A FULLY-AUTOMATED SOLVER FOR MULTIPLE SQUARE JIGSAW PUZZLES USING HIERARCHICAL CLUSTERING

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Zayd Hammoudeh

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by

Zayd Hammoudeh

APPROVED FOR THE DEPARTMENT OF COMPUTER SCIENCE

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Dr. Chris Pollett	Department of Computer Science
Dr. Thomas Austin	Department of Computer Science
Dr. Teng Moh	Department of Computer Science

ABSTRACT

A Fully-Automated Solver for Multiple Square Jigsaw Puzzles Using Hierarchical Clustering

by Zayd Hammoudeh

Square jigsaw puzzle solving is a variant of a standard jigsaw puzzles, wherein all pieces are equal-sized squares that must be placed adjacent to one another to match an original image. This thesis proposes the first algorithm that can solve multiple jigsaw puzzles simultaneously without any information provided to it beyond the set of pieces. The algorithm has been able to assemble up to 10 images simultaneously with results comparable to the state-of-the-art solver working on each image individually.

This thesis also defines the first set of metrics specifically tailored for multiple puzzle solvers. It also outlines the first visualization standards for viewing solver solutions.

DEDICATION

To my mother.

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CHAPTER 1

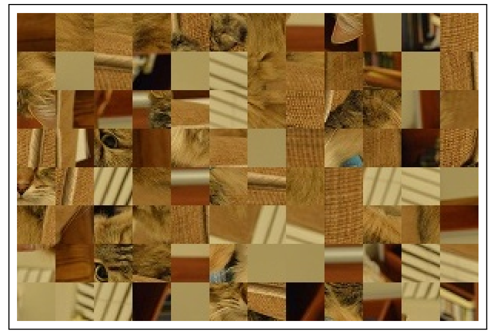
Introduction

Jigsaw puzzles were first introduced in the 1760s when they were made from wood; their name derives from the jigsaws that were used to carve the wooden pieces. The 1930s saw the introduction of the modern jigsaw puzzle where an image was printed on a cardboard sheet that was cut into a set of interlocking pieces [1, 2]. Although jigsaw puzzles had been solved by children for two centuries, it was not until 1964 that the first automated jigsaw puzzle solver was proposed by Freeman & Gardner [3]. While an automated jigsaw puzzle solver may seem trivial, the problem has been shown by Altman [4] and Demaine & Demaine [5] to be strongly NP-complete when pairwise compatibility between pieces is not a reliable metric for determining adjacency.

Jig swap puzzles are a specific type of jigsaw puzzle where all pieces are equally sized, non-overlapping squares; an example of these puzzles is shown in Figure **??**. These puzzles are substantially more challenging to solve than standard jigsaw puzzles since piece shape cannot be considered when determining affinity between pieces. Rather, only the image information on each individual piece is used when solving the puzzle.

Solving a jigsaw puzzle simplifies to reconstructing an object from a set of component pieces. As such, techniques developed for jigsaw puzzles can be generalized to many practical problems. Examples where jigsaw puzzle solving strategies have been used include: reassembly of archaeological artifacts [6, 7], forensic analysis of deleted files [8], image editing [9], reconstruction of shredded

(a) Ground-Truth Image



(b) Randomized Jig Swap Puzzle

Figure 1: Jig Swap Puzzle Example

documents [10], DNA fragment reassembly [11], and speech descrambling [12]. In most of these practical applications, the original, also known as "ground-truth," input is unknown. This significantly increases the difficulty of the problem as the structure of the complete solution must be determined solely from the bag of component pieces.

This thesis outlines the first fully-automated jigsaw puzzle solver algorithm for multiple puzzles; unlike all previous solvers, this thesis' implementation is provided no no makes no assumptions regarding the set of input pieces, including the number of ground-truth (i.e., original) puzzles. What is more, it defines a set of new metrics specifically tailored to quantify the quality of outputs of multi-puzzle solvers.

CHAPTER 2

Previous Work

Computational jigsaw puzzle solvers have been studied since the 1960s when Freeman & Gardner proposed a solver that relied only on piece shape and could puzzles with up to nine pieces [3]. Since then, the focus of research has gradually shifted from traditional jigsaw puzzles to jig swap puzzles.

Cho *et al.* [13] proposed in 2010 one of the first modern computational jig swap puzzle solvers; their approach relied on a graphical model built around a set of one or more "anchor piece(s)," which are pieces whose position is fixed in the correct location before the solver began. Cho *et al.*'s solver required that the user specify the puzzle's actual dimensions. Future solvers would improve on Cho *et al.*'s results while simultaneously reducing the amount of information (beyond the set of pieces) passed to the solver.

A significant contribution of Cho *et al.* is that they were first to use the LAB (Lightness and the A/B opponent color dimensions) colorspace to encode image pixels. LAB was selected due to its property of normalizing the lightness and color variation across all three pixel dimensions. Cho *et al.* also proposed a measure for quantifying the pairwise distance between two puzzle pieces that became the basis of most of the future work.

Pomeranz *et al.* [14] proposed an iterative, greedy jig swap puzzle solver in 2011. Their solver did not rely on anchor pieces, and the only information passed to the solver were the pieces, their orientation, and the size of the puzzle. Pomeranz *et al.* also generalized and improved on Cho *et al.*'s piece pairwise distance measure by proposing a "predictive distance measure." Finally, Pomeranz *et al.* introduced the concept of "best buddies". Side s_x of puzzle piece p_i is best buddies with side s_y of piece p_j if and only if they satisfy the Equation erefeq:bestBuddies. $C(p_i, p_j, s_x, s_y)$ represents the compatibility (i.e., similarity) between side s_x of piece p_i and side s_y of piece p_j.

$$\begin{gathered} \forall x_k \forall s_c, C(x_i, s_a, x_j, s_b) \geq C(x_i, s_a, x_k, s_c) \\ \text{and} \\ \forall x_k \forall s_c, C(x_j, s_b, x_i, s_a) \geq C(x_j, s_b, x_k, s_c) \end{gathered} \tag{1}$$

Best buddies have served as both an estimation metric for the quality of solver result as well as the foundation of some solvers' assemblers [15].

An additional key contribution of Pomeranz *et al.* is the creation of three image benchmarks. The first benchmark is comprised of twenty 805 piece images; the sizes of the images in the second and third benchmarks are 2,360 and 3,300 pieces respectively.

In 2012, Gallagher [16] formally categorized jig swap puzzles into four primary types. The following is Gallagher's proposed terminology; his nomenclature is used throughout this thesis.

- **Type 1 Puzzle**: The dimensions of the puzzle (i.e., the width and height of the ground-truth image in number of pixels) is known. The orientation of each piece is also known, which means that there are exactly four pairwise relationships between any two pieces. A single anchor piece, with a known, correct, location is required with additional anchor pieces being optional. This type of puzzle is used by [13, 14].

- **Type 2 Puzzle**: This is an extension of a Type 1 puzzle, where pieces may be rotated in 90° increments (e.g., 0°, 90°, 180°, or 270°); in comparison to a Type 1 puzzle, this change alone increases the number of possible solutions by a factor of 4^n, where n is the number of puzzle pieces. What is more, no piece locations are known in advance; this change eliminates the use of anchor piece(s). Lastly, the dimensions of the ground-truth image may be unknown.

- **Type 3 Puzzle**: All puzzle piece locations are known and only the rotation of the pieces is unknown. This is the least computationally complex of the puzzle variants and is generally considered the least interesting. Type 3 puzzles are not explored as part of this thesis.

- **Mixed-Bag Puzzles**: The input set of pieces are from multiple puzzles, or there are extra pieces in the input set that belong to no puzzle. The solver may output either a single, merged puzzle, or it may separate the input pieces into disjoint sets that ideally align the set of ground-truth puzzles. This type of puzzle is the primary focus of this thesis.

Sholomon *et al.* [17] in 2013 proposed a genetic algorithm based solver for Type 1 puzzles. By moving away from the greedy approach used by Pomeranz *et al.*, Sholomon *et al.*'s approach is more immune to suboptimal decisions early in the placement process. Sholomon *et al.*'s algorithm is able to solve puzzles of significantly larger size than previous techniques (e.g., greater than 23,000 pieces). What is more, Sholomon *et al.* defined three new large image (e.g., 5,015, 10,375, and 22,834 piece) benchmarks [18].

Paikin & Tal [15] published in 2015 a greedy solver that handles both Type 1 and Type 2 puzzles, even if those puzzles are missing pieces. What is more, their algorithm is one of the first to support Mixed-Bag Puzzles. Their algorithm is used as the assembler for this thesis' algorithm as described in Section 3.1.

CHAPTER 3

Mixed-Bag Solver Overview

This thesis' Mixed-Bag Solver supports Type 1, Type 2, and Mixed-Bag Puzzles. It consists of five distinct stages namely: segmentation, stitching piece solving, hierarchical clustering of segments, seed piece selection, and final assembly. The flow of the algorithm is shown in Figure 2.

The following subsections describe each of the components solver architecture. It also discusses the assembler which is an independent but associated component of the architecture.

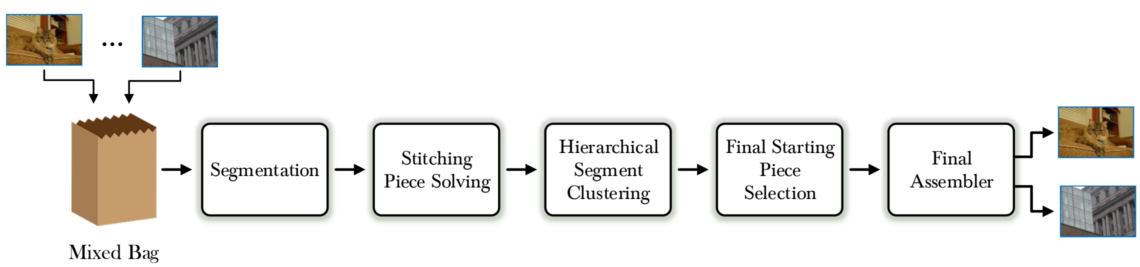


Figure 2: Components of the Mixed-Bag Puzzle Solver

3.1 Assembler

The role of the assembler assigns the placement (and optionally rotation) of the puzzle pieces in the solved puzzle. This solver architecture is largely independent of the assembler used. Hence, as assemblers improve, they can be incorporated into this solver to improve the overall performance. The same also applies if particular assembler(s) perform better for particular types of puzzles. This enables significant flexibility to balance competing concerns, while maintaining upgradability.

For all experiments in this thesis, we used the solver proposed by Paikin & Tal [15]. It was selected because it the current state-of-the-art. What is more, since it supports mixed bag puzzles, it can be used for direct comparison of performance.

3.2 Segmentation

As the first stage in the Mixed-Bag Solver, segmentation takes as input only the bag of puzzle pieces created from the original images; unlike all other solvers to date, this algorithm takes no other inputs. The role of stage is to provide structure to the unordered input. This is done by partitioning the pieces into disjoint sets, referred to here as segments. These segments are groups of puzzle pieces where there is a high degree of confidence that the pieces are assembled correctly.

Algorithm 1 outlines the basic segmentation framework. The algorithm is iterative and will have one or more rounds. In each round, all pieces that have not yet been assigned to a segment are assembled as if they all belong to a single ground truth image. This is done as it eliminates the need to make any assumptions regarding the input at this early stage of the solver. While it may lead to a single puzzle being divided into multiple segments, these segments can be merged together later in a later stage of the solver.

After the single puzzle is assembled in each round, the solved puzzle is then divided into segments; the procedure for this is described in Section . Assuming the largest segment exceeds the minimum allowed size[1], it is passed to the next stage of the Mixed-Bag Solver.

The term "α" in Algorithm 1 defines which segments other than the largest

[1]This thesis found that a minimum segment size of 7 provided the best balance between solution quality and algorithm execution time.

one constructed puzzle are immediately passed to the next solver stage. In this thesis, α was set to 0.5, meaning any segment that was at least half the size of the largest segment in that round (and larger than the minimum segment size) is saved. This scalar value provides sufficient balance between ensuring the largest segments for analysis with limiting the execution time of this stage.

Once a piece is assigned to a saved segment, it is removed from the set of unassigned pieces. Hence, those pieces will not be placed in the next segmentation round. Segmentation continues until all pieces have been assigned to sufficiently large segments, or no segment exceeds the minimum allowed segment size.

Algorithm 1 Pseudocode for the Segmentation Algorithm

```
function SEGMENTATION(all_pieces)
    solved_segments ← {}
    unassigned_pieces ← {all_pieces}
    repeat
        solved_puzzle ← run_single_assembly(unassigned_pieces)
        puzzle_segments ← segment(solved_puzzle)
        max_segment_size ← maximum size of segment in solved_segments
        for each segment ∈ puzzle_segments do
            if |segment|>α × max_segment_size then
                add segment to solved_segments
                remove pieces in segment from unplaced_pieces
    until max_segment_size>smallest_allowed and |unplaced_pieces|>0
    return solved_segments
```

3.2.1 The segment Function

The `segment` function shown in Algorithm 2 is adapted from the kernel growing segmentation procedure modified by Pomeranz *et al.*, where it was shown to have greater than 99.7% accuracy identifying genuine neighbors [14]. The kernel of each segment is a single seed piece.

Algorithm 2 Pseudocode for the segment Function

```
function SEGMENT(solved_puzzle)
    puzzle_segments ← {}
    unassigned_pieces ← {all pieces in solved_puzzle}

    while |unassigned_pieces| > 0 do
        segment ← new empty segment
        seed_piece ← next piece in unassigned_pieces
        queue ← [seed_piece]

        while |queue| > 0 do
            piece ← next piece in queue
            add piece to segment

            for each neighbor_piece of piece do
                if is_best_buddies(neighbor_piece, piece) then
                    add neighbor_piece to queue
                    remove neighbor_piece from unassigned_pieces

        articulation_points ← find_articulation_points(segment)
        remove articulation_points from segment

        disconnected_pieces ← find_disconnected_pieces(segment, seed_piece)
        add segment to puzzle_segments

        remove disconnected_points from segment
        add articulation_points and disconnected_pieces to unassigned_pieces

    return puzzle_segments
```

Whenever a piece is added to a segment, it is removed from the set of unassigned pieces. What is more, the algorithm check's all pieces directly adjacent to the added piece. If the adjacent piece and the added piece are "best buddies" (i.e., each is more similar to the other on their respective sides than they are to any other piece as defined by [14] and [15]), then the adjacent piece is also added to the segment. This process continues until no there are no pieces adjacent to a segment member that fulfill the best buddy criteria.

If Pomeranz *et al.*'s original segmentation algorithm is used for Mixed-Bag puzzles, two correctly assembled segments from different input puzzles can be merged into a single segment. This is usually in the form of narrow bridges no wider than a single piece. This necessitates that each segment goes through post-processing to identify and remove these single point bridges; the procedure for doing this is described in Section 3.2.2.

Once these single piece bridges have been broken, one or more pieces become disconnected from the segment. These pieces are then returned to the set of unassigned pieces to be assigned to a different segment. At the end of segment, each piece in the puzzle will be assigned to exactly one segment.

3.2.2 Articulation Points

A segment can be model as a single connected graph, with the vertices being the puzzle pieces and the edges being the best buddy relationships. An articulation point is any vertex (*i.e.*, puzzle piece) whose removal increases the number of connected components. The Mixed-Bag Solver uses the algorithm proposed by **??** for identifying articulation points. While most implementations of this algorithm are recursive, this thesis instead uses an iterative approach as segment can be several thousand pieces in sizes. As such, a recursive implementation is prone to stack overflows.

3.3 Stiching Piece Solving

As defined previously, a segment represents a partial assembly where there is a particularly high degree of confidence that the placement is correct. During the segmentation stage, puzzle pieces from a single ground truth image segment may be partitioned into multiple segments. If two segments are similar, it is likely that if one segment is allowed to grow that it would eventually merge with its adjacent segment. However, uncontrolled segment expansion can cause a segment to grow beyond the boundaries of its ground truth image and merge with a segment from a different input image. This thesis controls segmentation growth through the use of multiple "mini solvers," which each have a "stitching piece" as their seed. The output from these mini solvers are used to determine inter-segment similarity as described in the following subsection.

3.3.1 Definition of a Stitching Piece

As the name indicates, the set of stitching pieces assist in the "stitching" together of associated segments. Since segments are disjoint, a segment will need to grow in order to join with another segment.

3.3.2 defi

An "open location" with respect to a segment is any puzzle location that is not populated with a member of that segment.

3.3.3 Determine the

Stitching pieces should be near the edge of a segment to increase the likelihood of growth towards a neighboring segment. However, if a stitching piece is too close to the edge of the segment, the algorithm may make erroneous segmentation associations.

As outlined in Algorithm **??**, the Mixed-Bag Solver uses iterative boundary tracing to determine each stitching piece's distance to the nearest puzzle location that is not populated with a piece that is a member of the stitching piece's segment..

The algorithm begins by finding set of puzzle locations adjacent to pieces in the segment that are not filled by segment members. These locations represent the .

Algorithm 3 Pseudocode for Determining a Segment Point's Manhattan Distance to the Nearest Open Location

```
procedure FindPieceDistanceToOpen(segment_pieces)
    explored_pieces ← {}
    locations_at_prev_dist ← {segment_pieces}
    distance_to_open ← 1

    while |explored_pieces| > 0 do
        locations_at_current_dist ← {}

        for each prev_dist_loc ∈ locations_at_prev_dist do
            for each adjacent_loc of prev_dist_loc do
                if ∃ piece at adjacent_loc and piece ∉ explored_pieces then
                    set distance_to_open for piece
                    remove piece from explored_pieces
                    add adjacent_loc to locations_at_current_dist

        locations_at_prev_dist ← locations_at_current_dist
        distance_to_open ← distance_to_open + 1
```

Algorithm 3 is run for each of the segments. A single iteration of the algorithm has time complexity of $O(n)$, where n is the number of pieces in the segment. What is more, the algorithm is robust enough to handle voids within a segment as well as potential necking within the segment where two large segment components are joined by a narrower bridge.

3.3.4 Stitching Pieces

If stitching pieces are placed to close together, the solver will perform many redundant solvings. While this may not have a deleterious effect on the solver output, it can significantly increase the execution time of this stage. In contrast, if stitching pieces are placed too far apart, subtle segment pairings may be missed. To achieve

Figure **??** shows an original image along with an segment found during the segmentation. This figure also shows the location of the stitching pieces (denoted with a white cross).

3.3.5 Quantifying Inter-Segment Similarity

3.4 Hierarchical Clustering of Segments

Agglomerative hierarchical clustering is a bottom-up clustering algorithm where in each clustering round, two clusters are merged. Algorithm 4 shows the basic flow of the hierarchical clustering algorithm of the Mixed-Bag Solver; it is adapted from [19].

The only inputs to the hierarchical clustering algorithm are the segments found in the segmentation stage and the Segment Overlap Matrix from the Stitching stage.

Algorithm 4 Pseudocode for the Hierarchical Clustering Algorithm

```
function PerformHierarchicalClustering(solved_segments, overlap_matrix)
    segment_clusters = {}
    for each segment_i ∈ solved_segments do
        add new segment cluster Φ_i containing segment_i to segment_clusters
    Compute the similarity matrix, Γ

    while maximum similarity in Γ > min_cluster_similarity do
        Merge the two most similar clusters Φ_i and Φ_j in segment_clusters
        Update the similarity matrix, Γ for the merged clusters

    return cluster_segments
```

3.4.1 Calculating the Initial Similarity Matrix

The Segment Overlap Matrix is a form of hollow matrix, where all elements in the matrix, except those along the diagonal, are populated with meaningful values. In contrast, hierarchical clustering merges segments using a triangular, similarity matrix. Equation (2) defines the similarity, $\omega_(i, j)$ between any two clusters Φ_i to Φ_j.

$$\omega_{i,j} = \frac{Overlap(\Phi_i, \Phi_j) + Overlap(\Phi_j, \Phi_i)}{2} \tag{2}$$

If there are n solved segments found during segmentation, then the initial similarity matrix Γ is size n by n. Each element in Γ is defined by Equation (3). Both i and j are bounded between 1 and n inclusive. What is more, all elements in Γ are normalized between 0 and 1, also inclusive.

$$\Gamma = \begin{cases} 0 & j >= i \\ \omega_{i,j} & i < j \end{cases} \tag{3}$$

3.5 Updating the Similarity Matrix via Single Linking

The Mixed-Bag Solver uses the Single Link version of hierarchical clustering. Hence, the similarity between any two cluster segments is defined as the similarity between the two most similar segments in either cluster. This approach is required because two segments clusters may only be adjacent along the border of two of the composite segments.

Equation (4) defines the similarity between any a merged cluster containing segment clusters, Ψ_x and Ψ_y, and any other segment cluster Ψ_z. Note that segment Φ_i is a member of the union of Ψ_x and Psi_y while segment Φ_j is a member of segment cluster Ψ_z.

$$\omega_{x \cup y, z} = \underset{\Phi_i \in \Psi_x \cup \Psi_y}{\arg\max} \left(\underset{\Phi_j \in \Psi_z}{\arg\max}\, \omega_{i,j} \right) \tag{4}$$

3.6 Terminating Hierarchical Clustering

Unlike traditional hierarchical clustering, the Mixed-Bag Solver does not always merge all segment clusters until only a cluster remains. Instead clustering continues until the similarity between any of the remaining clusters drops below a predefined threshold. In this thesis, a minimum similarity of 0.1 provided sufficient clustering accuracy without merging unrelated segments.

3.7 Final Assembly

CHAPTER 4

Quantifying the Quality of a Mixed-Bag Solver Output

Modern jig swap puzzle solvers are not able to perfectly reconstruct the ground-truth input in most cases. As such, quantifiable metrics are required to objectively compare the quality of outputs from different solvers. Cho *et al.* [13] defined two such metrics namely: direct accuracy and neighbor accuracy. These metrics have been used by others including [17, 14, 15, 20, 16]. This section describes the existing quality metrics, their weaknesses, and proposes enhancements to these metrics to make them more meaningful for Type 2 and Mixed-Bag puzzles.

In the final subsection, tools developed as part of this thesis to visualize the solver output quality are discussed.

4.1 Direct Accuracy

Direct accuracy is a relatively naïve quality; it is defined as the fraction of pieces placed in the same location in the ground-truth (i.e., original) and solved image with respect to the total number of pieces. Equation (5) shows the formal definition of direct accuracy (DA), where n is the total number of pieces and c is the number of pieces placed in their original (i.e., correct) location.

$$DA = \frac{c}{n} \tag{5}$$

Direct accuracy is vulnerable to shifts in the solved image where even a few misplaced pieces can cause a significant decrease in accuracy. As shown in Section 4.1.2, this can be particularly true when the ground-truth image's dimensions are not known by the solver.

This thesis proposes two new direct accuracy metrics namely: Enhanced Direct Accuracy Score (EDAS) and Shiftable Enhanced Direct Accuracy Score (SEDAS). They are described in the following two subsections; the complementary relationship between EDAS and SEDAS is described in the third subsection.

4.1.1 Enhanced Direct Accuracy Score

The standard direct accuracy metric does not account for the possibility that there may be pieces from multiple puzzles in the same solver output. For a given a puzzle P_i in the set of input puzzles P (where $P_i \in P$) and a set of solved puzzles S where S_j is in S, Enhanced Direct Accuracy Score (EDAS) is defined as shown in Equation (6).

$$EDAS_{P_i} = \underset{S_j \in S}{\arg\max} \frac{c_{i,j}}{n_i + \sum_{k \neq i}(m_{k,j})} \tag{6}$$

c_{ij} is the number of pieces from input puzzle P_i correctly placed (with no rotation for Type 2 puzzles) in solved puzzle S_j while n_i is the number of pieces in puzzle P_i. $m_{k,j}$ is the number of pieces from an input puzzle P_k (where $k \neq i$) that are also in S_j.

When solving only a single puzzle, EDAS and standard direct accuracy as defined Equation (5) are equivalent. When solving multiple puzzles simultaneously, EDAS necessarily marks as incorrect any pieces from P_i that are not in S_j by dividing by n_i. What is more, the summation of $m_{k,j}$ in EDAS is used to penalize for any puzzle pieces not from P_i. Combined, these two factors enable EDAS to penalize for both extra and misplaced pieces.

It is important to note that EDAS is a score and not a measure of accuracy.

(a) Ground-Truth Image (b) Solver Output

Figure 3: Solver Output where a Single Misplaced Piece Catastrophically Affects the Direct Accuracy

While its value is bounded between 0 and 1 (inclusive), it is not specifically defined as the number of correct placements divided by the total number of placements since the denominator of Equation (6) is greater than or equal to the number of pieces in both P_i and S_j.

4.1.2 Shiftable Enhanced Direct Accuracy Score

As mentioned previously, the direct accuracy decreases if there are shifts in the solved image. In many cases such, direct accuracy is overly punitive.

Figure 3 shows a ground-truth image and an actual solver output when the puzzle boundaries were not fixed. Note that only a single piece is misplaced; this shifted all other pieces to the right one location causing the direct accuracy to drop to zero. Had this same piece been misplaced along either the right or bottom side of the image, the direct accuracy would have been largely unaffected. The fact that direct accuracy can give such vastly differing results for essentially the same error shows that direct accuracy has a serious flaw. This thesis proposes Shiftable Enhanced Direct Accuracy Score (SEDAS) to address the often misleadingly punitive nature of direct accuracy.

Let d_{min} be the Manhattan distance between the upper left corner of the solved image and the nearest placed puzzle piece. Also let L be the set of all puzzle piece locations within distance d_{min} of the upper left corner of the image. Given that l is a location in L that is used as the reference point for determining the absolute location of all pieces, then SEDAS is defined as shown in Equation (7).

$$SEDAS_{P_i} = \operatorname*{arg\,max}_{l \in L} \left(\operatorname*{arg\,max}_{S_j \in S} \frac{c_{i,j,l}}{n_i + \sum_{k \neq i}(m_{k,j})} \right) \tag{7}$$

In the standard definition of direct accuracy proposed by Cho *et al.*, l is fixed at the upper left corner of the image. In contrast, SEDAS shifts this reference point within a radius of the upper left corner of the image in order to find a more meaningful value for direct accuracy.

Rather than defining SEDAS based off the distance d_{min}, an alternative approach is to use the point anywhere in the image that maximizes Equation (7). However, that approach can be significantly more computationally complex in particular in large puzzles with several thousand pieces. Hence, this thesis' approach balances finding a meaningful direct accuracy score with computational efficiency.

4.1.3 The Necessity to Use Both EDAS and SEDAS

While EDAS can be misleadingly punitive, it cannot be wholly replaced by SEDAS. Rather, EDAS and SEDAS serve complementary roles. First, EDAS must necessarily be calculated as part of SEDAS since the upper left corner location is inherently a member of the set L. Hence, there is no additional time required to calculate EDAS. What is more, by continuing to use EDAS along with SEDAS, some shifts in the solved image may be quantified; this would not be possible if SEDAS was used alone.

4.2 Neighbor Accuracy

Cho *et al.* [13] defined neighbor accuracy as the ratio of the number of puzzle piece sides adjacent to the same piece's side in both the ground-truth and solved image versus the total number of puzzle piece sides. Formally, let q be the number of sides each piece has (i.e., four in a jig swap puzzle) and n be the number of pieces. If a is the number of puzzle piece sides adjacent in both the ground-truth and solved images, then the neighbor accuracy, NA, is defined as shown in Equation (8).

$$NA = \frac{a}{n\ q} \tag{8}$$

Unlike direct accuracy, neighbor accuracy is largely unaffected by shifts in the solved image since it considers only a piece's neighbors and not its absolute location. However, the standard definition of neighbor accuracy cannot encompass the case where pieces from multiple input puzzles may be present in the same solver output.

4.2.1 Enhanced Neighbor Accuracy Score

Enhanced Neighbor Accuracy Score (ENAS) improves the neighbor accuracy metric by providing a framework to quantify the quality of Mixed-Bag solver outputs.

Let n_i be the number of puzzles pieces in the input puzzle P_i and $a_{i,j}$ be the number of puzzle piece sides adjacent in P_i and S_j. If $m_{k,j}$ is the number of puzzle pieces from an input puzzle P_k (where $k \neq i$) in S_j, then the ENAS for P_i is defined as shown in Equation (9).

$$ENAS_{P_i} = \underset{S_j \in S}{\arg\max} \frac{a_{i,j}}{q\ (n_i + \sum_{k \neq i}(m_{k,j})} \tag{9}$$

In the same fashion as the technique described for EDAS in Section 4.1.1, ENAS divides by the number of pieces n_i in input puzzle P_i. By doing so, it effectively marks as incorrect any pieces from P_i that are not in S_j. What is more, by including a summation of all $m_{k,j}$ in the denominator of (9), ENAS marks as incorrect any pieces not from P_i that are in S_j. The combination of these two factors allows ENAS to account for extra and misplaced pieces.

4.3 Visualizing Solver Output Quality

In images with thousands of pieces, it is often difficult to visually determine the location of individual pieces that are incorrectly placed. What is more, visual tools help developers quickly detect and fix latent bugs.

The following two subsections describe the tools developed as part of this thesis for visualizing direct accuracy and neighbor accuracy.

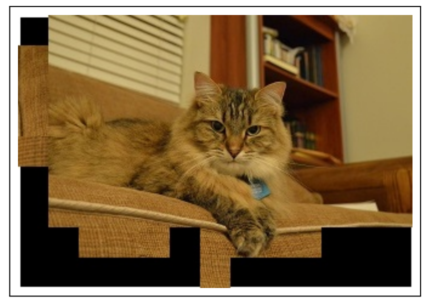
4.3.0.1 Visualizing EDAS and SEDAS

In standard direct accuracy, EDAS, and SEDAS, each puzzle piece is assigned a single value (i.e., correct or incorrect). Due to that, the direct accuracy visualization represents each puzzle by a square filled with a solid color. One additional refinement used in this thesis is to subdivide the “incorrect” placements into a set of subcategories; they are (in order of precedence): wrong puzzle, wrong location, and wrong rotation. Table 1 shows the colors assigned to puzzle pieces depending on their direct accuracy classification.

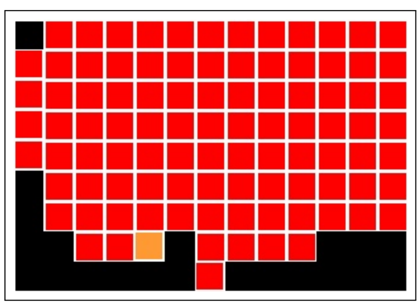
Wrong Puzzle	Wrong Location	Wrong Rotation	Correct Location	No Piece Present

Table 1: Color Scheme for Puzzles Pieces in Direct Accuracy Visualizations

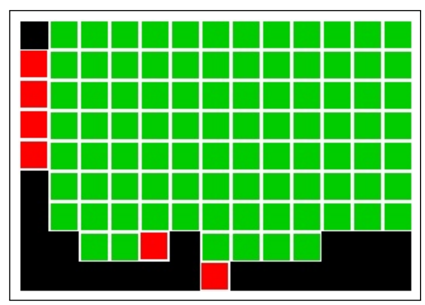
(a) Ground-Truth Image



(b) Type 2 Solver Output



(c) EDAS Visualization



(d) SEDAS Visualization

Figure 4: Example Solver Output Visualizations for EDAS and SEDAS

Figure 4 shows a Type 2 solver output along with the associated EDAS and SEDAS visualizations. Since four puzzle pieces were erroneously placed on the left of the image, almost all pieces had the wrong location according to EDAS; the only exception is a single piece that had the right location but wrong rotation. In contrast, almost all pieces have the correct location in the SEDAS representation; note that the piece in the correct location but with wrong rotation in EDAS has the wrong location in SEDAS.

4.3.1 Visualizing ENAS

The visualization for neighbor accuracy is very similar to the techniques described in Section **??** for visualizing best buddies where each puzzle piece is divided into four equal-sized isosceles triangles (one for each side). The triangles are assigned colors depending on whether their neighbors in the solver output and ground-truth image match. The visualization includes a subcategory known as "wrong puzzle" which is a special case that occurs when solving Mixed-Big puzzles and some of the pieces in the solved puzzle are not from the puzzle of interest, P_i. Table 2 defines the colors used to represent the different classifications of puzzle piece sides in neighbor accuracy visualizations.

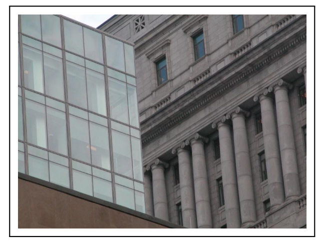
Wrong Puzzle	Wrong Neighbor	Correct Neighbor	No Piece Present

Table 2: Color Scheme for Puzzles Piece Sides in Neighbor Accuracy Visualizations

Figure 5 shows an actual output when solving a Mixed-Bag puzzle with two images. Note that that the puzzle of interest P_i is the glass and stone building while the other puzzle P_k is a rainforest house.

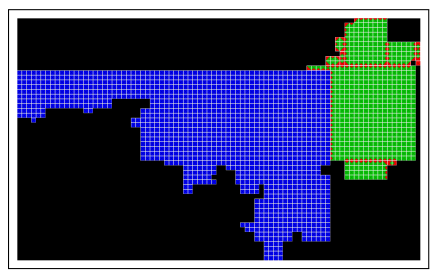
All pieces that came from the rainforest house image are shown as blue despite being assembled correctly; this is because they are not from the puzzle of interest. All neighbors from the puzzle of interest (i.e., the glass and stone building) that are placed next to their original neighbor are represented by green triangles while all incorrect neighbors, such as those bordering the rainforest house image, are shown with red triangles.

(a) Input Image # 1 - Rainforest House [21]



(b) Input Image # 2 - Building Exterior [22]

(c) Solver Output



(d) ENAS Visualization

Figure 5: Example Solver Output Visualization for ENAS

CHAPTER 5

Experimental Results

CHAPTER 6

Conclusions

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APPENDIX

Example Output of a Single Segmentation Round