Final Project Report

Analyzing and Designing Control System for an Inverted Pendulum on a Cart

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Course:
MECH 6021

DEPARTMENT OF MECHANICAL ENGINEERING

Fall 2022

Abstract

This paper presents some results obtained in the development of the control of an inverted pendulum which is a classical control problem with single input and multiple outputs (SIMO). The controller's objectives in this situation are to move the cart and pendulum to the desired position and maintain the pendulum's upright balance. In this project some of the control methods such as PID tuning, Pole placement method and full state feedback method are used and compared to each other. Also models and simulations are derived from MATLAB and SIMULINK. In the end, a comparison between methods is explained in order to choose an optimal option for controlling.

Key Words: Inverted pendulum, SIMO, PID control, Pole placement, full state feedback, Estimator.

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1-Introduction

The inverted pendulum is a highly nonlinear unstable system. Cart-single inverted pendulum system is the most basic example of this system. It has many excellent practical uses, including segways, human walking, and missile launchers. The dynamics of the system in this case are similar to those of a missile or rocket launcher, thus control engineers are interested in it. The controller design is difficult due to the unstable and nonlinear nature of the system.[1]

This project has been conducted on an "Inverted pendulum" which contains an upside-down pendulum mounted on a moving cart. The real application for this pendulum could have 2 or more degrees of freedom but for the sake of simplicity, the system was analyzed in Two-dimensional space with 1 degree of freedom.

Being inverted makes a pendulum inherently unstable (as it is derived by the open-loop approach in the upcoming sections) and it needs to be actively balanced to stay upright. This can be achieved by moving the cart (with the pivot point of the pendulum) horizontally as part of a feedback system. Other than cart position, cart velocity, and pendulum angular velocity, the feedback is derived from the pendulum's angle with respect to the vertical axis. The goal of the controller is to adjust the angle with the right amount of input force causing the pendulum to not tip over. Therefore, this is a SIMO output system.[2]

The dynamic analysis of the system is done with the help of the Euler–Lagrange equation. Then the transfer function is obtained with the use of the Laplace transform of the system equations assuming zero initial conditions. Meanwhile, the state-space equations are extracted from dynamic equations. The equations mentioned were used to analyze the open-loop and close-loop approach with the gain of 1 which both of them were unstable.

In the last section, simulations and block diagrams for Root-locus analysis, Step response, and Impulse response are mentioned.

2-Mathematical Calculations

As it is shown in figure 1, θ difined as angle of the rod from vertical line, also center of gravity of the pendulum (x_m, y_m) is difined as:

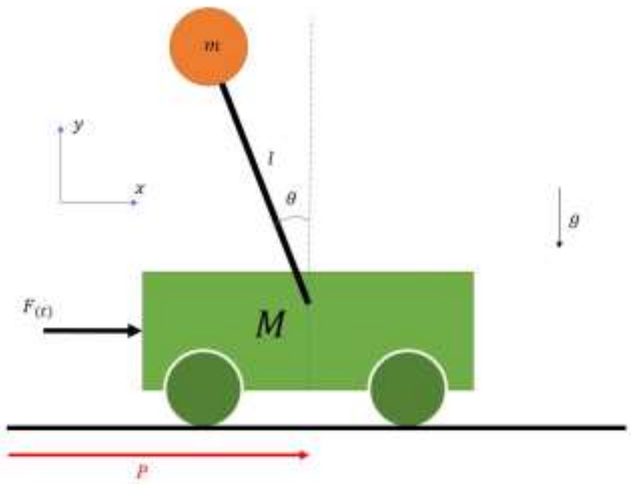

Figure 1- inverted pendulum on cart

$$x_m = x - l.\sin\theta \tag{1}$$

$$y_m = l.\cos\theta \tag{2}$$

As it was mentioned in introduction, Euler-Lagrange method is used for obtaining motion equations. Euler-Lagrange's formula is shown in Eq. 3:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{3}$$

Which L in this formula is defined as:

$$L = T - V \tag{4}$$

T in Eq.4 is Potential Energy of system which is:

$$V = mgY_m = mgl\cos\theta \tag{5}$$

And is V is Kinetic Energy of system which is:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2{}_m + \dot{y}^2{}_m\right) = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos\theta \tag{6}$$

Now Eq.4 can be obtained as:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2{}_m + \dot{y}^2{}_m\right) = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta - mgl\cos\theta \tag{7}$$

Using equation 3 and 6, Equation of motion of system can be obtained:

$$x: (M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = f_{(t)} - b\dot{x} \tag{8}$$

$$\theta: (I+ml^2)\ddot{\theta} - ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \tag{9}$$

"Since we must keep the inverted pendulum vertical, we can assume that θ and $\dot{\theta}$ are small quantities such that $sin\theta \approx \theta$, $cos\theta \approx 1$ and $\theta\dot{\theta}^2 \approx 0$. Then equation 8 and 9 can be linearized. The linearized equations are:" (Ogata, 2002) [3]

$$(M+m)\ddot{x} - ml\ddot{\theta} = f_{(t)} - b\dot{x} \tag{10}$$

$$(I+ml^2)\ddot{\theta} - ml\ddot{x} = mgl\theta \tag{11}$$

Laplace transform is used to obtain Eq.10 and 11:

$$(M+m)s^2X - mls^2\Theta - bsX = F \tag{12}$$

$$(I+ml^2)s^2\Theta - mls^2X - mgl\Theta = 0 \tag{13}$$

Matrix form of Eq. 12 and 13 is obtained in Eq. 14:

$$\begin{bmatrix} (M+m)s^2 + bs & -mls^2 \\ -mls^2 & (I+ml^2)s^2 - mgl \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{14}$$

We take inverse from equation 14:

$$\begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \begin{bmatrix} (M+m)s^2 + bs & mls^2 \\ -mls^2 & (I+ml^2)s^2 - mgl \end{bmatrix}^{-1} \tag{15}$$

So we could extract transfer function of the project from equation 15.

$$\frac{2000s^2-49000}{1100s^4+202s^3-34300s^2-4949s}=\frac{P}{U} \tag{16}$$

$$\frac{5000s}{1100s^3+202s^2-34300s-4949}=\frac{\Theta}{U} \tag{17}$$

On the other hand, if we put quantities on the equation (10) & (11) we could get:

$$0.7p'' - 0.06\theta'' + 0.101p' = u \tag{18}$$

$$0.036\theta'' - 0.06p'' = 0.588\theta \tag{19}$$

We could set the states as bellow:

$$\theta = x_1 \qquad \theta' = x_2$$
$$p = x_3 \qquad p' = x_4$$

now we extract state matrix:

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -38.1 & 0 & 0 & 0.55 \\ 0 & 0 & 0 & 1 \\ -32.66 & 0 & 0 & 0.33 \end{vmatrix} \tag{20}$$

$$B=\begin{bmatrix} 0 \\ -5.55 \\ 0 \\ -3.33 \end{bmatrix} \tag{21}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

$$D=[0] \tag{23}$$

And the general state space form equation is:

$$X' = AX + BU$$

$$Y = CX + DU$$

3-Simulation and results

3-1-Root-locus Map of Uncompensated Systems:

Fig.2 illustrate the Root-locus map of $\frac{P}{U}$ system. This system has a pole and a zero in right half side of s-plane. Thus, this system in unstable.

This system has four poles in $s = \pm 5.6, -0.144, 0$ and two zero in $s = \pm 4.95$.

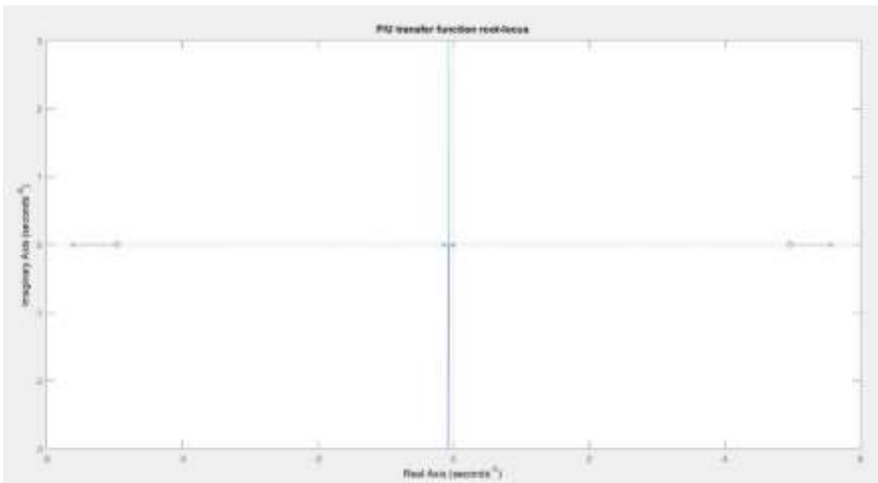

Figure 2- root-locus map of $\frac{P}{U}$

Fig.3 illustrate the Root-locus map of $\frac{\Theta}{U}$ system. This system has a pole and in right half side of s-plane. Thus, this system in unstable.

This system has three poles in $s = \pm 5.6, -0.144$ and a zero in $s = 0$.

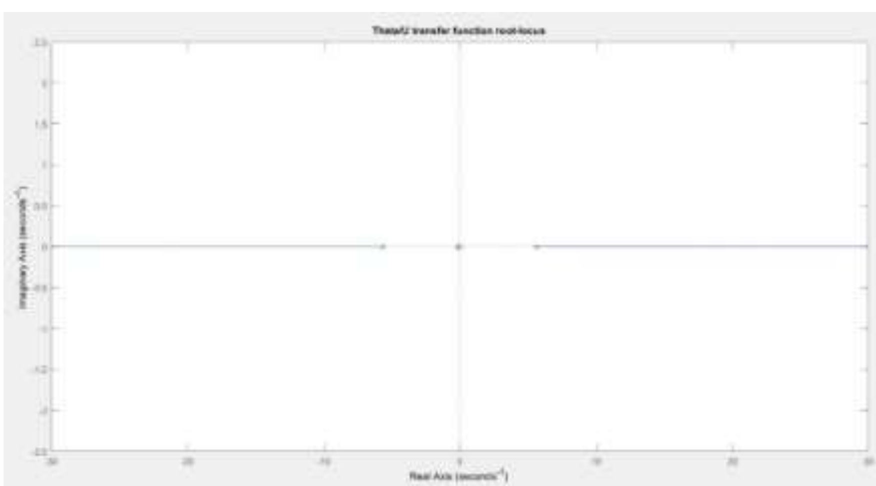

Figure 3- root locus map of $\frac{\Theta}{U}$

3-2-Block diagram:

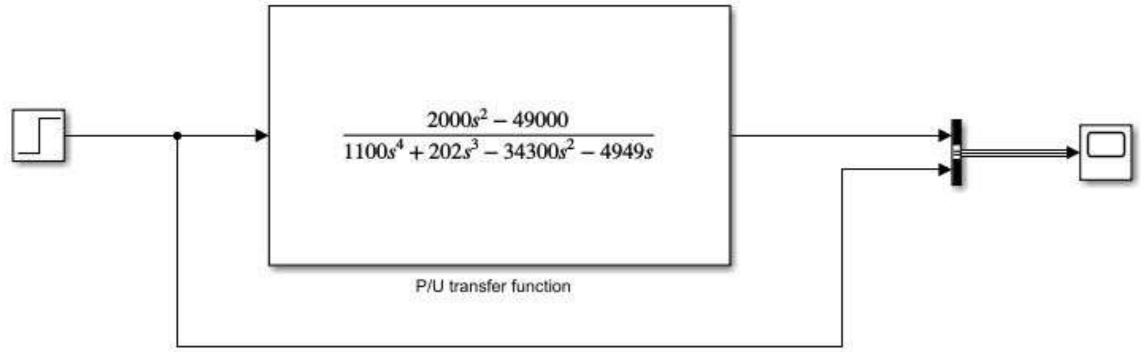

Figure 4- Open-loop block diagram of $\frac{P}{U}$ for step response

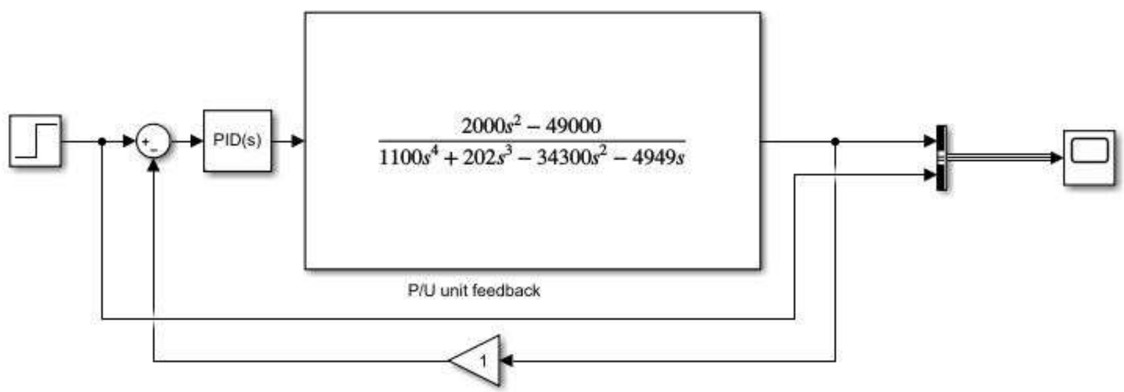

Figure 5- Close-loop block diagram of $\frac{P}{U}$ for step response

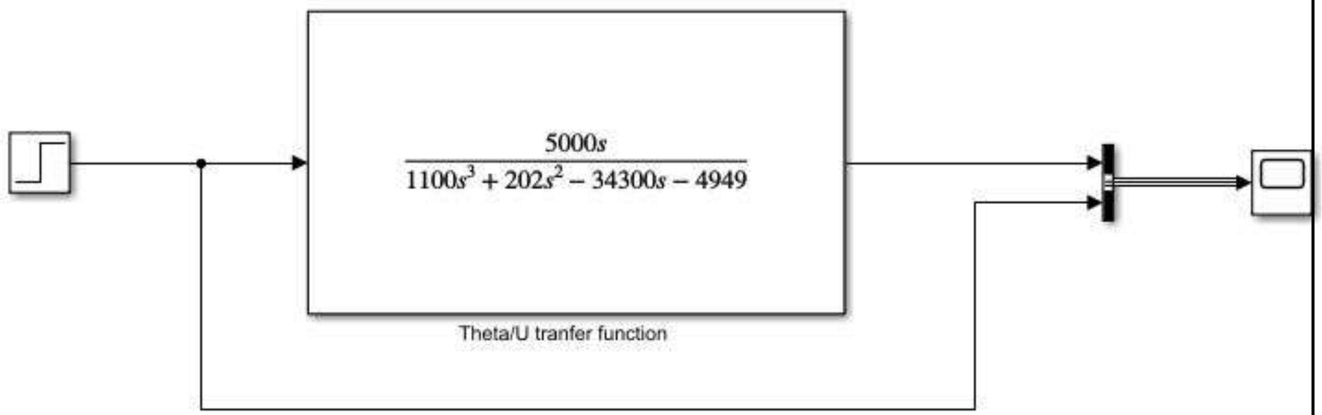

Figure 6- Open-loop block diagram of $\frac{\theta}{U}$ for step response

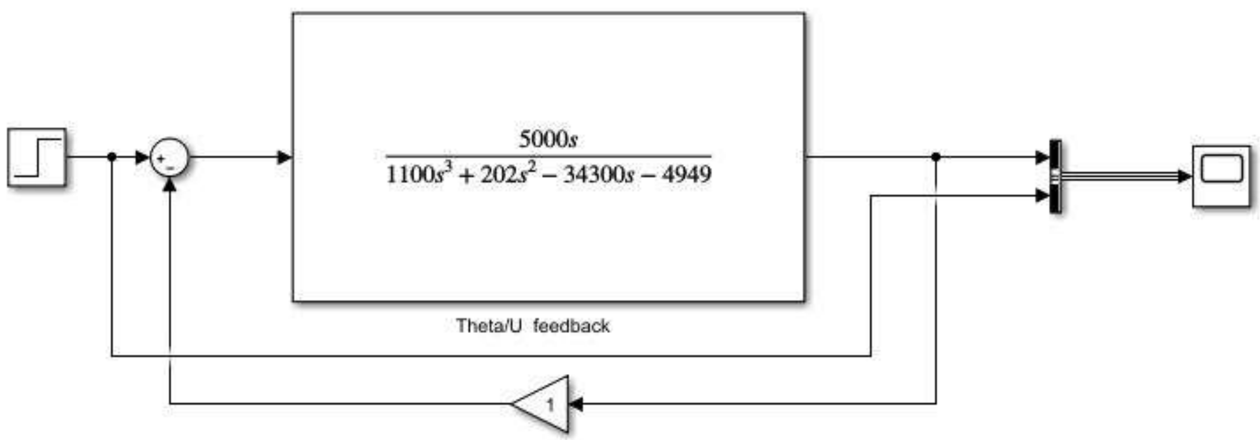

Figure 7- Close-loop block diagram of $\frac{\theta}{U}$ for step response

3-3-Step Response:

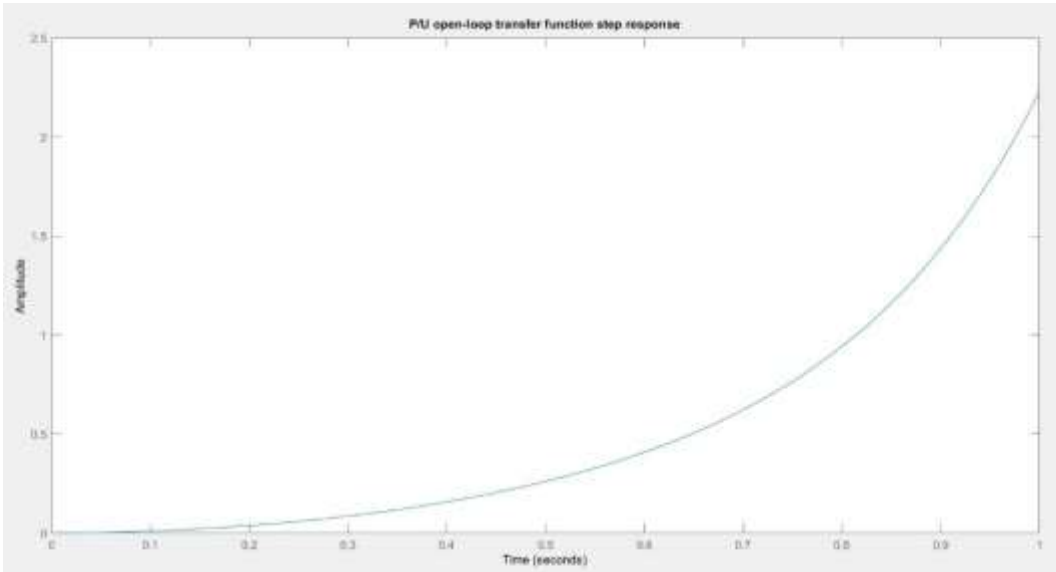

Figure 8- $\frac{P}{U}$ open-loop step response

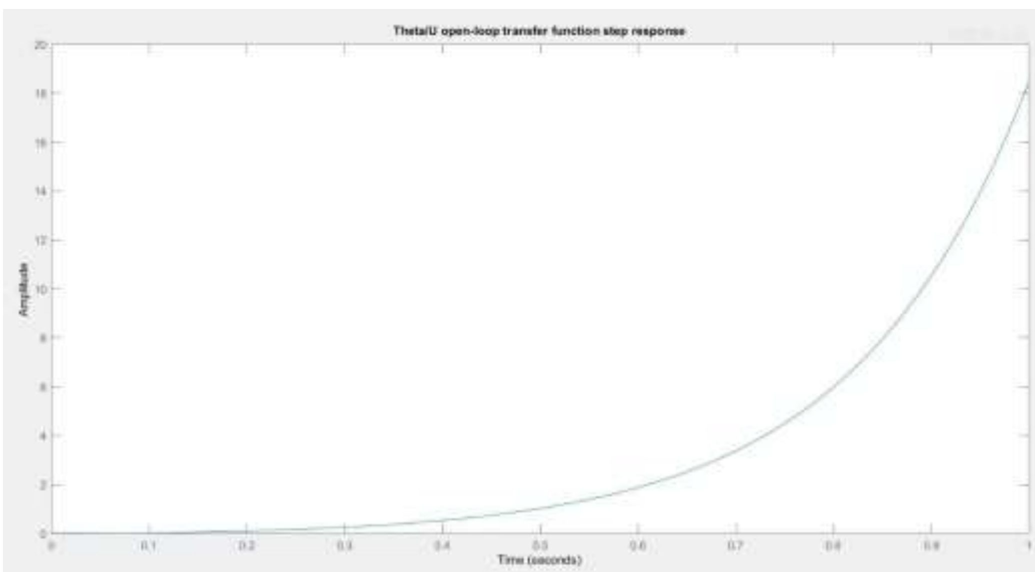

Figure 9-$\frac{\theta}{U}$ open-loop step response

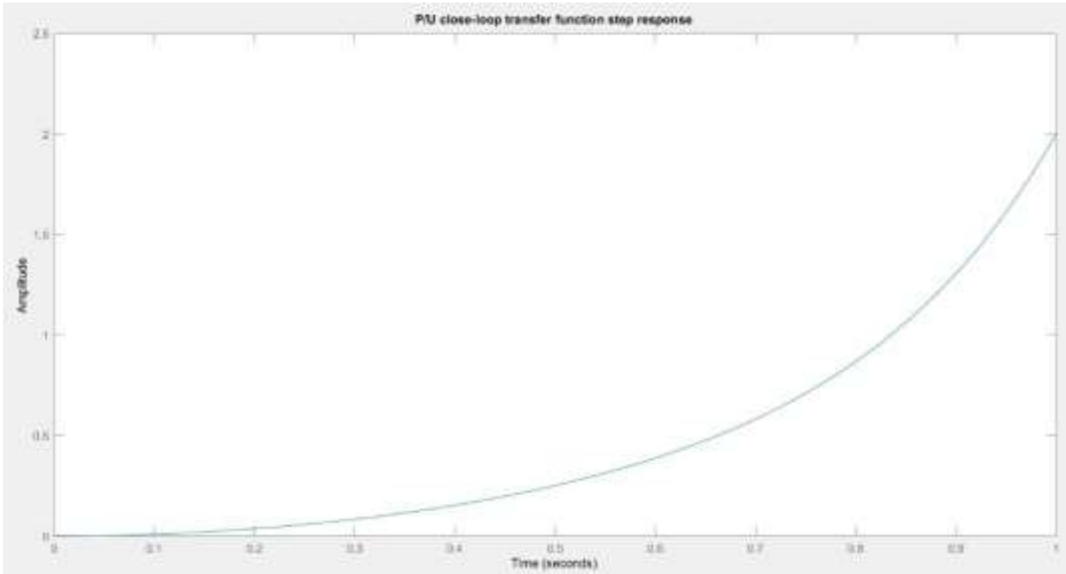

Figure 20- $\frac{P}{U}$ close-loop step response

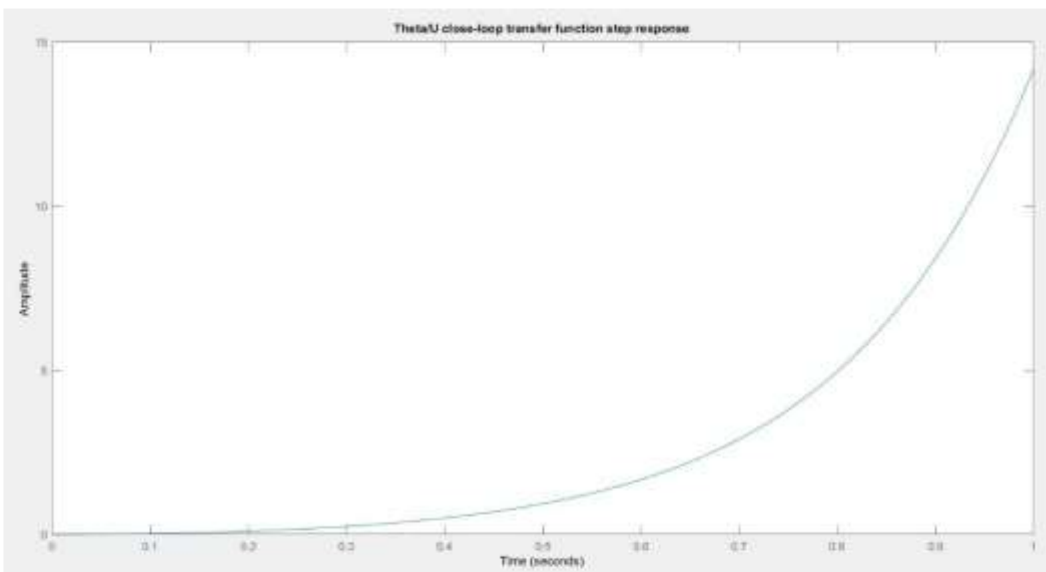

Figure 11-$\frac{\theta}{U}$ close-loop step response

3-4-Impulse response:

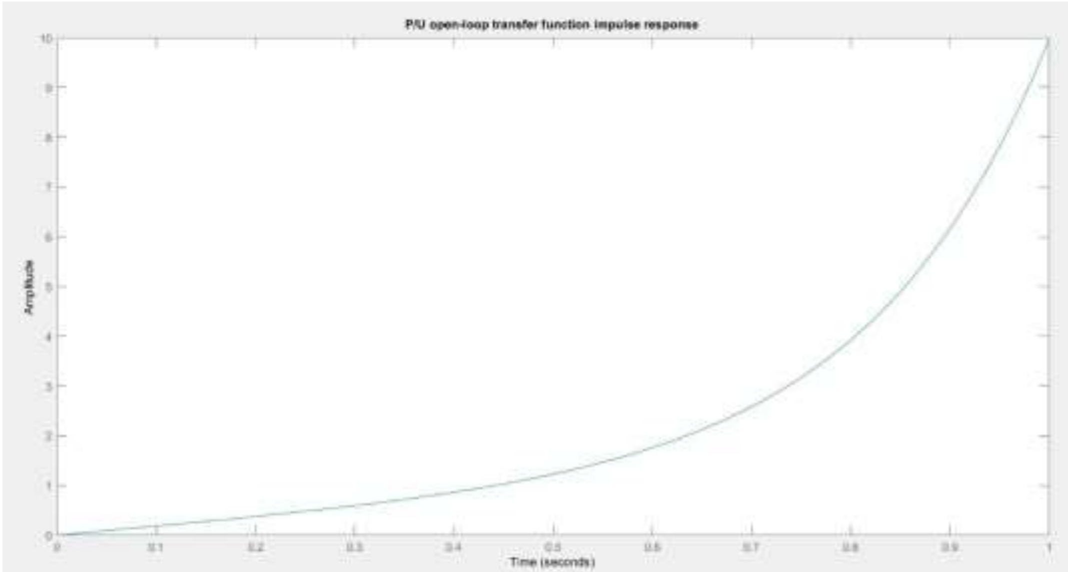

Figure 12- $\frac{P}{U}$ open-loop impulse response

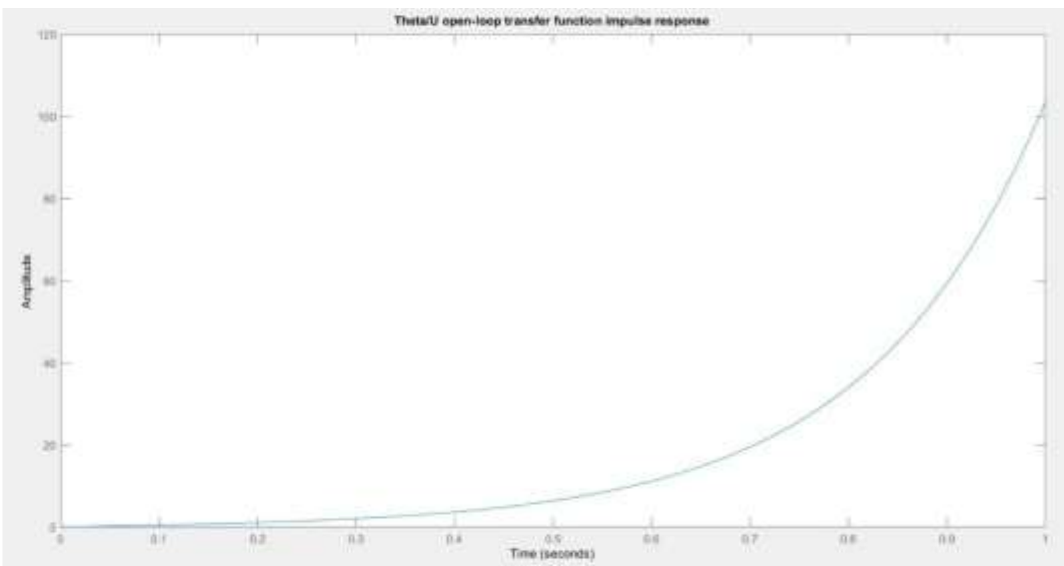

Figure 13--$\frac{\theta}{U}$ open-loop impulse response

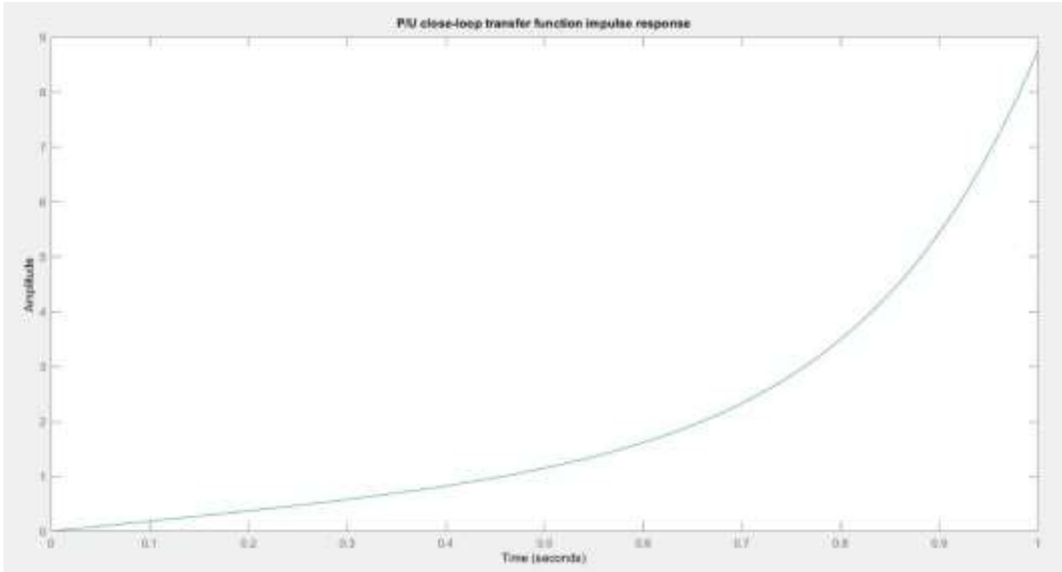

Figure 14-$\frac{P}{U}$ close-loop impulse response

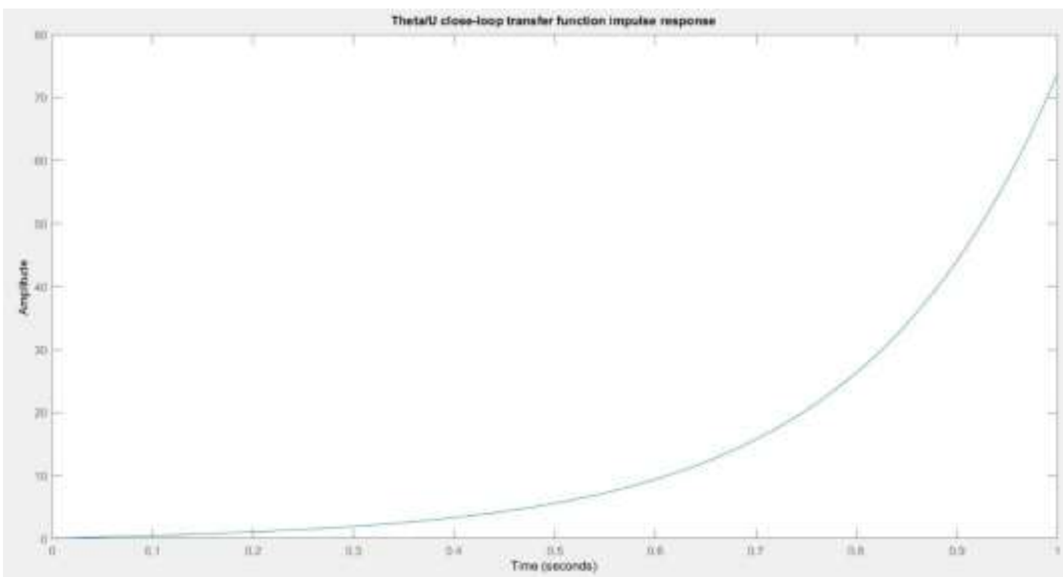

Figure 15--$\frac{\theta}{U}$ close-loop impulse response

4-Analysis

As we could see from the R-locus, the system is not stable, so we have to find an appropriate controller. In order to achieve that, we must check if our transfer function is controllable. Also, for the estimator, we must check if our system is observable.

C_x is $[B \quad AB \quad A^2B \quad A^3B]=\begin{bmatrix} 0 & -5.55 & -1.8315 & 210.85 \\ -5.55 & -1.8315 & 210.850 & 169.27 \\ 0 & -3.33 & -1.0989 & 180.4 \\ -3.33 & -1.098 & 180.9 & 119.51 \end{bmatrix}$

$$O_x \text{ is} \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -38.1 & 0 & 0 & 0.55 \\ -32.66 & 0 & 0 & 0.33 \\ -17.963 & -38.1 & 0 & 0.18 \\ -10.778 & -32.66 & 0 & 0.1089 \end{bmatrix}$$

Both matrixes are full rank (rank=4) therefore the overall system is controllable and observable. To obtain controllable form we must extract co-efficient of nominator and denominator of transfer function in eq. (17) in standard form.

$$\frac{4.54S}{S^3+0.18S^2-31.18S-4.99}=\frac{\Theta}{U} \tag{24}$$

Controllable form:

$$\begin{bmatrix} X_1{}' \\ X_2{}' \\ X_3{}' \end{bmatrix} = \begin{bmatrix} -0.18 & 31.18 & 4.49 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U(t) \tag{25}$$

$$Y(t) = [0 \quad 4.54 \quad 0] . \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

And for observable form we have:

$$\begin{bmatrix} X_1{}' \\ X_2{}' \\ X_3{}' \end{bmatrix} = \begin{bmatrix} -0.18 & 1 & 0 \\ 31.18 & 0 & 1 \\ 4.49 & 0 & 0 \end{bmatrix} . \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4.54 \\ 0 \end{bmatrix} U(t) \tag{26}$$

$$Y(t) = [1 \quad 0 \quad 0] . \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

4-1-Control methods

As it is shown in the Figures on the last section, the inverted pendulum system is unstable. Therefore, we need to stabilize this system in order to control it. To address this problem, three methods is suggested which are PID tuning, Pole placement method and Full state feedback method. According to the original paper, the desired outcome would be:

- Settling should be less than 5 seconds.
- Pendulum angle θ should not exceed 0.05 radians from the vertical axis.

In order to achieve these requirements, we could use equations (27) and (28) to obtain the dominant poles:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.05 \tag{27}$$

$$\frac{4}{\zeta\omega_n} = 5 \tag{28}$$

So, damping ratio and natural frequency of dominant poles are:

$$s^2 + 2\zeta\omega_n + \omega_n^2 = 0 \tag{29}$$

$$\zeta = 0.69 \tag{30}$$

$$\omega_n = 1.16 \tag{31}$$

So, using equation 29, our dominant poles are:

$$s = -0.8 \pm \ 0.84\, j \tag{32}$$

1. PID tuning

To stabilize the system, a PID controller with Ziegler-Nichols method is designed in article.
As it is shown in figure 16, in order to achieve a desirable result, PID controller is located in feedback path in this article.

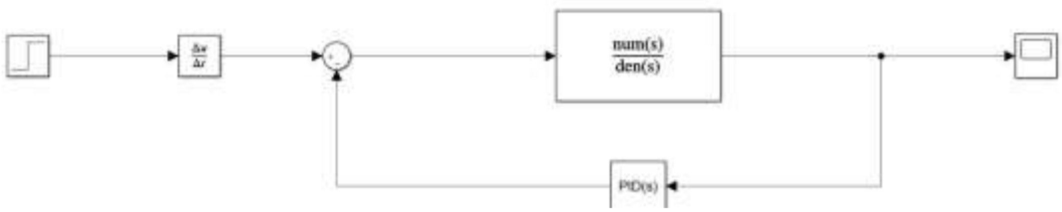

Figure 16- PID system block diagram

The PID controller coefficients are 100, 1, and 20 respectively. The impulse response of the stabilized pendulum system is shown if figure (17):

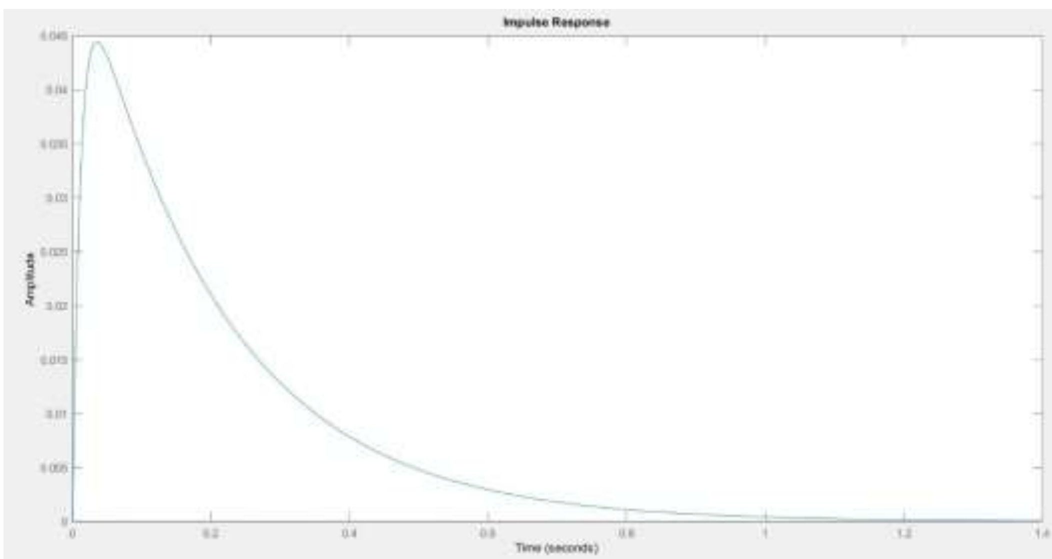

Figure 17-Impulse response for PID with 100, 1, 20 Coefficients

As it can be seen with these coefficients, settling time is much less than problem’s requirements, so we can achieve these requirements with less PID coefficients. Our suggested PID coefficients are 25, 1, 20 respectively and the impulse response of the system with the new PID is shown in figure (18):

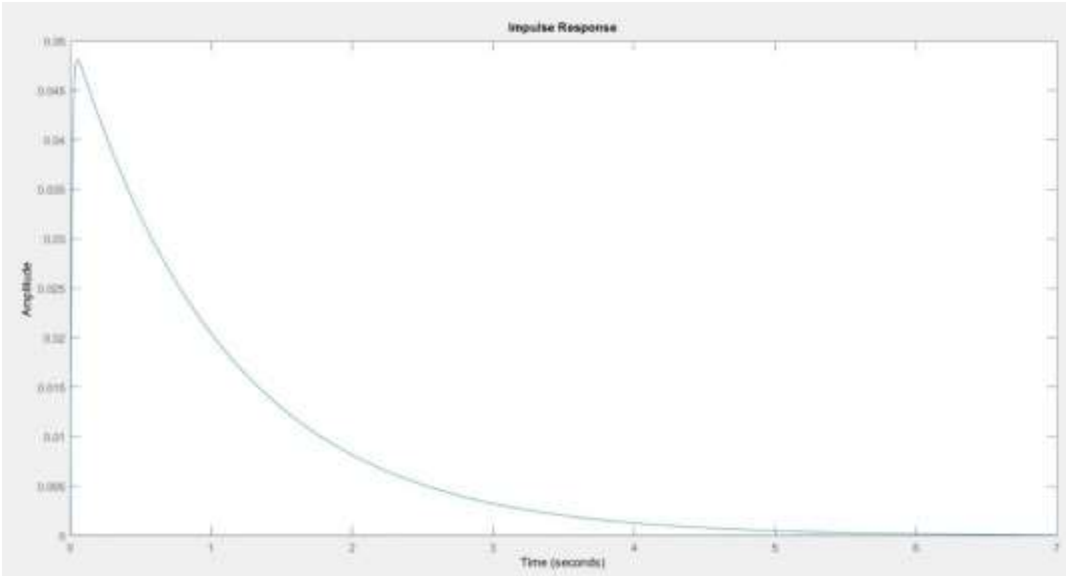

Figure 18-Impulse response for PID with 25, 1, 20 Coefficients

As it is shown in figure (18), our PID meets the 5 seconds of settling time and 0.05 rad of maximum pendulum's angle.

2. Pole placement

In this method, poles are simply placed into desired locations. From equation (33) we could get the overall number of poles to satisfy Diophantine equation.

$$n = 3 \rightarrow \ m = n - 1 = 2$$

In order to get a unique answer from Diophantine equation, the degree D_0 should be n + m = 5. We previously calculated the dominant poles on equation (32), the other three poles can be placed farther negative in X-Y axes to get a faster response. $s = \ -3 \pm 4j$, and $s = \ -5$ are chosen for other desired poles, therefore D_0 would be:

$$D_0(s) = (s + 0.8 + 0.84j)(s + 0.8 - 0.84j)(s + 3 + 4j)(s + 3 - 4j)(s + 5) \qquad (33)$$

$$D_0(s) = s^5 + 14.6s^4 + 87.1s^3 + 246.5s^2 + 287.4s + 168.2 \qquad (34)$$

$$S_m = \begin{vmatrix} -4949 & 0 & 0 & 0 & 0 & 0 \\ -34300 & 5000 & -4949 & 0 & 0 & 0 \\ 202 & 0 & -34300 & 5000 & -4949 & 0 \\ 1100 & 0 & 202 & 0 & -34300 & 5000 \\ 0 & 0 & 1100 & 0 & 202 & 0 \\ 0 & 0 & 0 & 0 & 1100 & 0 \end{vmatrix}$$

$$F = [168.2 \quad 287.4 \quad 246.5 \quad 87.1 \quad 14.6 \quad 1]'$$

S_m and F are the inputs of these equation. From the equation below, we could extract the overall controller of system.

$$C = Sm^{-1}F \tag{35}$$

$$C = \frac{0.031\, S^2 + 0.141\, s - 0.162}{0.0009\, s^2 + 0.013\, s - 0.034} \tag{36}$$

The figure (19) shows the impulse response of the system with Pole placement method.

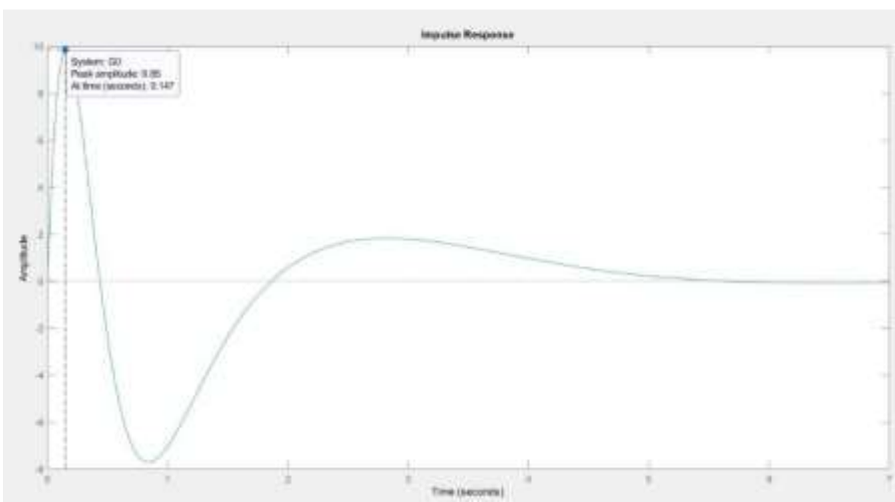

Figure 19- Impulse response of stabilized system with Pole Placement

We modeled the system with unit feedback as can be seen, although the model is stable and damped over time, it does not meet the problem criteria. In order to reduce the peak, we use gain $\frac{0.05}{9.85}$ as a pre-compensator to our system to meet the $\theta_{max} = 0.05$.

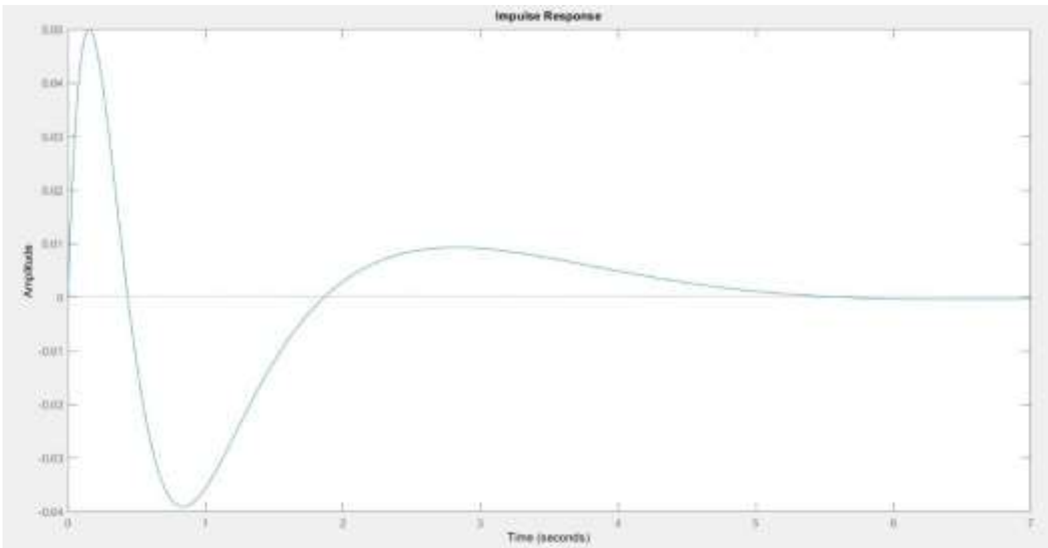

Figure 20- Impulse response of stabilized system with Pole Placement and pre compensator added

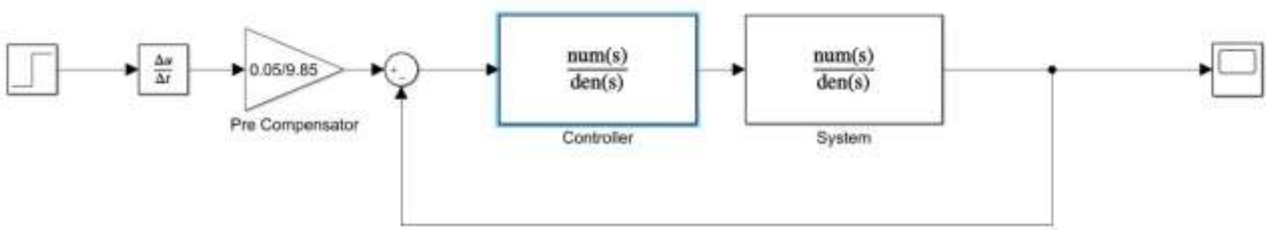

Figure 21- Pole placement system block diagram

3.1 Full state feedback

The first two Eigen values are the dominant poles in equation (32). We placed the third pole on the left side of the dominant poles in order to get faster response.

Eigen values = [-0.8± 0.84 j, -5]

As we derived from controllable matrix, Eigen values and Ackermann function, we can achieve K matrix:

$$K = [6.41 \quad 40.52 \quad 11.22] \tag{37}$$

We can obtain the overall transfer function of the system by using equation (38).

$$G_0(s) = c(sI - A + bk)^{-1}b \tag{38}$$

The Impulse response of this system is shown in figure (21):

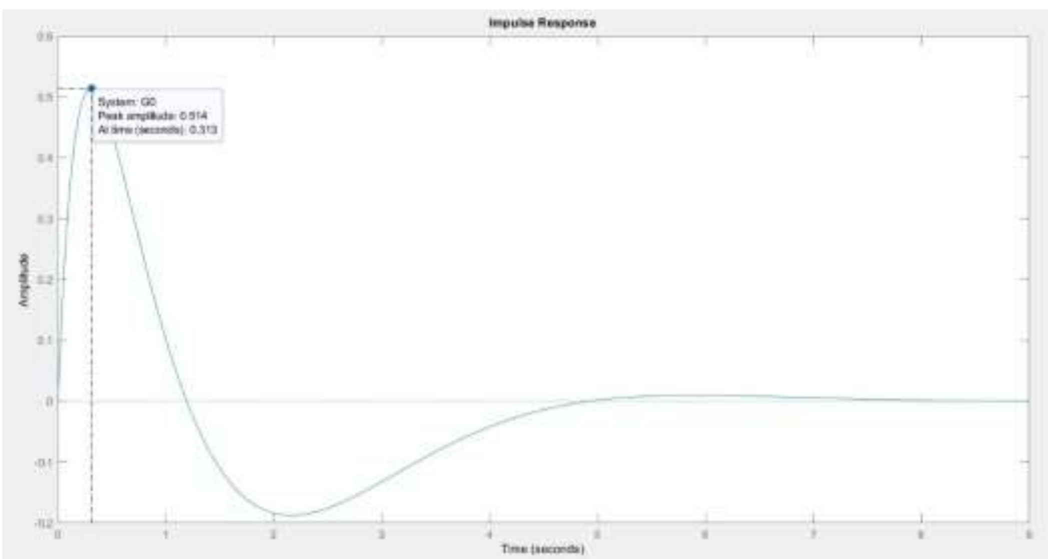

Figure 21- Impulse response full state feedback

We modeled the system with unit feedback as can be seen, although the model is stable and damped over time, it does not meet the problem criteria. In order to reduce the peak, we use gain $\frac{0.05}{0.514}$ as a pre-compensator to our system to meet the $\theta_{max} = 0.05$.

So the impulse response is shown in figure (22):

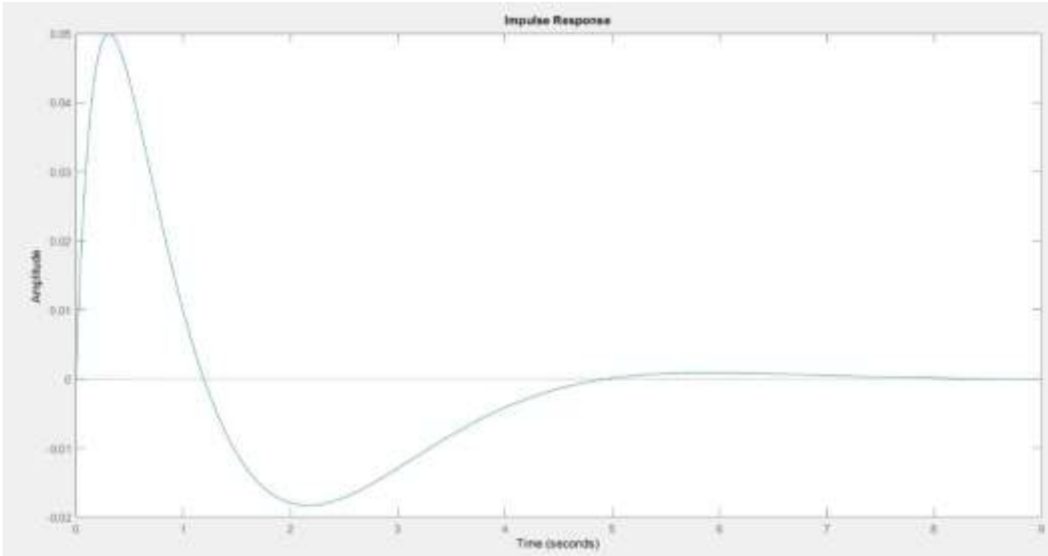

Figure 22-Impulse response full state feedback with pre compensator

3.2 Full state feedback with estimator.

In the real world most of the full state feedback come with an estimator. Estimators reduce cost of implementation of the whole plant. In this problem, we could use estimators because our system is observable. In general, to find the observer matrix (l) we have to multiply Eigen values by two or three times, so our Eigen values become s = 3× (-0.8 ± 0.84 j, -5).

$$l = [24.57 \quad 4.31 \quad 9.10]^T \tag{39}$$

To obtain estimator transfer function we could use equation (40):

$$G_{es}(s) = k(sI - A + bk + lc)^{-1} l \tag{40}$$

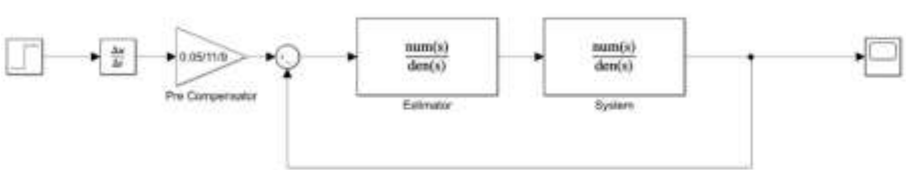

Figure 23-Full state feedback system with estimator block diagram

To calculate the overall transfer function, we use equation(41):

$$G_0(s) = \frac{G_{es} * G}{1 + G_{es}G} \tag{41}$$

So the impulse response is shown in figure (23):

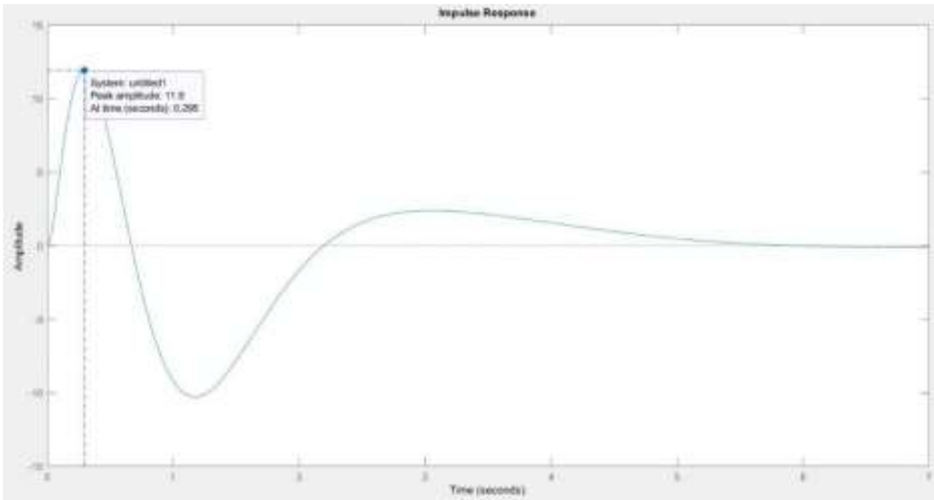

Figure 24-Impulse response full state feedback with estimator

We modeled the system with unit feedback as can be seen, although the model is stable and damped over time, it does not meet the problem criteria. In order to reduce the peak, we use gain $\frac{0.05}{11.9}$ as a pre-compensator to our system to meet the $\theta_{max} = 0.05$.

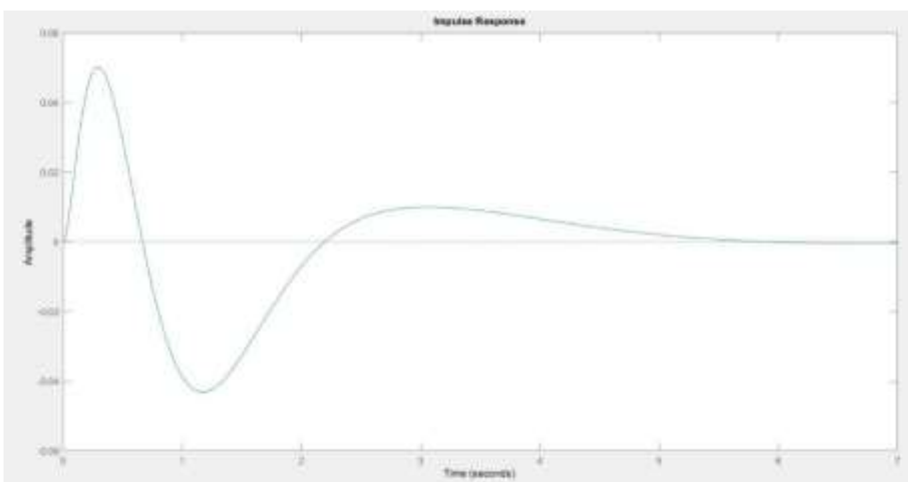

Figure 25-Impulse response full state feedback with estimator and pre compensator

5-Comparison:

We tuned the PID controller of the project with lower coefficients than the main paper's PID, therefore the total cost of the system was reduced due to this matter. As it is shown in fig (26), our PID controller's peak overshoot and settling time is higher than the main paper's PID controller but still meets the 5 seconds of settling time and 0.05 rad of maximum pendulum's angle with lower coefficients.

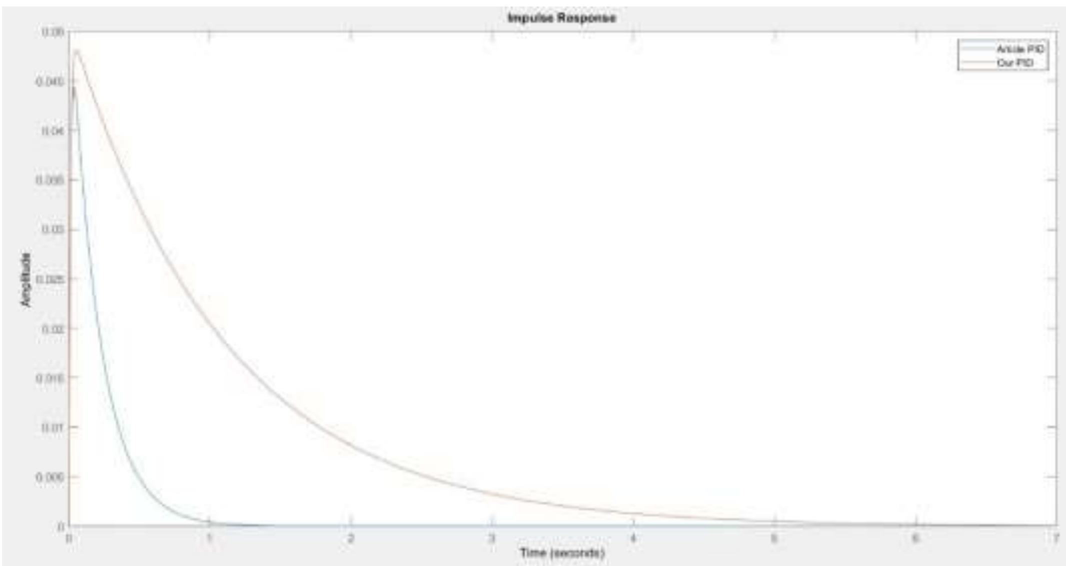

Figure 26- Comparison between our PID and article PID impulse response

In addition of PID tuning, modern control methods such as Pole placement and full state feedback (with Estimator) was used for the system and demonstrated the desired results. As it is shown in fig (27), all the other control methods meet the 5 seconds of settling time and 0.05 rad of maximum pendulum's angle as well.

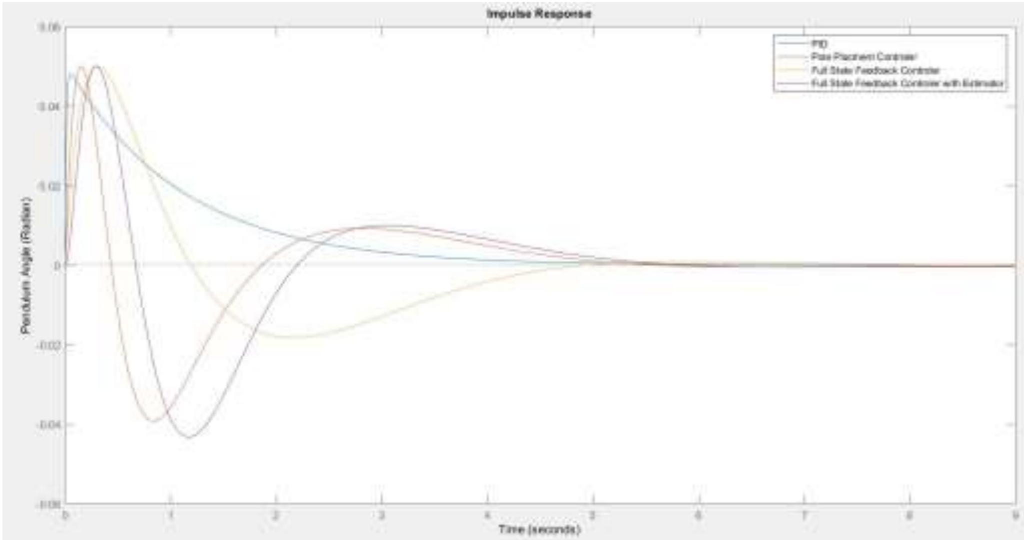

Figure 27-Comparison of the four control methods used for this project

According to fig (27), all control methods almost have an equal amount of peak overshoot and settling time but when it comes the rise time of PID controller is the lowest and full state feedback controller is highest of all controlling methods respectively.

Unlike the other methods, the PID controller does not have an undershoot peak while full state feedback controller with estimator has the highest undershoot.

6-Discussion

In general, it is preferred that the dynamical systems similar to inverted pendulum such as Segways, missiles etc. being controlled, perform optimally. Due to the coupling between the dynamics of the pendulum angle and cart position, any change in a controller parameter will have an impact on both the pendulum angle and cart position, making tuning difficult. There are many optimization and optimal control techniques but in this case, by using the trial-and-error process, the controller parameters in PID tuning such as K_P, K_I and K_D are adjusted until the SIMULINK model responses become optimal. [4] In our project the trial and error process for K_P, K_I and K_D is used until the outcome was reasonably optimal.

Model simplifications could give rise to errors or inaccuracies in some cases. In this project the friction force is modeled as a damper but in reality air resistance would be effective too. Also for the sake of simplicity the mass of the rod attached to the pendulum ball is neglected. At last, the initial conditions for Laplace transform calculation is assumed zero.

7-Conclusion

The goal of this project was to control an inverted pendulum. Our first step was deriving the mathematical modeling of the dynamic system. We obtained that the dynamical equations of the inverted pendulum is non-linear therefore linearization was necessary. It was established in the Root-Locus map that the system has a RHP pole that makes the system unstable. The instability of the system was visible in the step and impulse response as well.

In order to control and stabilize the system, PID tuning, Pole placement and full state feedback (with and without estimator) is suggested. The comparison of these methods has been done and it was obtained that all systems meets the desired requirements.

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Analyzing and Designing Control System for an Inverted Pendulum on a Cart

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Doi: 10.19044/esj.2018.v14n6p387 URL:http://dx.doi.org/10.19044/esj.2018.v14n6p387

Abstract

It is a collection of MATLAB functions and scripts, and SIMULINK models, useful for analyzing Inverted Pendulum System and designing Control System for it. Automatic control is a growing field of study in the field of Mechanical Engineering. This covers the proportional, integral and derivative (PID). The principal reason for its popularity is its nonlinear and unstable control. The reports begin with an outline of research into inverted pendulum design system and along with mathematical model formation. This will present introduction and review of the system. Here one dimensional inverted pendulum is analyzed for simulating in MATLAB environment. Control of Inverted Pendulum is a Control Engineering project based on the flight simulation of rocket or missile during the initial stages of flight. The aim of this study is to stabilize the Inverted Pendulum such that the position of the carriage on the track is controlled quickly and accurately so that the pendulum is always erected in its inverted position during such movements.

Keywords: MATLAB, Inverted pendulum, PID Controller, Simulation

Introduction

An inverted pendulum is a pendulum which has its center of mass above its pivot point (Said,L., Latifa, B.,, 2012). It is often implemented with the pivot point mounted on a cart that can move horizontally and may be called a cart and pole. Most applications limit the pendulum to 1 degree of freedom by affixing the pole to an axis of rotation. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright; this can be done either by applying a torque at the pivot point, by moving the pivot point horizontally as part of a feedback system, changing the rate of rotation of a mass mounted on the pendulum on an axis parallel to the pivot axis and thereby generating a net torque on the pendulum, or by oscillating the pivot point vertically. A simple demonstration of moving the pivot point in a feedback system is achieved by balancing an upturned broomstick on the end of one's finger. The inverted pendulum is a classic problem in dynamics and control theory and is used as a benchmark for testing control strategies. Can anyone balance a ruler upright on the palm of his hand? If he concentrates, he can just barely manage it by constantly reacting to the small wobbles of the ruler (Irza M. A., Mahboob I., Hussain C.,, 2001). This challenge is analogous to a classic problem in the field of control systems design: stabilizing an upside-down ("inverted") pendulum.

Simulation is the imitation of the operation of a real-world process or system over time (Banks J., Carson J., Nelson B., Nicaol D., 2001). The act of simulating something first requires that a model be developed; this model represents the key characteristics or behaviors/functions of the selected physical or abstract system or process. The model represents the system itself, whereas the simulation represents the operation of the system over time. The inverted pendulum is among the most difficult systems to control in the field of control engineering. Due to its importance in the field of control engineering, it has been a task of choice to be assigned to control engineering students to analyze its model and propose a linear compensator according to the control law. Being an unstable system, it creates a problem in case of controlling (O., 2012).The reasons for selecting the Inverted Pendulum as the system are:

- It is the most easily available system
- It is a nonlinear system, which can be treated to be linear, without much error (Maravall D., Zhou C., Alonso J., 2005).
- Provides good practice for prospective control engineering.

Theory

The system involves cart, able to move backwards and forwards. And a pendulum hinged to the cart at the bottom of its length such that the pendulum can move in the plane as the cart moves. That is, the pendulum mounted on the cart is free to fall along the cart's axis of rotation. The system is to be controlled so that the pendulum remains balanced and upright. If the pendulum starts off-center, it will begin to fall. The pendulum will move to opposite direction of the cart movement. It is a complicated control system because any change to a part will cause change to another part. We only take feedback from the angle of the pendulum relative to vertical axis other than state of being carriage position, carriage velocity and pendulum angular velocity. The cart undergoes linear translation and the link is unstable at the inverted position. So, briefly the inverted pendulum is made up of a cart and a pendulum. The goal of the controller is to move the cart to its commanded position causing the pendulum without tip over. In open loop the system is unstable. This is a SIMO output system.

Basic block diagram for the feedback control system:

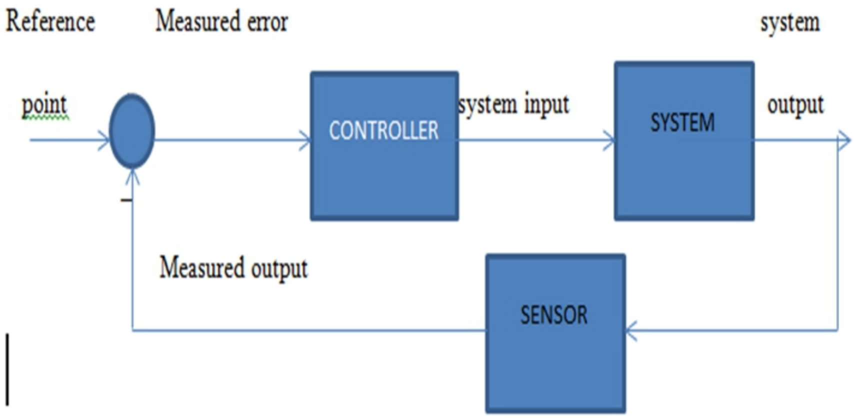


Figure-01: Feedback control system

Analysis Of Inverted Pendulum With Cart System

The inverted pendulum on a cart is representative of a class of system that includes stabilization of a rocket during launch. The position of the cart is P, the angle of rod is θ, the force input to the cart is F, the cart mass is M, the mass of the bob is m, the length of the rod is L, the coordinate of the bob is (P_2, Z_2).

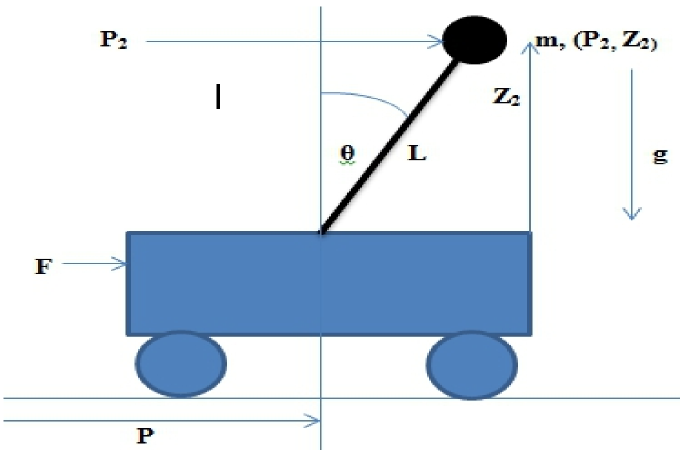


Figure-02: Inverted pendulum on a cart

In the following, the differential equations which describe the dynamics of the inverted pendulum using lagrangian's equation because these deal with the scalar energy functions rather than vector forces and acceleration in case of Newtonian approach, thus it minimizes error (Ogata, 2002).
The partial differential equations (F., 1994) yield:

$(M+m)^* \ddot{P} + m^*L^* \ddot{\theta}^*\cos\theta - m^* L ^*\dot{\theta}^2 {}^*\sin\theta = F$ ………………(1)
$M^*L^* \ddot{P}{}_* \cos\theta + (I + m^*L^2)\ {}^*\ddot{\theta} + m^*g^*L^* \sin\theta = 0$ ……......…… (2)
$(M + m)\ddot{p} - m L\ddot{\varnothing} = u$…………………….......................... (3)
$(I+mL^2)\ \ddot{\varnothing} + m L\ \ddot{p} = mg L\varnothing$………….……….......................(4)

If friction force is considered the equation converts to:

$(M + m)\ddot{p} - m L\ddot{\varnothing} + b\dot{p} = u$………………………………….(5)
$(I+mL^2)\ \ddot{\varnothing} - m L\ \ddot{p} = mg L\varnothing$……………………………….(6)

To obtain the transfer functions of the linearized system equations, the Laplace transform of the system equations assuming zero initial conditions has been taken. The resulting transfer function for pendulum position becomes:

$$P_{pend}(s) = \frac{\varnothing(s)}{u(s)} = \frac{\frac{mLs^2}{q}}{s^4 + \frac{b(I+mL^2)s^3}{q} - \frac{(M+m)mgls^2}{q} - \frac{bmgLs}{q}} \text{ (rad/N)………………(7),}$$

where, $q=[(M+m)(I+mL^2)-(mL)^2]$

Again for transfer function for cart position as follow:

$$P_{cart}(s) = \frac{p\ (s)}{u(s)} = \frac{(I+ml^2)s^2 - gml}{s^4 + \frac{b(I+mL^2)s^3}{q} - \frac{(M+m)mgls^2}{q} - \frac{bmgLs}{q}} \text{ (m/N)……… …….(8)}$$

For this example, assuming the following quantities:

Mass of the cart, (M) = 0.5 kg, Mass of the pendulum, (m) = 0.2 kg, Coefficient of friction for cart, (b) = 0.101N/m/sec,

Length to pendulum center of mass, (l)= 0.3 m , Mass moment of inertia of the pendulum, (I)= 0.006 kg.m^2, Force applied on the cart = F (N) , Cart position coordinate = x (m), Initial Pendulum angle from vertical downward = theta

For the PID, root locus, and frequency response sections of this problem, it will be interested only in the control of the pendulum's position. This is because the techniques used in these sections are best-suited for single-input, single-output (SISO) systems. Therefore, none of the design criteria deal with the cart's position. It will, however, be investigated the controller's effect on the cart's position after the controller has been designed. For these sections, the design of a controller to restore the pendulum to a vertically upward position after it has experienced an impulsive "bump" to the cart. Specifically, the design criteria are that the pendulum returns to its upright position within 5 seconds and that the pendulum never moves more than 0.05 radians away from vertical after being disturbed by an impulse of magnitude 1 Nsec. The pendulum will initially begin in the vertically upward equilibrium, $\theta = \pi$. In summary, the design requirements for this system are:

- Settling time for θ of less than 5 seconds
- Pendulum angle θ never more than 0.05 radians from the vertical

Pole Zero Map of Uncompensated Open Loop System: The poles position of the linearized model of Inverted Pendulum (in open loop configuration) shows that system is unstable, as one of the poles of the transfer function lies on the Right Half Side of the s-plane. Thus the system is absolutely unstable.

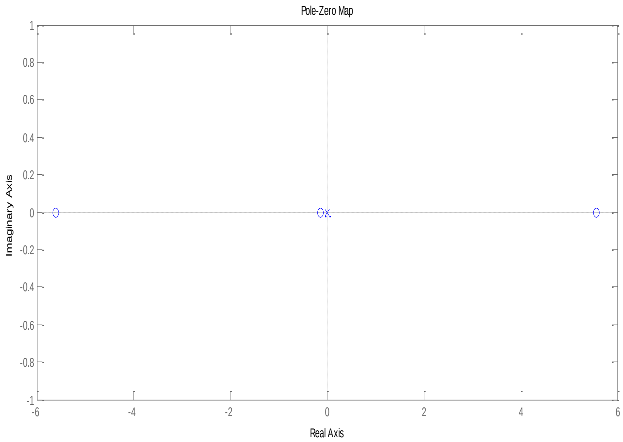


Figure-03: zeros and poles of pendulum position.

From figure,
Zeros =0
Poles =5.5651, -5.6041, -0.1428

Likewise, the zeros and poles of the system where the cart position is the output are found as follows:

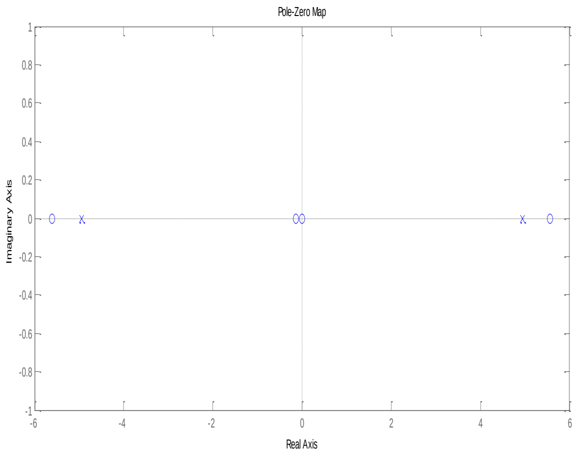


Figure-04: zeros and poles of cart position

The clear results are:
Zeros = 4.9497, -4.9497
Poles =0, 5.5651, -5.6041, -0.1428
As predicted, the poles for both transfer functions are identical. The pole at 5.5651 indicates that the system is unstable since the pole has positive real part (V., 1991). In other words, the pole is in the right half of the complex s-plane. This agrees with what we observed above.

Step Response of Uncompensated Open Loop System:

Since the system has a pole with positive real part its response to a step input will also grow unbounded. The verification of this using the “lsim” command which can be employed to simulate the response of LTI models to arbitrary inputs. In this case, a 1-Newton step input will be used. Adding the MATLAB code to “m-file” and running it in the MATLAB command window generates the plot given below:

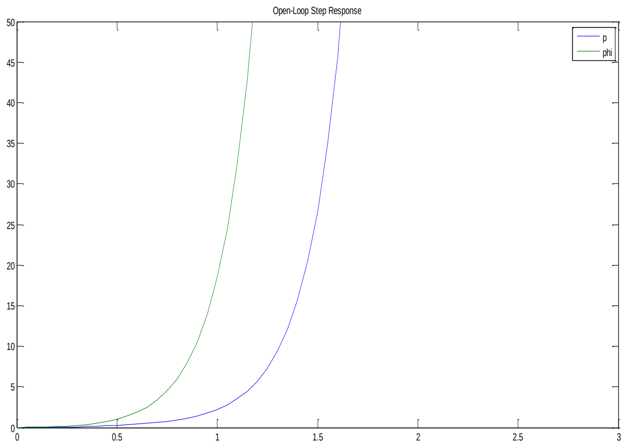


Figure-05: Step Response of Uncompensated Open Loop System

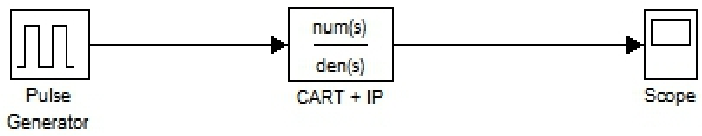
The above results confirm the expectation that the system's response to a step input is unstable.

It is apparent from the analysis above that some sort of control is needed to be designed to improve the response of the system. PID, root locus, frequency response, and state space are the controllers can be used but here PID controller is designed.

Simulink Model for the Open Loop Impulse Response of the Inverted Pendulum System

SIMULATION PARAMETERS:
Impulse is applied for 0.5 s
Start Time: 0

Stop Time: 1.5
Solver Algorithm: Variable-step ODE45 (Dormand-Prince), Maximum Step Size: 0.03



The impulse response of open-loop uncompensated system is given below:

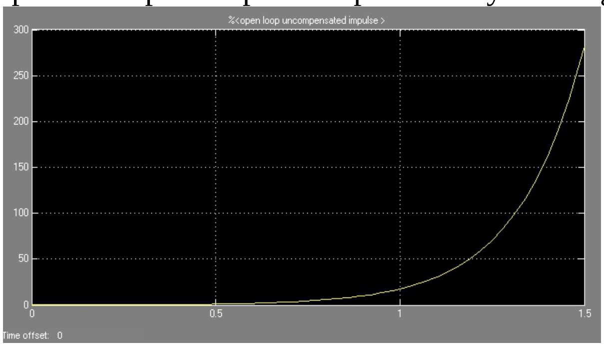


Figure-06: open loop impulse response (Scope view)

This model shows that impulse response of inverted pendulum. This model is highly unstable as theta diverges rapidly with time. Applying step for 1s reveals that the pendulum remains upright, but becomes highly unstable as step comes.

Simulink Modelling And Pid Controller

Nonlinear Simscape Model: SimMechanics software is a block diagram modeling environment for the engineering design and simulation of rigid body machines and their motions, using the standard Newtonian dynamics of forces and torques, instead of representing a mathematical model of the system (Said,L., Latifa, B.,, 2012). The inverted pendulum model using the physical modeling blocks of the Simscape extension to Simulink has been built. The blocks in the Simscape library represent actual physical components; therefore, complex multi-body dynamic models can be built without the need of mathematical equations from physical principles by applying Newton's laws. Establishing and saving SimMechanics model of the inverted pendulum and cart, the animated view of the physical system is created which is given below:

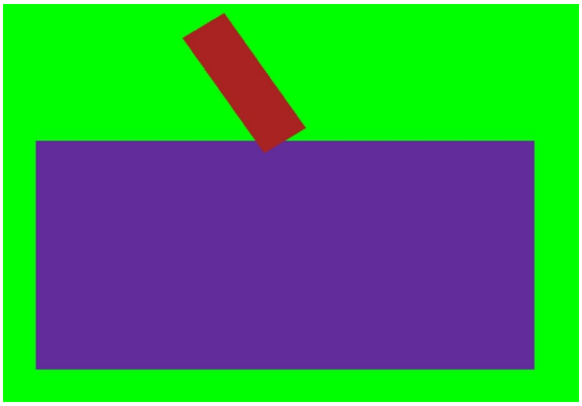


Figure-07: System animation without controller (20 degree displace with vertical).

In the Scope, clicking the **Autoscale** button, the following output for the pendulum angle and the cart position has been found which is nonlinear in practice.

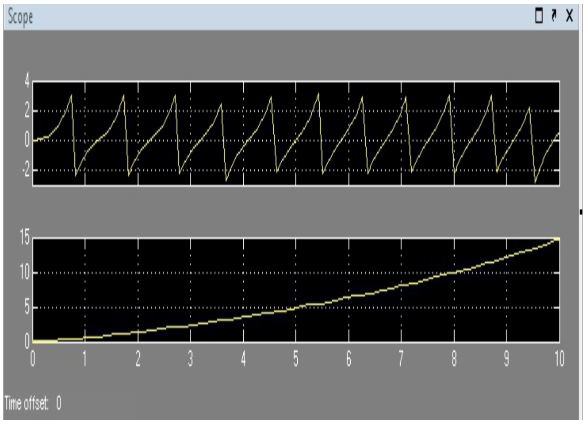


Figure-08: Scope condition of pendulum angle and cart position.

The pendulum repeatedly swings through full revolutions where the angle rolls over 360 degrees. Furthermore, the cart's position grows unbounded, but oscillates under the influence of the swinging pendulum.

PID control design: In the design process, it has been assumed a single-input, single-output plant as described by the transfer function (Kumar R., Singh B., Das J., 2013). Closed loop impulse with PID controller is given below:

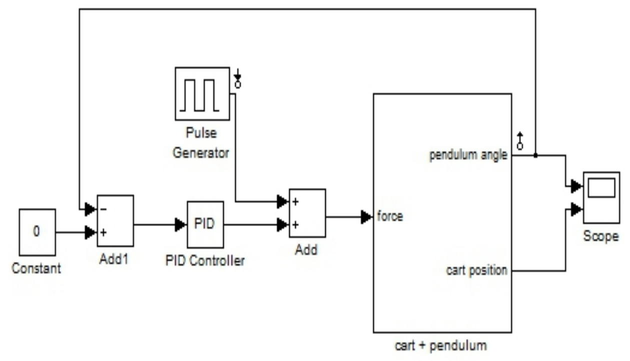


Figure-09: Feedback control system for the Iverted Pendulum

To design a compensator using the automated PID Ziegler-Nichols open-loop tuning algorithm, this tuning method computes the proportional, integral, and derivative gains using the Chien-Hrones-Resnick (CHR) setting with a 20% overshoot. The response of the closed-loop system to an impulse disturbance for this initial set of control gains: Kp = 1; Ki = 1;Kd = 1;

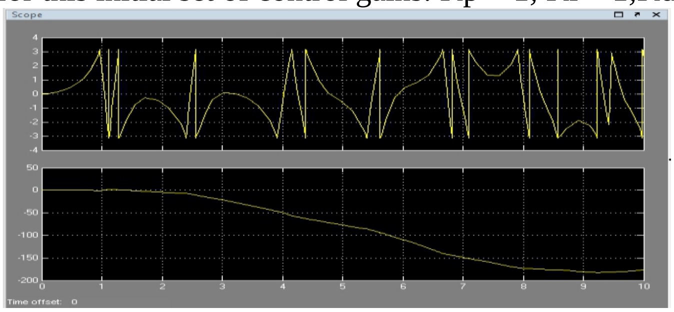


Figure-10: Zigzag movement of pendulum K_p=1;K_i=1;K_d =1

This response is still not stable. To modify the response, an iteration process is followed by manipulating proportional, integral and derivative gain.

Results

The Inverted Pendulum was given an initial angle inclination, as indicated by an initial 20 magnitude of pendulum's angular displacement. the system is completely controlled under operating condition. The design criteria of the PID controller are: K_p=100; K_i=1; K_d=20; Increasing amplitude of impulse, it is seen error increases further. As is shown in the plot, the settling time of the system is less than 5 seconds. Impulse response, scope view and final animation are given below:

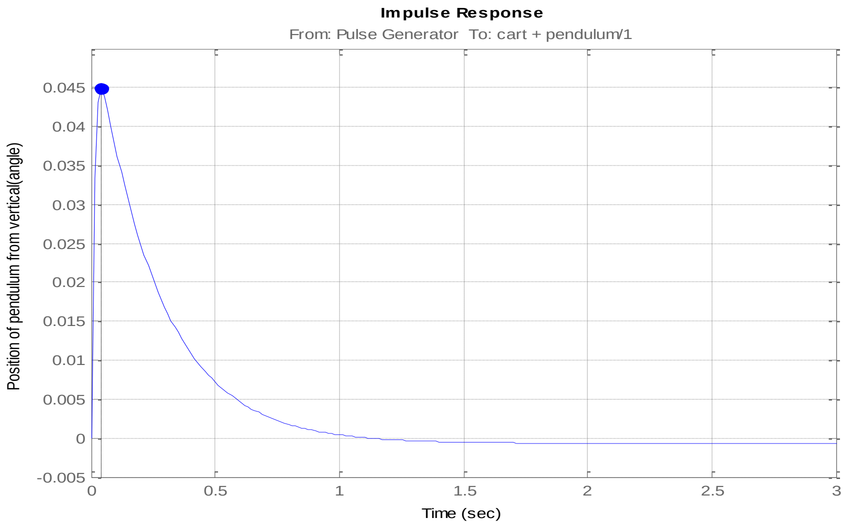


Figure-11: Final simulation result (LTI view).

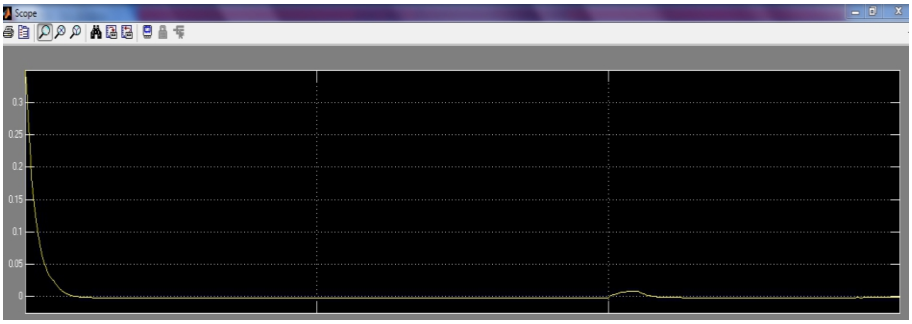


Figure-12: Actual movement of pendulum (scope)

A video of system simulation is extracted using recording options. the cart moves in the negative direction with approximately constant velocity.

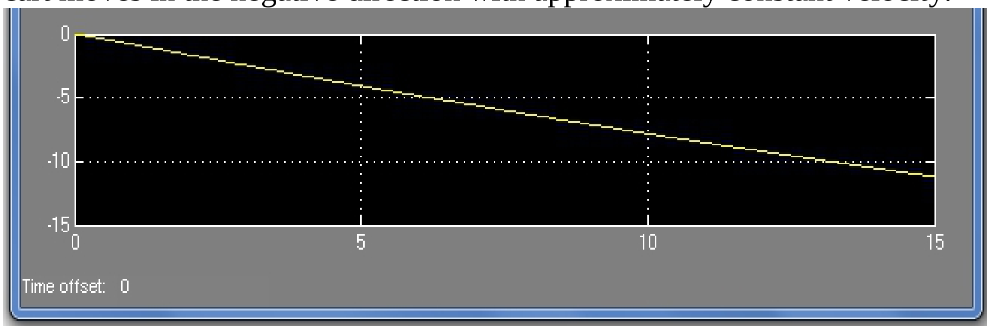


Figure-13: cart position with time.

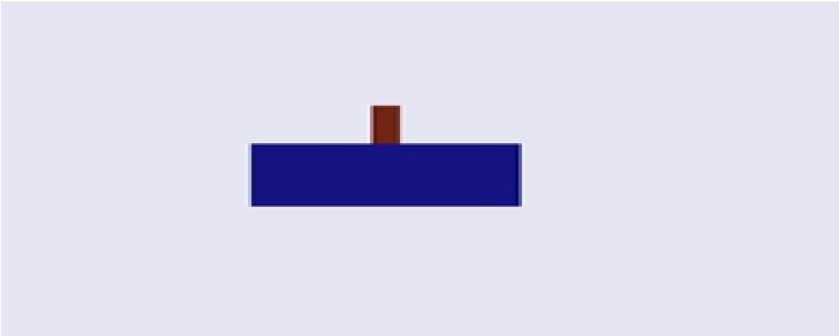


Figure-14: Final animation of model of inverted pendulum and cart.

Conclusion

From the analysis, it creates an ability of the control of a nonlinear model by any linear feedback control system. PID controller designed here is followed an iteration process. In the project only friction force is assumed as external impedance but in reality there would have air impedance. The cart velocity decreases in the negative direction. The actuator needs a very small effort and power to stable the pendulum as it quickly stabilizes without too much fluctuations. In the project a unit feedback gain is considered as it becomes simple and avoiding so much complexity of calculation. System properties are taken reasonable as the simulation is completed in SimMechanics model by breaking mechanical elements into building blocks. Control System Toolbox provides an app and functions for analyzing linear models. Impulse response plot is used and settling time, peak amplitude or maximum overshot are defined using linearized tools. Applying controller to cart position, the system would be implemented. Root locus method has been used to define that the system is unstable. ODE45 is based on an explicit Runge-Kutta (4, 5) formula, the Dormand-Prince pair. It is a one-step solver; that is, in computing $y(t_n)$, it needs only the solution at the immediately preceding time point, $y(t_{n-1})$. In general, ode45 is the best solver to apply as a first try for most problems. For this reason, ode45 is the default solver used for models with continuous states and been used for this problem.

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