Square Jigsaw Puzzle Solver Literature Review

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Introduction

- "Jigsaw Puzzle Problem"
  - **Problem Statement:** Reconstruct an image from a set of image pieces
  - **Problem Complexity:** NP-Complete (via the set partition problem) when the pairwise affinity of pieces is unreliable [1]

- **Problem Formulation**: Set of square, non-overlapping pieces
  - "*Type 1*" (also know as "jig swap") Puzzle: Has fixed, known orientation of pieces [19]
  - *Type 2 Puzzles*: Correct rotation of pieces is unknown [19]

- **A Key Difference with Standard Jigsaw Puzzle Solving:** The source image you are trying to reconstruct is unknown.

Square Jigsaw Puzzle Example

- Source image (left) is divided into 81 (9x9) uniform, square pieces (center). The goal is to organize the pieces to reconstruct the source image (right).

Jigsaw Puzzle Solver Applicability

- Possible and existing applications of the jigsaw puzzle problem include:
  - **Computer Forensics**: Reconstructing deleted JPEG, block-based images [2]
  - **Document Investigation**: Reconstruct shredded documents [3]
  - **Bioinformatics**: DNA/RNA modelling and reconstruction [4]
  - **Archeology**: Reconstruction of damaged relics [5]
  - **Audio Processing**: Voice descrambling [6]

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Additional Variants of Problem

Some of the possible variants to the jigsaw puzzle problem include:

- Missing pieces
- Extra pieces
- Three dimensional puzzles
- Unknown puzzle dimension
- Multiple puzzles mixed into a single set of pieces.

Quantifying Piece to Piece Similarity

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Pairwise Affinity

- **Definition:** Quantifies the similarity/compatibility between two pieces.
- Between two pieces x_i and x_j, there are 4 pairwise affinity values when rotation is not allowed and 16 when rotation is allowed.
- Metrics of particular interest in the literature are divided into two categories.
  - Boundary/Edge Based:
    - Normalized and Unnormalized Dissimilarity-based Compatibility
    - Mahalanobis Gradient Distance [12]
    - Prediction-based Compatibility
  - Statistical based using the entire piece and its statistical properties [14]

Dissimilarity-Based Compatibility

- Proposed in Cho *et. al.* [7]

- Uses the LAB (lightness and a/b color opponent dimensions), which is three (3) dimensions.

- Given two pieces x_i and x_j that are size K pixels by K pixels, then the left-right (LR) dissimilarity (where x_j is to the right of x_i) is:

$$D_{LR}(x_i, x_j) = \sum_{l=1}^{K} \sum_{d=1}^{3} \left(x_i(l, K, d) - x_j(l, 1, d)\right)^2$$

Where $x_m(r, c, d)$ is value for the pixel in row r and column c of piece x_m at dimension d.

- **Disadvantage of this Approach:**
  - Severely penalizes boundary differences between pieces which *do* occur in actual images [10].
  - It is common that actual image does **not** the minimum dissimilarity. Hence, this "*better than perfect score*" where the solved solution has a lower score than the original is a type of overfitting [9].

$\left(L_p\right)^q$ Dissimilarity-Based Compatibility

- Proposed by Pomeranz *et. al*. in [10]. *Generalizes* the dissimilarity metric from [7] with the $\left(L_p\right)^q$ norm.

$$D_{p,q}\left(x_i, x_j\right) = \left(\sum_{l=1}^{K}\sum_{d=1}^{3}\left|x_i(l, K, d) - x_j(l, 1, d)\right|^p\right)^{\frac{q}{p}}$$

Hence, [7]'s metric is essentially the $(L_2)^2$ norm.

- While q has no effect on the piece pairwise classification accuracy, [10] observed it had an effect on their solver's performance

Prediction-Based Compatibility

- The dissimilarity based approach measured the difference between two pieces.
  - Prediction based attempts to predict the boundary pixel value of the neighboring piece.

- **First-Order Example**:
  - Use the last two pixels of each piece to predict the neighboring piece's value.
  - Gradient between two right edge pixels for piece x_i in row l for dimension d:

$$x_i(l, K, d) - x_i(l, K-1, d)$$

  - Gradient between two left edge pixels for piece x_j row l for dimension d:

$$x_j(l, 1, d) - x_i(l, 2, d)$$

Prediction-Based Compatibility (Continued)

- The two pixel gradient can be combined with the dissimilarity-based compatibility as shown below for piece x_i's right edge:

$$\big(x_i(l,K,d) - x_j(l,1,d)\big) + \big(x_i(l,K,d) - x_i(l,K-1,d)\big)$$

which is equivalent to:

$$\big(2 * x_i(l,K,d) - x_i(l,K-1,d)\big) - x_j(l,1,d)$$

- If the $\left(L_p\right)^q$ dissimilarity is used, the entire prediction based compatibility for the left-right boundary of x_i and x_j is:

$$\sum_{l=1}^{K}\sum_{d=1}^{3}\Big(\left|\big(2 * x_i(l,K,d) - x_i(l,K-1,d)\big) - x_j(l,1,d)\right|^p$$
$$+ \left|\big(2 * x_j(l,1,d) - x_j(l,2,d)\big) - x_i(l,K,d)\right|^p\Big)^{\frac{q}{p}}$$

- **Advantage of this Approach:** Incorporates a predictor of the pairwise change which may better estimate pairwise affinity.

Accuracy Comparison of the Compatibility Metrics

- Pomeranz *et. al.* in [10] compared the accuracy of the three compatibility metrics on 20 images in a test dataset.
- Using the $(L_p)^q$ norm resulted in a 7% to 10% improvement in selecting the correct neighbor.
- The impact of using the prediction-technique varied from no change up to a 3% improvement.

Puzzle Size	Dissimilarity-Based	$(L_{3/10})^{1/16}$	Prediction-Based
432 Pieces	78%	**86%**	**86%**
540 Pieces	76%	85%	**88%**
805 Pieces	74%	84%	**86%**

Comparison of Pairwise Similarity Metric Accuracy

Asymmetric Dissimilarity

- Proposed by Paikan and Tal [20] and consists of a two parts.
- The previous definitions of pairwise affinity have been symmetrically similar such that:

$$D\left(p_i, p_j, right\right) = D(p_j, p_i, left)$$

- [20] proposes using an asymmetric dissimilarity such that equality in the above equation does not hold.
- **Part #1:** Paikan and Tal use a one sided, L_1 version of Pomeranz *et. al.*'s prediction based distance as shown below:

$$D\left(x_i, x_j, right\right) = \sum_{l=1}^{K} \sum_{d=1}^{3} \left\| \left(2 * x_i(l, K, d) - x_i(l, K-1, d)\right) - x_j(l, 1, d) \right\|$$

Benefits of Asymmetric Dissimilarity

- Three times faster due to the elimination of the exponent (80% of runtime is in distance calculations)
  - Additional speedup can be gained if when the asymmetric dissimilarity is sufficiently large (i.e. no chance of a pairing), the calculation is stopped and the distance set to infinity.

- Number of correct “best buddies” increased

- Number of incorrect decreased

- Using the benchmark in [17], the number of correctly solved puzzles increased from 25 to 37.

Confident Compatibility

- In smooth areas, every piece has a small dissimilarity to every other piece in the region.
  - Hence, having a small dissimilarity by itself does not tell the full story.

- **Part #2:** If a piece's dissimilarity to its *closest* neighbor is far less than the distance to second closest neighbor, then we can have higher confidence they are actually neighbors.
  - Paikan and Tal use that as the basis for their confident compatibility measure.

$$C(p_i, p_j, r) = 1 - \frac{D(p_i, p_j, r)}{secondD(p_i, r)}$$

- r – Spatial relationship (e.g. left, right, top bottom) between pieces p_i and p_j
- $D(p_i, p_j, r)$ - Asymmetric dissimilarity between pieces p_i and p_j
- $secondD(p_i, r)$ – Second best similarity between piece p_i and all other pieces with relation r

- **Goal:** Maximize the value of $C(p_i, p_j, r)$.

Quantifying Solution Quality

Measuring Solution Quality

- **Problem Statement:** There is no uniform technique for grading the final output of a square jigsaw puzzle solver.

- **Two Divergent Approaches:**
  - *Performance Metrics:* Use the original image to grade solution quality.
    - Direct Comparison [7]
    - Neighbor Comparison [7]

  - *Estimation Metrics:* Evaluates the quality of a solution without reference to the original image [10].
    - "Best Buddies" Ratio

Performance Metrics

- **Summary:** Evaluate the performance of a jigsaw puzzle solver against the original (correct) image.

- Cho *et. al.* proposed three performance metrics, but only two are generally relevant. They are:
  - *Direct Comparison Method:* Most naïve approach. The ratio of the number of pieces in their correct locations versus the total number of pieces.
    - *Disadvantage:* Susceptible to shifts
  - *Neighbor Comparison Method:* For each piece, calculate the fraction of its four neighbors that are correct. The total accuracy is the average neighbor accuracy of all pieces.

"Best Buddies"

Definition: Two pieces are *best buddies* if they are more similar to each other on their respective sides than they are two any other pieces [10].

Hence, two pieces, x_i and x_j, are said to be "best buddies" for a spatial relationship R. if and only if, two conditions hold:

$$\forall x_k \in \{Patches\}, C\left(x_i, x_j, r_1\right) \geq C(x_i, x_k, r_1)$$

$$\forall x_k \in \{Patches\}, C\left(x_j, x_i, r_2\right) \geq C(x_j, x_k, r_2)$$

Where:

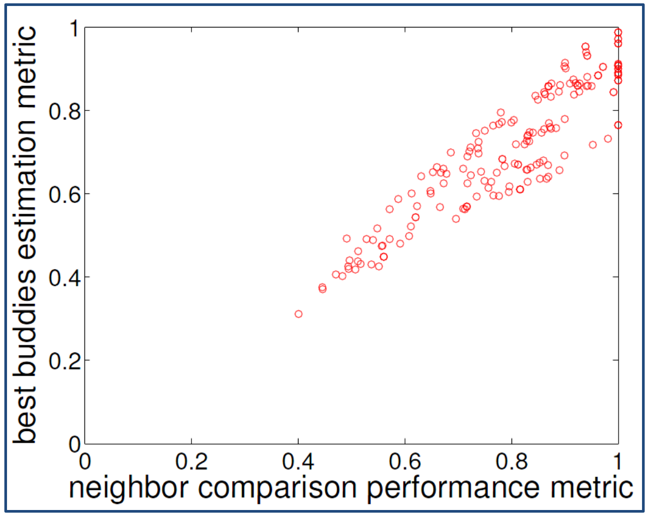
- $C(x_i, x_j, R_1)$ – Compatibility between pieces x_i and x_j on side R_i of x_i
- $\{Patches\}$ – Set of all pieces in the puzzle
- r_1 – Spatial relationship (e.g. top, bottom, left, right) of x_i where x_j will be placed assuming no rotation.
- r_2 - Given x_i and r_1, this represents the complementary side of x_j. For example if r_1 is "left", then r_2 would be "right"

“Best Buddies” Estimation Metric

- **Definition:** Ratio of the number of neighbors who are said to be “best buddies” to the total number of best-buddy neighbors [10].

- Correlation between the “Best Buddies” Estimation Metric and Cho *et. al.*’s two performance metrics:
  - *Direct Comparison Metric*: Little to no correlation since direct comparison method is not based on pairwise accuracy.

  - *Neighbor Comparison Metric*: Stronger correlation Graph below is for 20 images tested 10 times each (for 200 total points)



Scatterplot of “Best Buddy” Metric versus Neighbor Comparison Metric

Existing Jigsaw Puzzle Solver Approaches

- Dynamic Programming and the “Hungarian” Procedure [13]
- Patch Transform using a Low Resolution “Solution Image” [8]
- “Dense and Noisy” or “Sparse and Accurate” with Loopy Belief Propagation [7]
- Particle Filter-Based Solver [11]
- Greedy Algorithm [10]
- Genetic Algorithm [9]
- Loop Constraint Solver [19]

Cho *et. al.* – The Patch Transform and its Application to Image Editing (2008)

Patch Transform

- Introduced by Cho *et. al.* in [8]

- **Overview of the Patch Transform**: Segment a source image into a set of non-overlapping "patches" and rearrange these patches and reorganize the image in the "patch" domain.
  - *Intended Usage*: Image editing

- **"Inverse" Patch Transform**: Reconstruct an image from a set of patches. This requires two components:
  - A patch compatibility function
  - An algorithm that places all patches

- Uses a provided low resolution image as part of the patch placement algorithm.

Markov Random Field

- Use a Markov Random Field (MRF) to enforce three rules:
  - Adjacent pieces should fit plausibly together
  - A patch should "*never*" (or in the loosened case "*seldomly*") be reused.
  - User constraints (e.g. board size) on patch placement.

- Consider each possible patch location as a node in the MRF. The key notation definitions:
  - x_i – Undetermined state for the node i^{th} in the MRF.
  - $\psi_{i,j}(k,l)$ – Compatibility between patches k and l at adjacent MRF locations i and j
  - X – Vector of N determined patch indices, x_i
  - Y – Low resolution version of the original image.

Maximizing the Patch Assignment Probability

For a given patch assignment X, the probability of that assignment is defined as:

$$P(X) = \frac{1}{Z} \prod_{i} \phi_i(x_i) \prod_{j \in \zeta(i)} \left(\psi_{ij}(x_i, x_j) * E(x)\right)$$

- i : i^{th} node in the MRF/board
- N : Number of nodes in the MRF/board.
- $\phi_i(x_i)$: User constraints (e.g. board size)
- $\psi_{ij}(x_i, x_j)$: Patch to patch compatibility
- $\zeta(i)$: Markov blanket of node i
- $E(X)$: Exclusion term that discourages patches being used more than once.
- Z : Normalization term to ensure $\int P(X)\, dX = 1$

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Loopy Belief Propagation Solver

- Maximizes the preceding probability function using loopy belief propagation.
- Susceptible to local maxima so random restarts may be performed.
- **Segue Question:** What if I do not have access to a low resolution version of the original image? Can I make one or use a substitute?

Cho *et. al.* – A Probabilistic Jigsaw Puzzle Solver (2010)

“Dense and Noisy” Estimation

- Proposed by Cho *et. al.* in [7] in 2010.

- **Review:** In Cho *et. al.*’s work in [8], they assumed access to a correct, low resolution version of the original image.
  - In many real world applications, such a low resolution image is not available.

- **Solution:** Estimate a low resolution image from a “bag of patches.” The simplified procedure is:
  - Creating a histogram of the bag of patches
  - “Estimate” a low resolution version by comparing the histogram to a set of K centroids with predefined low resolution images.

“Dense and Noisy” Clustering and Histogram Generation

- **Training Set:** 8.5M patches from 15,000 images.
  - *Patch Size:* 7px by 7px by 3 (LAB) for 147 total, original dimensions. This dimensionality is reduced via PCA.

- **Clustering the Patches**
  - *Step #1:* Cluster each image’s patches into L (e.g. 20) centroids.
  - *Step #2*: Re-cluster the L centroids from all images into N (e.g. 200) centroids.

- **Creating the Histogram:** For a given image, assign each patch to its closest centroid.

"Dense and Noisy" – Generating the Low Res. Image

- **Theoretical Motivation:** Different colors are more likely to be at different places in an image.
  - *Example***:** Blue (sky) is more likely to be towards the top of the image while brown (soil) tends to be in the image foreground.

- **Mapping Bins to the Image**: Use the training set to generate probability density maps for each histogram bin.

- **Use the Histogram to Create the Low Resolution Image**: Use a trained, linear regression function to map the "bag of patches" histogram to the training images (i.e. use prior knowledge).

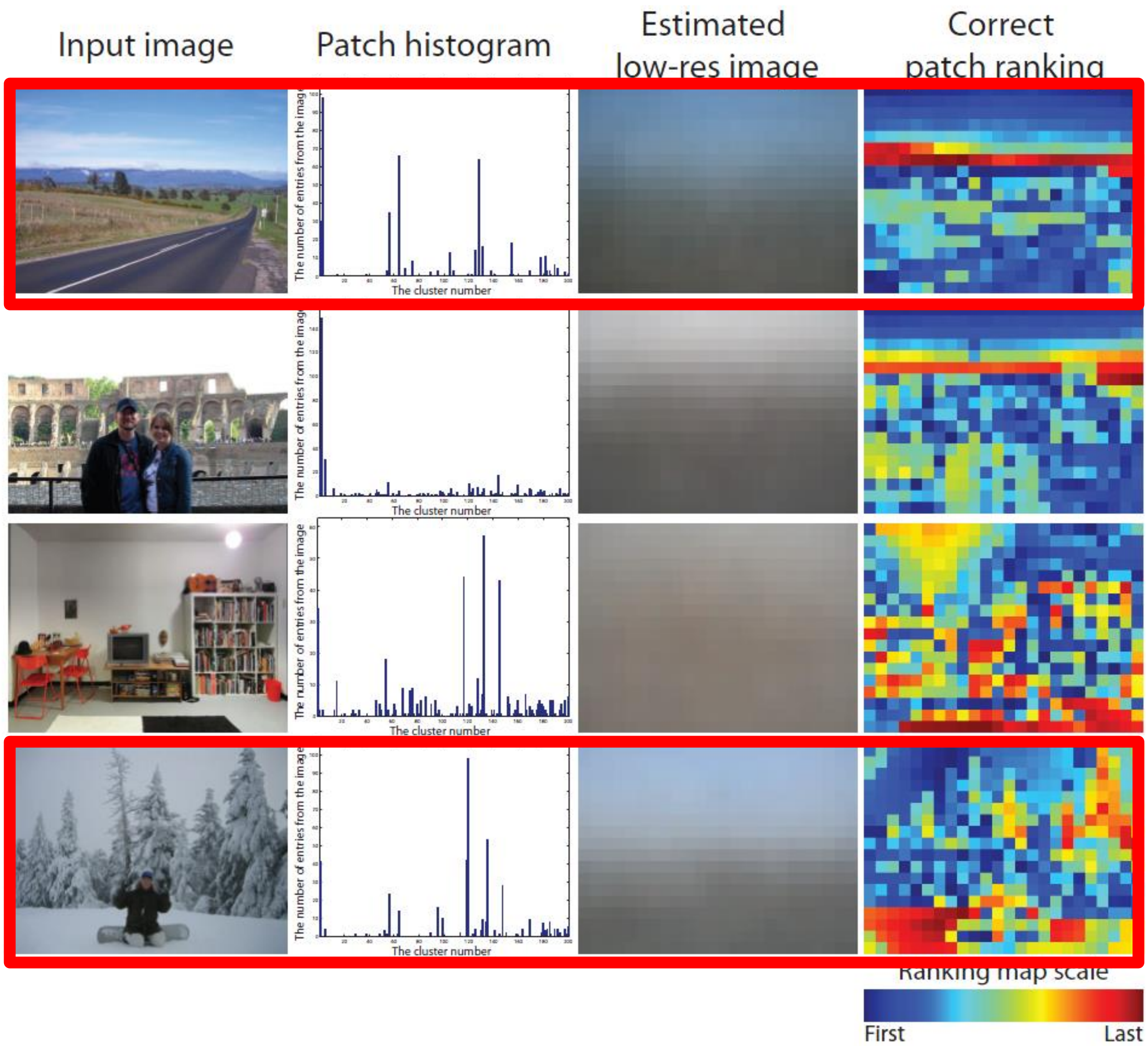
“Dense and Noisy” Results

- **Summary:** Patch histogram can “coarsely predict” a low resolution of the original image.
  - *Possible Explanation*: There is enough “structural regularity” in images that a bag of patches provides spatial information.

- **Patch Rank Map:** For each pixel in the low resolution images, patches are ranked from least likely to most likely to reside in that location.
  - *Ideal Case*: The set of patches that map to the low resolution will have the best rank (i.e. 1)

  - *Worst Case*: The matching set of patches will have rank N (where N is the number of patches in the image).

“Dense and Noisy” End to End Example



Best Results

Worst Results

Confused snow for sky

“Sparse and Accurate”

- Proposed by Cho *et. al.* in [7]

- **Common Human Approach to Solving Puzzles**: “Outside-in”
  - Find the puzzle’s four corner pieces.
  - Build from the corner pieces until all four sections converge.

- “Sparse and accurate” is based off the “outside-in” technique.
  - **Definition on an “Anchor Patch”**: A puzzle patch that is placed in its correct location and orientation.
  - **Summary of the Approach:** Place a set of N anchor patches and then solve the puzzle.

- Two most important criteria of anchor patches
  - Quantity
  - Uniform Spatial Distribution of the Anchors

Pomeranz *et. al.* – A Fully Automated Greedy Square Jigsaw Puzzle Solver (2011)

Generalized Greedy Algorithm

- Proposed by Pomeranz *et. al.* in [10] in 2011.

- **Goal:** Provide a computational framework for handling square jigsaw puzzles in reasonable time that does not rely on any prior knowledge or human intervention.

- Solver divides the puzzle reconstruction into three subproblems:
  - *Placement*: Given a single piece or partially-placed set of pieces, place the remaining pieces.
  - *Segmentation*: Given a fully-placed board, segment the board into *trusted* subcomponents that are *believed* to be placed correctly.
  - *Shifting*: Given a set the *trusted* segments, relocate entire segments and individual pieces to improve solution quality.

Overview of the Greedy Placement Phase

- Given a partially assembled board (either a single piece or set of pieces), continue applying the greedy choice until all pieces are placed.
- **Overview of the Greedy Choice**:
  - Board dimensions are known in advance and fixed
  - Board locations with a higher number of occupied neighbors are preferred as the choice of the next piece is more informed.
  - Piece selection criteria:
    - *Primary Criteria*: Prefer a "best buddy" first.
    - *Secondary Criteria*: If no or multiple pieces satisfy the primary criteria, select the piece with the highest compatibility score.
- **Question:** Why is a placer not enough?
- **Answer:** A greedy placer works solely on local information. To get the best results, we must also look at the entire global solution.

Segmenter Phase

- **Definition of "Segments"**: Areas of the puzzle that are (or "are believed to be") assembled correctly.

- **Procedure**: Using random seeds and a segmentation predicate based on the "best buddies" metric, grow the segments via "region growing segmentation algorithm" described in [15].

- **Accuracy of the Segmenter**: 99.7%

Pomeranz's Complete Algorithm

- ***Step #1***: Select a single puzzle piece as the seed to placement phase.
- ***Step #2***: Perform the placement phase around the seed.
- ***Step #3***: Use the segmenter to partition the board.
- ***Step #4***: Calculate the "best buddies" ratio. If you are at a local maximum, stop.
- ***Step #5***: Select the largest segment from step #3 and use it as the seed of the placement phase. Return to step #2.
  - Performing this step is similar to shifting the largest segment.

Sholomon *et. al.* – A Genetic Algorithm-Based Solved for Very Large Jigsaw Puzzles (2013)

Genetic Algorithm (GA) Solver

- Proposed by Sholomon *et. al.* in 2013 [9].
  – A genetic algorithm puzzle solver was first proposed in [16] in 2002.

- **Genetic Algorithm Review**
  – Based off the biological theory of natural selection.
  – GAs are divided into a series of stages
    - Random generation of initial population
    - Successor selection
    - Reproduction
    - Mutation
  – Requires a “fitness function” that measures solution quality.

Sholomon's GA Implementation

- **Puzzle Type:** 1 (pieces have known orientation)
- **Chromosome (Solution) Representation:** N by M matrix where each cell represents one patch in the puzzle.
- **Population Size:** 1,000
- **Number of Generations:** 100
- **Number of Restarts:** 10
- **Successor Selection Algorithm:** Roulette Wheel
- ***Elitism***: Always pass the *four* best solutions to the next generation
- **Culling:** None
- **Mutation Rate:** 5%
- **Fitness Function:** Sum of the L_2 dissimilarity of all pieces in the puzzle
- **Color Space:** LAB

GA Crossover

- Takes two "*highly fit*" parents and returns one child.
  - Non-trivial as the crossover must ensure there are no duplicate/missing pieces in the solution.

- Correctly assembled segments may be at incorrect absolute locations. Hence, the crossover must allow for "*position independence*", which is the ability to shift segments.

- **Sholomon *et. al.*'s Approach:** Kernel-growing.

Sholomon's Kernel Growing Algorithm

- Start with a single puzzle piece that is "floating" in the board such that the puzzle can grow in any direction.
  - Boundary size (i.e. length by width) is fixed and known.
- **Piece Placement Algorithm:** When deciding on the next piece to place, the algorithm iterates through up to three phases.
  - *Phase #1*: In an available boundary location, place the piece where both parents agree on the neighbor.
  - *Phase #2*: Place a "best buddy" that *exists in one of the parents*.
  - *Phase #3*: Select a location randomly and pick the piece with the best pairwise affinity.
  - If in any phase there is a tie, the tie is broken randomly.
  - After a piece is placed, the placement algorithm returns to phase #1 for the next piece.
  - Once a piece is placed, it can never be reused.

Kernel Growing with Mutation

- Mutations in genetic algorithms are used to improve the quality of the final solution via increased population diversity.

- **Sholomon's Mutation Strategy:** During the first and third phase of placement, place a piece at random with some low probability (e.g. 5%)

A Possible Benchmark

- Sholomon *et. al.* provide three large puzzle datasets as well as their results for comparative benchmarking [17].
  - *Dataset Puzzle Sizes*: 5,015, 10,375, and 22,834

- Unfortunately the website seems to no longer exist. I will separately send an email to the authors about why the removed the content.

- Used as a benchmark in [20].

Algorithm Runtime Comparison

To improve execution time, Sholomon *et. al.* precompute and store all pairwise dissimilarity values.

# of Pieces	Sholomon *et. al.*	Pomeranz *et. al.*
432	48.3s	1.2min
540	64.1s	1.9min
805	116.2s	5.1min
2,360	17.60min	N/A
3,300	30.24min	N/A
5,015	61.06min	N/A
10,375	3.21hr	N/A
22,834	13.19hr	N/A

Comparison of the Algorithm Execution Time
for Sholomon *et. al.* and Pomeranz *et. al.*

Son *et. al.* – Solving Square Jigsaw Puzzles with Loop Constraints (2014)

Solving Jigsaw Puzzles with Loop Constraints

- Proposed by Son *et. al.* in [19].

- Best buddies can be viewed as a loop of two pieces that agree on one boundary.
  - Son *et. al.* propose using a larger loop of 4 pieces (2x2) that agree on four boundaries.

- Other work on the puzzle problem has either ignored or explicitly avoided cycles [12].
  - By using cycles, you are able to achieve a type of outlier rejection.

Small Loops

- **Notation:**
  - SL_i – Small loop of size i by i pieces.
  - SL_N – Maximum size of a small loop.

- The term "small loop" is used to emphasize that the algorithm focuses on the shortest possible cycle at each stage. **Benefits of shorter loops include**:
  - Longer loops are less likely to be made of entirely correct pairwise matches.
  - The number (i.e. permutations) of different cycles increases exponentially with the length of the cycle.
  - Longer loops can be constructed by assembling multiple smaller loops.

- Smaller loops are merged to form larger loops.
  - *Example:* Four 2x2 loops are merged to form one 3x3 loop.

Representing Pieces and Puzzles

- Each piece in the puzzle is represented by a complex number.
  - **Real Component**: A unique piece ID between 1 and the total number of pieces in the board.
  - **Imaginary Component:** A whole number in the set $\{0, 1, 2, 3\}$ with the number representing the number of counter clockwise piece rotations.
    - For type 1 puzzles, there is no imaginary component.
- Structures (e.g. small loops, even the entire puzzle) are represented as complex value matricies.

Relationships between the Complex Matrices

- If two complex-valued matrices, U and V, do not share at least two of the same ID pieces in complementary locations, they are considered *unrelated* ($U||V$).

- If U and V that share at least two of the same ID pieces, they can be considered *geometrically consistent* ($U{\sim}V$).

- Types of geometric conflicts that make two matrices, U and V, *geometrically inconsistent* ($U \perp V$) are:
  - Overlap with different complex numbers (i.e. ID or rotation)
  - Existing of the same ID (real) in a non-shared region.

- If two matrices, U and V, are geometrically consistent, they can be *merged* ($U \bigoplus V$).

Managing Piece-wise Computations

- If for a given pair of pieces the distance is above some threshold, the two pieces are consider not pair worthy and ignored with respect to each other.
  - Each piece will have a maximum number (e.g. 10) of pair worthy neighbors.
- Pairwise compatibility is stored in a K by K by 16 matrix (M) where K is the number of pieces and 16 represents the number of possible rotations for each piece in a Type-2 puzzle.
  - If $M(x, y, z) = 1$, then pieces x and y are compatible with configuration (rotation and side) z.

Creating Larger Small Loops

- Larger "small loops" are build iteratively.

- In the first iteration, SL_2 (i.e. two piece by two piece) loops are formed.
  – Consistency between all loops is them check.

- In the next iteration, four consistent SL_2 loops can be merged to form SL_3 loops.

- Hence, the algorithm constructs SL_i loops using SL_{i-1} loops.

- This process continues until no higher order loops can be built and some highest order loop (SL_N) is found.

Managing Structure-Wise Computations

- $\Omega_i = \{\omega_{i1}, \omega_{i2}, \ldots, w_{iK_i}\}$ represents all of the SL_i dimension structures
  - Similar to what was done for piece-wise compatibility, structure-wise compatibility is stored in a K_i by K_i by 16 matrix (where K_i is the number of structures of dimension SL_i).

- Structures that are consistent and overlap on more than two pieces are merged.
  - If two structures both align at a given location, the one with the superior pairwise matching is prefered.

Paikan and Tal – Solving Multiple Square Jigsaw Puzzles with Missing Pieces (2015)

Managing Missing Pieces and Multiple Puzzles

- Proposed by Paikin and Tal in [20].
- Inspired by Pomeranz *et. al.*'s greedy algorithm [10] with **three additional requirements**:
  - *New Requirement #1*: A modified compatibility function
  - *New Requirement #2*: Superior initial seed selection.
  - *New Requirement #3*: Rather than making the "best"/ "closest matching" selection at each iteration, make the selection with the lowest chance of erring regardless of location.
    - This makes their algorithm deterministic eliminating the need for restarts.
- **Accuracy:** 97.7% on dataset in [17]

Puzzle Problem Requirements

Paikan's & Tal's jigsaw puzzle problem definition (as enumerated below) is the most difficult presented to date.

- Size of the puzzle(s) is unknown and may be different
- Orientation of the pieces is unknown
- Pieces may missing
- Input may contain pieces from multiple puzzles

Only Input to the Algorithm: Number of puzzles to be solved.

Overview of Paikan and Tal's Algorithm

- Similar to Pomeranz *et. al.*, Paikan and Tal use a greedy strategy.

- With greedy algorithms, early suboptimal decisions can lead to major divergences in the future.
  - To reduce the likelihood such poor decisions, Paikan and Tal's algorithm focuses on delaying potentially poor decisions.

- **Phase #1:** Calculate and store all piece to piece the *confident* compatibility values.

Phase #2 – Initial Piece Selection

- Previous work by [9] and [10] selected a random piece as the seed for their placer
  - This spawns the need to run their algorithms multiple times to get better results.

- Paikan and Tal select the *most distinctive piece* in the *most distinctive region* as their algorithm's initial seed.

- **Picking the Most Distinctive Piece:** Select as the initial seed the piece that has four best buddies as its neighbors and whose neighbors also have four best buddies.
  - This approach helps ensure both the piece and region are distinctive
  - **Note:** Best buddies is defined based off the confident compatibility unlike how it is defined in Pomeranz *et. al.* [10].

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Phase #2 – Mutual Compatibility

- If multiple pieces satisfy the "most distinctive" piece criteria, then select the piece with the "strongest" best buddies in all four directions.

- **Paikan and Tal's approach:** Maximize the mutual compatibility with all four neighbors.

$$\tilde{C}(p_i, p_j, r_1) = \tilde{C}(p_j, p_i, r_2) = \frac{C(p_i, p_j, r_1) + C(p_j, p_i, r_2)}{2}$$

- $\tilde{C}(p_i, p_j, r_1)$ – *Mutual* compatibility between pieces p_i and p_j for spatial relation r_1
- $C(p_i, p_j, r_1)$ - *Confident* dissimilarity between pieces p_j and p_i for spatial relation r_1
- r_2 - Complementary spatial relationship with r_1. For example, if r_1 is "right", then r_2 is "left".

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Phase #3: Basic Placement Algorithm

```
While there are unplaced pieces

    if the pool is not empty
          Extract the best candidate from the pool
    else
        Recalculate the compatibility function
        Find the best neighbors (not best buddies)

    Place the above best piece.

    Add the best buddies of the placed piece to
    the pool
```

Phase #3: Placement Overview

- If the placement pool is not empty, then the "best candidate" is defined as the one in the pool with the highest mutual compatibility.
  - Unlike best buddies which used asymmetric dissimilarity, the greedy placer uses mutual compatibility.

- If the pool is empty, the mutual compatibility values are recalculated using only the unplaced pieces *and the border pieces in the puzzle*.
  - The piece with the highest mutual compatibility is then placed onto the board
  - The newly placed piece's best buddies (if any) are placed into the pool.

Phase #3: Handling Multiple Puzzles

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- Other than the pieces themselves, the only input into Paikin and Tal's algorithm is the number of puzzles

- **Modified Approach for Multiple Boards:** When the mutual compatibility *between placed and unplaced pieces* drops below a specified threshold (e.g. 0.5), the candidate pool is cleared, and a new puzzle is started.
  - The seed of the new puzzle uses the same approach that was used for the first puzzle.
  - New puzzles can be created up to the specified input number.
  - Placement goes on simultaneously across all puzzles.

Phase #3: Handling Missing Pieces

- Unlike previous attempts at the problem, Paikan and Tal never specifically try to fill a particular slot in the puzzle.

- Rather Paikan and Tal always try to fill the slot in which they have the most confidence.

- This allows their algorithm to handle missing puzzle pieces.

Puzzle Piece Size

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Comparison of Piece Sizes

Reference	Piece Size
Cho *et. al.* (2010)	7px by 7px
Pomeranz *et. al.* (2010)	28px by 28px
Sholomon *et. al.* (2013)	28px by 28px
Wu (SJSU Thesis) [20]	25px by 25px

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