0$, edge P_1P_2 of the control polygon is tangent to the curve at the midpoint of P_1P_2 and the curve (4) *always lies inside the control polygon see* Figure 3.

Proposition 4. *The rational cubic Bézier-like curve* (4) *represents a conic segment if and only if the weight of curve* (4) *is*

$$w = \left(\frac{m-2}{m}\right)\frac{(P_3 - P_0)\cdot(P_2 - P_1)}{(P_2 - P_1)\cdot(P_2 - P_1)}. \tag{6}$$

Proof. Let $s(u) = \sum_{i=0}^{3} B_i P_i w_i$ be the rational cubic polynomial in (4). Since the middle weights are equal, the denominator is quadratic, then for $s(u)$ to be a conic, by

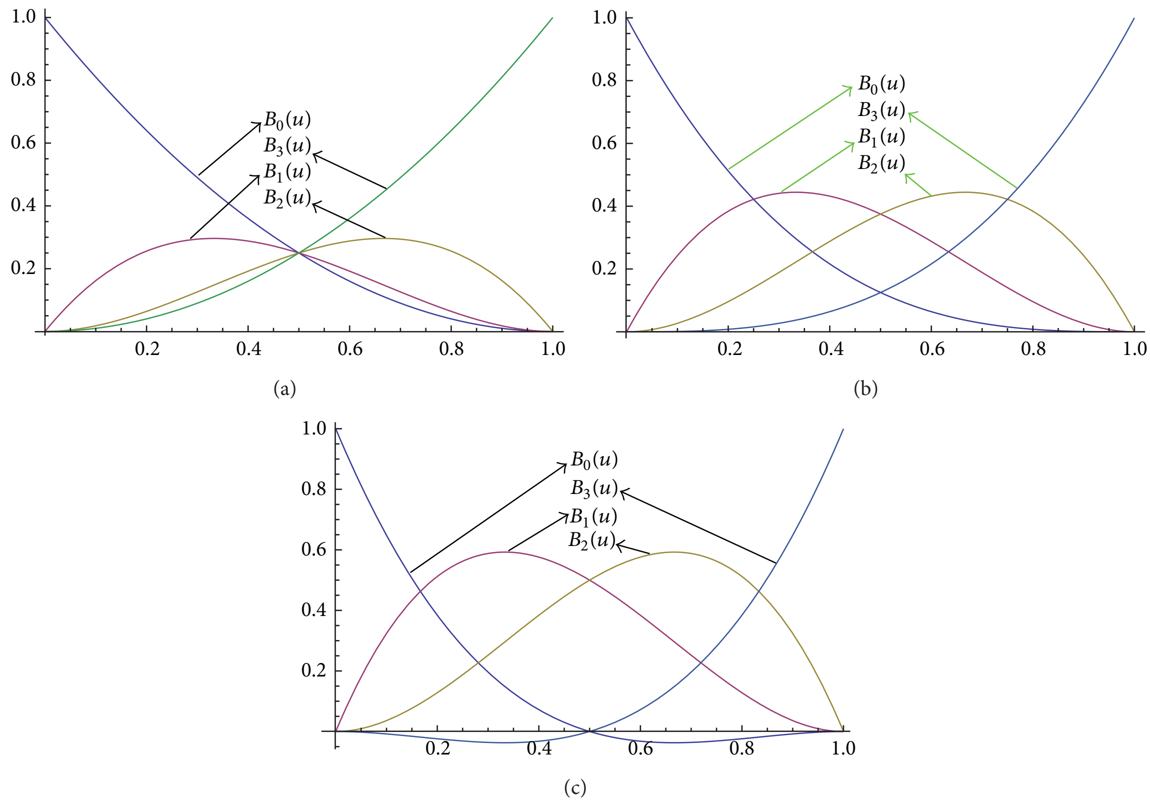

FIGURE 1: (a) Ball cubic basis functions for $m = 2$, (b) Bernstein cubic Bézier basis functions for $m = 3$, and (c) Timmer cubic basis function for $m = 4$.

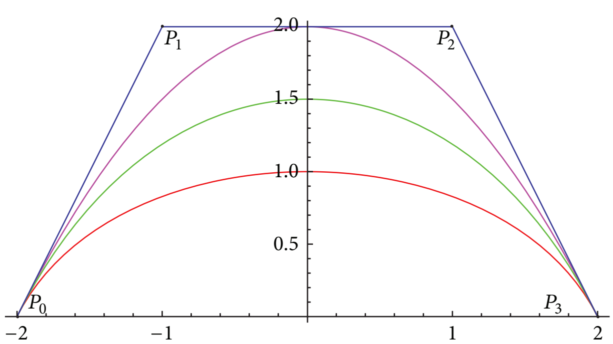

FIGURE 2: Cubic Said-Ball curve for $m = 2$ (red), cubic Bézier curve for $m = 3$ (green), and Timmer PC curve for $m = 4$ (magenta) with arbitrary control points P_i.

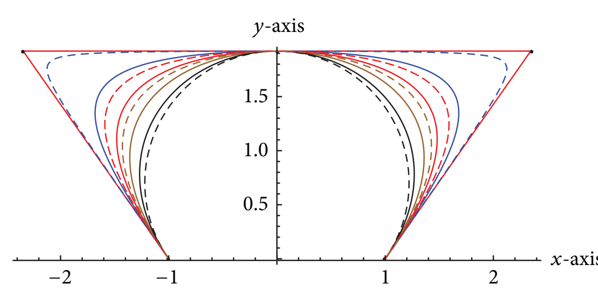

FIGURE 3: Rational cubic Timmer curve for $m = 4$ with different local values of weight $w = 100$ (blue dashed), $w = 5.5$ (blue), $w = 3.5$ (red dashed), $w = 2$ (red), $w = 1.5$ (brown dashed), $w = 1$(brown), $w = 0.5$ (black), and $w = 0.22$ (black dashed).

taking the condition that the numerator of (4) to be quadratic we get

$$\begin{gathered}(2-m)\left(P_0-P_3\right)+m\left(P_1-P_2\right)w=0,\\ \left(P_2-P_1\right)w=\frac{m-2}{m}\left(P_3-P_0\right),\\ w=\left(\frac{m-2}{m}\right)\frac{\left(P_3-P_0\right)}{\left(P_2-P_1\right)}.\end{gathered} \tag{7}$$

Showing that $P_2 - P_1 \parallel P_3 - P_0$,

$$w=\left(\frac{m-2}{m}\right)\frac{\left(P_3-P_0\right)\cdot\left(P_2-P_1\right)}{\left(P_2-P_1\right)\cdot\left(P_2-P_1\right)}. \tag{8}$$

For $m = 4$,

$$w=\frac{1}{2}\frac{\left(P_3-P_0\right)\cdot\left(P_2-P_1\right)}{\left(P_2-P_1\right)\cdot\left(P_2-P_1\right)}. \tag{9}$$

□

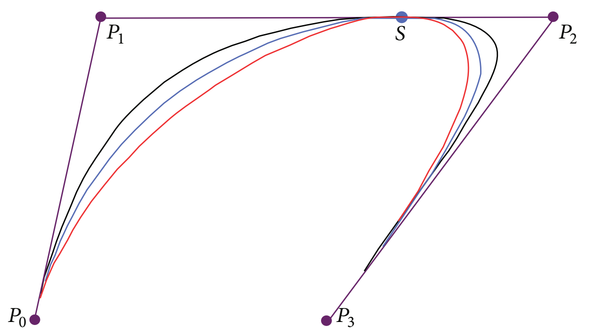

FIGURE 4: Different rational cubic Timmer curves with different values of w_2.

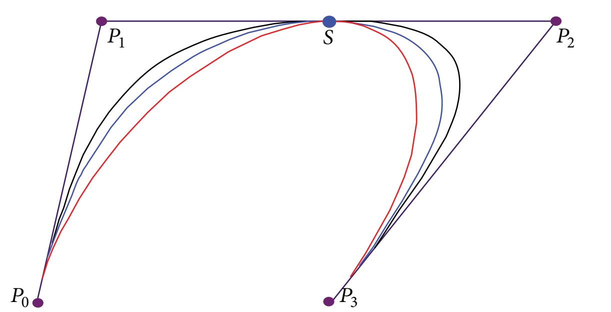

FIGURE 5: Family of rational cubic Timmer with midpoint S and red curve is a conic.

4. A Rational Cubic Timmer (RCT) Curve as a Conic Segment

In this section, we illustrate simple determination of middle weights for a curve to interpolate the intermediate point S. Sarfraz [17] has shown the construction using rational Bézier cubic. It is easy to determine the weights using RCT instead of Bézier because S is the intermediate point of RCT curve (4) at point $t = 1/2$ such that

$$r(0.5) = \frac{w_1 P_1 + w_2 P_2}{w_1 + w_2}. \tag{10}$$

Since S is known intermediate point of Timmer; thus from (9), we obtain

$$S = \frac{w_1 P_1 + w_2 P_2}{w_1 + w_2}. \tag{11}$$

From (11), we get

$$\frac{w_1}{w_2} = k, \tag{12}$$

where

$$k = \frac{(P_2 - S)\cdot(P_2 - S)}{(S - P_1)\cdot(P_2 - S)}. \tag{13}$$

As long as the ratio does not change the curve passes through S. Figure 4 illustrates the different RCT curves.

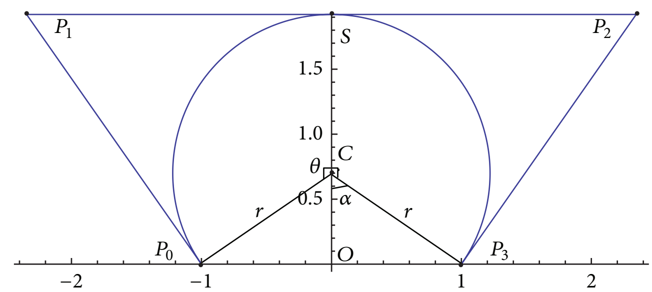

FIGURE 6: Rational cubic Timmer circular arc for $m = 4$.

From (12), if S is the midpoint, then the ratio is one. If w is equal to (9), then the RCT is a conic. Figure 5 shows that the family of RCT with S is the midpoint of P_1P_2.

Sometimes only the end points, end tangents, and the intermediate point are given. The middle control points are obtained by taking the intersection points when the line through S is parallel to the line defined by two end points intersecting with end tangents.

5. A Representation of Circular Arc

In this section, the representation of circular arc using RCT curve (4) is discussed. Let $r(u) = (x, y)$ be a RCT curve centered at C with θ as a turning angle $\theta \in (0, 2\pi)$ with radius $r = \csc\alpha$, where $\alpha = \pi - \theta/2$ is given. Let T_0 and T_1 be the end point tangents to this circular arc. Since $P_2 - P_1 \parallel P_3 - P_0$, referring to Figure 6, the control points of $r(u)$ are

$$\begin{aligned} &P_0(-1, 0), \\ &P_1\left(-r\cot\frac{\alpha}{2}, \tan\left(\frac{\pi-\alpha}{2}\right)\right), \\ &P_2\left(r\cot\frac{\alpha}{2}, \tan\left(\frac{\pi-\alpha}{2}\right)\right), \\ &P_3(1, 0), \end{aligned} \tag{14}$$

with equal internal weights

$$w = \left(\frac{m-2}{m}\right)\frac{(P_3 - P_0)\cdot(P_2 - P_1)}{(P_2 - P_1)\cdot(P_2 - P_1)}. \tag{15}$$

The parametric coordinates of the curve (4) are

$$\begin{aligned} x(u) &= \frac{-1 + 2u}{1 + 2(-1+u)u + (-2+m)(-1+u)u\operatorname{Cos}\alpha}, \\ y(u) &= -\frac{(-2+m)(-1+u)u\operatorname{Sin}\alpha}{1 + 2(-1+u)u + (-2+m)(-1+u)u\operatorname{Cos}\alpha}. \end{aligned} \tag{16}$$

Theorem 5. *A RCT curve $r(u) = (x, y)$ with control points P_i, $i = 0, 1, 2, 3$, and weight w represents the circular arc with central angle $\theta \in (0, 2\pi)$ if and only if $x^2(u) + (y(u) - \cot\alpha)^2 = \csc^2\alpha$.*

Proof. Let $r(u)$ be a circular arc with control points P_i, $i = 0, 1, 2, 3$. The coordinates of any point $(x(u), y(u))$ on the curve (4) with equal internals weights w are given by

$$x(u) = \frac{(1-2u)}{2(1-u)u(1+\operatorname{Cos}\alpha)-1},$$
$$y(u) = -\frac{2u(1-u)\operatorname{Sin}\alpha}{2u(1-u)(1+\operatorname{Cos}\alpha)-1}. \tag{17}$$

Then

$$\begin{aligned}
x^2(u) + (y(u) - \cot\alpha)^2 &= \left(\frac{1-2u}{2u(1-u)(1+\cos\alpha)-1}\right)^2 \\
&\quad + \left(-\frac{2u(1-u)\sin\alpha}{2u(1-u)(1+\cos\alpha)-1} - \cot\alpha\right)^2 \\
&= \Big((1-2u)^2\sin^2\alpha \\
&\quad + \big(2u(1-u)\sin^2\alpha + \cos\alpha \\
&\quad \times (2u(1-u)(1+\cos\alpha)-1)\big)^2\Big) \\
&\quad \times \Big((2u(1-u) \\
&\quad \times (1+\cos\alpha)-1)^2\sin^2\alpha\Big)^{-1} \\
&= (2u(1-u)(1+\cos\alpha)-1)^2 \\
&\quad \times \Big((2u(1-u) \\
&\quad \times (1+\cos\alpha)-1)^2\sin^2\alpha\Big)^{-1} \\
&= \csc^2\alpha \\
&= r^2.
\end{aligned} \tag{18}$$

□

Proposition 6. *Given $m = 4$ and $\theta \in (0, 2\pi)$ as the central angle of the circular arc $r(u)$ with control points P_i, then the weight is $w = \sin^2(\alpha/2)$.*

This is the outcome of (9).

Proposition 7. *For $w = 0$ the RCT curve* (4) *becomes a straight line which is parallel to both x-axis and P_0P_3.*

6. Numerical Examples

We apply our representation method with examples that are shown in Figures 7 and 8 using different values of turning angle θ and values of weight w.

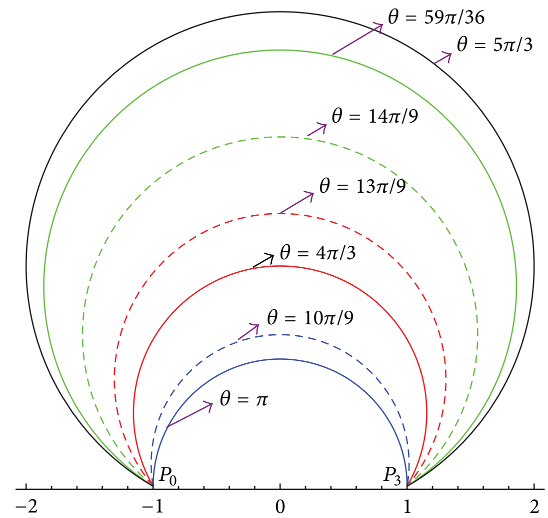

Figure 7: The representation of circular arcs with different turning angles and weights $w = 0.5$ (blue), $w = 0.413$ (blue dashed), $w = 0.25$ (red), $w = 0.178$ (red dashed), $w = 0.116$ (green dashed), $w = 0.078$ (green), and $w = 0.066$ (black).

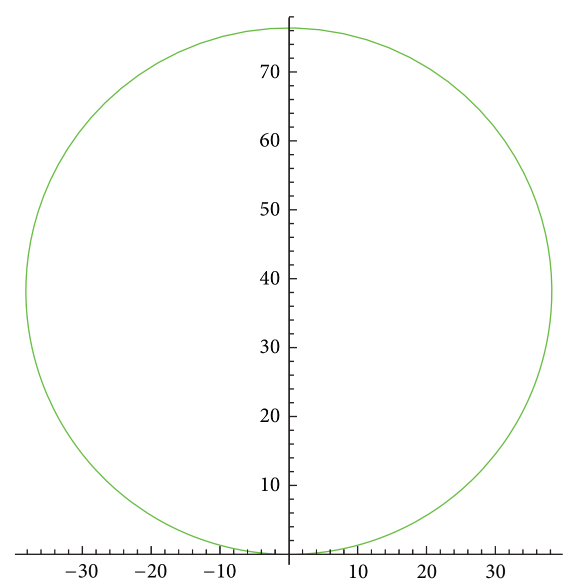

Figure 8: RCT curve as a circular arc with weight $w = 0.00017$ and turning angle $\theta = 119\pi/60$.

7. Conclusion

In this paper, the representation of circular arcs using RCT curves was presented. The largest turning angle of a rational cubic Bézier and rational cubic Ball circular arcs is still not more than $4\pi/3$ and π, respectively. Therefore, we have increased the range for representation of circular arc up to 2π (but not including 2π) without resorting to negative weights using RCT curve. Conic segments have also represented with RCT curves, thus, studying the necessary and sufficient conditions for RCT representation of circular arc is worthy.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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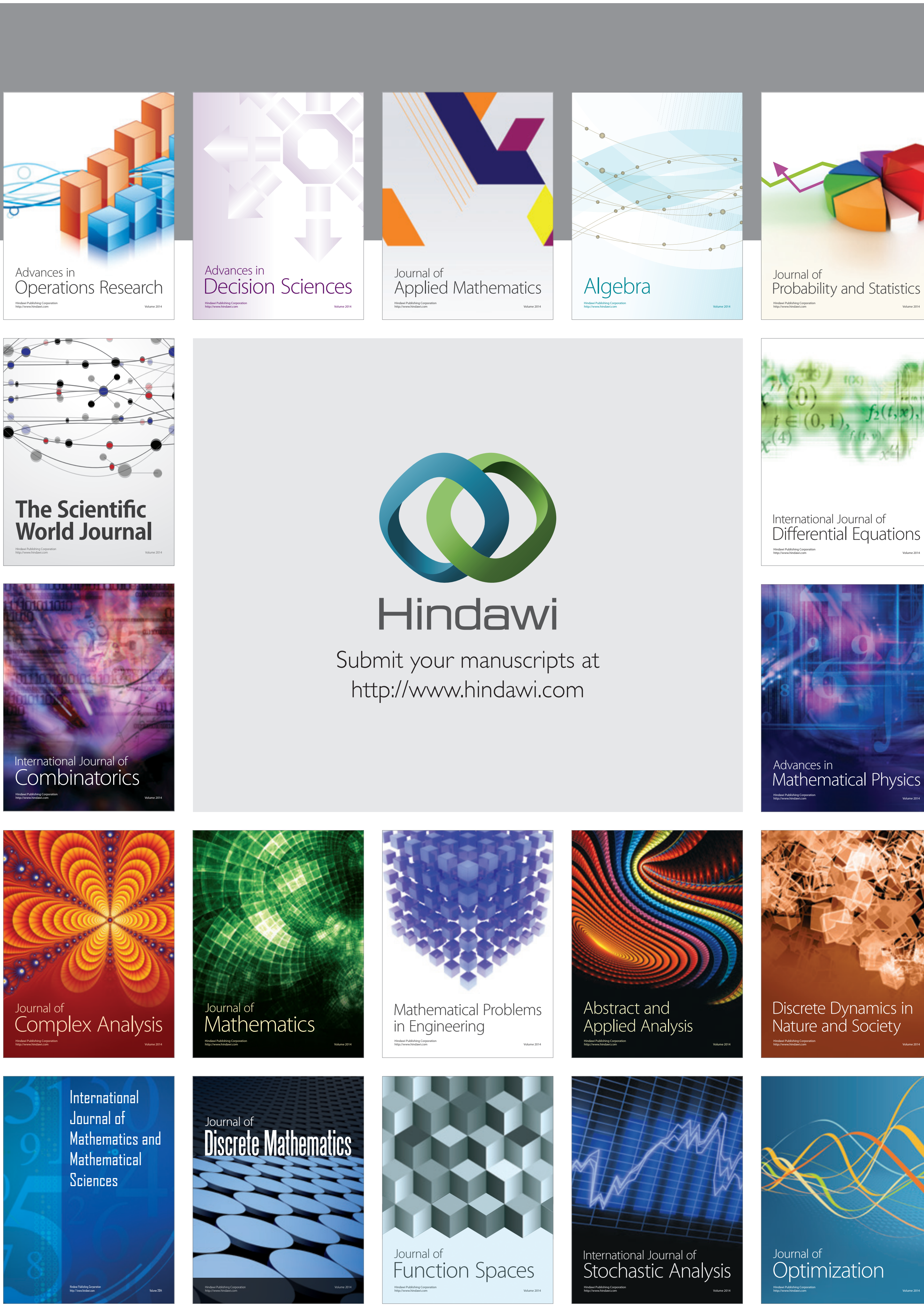

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