Stochastic Process

* Stationary distribution :- Initially what distribution and still remain same distribution / fluctuates and converge to one distribution.

* Limiting distribution :- After long run what will be the distribution.

* Transient :- $Pr(\tau_i < \infty) < 1$ or $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$,

* Recurrent :- $Pr(\tau_i < \infty) = 1$ or $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$.

→ P = Transition matrix, probability distribution $(\pi) = (\pi_1, \pi_2, ---)$

$\pi P = \pi$ { Stationary distribution is the π }

Note :- ① For a irreducible Markov chain;
If a stationary distribution exists,
Then it is unique

② If state space (S) is finite and Markov chain is irreducible,
Then unique stationary distribution (π) exists

③ Let i be a current state.

$E_i[\tau_i] < \infty$: positive recurrent; { Gambler's game :- $p < \frac{1}{2}$ }

$E_i[\tau_i] = \infty$: null recurrent; { Gambler's game :- $p = \frac{1}{2}$ }

④ An irreducible Markov chain $\{X_n\}$ has a unique stationary distribution π $\Leftrightarrow$ $\{X_n\}$ is positive recurrent

⑤ gcd → 1 : aperiodic
gcd → >1 : periodic

Irreducible, positive recurrent, aperiodic Markov chain;
$P_{ij}^{(n)} \longrightarrow \pi_j$ as $n \to \infty$ { limiting ≡ stationary }

(6) Stationary distribution $= \pi$, states $= i, j$.

$\tau_i =$ return time to state i,

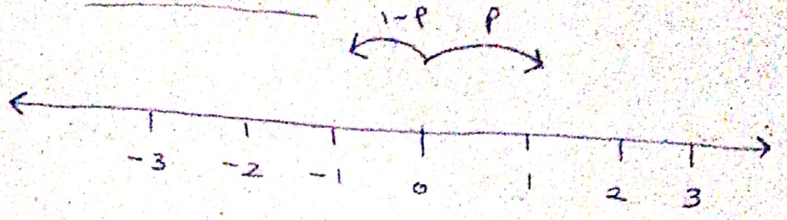
$N_j =$ Number of visits to j between conservetive visits to i.

$$E_i[\tau_i] = \frac{1}{\pi_i} \quad \text{and} \quad E_i[N_j] = \frac{\pi_j}{\pi_i}$$

mean recurrence time

$\longleftrightarrow$ geometric dis^n $:\rightarrow$ Expected value $= \dfrac{1-p}{p}$

Random Walk



$$P_{ii}^{(2n-1)} = 0$$

odd number

$$P_{ii}^{(2n)} = \binom{2n}{n} p^n \cdot (1-p)^n = \frac{(2n)!}{(n!)^2} \cdot [p(1-p)]^n.$$

Stirling's formula $\Rightarrow n! \sim n^n \cdot \sqrt{n} \cdot e^{-n} \cdot \sqrt{2\pi}$

$$\therefore P_{ii}^{(2n)} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$

Where, $p = \frac{1}{2}$:- $\dfrac{1}{\sqrt{\pi n}}$ $\boxed{\text{is infinite}}$ $\Rightarrow \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent $\Rightarrow$

$\sum_{n=1}^{\infty} P_{ii}^{(2n)} = \infty \ \{\frac{1}{2} < 1\}$

where, $p \neq \frac{1}{2}$:- $\boxed{\text{is finite}}$ $\Leftarrow$ convergent

$\therefore$ Simple random walk is $\rightarrow$ null reccurent for $p = \frac{1}{2}$,

$\rightarrow$ transient for $p \neq \frac{1}{2}$.

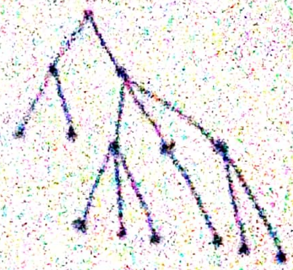
$\rightarrow$ 2 dimension $\equiv$ 1dimension

$\rightarrow$ 3, 4, 5 dimension $\Rightarrow$ Always transient //

Branching Process

pgf : $E[t^X]$

mgf : $E[e^{tX}]$



mgf $\xrightarrow{t=\ln t}$ pgf $\quad\quad \rightarrow G_n(s) = G_{n-1}(G(s)) = \ldots$

$P(E) = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} G_n(0)$

Example 1 $X = \begin{cases} 0 & \text{with probability } 1/4 \\ 2 & \text{with probability } 3/4 \end{cases}$

$G(s) = \frac{1}{4}\cdot s^0 + \frac{3}{4}\cdot s^2 = \frac{1}{4} + \frac{3}{4}s^2$

$\boxed{E[t^X] = \sum t^x \cdot P(x) = t^0 \cdot \frac{1}{4} + t^2 \cdot \frac{3}{4} = \frac{1}{4} + \frac{3}{4}t^2}$

$G(0) = \frac{1}{4}$

$G_2(s) = G(G(s)) = \frac{1}{4} + \frac{3}{4}\left(\frac{1}{4} + \frac{3}{4}s^2\right)^2$

$G_2(0) = \frac{19}{64}$

$\frac{1}{4} + \frac{3}{4}t^2 = t \Rightarrow 3t^2 - 4t + 1 = 0 \Rightarrow (3t-1)(t-1) = 0 \Rightarrow$

$t = \frac{1}{3}$ or $1 \Rightarrow \therefore \frac{1}{3}$ is the probability of extinct.

$\rightarrow E[Z_n] = \begin{cases} \rightarrow 0 & ; \mu < 1 \\ 1 & ; \mu = 1 \\ \rightarrow \infty & ; \mu > 1 \end{cases} \qquad V[Z_n] = \begin{cases} \rightarrow 0 & ; \mu < 1 \\ \rightarrow \infty & ; \mu = 1 \\ \rightarrow \infty & ; \mu > 1 \end{cases}$

$\therefore \mu \le 1 \Rightarrow P(E) = 1, \quad \mu > 1 \Rightarrow P(E) < 1.$

Poisson Process

exponential :- $\lambda \cdot e^{-\lambda t}$

$E : \frac{1}{\lambda}$

$Pr(t < T) = 1 - e^{-\lambda T}$

Poisson :- $\frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$E :- \lambda$

$Pr(x = 0) = e^{-\lambda}$

construction by exponential inter arrival times:

$$P(S_n > t) = P(N(t) < n)$$
$$= \sum_{j=0}^{n-1} e^{-\lambda t} \cdot \frac{(\lambda t)^{j}}{j!}$$

Queue Theory ①

* Customers, Servers, Birth rate, Death rate, jokey, balk, ...

FCFS = First come First Serve

LCFS = Last Come First serve

SIRO = Serve In Random order

Exponential :- $\boxed{\lambda \cdot e^{-\lambda t}}$ Expected value = $\frac{1}{\lambda}$

$$P(t \le T) = 1 - e^{-\lambda t}$$

Poisson :-

λT for time = T is Expected value

$$\frac{(\lambda T)^n \cdot e^{-\lambda T}}{n!} \rightarrow \text{Birth}$$

$$\frac{(\lambda T)^{N-n} \cdot e^{-\lambda T}}{(N-n)!} \rightarrow \text{Death}.$$

	Exponential (cts)	Poisson (Discrete)
1)	$t \geq 0$	$n = 0, 1, 2, ---$
2)	$\frac{1}{\lambda}$	λT for time = T
3)	Time between next successful arrival	Number of arrivals with in Time = T

* Define,

n = Number of customers in queue and service,

λ_n = Arrival rate (given n customers in system),

μ_n = Departure rate (given n customers in system),

P_n = steady-state probability (of n customers in system).

$\lambda_0 \cdot P_0 = \mu_1 \cdot P_1$

$\therefore P_1 = \left(\frac{\lambda_0}{\mu_1}\right) \cdot P_0 \Rightarrow \boxed{P_n = \frac{(\lambda_{n-1} \cdot \lambda_{n-2} \cdot \lambda_{n-3} \cdot \cdots \cdot \lambda_0)}{(\mu_n \cdot \mu_{n-1} \cdot \mu_{n-2} \cdot \cdots \cdot \mu_1)} \cdot P_0}$

$\Uparrow$

$\therefore \boxed{\sum_{n=0}^{\infty} P_n = 1} \rightarrow P_0$ get to know

e.g :- Problem 6 ;

$\lambda = 4$ (per hour)

$\mu = \frac{60}{15} = 4$ (per hour)

i)

0	1	2	3	4
4 can come	3 can come	2 can come	1 can come.	0 can come

$\lambda_n = \begin{cases} \lambda = 4 ; & n = 0, 1, 2, 3 \\ 0 ; & n \geq 4 \end{cases}$

$\mu_n = \mu = 4$

P_0, $P_1 = \left(\frac{\lambda_0}{\mu_1}\right) \cdot P_0 = \frac{4}{4} \cdot P_0 = P_0$,

$P_2 = \left(\frac{\lambda_1 \cdot \lambda_0}{\mu_2 \cdot \mu_1}\right) \cdot P_0 = \frac{4 \cdot 4}{4 \cdot 4} = P_0$,

$P_3 = \frac{(\lambda_2 \cdot \lambda_1 \cdot \lambda_0)}{(\mu_3 \cdot \mu_2 \cdot \mu_1)} \cdot P_0 = \frac{4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4} \cdot P_0 = P_0$,

$P_4 = P_0$,

$P_5 = 0$, $P_6, P_7, \cdots = 0$.

But, $\sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + P_0 + P_0 + P_0 + P_0 + 0 = 1$

$\therefore P_0 = \frac{1}{5} = 0.2$ //

ii) Expected # customers in shop = $0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 + 5 \cdot P_5 + 6 \cdot P_6 + \ldots$

$= (0+1+2+3+4) \cdot P_0 + 0$

$= \frac{4 \times 5}{2} \times 0.2$

$\therefore \quad = 2$ //

iii) 0.2.

SPECIALIZED POISSON QUEUES.

(a|b|c) : (d|e|f)

where,

a = Arrivals distribution

b = Departures distribution

c = Number of parallel servers ($c = 1, 2, \ldots, \infty$)

d = Queue discipline

e = Maximum number allowed in (queue + service) system

f = Size of the calling source

Measures of Performance : $L_s, L_q, W_s, W_q, \bar{c}$.

$$L_s = \sum_{n=1}^{\infty} n \cdot P_n$$

$$L_q = \sum_{n=c+1}^{\infty} (n-c) \cdot P_n \qquad W_s = W_q + \frac{1}{\mu}$$

$$L_s = \lambda_{eff} \cdot W_s \qquad \bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu}$$

$$L_q = \lambda_{eff} \cdot W_q$$

Facility utilization $= \frac{\bar{c}}{c}$

Example 15.6 - 1 //

last probability

* $\lambda_{eff} = \lambda - \lambda \cdot p_8 = \lambda(1 - p_8)$

* $L_s = \sum n \cdot p_n$

* $W_s = \dfrac{L_s}{\lambda_{eff}}$

* $W_q = W_s - \dfrac{1}{\mu}$

* $\bar{c} = L_s - L_q = \dfrac{\lambda_{eff}}{\mu}$

* utilization $= \dfrac{\bar{c}}{c}$

order

1) Single server Models ; (M|M|1) : (GD|∞|∞)

$\lambda_n = \lambda$ $\quad \lambda_{eff} = \lambda$ and $\lambda_{lost} = 0$

$\mu_n = \mu$

Let, $\rho = \dfrac{\lambda}{\mu} \Rightarrow \boxed{p_n = \rho^n \cdot p_0}$

$p_0 + p_1 + p_2 + \cdots = 1$

$p_0(1 + \rho^1 + \rho^2 + \cdots) = 1$

$p_0 = 1 - \rho$

$\boxed{p_n = \rho^n (1 - \rho)}$

* $L_s = \dfrac{\rho}{1-\rho}$ $\quad$ * $W_s = \dfrac{1}{\mu - \lambda}$ $\quad$ * $W_q = \dfrac{\rho}{\mu(1-\rho)}$

* $L_q = \dfrac{\rho^2}{1-\rho}$ $\quad$ * $\bar{c} = \rho$

2) $(M/M/1):(GD/N/\infty)$ ← single server with finite system (N)

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, \ldots, N-1 \\ 0, & n = N, N+1 \end{cases}$$

$\mu_n = \mu$ for $n = 0, 1, 2, \ldots$

$$\rho = \frac{\lambda}{\mu} \Rightarrow p_n = \begin{cases} \rho^n p_0, & n \le N \\ 0, & n > N \end{cases}$$

$$\boxed{p_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}}, & \rho \ne 1 \\ \dfrac{1}{N+1}, & \rho = 1 \end{cases}}$$

$\lambda_{lost} = \lambda p_N$

$\lambda_{eff} = \lambda(1 - p_N)$

Example: using excel sheet by Sir...

3) Multiple servers → $(M/M/c):(GD/\infty/\infty)$

$\lambda_n = \lambda; \; n \ge 0$

$$\mu_n = \begin{cases} n\mu; & n < c \\ c\mu; & n \ge c \end{cases}$$

$$\boxed{p_n = \begin{cases} \dfrac{\rho^n}{n!} p_0; & n < c \\ \dfrac{\rho^n}{c!\, c^{n-c}} p_0; & n \ge c \end{cases}}$$

4) Multiple servers - finite queue length → $(M/M/c):(GD/N/\infty)$

5) Infinite servers → $(M/M/\infty):(GD/\infty/\infty)$

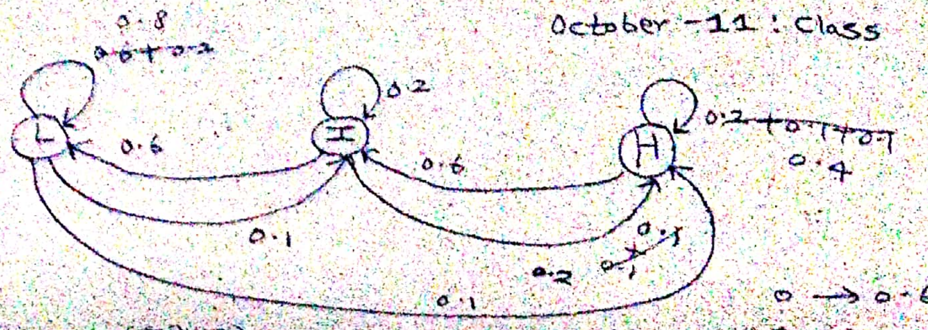
Stochastic Process - Questions 1

$\downarrow$

October-11 : Class

ex4.

(a)



$0 \rightarrow 0.6$

$1 \rightarrow 0.2$

$2 \rightarrow 0.1$

$>2 \rightarrow 0.1$

(b) $\frac{1-\overset{\text{Pr(failure)}}{p}}{p} = \frac{1-0.2}{0.2} = 4$ years

(c) $E_H[T_H] = \frac{1}{\pi_H}$; $\pi = (\pi_L \ \pi_I \ \pi_H)$

$\pi \cdot P = \pi \Rightarrow \pi_H \Rightarrow \checkmark$

$$P = \begin{array}{c|ccc} & L & I & H \\ L & 0.8 & 0.1 & 0.1 \\ I & 0.6 & 0.2 & 0.2 \\ H & 0 & 0.6 & 0.4 \end{array}$$

ex5.

(a) 4

(b) gcd $(4, 6, \ldots) = 2$

(c) No, $\pi = \pi \cdot P$; $\pi = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5]$

$\Downarrow$

$\pi = \checkmark$

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

ex2.

$Pr(\underset{\text{Day1}}{S}, \underset{\text{Day2}}{S}) = 0.7$

$Pr(S, R) = 0.1$

$Pr(R, S) = 0.1$

$Pr(R, R) = 0.1$

$$P = \begin{array}{c|cc} & S & R \\ S & Pr(S|S) & Pr(R|S) \\ R & Pr(S|R) & Pr(R|R) \end{array} = ?$$

$Pr(S|S) = \frac{Pr(S, S)}{Pr(S)} = \frac{0.7}{0.7+0.1} = \frac{7}{8}$,

$Pr(R|S) = \frac{Pr(S, R)}{Pr(S)} = \frac{0.1}{0.7+0.1} = \frac{1}{8}$,

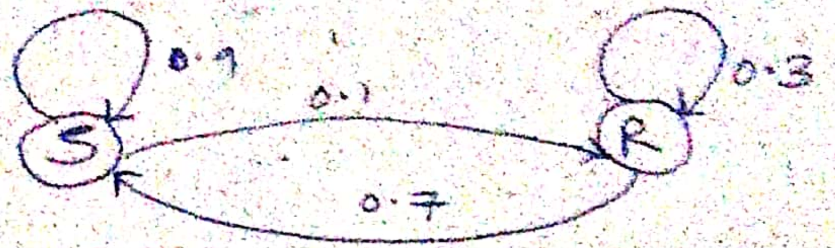
$Pr(S|R) = \frac{Pr(R, S)}{Pr(R)} = \frac{0.1}{0.2} = 0.5$,

$Pr(R|R) = \frac{Pr(R, R)}{Pr(R)} = \frac{0.1}{0.2} = 0.5$.

$\therefore P = \begin{bmatrix} 7/8 & 1/8 \\ 1/2 & 1/2 \end{bmatrix}$

ex 1.

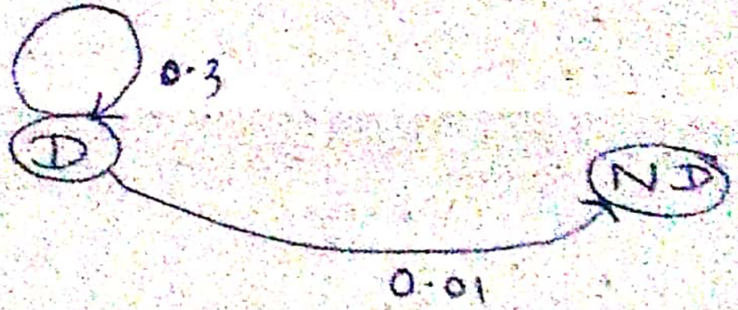
(a)



(b) $0.9 \times 0.9 + 0.1 \times 0.7 = 0.81 + 0.07 = 0.88$

(c) 0.81

ex 3.



what ?

Stochastic Process – Questions [1]

Between 10 - 11 am

Q1) $\frac{e^{-\lambda} \lambda^x}{x!}$; $\lambda = 2/\text{week} = \frac{2}{7}/\text{day}$

(a) $\frac{e^{-2/7} (2/7)^0}{0!} = e^{-2/7}$

(b) $\lambda = 2/\text{week} = 8/\text{month}$ ⇒ 10% rate ⇒ $\lambda = 0.8$ ...

$\frac{e^{-0.8} \cdot 0.8^1}{1!} = 0.8 e^{-0.8}$ ← probability that each 1 accident

$\therefore Pr(X \geq 1) = 1 - Pr(X=0) = 1 - e^{-0.8}$

(c) Binomial $(n, p) = \binom{52}{r} p^r (1-p)^{n-r}$, $n = 52$

Q2)

↓ ↑

$\lambda_S = 2/\text{min}$ $\lambda_N = 3/\text{min}$

(a) $\lambda = \lambda_S + \lambda_N = 5/\text{min}$ ⇒ $\frac{e^{-5} \cdot 5^2}{2!} = \frac{25}{2} e^{-5}$

(b) $\left(\frac{e^{-2} \cdot 2^1}{1!}\right) \cdot \left(\frac{e^{-3} \cdot 3^1}{1!}\right) = 6 e^{-5}$

(c) $Pr(N=2 \mid Tot=2) = Pr(N=2, S=0) / Pr(N+S=2)$

$= \left(\frac{e^{-3} \cdot 3^2}{2!}\right) \cdot \left(\frac{e^{-2} \cdot 2^0}{0!}\right) \Big/ \frac{e^{-5} \cdot 5^2}{2!}$

$= \frac{9}{25} \frac{e^{-5}}{e^{-5}}$ $\boxed{9/25}$

(d) $\frac{[Pr(N+S=2 \text{ in } 5 \text{ mins})]^2}{Pr(N+S=4 \text{ in } 10 \text{ mins})}$ $\lambda = 25$, $\lambda = 50$

$= \frac{\left(e^{-25} \cdot 25^2/2!\right)^2}{e^{-50} \cdot 50^4/4!}$

$= \frac{25^4 \cdot 4!}{50^4 \cdot (2!)^2} = \frac{6}{16} = \frac{3}{8}$

Q3>

~~(a) Pr(t > 2) = 1 − Pr(t = 1) − Pr(t = 0)~~ $\exp(\lambda) \sim t$;

(a) $Pr(t \geq 2) = e^{-2\lambda}$ $\{1 - [1 - e^{-2\lambda}]\}$

(b)

Case I :- 2 in (1,3), 3 in (4,5) $\dfrac{e^{-2\lambda}\cdot(2\lambda)^2}{2!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^0}{0!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^3}{3!}$ → ①

Case II :- 1 in (1,3), 1 in (3,4), 2 in (4,5) $\dfrac{e^{-2\lambda}\cdot(2\lambda)^1}{1!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^1}{1!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^2}{2!}$ → ②

Case III :- 2 in (3,4), 1 in (4,5) → $\dfrac{e^{-2\lambda}\cdot(2\lambda)^0}{0!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^1}{1!} \cdot \dfrac{e^{-\lambda}\cdot\lambda^1}{1!}$

∴ Probability = ① + ② + ③ //

Q4>

(a) $Pr(M = 10 \mid W = 10) = \dfrac{Pr(M = 10, W = 10)}{Pr(W = 10)}$ $\lambda_m = 5$, $\lambda_w = 5$.

$$= \frac{\dfrac{e^{-5}\cdot 5^{10}}{10!} \cdot \dfrac{e^{-5}\cdot 5^{10}}{10!}}{\dfrac{e^{-5}\cdot 5^{10}}{10!}} = \frac{e^{-5}\cdot 5^{10}}{10!} \; //$$

(b) $Pr(M + W \geq 20) = 1 - \sum_{i=1}^{19} \dfrac{e^{-10}\cdot 10^i}{i!}$

Q5> $\lambda = 10$/hour, $p = 1/10$, $n = 2$.

$$P(S_n > t) = \sum_{j=0}^{n-1} \frac{e^{-\lambda t}\cdot(\lambda t)^j}{j!}$$

$$P(S_n > 2) = \sum_{j=0}^{1} \frac{e^{-1}\cdot 1^j}{j!} = e^{-1} + e^{-1} = 2\cdot e^{-1} \; //$$

[5]

Q6)

a)

b) $P(X_1=3, X_2=2, X_3=1) = P(X_3=1 \mid X_2=2, X_1=3)\, P(X_2=2, X_1=3)$

$= P(X_3=1 \mid X_2=2)\, P(X_2=2 \mid X_1=3)\, P(X_1=3)$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2}$

$\therefore \quad = \frac{1}{12}$ //.

c) Yes

d) Yes $\{ \gcd(1, 2, \dots) = 1 \}$

e) $\pi \cdot P = \pi$ where, $\pi = (\pi_1 \ \pi_2 \ \pi_3)$

$$(\pi_1 \ \pi_2 \ \pi_3) \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$$

$\pi_1 + \pi_2 + \pi_3 = 1$ — (1)

$\frac{\pi_1}{2} + \frac{\pi_2}{3} + \frac{\pi_3}{2} = \pi_1$ — (2)

$\frac{\pi_1}{4} + \frac{\pi_3}{2} = \pi_2$ — (3)

$\frac{\pi_1}{4} + \frac{2\pi_2}{3} = \pi_3$ — (4)

f) Yes

Q7)

← periodic

a)

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0.5 & 0.5 \\ 1 & 0.5 & 0 & 0.5 \\ 2 & 0.5 & 0.5 & 0 \end{array}$$

b) P^3

c) $\pi \cdot P = \pi$ where $\pi = (\pi_0 \; \pi_1 \; \pi_2)$

$$(\pi_0 \; \pi_1 \; \pi_2) \cdot \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = (\pi_0 \; \pi_1 \; \pi_2)$$