T.S

* $\mu_t = E[Y_t]$
* $\gamma_{t,s} = Cov(Y_t, Y_s) = E[Y_t Y_s] - \mu_t \mu_s$
* $\rho_{t,s} = Cor(Y_t, Y_s) = \dfrac{\gamma_{t,s}}{\sqrt{\gamma_{t,t} \cdot \gamma_{s,s}}}$

Random Walk ← Not stationary

* $e_1, e_2, e_3, \ldots$ i.i.d with mean $= 0$, variance $= \sigma_e^2$.
* $Y_t = Y_{t-1} + e_t$
* Mean $= \boxed{0}$
* $Var(Y_t) = \boxed{t\,\sigma_e^2}$
* $\gamma_{t,s} = Cov(Y_t, Y_s) = \boxed{t\,\sigma_e^2}$
* $\rho_{t,s} = \dfrac{t\,\sigma_e^2}{\sqrt{t\,\sigma_e^2 \cdot s\,\sigma_e^2}} = \boxed{\sqrt{\dfrac{t}{s}}}$

A Moving Average ← stationary

* $e_1, e_2, e_3, \ldots$ i.i.d with mean $= 0$, variance $= \sigma_e^2$
* $Y_t = \dfrac{e_t + e_{t-1}}{2}$
* Mean $= \boxed{0}$
* $Var(Y_t) = \boxed{0.5\,\sigma_e^2}$
* $\gamma_{t,s} = Cov\left(\dfrac{e_t + e_{t-1}}{2}, \dfrac{e_s + e_{s-1}}{2}\right)$
* $\rho_{t,s} = \dfrac{\gamma_{t,s}}{\sqrt{\gamma_{t,t} \cdot \gamma_{s,s}}}$

$$\boxed{\begin{array}{l} \gamma_{t,s} = \begin{cases} 0.5\,\sigma_e^2, & |t-s| = 0 \\ 0.25\,\sigma_e^2, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases} \\ \rho_{t,s} = \begin{cases} 1, & |t-s| = 0 \\ 0.5, & |t-s| = 1 \\ 0, & |t-s| > 1 \end{cases} \end{array}}$$

Note :-

1) $Cov(X, a) = 0$
2) $Cov(X, X) = Var(X)$
3) $Cov(X, Y) = Cov(Y, X)$
4) $Cov(aX, bY) = ab\, Cov(X, Y)$
5) $Cov(X+c, Y+d) = Cov(X, Y)$
6) $Cov(aX+c, bY+d) = ab\, Cov(X, Y)$
7) $Var(aX+b) = a^2\, Var(X)$

Stationarity

* Process / series do not change over time
  i) Mean function is constant over time
  ii) $\gamma_{t,\,t+k} = \gamma_{0,\,k}$ for all time t and lag k

$\boxed{Y_t = \mu_t + X_t}$ ← Trend Analysis → Deterministic trend / Stochastic trend

i) Estimation of a constant Mean

$Y_t = \mu + X_t$ $\qquad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$

ii) Linear and Quadratic trends in time

$\mu_t = \beta_0 + \beta_1 t$

* $Q(\beta_0, \beta_1) = \sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$

$$\boxed{\hat{\beta}_1 = \frac{\sum_{t=1}^{n} (Y_t - \bar{Y})(t - \bar{t})}{\sum_{t=1}^{n} (t - \bar{t})^2}} \quad , \quad \boxed{\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \cdot \bar{t}}$$

$\bar{t} = \frac{(n+1)}{2}$

R^2 = Coefficient of determination.

$\varepsilon_t \overset{i.i.d}{\sim} N(0, \sigma^2)$

* Disadvantage of trend analysis: Ignoring Correlation.

Models for Stationary Time series

1.1. Moving Average Processes

* Moving average of order q ⇒

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \dots - \theta_q e_{t-q}$$

* First order moving average process :-

$$Y_t = e_t - \theta \cdot e_{t-1} \; ; \; E[Y_t] = \boxed{0}, \; Var(Y_t) = \boxed{\sigma_e^2 (1+\theta^2)}$$

$$\gamma_k = \gamma_{t,t-k} = Cov(Y_t, Y_{t-k}) = \begin{cases} -\theta \cdot \sigma_e^2 & ; k=1 \\ 0 & ; k \geq 2 \end{cases}$$

$$\rho_k = \begin{cases} \dfrac{-\theta}{1+\theta^2} & ; k=1 \\ 0 & ; k \geq 2 \end{cases}$$

* Second order moving average process :-

$$Y_t = e_t - \theta_1 \cdot e_{t-1} - \theta_2 \cdot e_{t-2} \; ;$$

$$E[Y_t] = 0,$$

$$Var[Y_t] = (1+\theta_1^2+\theta_2^2) \cdot \sigma_e^2 = \gamma_0,$$

$$\gamma_1 = (-\theta_1 + \theta_1\theta_2) \cdot \sigma_e^2, \quad \gamma_2 = -\theta_2 \cdot \sigma_e^2.$$

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2} \qquad \rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

* MA(q) :- $Y_t = e_t - \theta_1 \cdot e_1 - \theta_2 \cdot e_2 - \dots - \theta_q \cdot e_{t-q}$

$$\boxed{\gamma_0} = Var[Y_t] = (1+\theta_1^2+\theta_2^2+\theta_3^2+\dots+\theta_q^2) \cdot \sigma_e^2$$

$$\boxed{\rho_k} = \begin{cases} \dfrac{-\theta_k + \theta_1 \cdot \theta_{k+1} + \theta_2 \cdot \theta_{k+2} + \dots + \theta_{q-k} \cdot \theta_q}{1+\theta_1^2+\theta_2^2+\theta_3^2+\dots+\theta_q^2} & ; k=1,2,\dots,q \\ 0 & ; k > q \end{cases}$$

1.2. Autoregressive Processes

- $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + \cdots + \phi_p Y_{t-p} + e_t$

- First-order Auto Regressive Processes

$\rightarrow Y_t = \phi_1 Y_{t-1} + e_t$

Mean = $\boxed{0}$, $\boxed{-1 < \phi < +1}$

Variance = $\sigma_0 = \boxed{\dfrac{\sigma_e^2}{1-\phi_1^2}}$, Covariance = $\sigma_k = \boxed{\phi^k \cdot \dfrac{\sigma_e^2}{(1-\phi_1^2)}}$,

Auto correlation = $\rho_k = \boxed{\phi^k}$.

$Y_t = e_t + \phi \cdot e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{p-1} \cdot e_{t-p+1} + \phi^p \cdot Y_{t-p}$

* if $|\phi| < 1$ then AR(1) will be stationary process.

- Second-order Auto Regressive Processes

Auto correlation $\Rightarrow \boxed{\rho_k = \phi_1 \cdot \rho_{k-1} + \phi_2 \rho_{k-2}}$

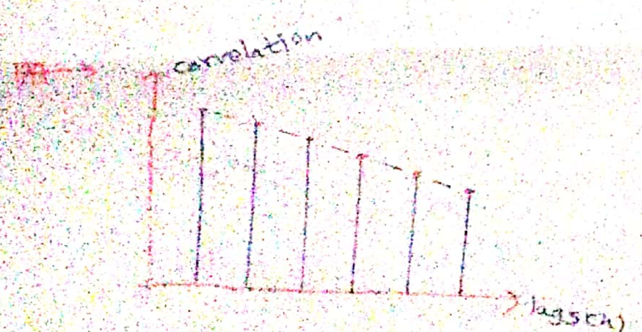
$k = 1 \Rightarrow \rho_1 = \phi_1 \cdot \rho_0 + \phi_2 \cdot \rho_{-1}$ $\{\rho_0 = 1, \ \rho_{-1} = \rho_1\}$

$\rho_1 = \phi_1 \cdot 1 + \phi_2 \cdot \rho_1$

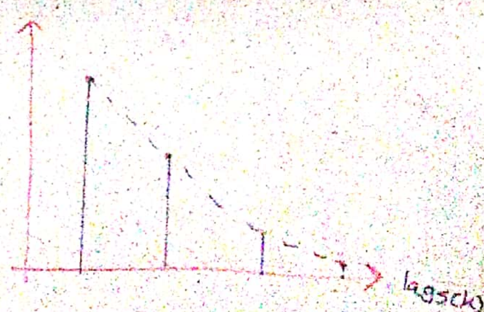
$\boxed{\rho_1 = \dfrac{\phi_1}{(1-\phi_2)}}$

* 1. $\phi_1 + \phi_2 < 1$
2. $\phi_2 - \phi_1 < 1$
3. $|\phi_2| < 1$

$\Big\}$ If these 3 conditions satisfies, then Stationary AR(2) model.



Linear decreasing :- Non-stationary



Non-linear decreasing :- Stationary

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1.3. Mixed Models ≡ ARMA

$$Y_t = \phi \cdot Y_{t-1} + e_t + (-\theta \cdot e_{t-1}) \leftarrow ARMA(1,1)$$

Invertibility

* → MA :- for θ and $\frac{1}{\theta}$ same auto correlation.

$\boxed{\text{If } |\theta| < 1 \text{ then } MA(1) \longrightarrow \text{Infinite-order autoregressive model}}$

MA(q) is invertible

2) Models for Non-stationary Time Series

* Random walk : $Y_t - Y_{t-1} = e_t \Rightarrow$ First difference is stationary
* ARIMA $(p, d, q) \equiv$ ARMA (p, q) ; practically $d = 1 | 2$.

1) ARIMA (p, d, q)
2) IMA (d, q)
3) ARI (p, d)

* Methods :- 1) Differencing

2) Transformations

- Log ; $\log(Y_t)$ ← get rid of increasing variability
- Power ;

$$g(x) = \begin{cases} \frac{x^{\lambda} - 1}{\lambda} & ; \lambda \neq 0 \\ \log(x) & ; \lambda = 0 \end{cases}$$ ← Box & Cox (1964)

- Percentage change

MODEL SPECIFICATION

* Box-Jenkins method ⟶ 1) What value for p, d, q ?
  2) What coefficients ?
  3) Check appropriateness.

Sample autocorrelation function ⇒ $r_k = \dfrac{\sum_{t=k+1}^{n} (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n} (Y_t - \bar{Y})^2}$ for $k = 1, 2, \ldots$

* ⟶ **MA** :- ACF (Auto Correlation Function),

AR :- PACF (Partial Auto Correlation Function),

$\phi_{kk} = \text{Corr}(Y_t, Y_{t-k} \mid Y_{t-1}, Y_{t-2}, \ldots, Y_{t-k+1})$

Mixed Models :- $\boxed{\text{EACF}}$ (Extended Auto Correlation Function), Corner method, SCAN method.

$\phi_{kk} = \text{Corr}(Y_t - \beta_1 Y_{t-1} - \beta_2 Y_{t-2} - \cdots - \beta_{k-1} Y_{t-k+1},\ Y_{t-k} - \beta_1 Y_{t-k+1} - \beta_2 Y_{t-k+2} - \cdots - \beta_{k-1} Y_{t-1})$.

⇓

$$\boxed{\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}}$$

* Over differencing creates new correlations ⇒
  check when differencing ⇒
  Hypothesis testing ≡ (Augmented)
  1) Dickey-Fuller Unit-Root Test - Stationary / Not-Stationary
  2) AIC [Akaike's Information Criterion]
  3) BIC [Bayesian Information Criterion]

* select correct model

* AIC = $-2 \log$ (maximum likelihood) + $2k$,
* AIC_c = AIC + $\frac{2(k+1)(k+2)}{(n-k-2)}$,
* BIC = $-2 \log$ (maximum likelihood) + $k \cdot \log(n)$.

PARAMETER ESTIMATION

* ② Least squared
* ③ Maximum Likelihood Estimation
* 1) Method of moments

* $y_t = \alpha \cdot y_{t-1} + \epsilon_t$ $\quad\quad y_1, y_2, ---, y_T$

$\hat{\alpha} = f(y_1, y_2, ---, y_T)$

③ Generally, MLE method : because to get small variability.

1) Method of moments

1.1) Auto Regressive models

$AR(1) :- \rho_1 = \phi \;,\; \rho_1 = r_1 \Rightarrow \boxed{\hat{\phi} = r_1}$

$AR(2) :- \rho_1 = \phi_1 + \rho_1 \phi_2$ and $\rho_2 = \rho_1 \phi_1 + \phi_2$

$r_1 = \phi_1 + r_1 \phi_2$ and $r_2 = r_1 \phi_1 + \phi_2$

$\therefore \boxed{\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2}}$ and $\boxed{\hat{\phi}_2 = \frac{r_2 - r_1^2}{1-r_1^2}}$

1.2) Moving Average models

$MA(1) :- \rho_1 = \frac{-\theta}{1+\theta^2} \Rightarrow \hat{\theta} = \frac{-1}{2r_1} \pm \sqrt{\frac{1}{4r_1^2} - 1}$

* $|\theta| < 1$ satisfies invertibility rule ⇒ So, positive one and $|r_1| < 0.5$

② Method of moments :- MA and ARMA : Hard

↳ $\epsilon_t = y_t - \alpha\, y_{t-1}$

$V(\epsilon_t) = V(y_t - \alpha y_{t-1}) = V(y_t) + \alpha^2 \cdot V(y_{t-1}) - 2 \cdot \alpha \cdot cov(y_t, y_{t-1})$

$\therefore V(\epsilon_t) = \gamma_0 + \alpha^2 \gamma_0 - 2\alpha \gamma_1$

2) Least Squares Estimation

2.1) Auto Regressive Models

- AR(1) : $Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$

Conditional Sum of Squares

$$S_c(\phi, \mu) = \sum_{t=2}^{n} [(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2$$

$$\hat{\phi} = \frac{\sum_{t=2}^{n} (Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\sum_{t=2}^{n} (Y_{t-1} - \bar{Y})^2}$$

2.2) Moving Average Models

$$Y_t = e_t - \theta \cdot e_{t-1}$$

$$\Downarrow$$

$$Y_t = -\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots + e_t$$

$$\boxed{e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \cdots + \theta^k Y_{t-k}}$$

3) Maximum Likelihood Estimation

But, time series data is not i.i.d?

MLEs : $\frac{\partial L}{\partial \mu} = 0$, $\frac{\partial L}{\partial \sigma^2} = 0$

$\Downarrow$

$\mu, \sigma^2 = \checkmark$

- AR(1) :

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \overset{i.i.d}{\sim} N(0, \sigma_\varepsilon^2)$$

$$f(Y_1, Y_2, Y_3, \ldots, Y_n) = f(Y_n | Y_{n-1}, Y_{n-2}, \ldots, Y_1) \cdot f(Y_{n-1}, \ldots, Y_1)$$

$$= f(Y_n | Y_{n-1}) \cdot f(Y_{n-1} | Y_{n-2}) \cdots f(Y_2 | Y_1)$$

$$f(Y_{n-1}, Y_{n-2}, \ldots, Y_1)$$

$$= \prod_{t=2}^{n} (Y_t | Y_{t-1}) \cdot f(Y_1)$$

Standard Error $= \sqrt{\hat{Var}(\hat{\phi})} \approx \sqrt{\frac{1 - \hat{\phi}^2}{n}} = \sqrt{\frac{1 - 0.91^2}{120}} = 0.04$

good

Quiz - 01

Q1) $\{X_t\} \rightarrow$ stationary disⁿ $\Rightarrow E[X_t] = \mu$; constant over time,

$Cov(X_t, X_{t-k}) = \gamma_k$ not depend on time.

$$Y_t = \begin{cases} X_t + 3 \; ; \text{ for } t \text{ odd} \\ X_t - 5 \; ; \text{ for } t \text{ even} \end{cases}$$

(a) $Cov(Y_t, Y_{t-k}) = E[Y_t \cdot Y_{t-k}] - E[Y_t] \cdot E[Y_{t-k}]$

case I : $t, t-k$ are odd :-

$$Cov(Y_t, Y_{t-k}) = E[(X_t + 3)(X_{t-k} + 3)] - (\mu+3)\cdot(\mu+3)$$

$$= E[X_t \cdot X_{t-k} + 3 \cdot X_t + 3 \cdot X_{t-k} + 9] - (\mu+3)^2$$

$$= (\gamma_k + \mu^2) + 3(\mu + \mu) + 9 - (\mu+3)^2$$

which is free from t.

Like this show it for all 4 cases

(b) No, Cov of Y_t is free from t but mean is not constant over time.

Q2)

(a) $W_t = \nabla Y_t = Y_t - Y_{t-1}$

$E[W_t] = E[Y_t - Y_{t-1}] = \mu - \mu = 0$ $\{\because Y_t$ is stationary$\}$

$$Cov(W_t, W_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k})$$

$$= Cov(Y_t, Y_{t-k} - Y_{t-1-k}) - Cov(Y_{t-1}, Y_{t-k} - Y_{t-1-k})$$

$$= (\gamma_k - \gamma_{k+1}) - (\gamma_{k-1} - \gamma_k)$$

$$\therefore \quad = 2(\gamma_k) - \gamma_{k+1} - \gamma_{k-1}$$

So, W_t is stationary

(b) Y_t is stationary $\Rightarrow \nabla Y_t$ is stationary

So, $\nabla^2 Y_t$ should be stationary

Q3) $\rho_1 = \frac{-\theta}{1+\theta^2}$

$$\frac{d(\rho_1)}{d\theta} = \frac{(1+\theta^2)(-1) - (-\theta)\,2\theta}{(1+\theta^2)^2} = \frac{-1-\theta^2+2\theta^2}{(1+\theta^2)^2} = 0$$

$\therefore \theta^2 - 1 = 0 \Rightarrow \therefore \theta = -1, +1$

$$\rho_1\Big|_{max} = \frac{-(-1)}{1+1} = \frac{1}{2} = 0.5,$$

$$\rho_1\Big|_{min} = \frac{-1}{1+1} = -\frac{1}{2} = -0.5.$$

Q4) $Cov(Y_t, Y_{t-k}) = Cov(e_{t-1} - 0.4\, e_{t-2},\ e_{t-1-k} - 0.4\, e_{t-2-k})$

$= Cov(e_{t-1},\ e_{t-1-k} - 0.4\, e_{t-2-k}) -$

$0.4 \cdot Cov(e_{t-2},\ e_{t-1-k} - 0.4\, e_{t-2-k})$

$k=1$:- $\qquad$ ($e_t \sim N(0, \sigma_e^2)$)

$Cov(Y_t, Y_{t-1}) = -0.4 \cdot \sigma_e^2$

$k \geq 2$:- $Cov(Y_t, Y_{t-2}) = 0$

$k=0$:- $Cov(Y_t, Y_t) = V(Y_t) = \sigma_e^2 + 0.4 \times 0.4 \cdot \sigma_e^2$

$= \sigma_e^2 (1 + 0.4^2)$

$= 1.16 \cdot \sigma_e^2$

$$\therefore Cov(Y_t, Y_{t-k}) = \begin{cases} 1.16 \cdot \sigma_e^2 ; & k=0 \\ -0.4 \cdot \sigma_e^2 ; & k=1 \\ 0 ; & k \geq 2 \end{cases}$$

Q5)

(a) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t - 0.8 e_{t-1}$

$\left.\begin{matrix}\phi_1 = 2 \\ \phi_2 = -1\end{matrix}\right\} \phi_1 + \phi_2 = 1 \not< 1.$

Y_t is not stationary

$Y_t - Y_{t-1} = Y_{t-1} - Y_{t-2} + e_t - 0.8 e_{t-1}$

$\nabla Y_t = \nabla Y_{t-1} + e_t - 0.8 e_{t-1}$

$\nabla Y_t - \nabla Y_{t-1} = e_t - 0.8 e_{t-1}$

$\nabla^2 Y_t = e_t - 0.8 e_{t-1}$

Now it is stationary

$\therefore d=2,\ p=0,\ q=1.$

ARIMA(0, 2, 1) = IMA(2, 1) //

(b) $Y_t = \underset{\uparrow \atop \phi_1}{0.25} Y_{t-1} + \underset{\uparrow \atop \phi_2}{0.75} Y_{t-2} + e_t - 0.5 e_{t-1} + 0.25 e_{t-2}$;

$\phi_1 + \phi_2 = 1 \not< 1$

Y_t is not stationary

$Y_t - Y_{t-1} = -0.75 Y_{t-1} + 0.75 Y_{t-2} + e_t - 0.5 e_{t-1} + 0.25 e_{t-2}$

$W_t = -0.75 W_{t-1} + e_t - 0.5 e_{t-1} + 0.25 e_{t-2}$

$|-0.75| < 1$

$\therefore$ Now, it is Stationary

$\therefore d = 1,\ p = 1,\ q = 2$

ARIMA (1, 1, 2) //