$$[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i (v_i, w_i)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i [(v_i, w_i)]_{/\simeq} = \tag{1}$$

$$\sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} = \tag{2}$$

$$\sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k}\right)\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}}^{l_1} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{/\simeq} \tag{3}$$

$$\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}}^{n} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq})_{/\simeq} \tag{4}$$

Then take $\sum_{i=1}^{n} (v_i, w_i)$. There are no v, w such that $[(v, w)]_{/\simeq} = [(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$. Thus for the element $[(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$ of $V \otimes W$ there are no v, w such that $v \otimes w = [(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$. However, since $[(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq} = [(v_1, w_1)]_{/\simeq} + \cdots + [(v_n, w_n)]_{/\simeq}$ it can be writen as $v_1 \otimes w_1 + \cdots + v_n \otimes w_n$.
Now we will make some futher observations on how $V \otimes W$ looks like.

is a vector space created from of elemets of form $\{v \otimes w : v \in V, w \in W\}$ such that for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $k \in K$ there hold

$$\begin{aligned} v \otimes w_1 + v \otimes w_2 &= v \otimes (w_1 + w_2) \\ v_1 \otimes w + v_2 \otimes w &= (v_1 + v_2) \otimes w \\ k(v \otimes w) &= (kv) \otimes w = v \otimes (kw) \end{aligned}$$

with basises $\{v_i\}_{i \in I}$, $\{w_j\}_{j \in J}$ with basis and operations defined

As algebra we will understand a vector space $\mathcal{H}$ with additional linear operation $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ called multiplication. Natural example are polinomials. Trivial example is multimlication within a field. Now we will introduce an important example of algebra of noncommutative variables. Later it will rise to Hopf algebra we will be using for describing inverse riffle shuffle.

In fact this is the purpose of map u to insert the copy of the field K into a K-algebra.

("$1a$" means a from $\mathcal{H}$ multiplicated by 1 from the field K.)

means exactly what was in the definition of u. Which mean that "counit does the same thing to comultiplication as unit to the multiplication" ;)

Since

$$\begin{aligned} (\Delta \otimes I)\Delta(c) &= (\Delta \otimes I)\left(\sum c_1 \otimes c_2\right) = \sum \Delta(c_1) \otimes c_2, \\ (I \otimes \Delta)\Delta(c) &= (I \otimes \Delta)\left(\sum c_1 \otimes c_2\right) = \sum c_1 \otimes \Delta(c_2). \end{aligned}$$

$(\mathcal{H}, m, u)$
$(\mathcal{H}, \Delta, \varepsilon)$

The canonical isomorphism between $K \otimes K$ and K for all $k_1, k_2 \in K$ taking a form $k_1 \otimes k_2 \overset{\cong}{\longmapsto} k_1 k_2$ will be writen as φ_K.

and because of that we will omit it and identify $K \otimes K$ with K (and, because of associativity of a field multiplication we will identify any power $K^{\otimes n}$ with K).

Note, that later, when there will be no risk of confusion, we will be still using "$\cdot$" as multiplication on algebra setted by "m" from that algebra.

Explanation. In fact for a given vector space $\mathcal{H}$ with both an algebra structure $(\mathcal{H}, m, u)$ and a coalgebra structure $(\mathcal{H}, \Delta, \varepsilon)$, m and u are morfisms of coalgebras iff Δ and ε are morfisms of algebras and these are equivalent to conjuction of contditions that

preceisly.

where I^m means $\overbrace{I \otimes \cdots \otimes I}^{m \text{ times}}$.

$$[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i (v_i, w_i)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i [(v_i, w_i)]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k}\right)\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left(\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}\right) = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}$$

$$
\begin{aligned}
[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i (v_i, w_i)\right]_{/\simeq} &= \sum_{i=1}^{n} \alpha_i [(v_i, w_i)]_{/\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k}\right)\right)\right]_{/\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{/\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left(\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}\right) \\
&= \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}
\end{aligned}
$$

$$
\begin{aligned}
(u\varepsilon * f)(c) &= \sum u\varepsilon(c_1) \cdot f(c_2) \\
&= \sum \varepsilon(c_1) 1_A \cdot f(c_2) \\
&= \sum 1_A \cdot \varepsilon(c_1) f(c_2) \\
&= 1_A \cdot \sum \varepsilon(c_1) f(c_2) \\
&= 1_A \cdot f(c) = f(c)
\end{aligned}
$$

$$
\begin{aligned}
(f * u\varepsilon)(c) &= \sum f(c_1) \cdot u\varepsilon(c_2) \\
&= \sum f(c_1) \cdot \varepsilon(c_2) 1_A \\
&= \sum f(c_1) \varepsilon(c_2) \cdot 1_A \\
&= \left(\sum f(c_1) \varepsilon(c_2)\right) \cdot 1_A \\
&= f(c) \cdot 1_A = f(c)
\end{aligned}
$$

Hence the name.

"with respect to coordinates"

in next steps

That means that for bialgebra $\mathcal{H}$, a linear map $S \in Hom(\mathcal{H}^C, \mathcal{H}^A)$ is an antipode iff it satisfies

$$\begin{aligned}
\Delta(x_{i_0} \dots x_{i_k}) &= \Delta m^{[k]}(x_{i_0} \otimes \dots \otimes x_{i_k}) \\
&= (m^{[k]} \otimes m^{[k]}) \left(\sum (x_{i_0})_1 \otimes \dots \otimes (x_{i_k})_1 \otimes (x_{i_0})_2 \otimes \dots \otimes (x_{i_k})_2 \right) \\
&= \sum (x_{i_0})_1 \dots (x_{i_k})_1 \otimes (x_{i_0})_2 \dots (x_{i_k})_2 \\
&= \sum_{S \subseteq \{i_0, \dots i_k\}} \prod_{j \in S} x_j \otimes \prod_{j \notin S} x_j.
\end{aligned}$$

Let $\nu = (\nu_1, \dots \nu_N) \in \mathbb{N}^N$ be our deck of N cards (we just pick N cards from $\mathcal{X}$ in gene

Let $\nu = (\nu_1, \dots \nu_N) \in \mathbb{N}^N$ be our deck of N cards (we just pick N cards from $\mathcal{X}$ in general case they can repeat

comes with certain probability.

we dont know how exactly these stacks looks like but

if we have two stacks (we will refer to them as (L)eft and (R)ight).

and we have with probability p s_1 on L and s on R and with probability $1-p$ s_2 on L and s on R is exactly the same situation as having

Note that for all $k_1, \dots, k_n \in K$, such that at least one of them is non-zero, an expression $\sum_{i=0}^{n} k_i($

anallogly

Let denote that pulling apart as a $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, which for all $x_{i_0}, \dots, x_{i_n}$ gives

$$\Delta(x_{i_0} \cdots x_{i_n}) = \sum_{\substack{S \subseteq \{i_0, \dots i_n\} \\ S = \{i_{j_1}, \dots, i_{j_l}\} \\ S^c = \{i_{k_1}, \cdots, i_{k_{n-l}}\}}} x_{i_{j_1}} \cdots x_{i_{j_l}} \otimes x_{i_{k_1}} \cdots x_{i_{k_{n-l}}}.$$

For putting two piles back together by placing left on the top let us write a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that is concatenation, which means, that for all $x_{i_1}, \dots, x_{i_k}, x_{j_1}, \dots, x_{j_l} \in \mathcal{X}$

$$m(x_{i_1} \dots x_{i_k} \otimes x_{j_1} \dots x_{j_l}) = x_{i_1} \dots x_{i_k} x_{j_1} \dots x_{j_l}.$$

Let denote that pulling apart as a $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, then for all $s \in \mathcal{X}^*$ it will give

$$\Delta(s) = \sum_{\substack{s_1 \prec s \\ s_2 = s/s_1}} s_1 \otimes s_2.$$

For putting two piles back together by placing left on the top let us write a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that is concatenation, which means, that for all $s_1, s_2 \in \mathcal{X}^*$

$$m(s_1 \otimes s_2) = s_1 s_2.$$

all $s = x_{i_1} \dots x_{i_n} \in \mathcal{X}^*$ that meet the condition:

$$\mathbb{P}F_{n+1} = s_2 \mid F_n = s_1 = \mathbb{I}_{\ltimes + \not\Vdash} = \sim_{\not\Vdash} \mid \mathbb{I}_{\ltimes} = \sim_{\not\vDash}.$$

, we percive it as a certain amount of stacks

\\ Indeed it turned out that a vector space of

At this point we can start to think that it is convinient to interprete a space of possible arragements of two stacks of cards as a $\mathcal{H} \otimes \mathcal{H}$.

So we need a vector space with basis made of pairs of basis vectors from $\mathcal{H}$ with actions on them that are linear to both coordinates. A tensor product $\mathcal{H} \otimes \mathcal{H}$ is the less-degenerated vector space with that properties.

We will now show, that what we just had defined indeed give the same results as standard model of inverse riffle shuffling. In inverse riffle shuffling

Which also equvalent to that matrixes of $(F_i)_{i \geqslant 0}$ and $(I_i)_{i \geqslant 0}$ are transpositions of each other.

a \ big! \ huge! \ giant!

We will be working of an examples of riffle shuffle and inverse riffle shuffle cards shuffling as our Markov chains.
How we will put them in the algebraic way?
First we will do this with inverse riffle shuffle, the forward riffle shuffle will then apear in a natural way.

Earlier it was said that comultiplication can be sometimes viewed as a sum of possible divisions into smaller objects. It happends naturally when we are working with graded bialgebras. Like in example of polinomials, where natural grading is by degree.

0.0.1 Graded, connected Hopf algebra of non-commuting variables

This is a main example of our interest.
Let K be a field with characteristic 0. Let $\mathcal{X} = \{x_1, \dots, x_N\}$ be a finite set. For every $n \in \mathbb{N}$ let H_n be a vector space having as a basis all words of length n made of elements of $\mathcal{X}$. Let $\mathcal{H} \coloneqq \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. Let $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ be concatenation of words, that is, for all $k, l \in \mathbb{N}$ for all $x_{i_0}, \dots, x_{i_k}, x_{j_0}, \dots, x_{j_l} \in \mathcal{X}$

$$m(x_{i_0} \dots x_{i_k} \otimes x_{j_0} \dots x_{j_l}) \coloneqq x_{i_0} \dots x_{i_k} x_{j_0} \dots x_{j_l}.$$

Let $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be defined for all elements from $\mathcal{X}$ as

$$\Delta(x_i) = x_i \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_i.$$

and extends lineary and multiplically
Lemma. Then $\mathcal{H}$ is the a graded, connected Hopf algebra that is cocomutative.

Proof. Associativity of m and coassociativity of Δ are obvious. Actions fit together, because we define them so. Algebra is graded and connnected because it is. $\square$

We can see $\Delta(x_{i_0}\dots x_{i_k})$ how looks like:

$$\begin{aligned}\Delta(x_{i_0}\dots x_{i_k}) &= \Delta m^{[k]}(x_{i_0}\otimes\dots\otimes x_{i_k})\\ &= (m^{[k]}\otimes m^{[k]})\left(\sum (x_{i_0})_1\otimes\dots\otimes(x_{i_k})_1\otimes(x_{i_0})_2\otimes\dots\otimes(x_{i_k})_2\right)\\ &= \sum (x_{i_0})_1\dots(x_{i_k})_1\otimes(x_{i_0})_2\dots(x_{i_k})_2\\ &= \sum_{S\subseteq\{i_0,\dots i_k\}}\prod_{j\in S}x_j\otimes\prod_{j\notin S}x_j.\end{aligned}$$

where S is a multiset, because some of the $i_0, \dot{i}_k$ can be the same. Form of that co-product is like that because when for all $x_i\in\mathcal{X}$ we have $\Delta(x_i) = x_i\otimes 1_{\mathcal{H}} + 1_{\mathcal{H}}\otimes x_i$ then

a ! ! ! ! !

1 Summarize

$\sum \bigoplus + \oplus$

2 Productise

$\prod \bigotimes \bigodot \times *$

To jest źle:

Please note, that at first it can be not clear that $h^*m\in\mathcal{H}^*\otimes\mathcal{H}^*$. For sure $h^*\in(\mathcal{H}\otimes\mathcal{H})^*$ but in general case we have $\mathcal{H}^*\otimes\mathcal{H}^*\subseteq(\mathcal{H}\otimes\mathcal{H})^*$ (with inclusion given by canonical injection) but not necessarily $\mathcal{H}^*\otimes\mathcal{H}^* = (\mathcal{H}\otimes\mathcal{H})^*$. Although we are in graded bialgebra $\mathcal{H}$ and because of that we know that $h^*m\in\left(\bigoplus_{n=0}^{\infty}\mathcal{H}_n\otimes\mathcal{H}_n\right)^* = \bigoplus_{n=0}^{\infty}(\mathcal{H}_n\otimes\mathcal{H}_n)^*$. Futher, for all $n\in\mathbb{N}$ since $\mathcal{H}_n$ is finite-dimentional we have that $(\mathcal{H}_n\otimes\mathcal{H}_n)^* = \mathcal{H}_n^*\otimes\mathcal{H}_n^*$. Hence

$$h^*m\in\bigoplus_{n=0}^{\infty}\mathcal{H}_n^*\otimes\mathcal{H}_n^* = \mathcal{H}^*\otimes\mathcal{H}^*.$$

Now we will describe how a dual algebra to $\mathcal{H}$ looks like.
Let $\mathcal{H}^*$ be the dual vector space to $\mathcal{H}$. (A space of linear functions from $\mathcal{H}$ to K). We define multiplication $\Delta^*:\mathcal{H}^*\otimes\mathcal{H}^*\to\mathcal{H}^*$ and comultiplication $m^*:\mathcal{H}^*\to\mathcal{H}^*\otimes\mathcal{H}^*$ as (for all $h_1^*, h_2^*, h^*\in\mathcal{H}^*$):

$$\begin{aligned}\Delta^*(h_1^*\otimes h_2^*) &= (h_1^*\otimes h_2^*)\Delta,\\ m^*(h^*) &= h^*m.\end{aligned}$$

Structure detail described in this paragraph will be important in chapter about eigenbasises. [GR89] shows that symmetrized sums of certain primitive elements form basis of a free associative algebra. It will turn out that this will be left eigenbasis of $m\Delta$. Now we will introduce methods for construction of that basis. Explanation why this is an eigenbasis will came in Chapter 4. This whole paragraph is exactly the same as in [?] I put it here because it is quite short and

$$\Psi^{[i]}(h) = m^{[i]}\Delta^{[i]}(h) =$$
$$m^{[i]}(\sum h_1 \otimes \cdots \otimes h_i) =$$
$$m^{[i]}(\underbrace{\overbrace{h \otimes 1_{\mathcal{H}} \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}} + \overbrace{1_{\mathcal{H}} \otimes h \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}} + \cdots + \overbrace{1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes h}^{i \text{ factors}}}_{i \text{ summands}}) =$$
$$\underbrace{m^{[i]}(\overbrace{h \otimes 1_{\mathcal{H}} \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}}) + m^{[i]}(\overbrace{1_{\mathcal{H}} \otimes h \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}}) + \cdots + m^{[i]}(\overbrace{1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes h}^{i \text{ factors}})}_{i \text{ summands}} =$$

For simplifying the notation we will write a symbol for algebra multiplication also for componentwise multiplication, so for all ${}^1h, \ldots, {}^nh \in \mathcal{H}$:

$$\Delta^{[m]}m^{[n]}({}^1h \otimes \cdots \otimes {}^nh) ==: \sum \tag{5}$$

Proof.

$$\Psi^{[a]}(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)}) =$$
$$m^{[a]}\Delta^{[a]}(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)}) =$$
$$m^{[a]}(\sum_{\sigma \in S_k} \Delta^{[a]}(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)})) =$$
$$m^{[a]}(\sum_{\sigma \in S_k} (x_{\sigma(1)} \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_{\sigma(1)}) \cdot \ldots \cdot (x_{\sigma(k)} \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_{\sigma(k)}) =$$

$\square$

$$\Delta([x,y]) = \Delta(x \cdot y - y \cdot x) = \Delta(x \cdot y) - \Delta(y \cdot x) = \Delta m(x \otimes y) - \Delta m(y \otimes x) = \sum x_1 \cdot y_1 \otimes x_2 \cdot y_2 - \sum y_1 \cdot$$

Proof. (of the lemma)
Proof will be as follows:
- we will show that if
- then we will show that $w \in \mathcal{H}_\nu$ iff $\lambda(l_1)\ldots\lambda(l_k) \in \mathcal{H}_\nu$, where $(l_1, \ldots, l_k)$ is Lyndon factorisation of w. $\square$

Let $|w|$ be the length of word w. For a word $w = a_1 \ldots a_{|w|}$ and permutation $\sigma \in S_{|w|}$ let $\sigma(w) := a_{\sigma(1)} \ldots a_{\sigma(|w|)}$.
Let $\sim$ be a relation on $\mathcal{X}^* \times \mathcal{X}^*$ such that for all $w, v \in \mathcal{X}^*$ such that $w = a_1 \ldots a_{|w|}$, $v = b_1 \ldots b_{|v|}$ for $a_1, \ldots, a_{|w|}, b_1, \ldots, b_{|v|} \in \mathcal{X}$

$$w \sim v \iff |w| = |v| = n \wedge \exists_{\sigma \in S_n} a_1 \ldots a_n = b_{\sigma(1)} \ldots b_{\sigma(n)}$$

Proof. For the forward riffle shuffle we want to every possible permutation have probability 1 exept of identity with probability $n-1$. Reachable permutations are the same because of form of actions (just the same) we litterally cut that deck and put in on the top. coefficients holds, because numbers of the number of occurences holds. Therefore its indeed the inverse riffle shuffling. $\square$

2.0.1 Alternative structure

Now we will describe an hopf algebra structure on $\mathcal{H}^{\mathrm{gd}*}$ (definition $V^{\mathrm{gd}*}$ for a given V can be found in 2.1.1).
We define multiplication $\Delta^* : \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*}$ and comultiplication $m^* : \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*}$ as (for all $h_1^*, h_2^*, h^* \in \mathcal{H}^{\mathrm{gd}*}$):

$$\Delta^*(h_1^* \otimes h_2^*) = (h_1^* \otimes h_2^*)\Delta,$$
$$m^*(h^*) = h^* m.$$

2.0.2 Alternative structure To do

Now we will describe an alternative Hopf algebra structure on $\mathcal{H}$ - a vector space spanned by finite words over fixed alphabet $\mathcal{X}$. It will be the structure of $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$ with actions induced by actions from Hopf algebra $\mathcal{H}$ so firstly we will introduce that structure on $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$. But since $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$ and $\bigoplus_{i=0}^{\infty} \mathcal{H}_i \simeq \bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$ as a linear spaces, it will work on $\mathcal{H}$ the same. We will denote a Hopf algebra with this structure as $\mathcal{H}^*$ and call it a graded dual to $\mathcal{H}$. (but it will NOT be isomorphic to $\mathcal{H}^*$ - vector space dual to $\mathcal{H}$). (in [?] that Hopf algebra is also denoted as $\mathcal{H}^*$).
It will turn out that it describes the structure of forward riffle shuffle.

Let denote $\mathcal{H}^{\mathrm{gd}*}$ for $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$. We define multiplication $\Delta^* : \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*}$ and comultiplication $m^* : \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*}$ as (for all $h_1^*, h_2^*, h^* \in \mathcal{H}^{\mathrm{gd}*}$):

$$\Delta^*(h_1^* \otimes h_2^*) = (h_1^* \otimes h_2^*)\Delta,$$
$$m^*(h^*) = h^* m.$$

$$\sum g_1 \cdot h_1 \otimes g_2 \cdot h_2 =$$
$$\sum \{g_1 \cdot h_1 \otimes g_2 \cdot h_2 : g_1 g_2 = g \text{ and } h_1 h_2 = h\} =$$
$$\sum \{\sum \{k : g_1 \prec k \text{ and } h_1 = k/g_1\} \otimes \sum \{k : g_2 \prec k \text{ and } h_2 = k/g_2\} : g_1 g_2 = g \text{ and } h_1 h_2 = h\} =$$
$$\sum \{\sum \{m_1 \otimes m_2 : k = m_1 m_2\} : g \prec k \text{ and } h = k/g\} =$$
$$\sum \{m^*(k) : g \prec k \text{ and } h = k/g\} =$$
$$m^* \left(\sum \{k : g \prec k \text{ and } h = k/g\}\right) =$$
$$m^* \Delta^*(g \otimes h)$$

interlace is different leads to a specific term

The pair of interlace and division from $m^*\Delta^*$ generates divisions of g and h as "that letters that went to the prefix" and "that letters that went to the suffix" and interlaces of that prefixes and suffixes of g and h that are primal interlace restricted to a part of word.

Having Now we will see that there is another method of introducing that structure The structure of $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$ with actions induced by actions from Hopf algebra $\mathcal{H}$ (let $\mathcal{H}^{\mathrm{gd}*} := \bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$):
multiplication $\Delta^* : \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*}$ and
comultiplication $m^* : \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*}$
such that for all $h_1^*, h_2^*, h^* \in \mathcal{H}^{\mathrm{gd}*}$:

$$\begin{aligned} \Delta^*(h_1^* \otimes h_2^*) &= (h_1^* \otimes h_2^*)\Delta, \\ m^*(h^*) &= h^* m. \end{aligned}$$

so firstly we will introduce that structure on $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$. But since $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$ and $\bigoplus_{i=0}^{\infty} \mathcal{H}_i \simeq \bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$ as a linear spaces, it will work on $\mathcal{H}$ the same. We will denote a Hopf algebra with this structure as $\mathcal{H}^*$ and call it a graded dual to $\mathcal{H}$. (but it will NOT be isomorphic to $\mathcal{H}^*$ - vector space dual to $\mathcal{H}$). (in [?] that Hopf algebra is also denoted as $\mathcal{H}^*$).
It will turn out that it describes the structure of forward riffle shuffle.

Let denote $\mathcal{H}^{\mathrm{gd}*}$ for $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$. We define multiplication $\Delta^* : \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*}$ and comultiplication $m^* : \mathcal{H}^{\mathrm{gd}*} \to \mathcal{H}^{\mathrm{gd}*} \otimes \mathcal{H}^{\mathrm{gd}*}$ as (for all $h_1^*, h_2^*, h^* \in \mathcal{H}^{\mathrm{gd}*}$):

$$\begin{aligned} \Delta^*(h_1^* \otimes h_2^*) &= (h_1^* \otimes h_2^*)\Delta, \\ m^*(h^*) &= h^* m. \end{aligned}$$

so firstly we will introduce that structure on $\bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$. But since $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$ and $\bigoplus_{i=0}^{\infty} \mathcal{H}_i \simeq \bigoplus_{i=0}^{\infty} \mathcal{H}_i^*$ as a linear spaces, it will work on $\mathcal{H}$ the same. We will denote a Hopf algebra with this structure as $\mathcal{H}^*$ and call it a graded dual to $\mathcal{H}$. (but it will NOT be isomorphic to $\mathcal{H}^*$ - vector space dual to $\mathcal{H}$). (in [?] that Hopf algebra is also denoted as $\mathcal{H}^*$).
It will turn out that it describes the structure of forward riffle shuffle.

Here we will provide a more specific probabilistic interpretation of spaces and actions in $\mathcal{H}$. We will do so by introduce algebraic structure on inverse riffle shuffle step-by-step. What we will end up with will be eventually exactly the non-commuting algebra. Note, that what is written below are only some of observations how structures of free-assiociative algebra and inverse riffle shuffle

Markov chain works togrther. It will not be proof that these structures are litterally the same nor that these arguments apply in general to all Hopf algebras and Markov chains.

$$\sum_{b\in\mathcal{B}} \frac{b^*\psi\psi^n\left(\sum_{i=1}^{n_0}\alpha_i b_i\right)}{s^{n+1}s_0} = \sum_{b\in\mathcal{B}} \frac{b^*\psi\left(\sum_{i=1}^{n_0}\alpha_i \psi^n(b_i)\right)}{s^{n+1}s_0} =$$

$$\sum_{b\in\mathcal{B}} \frac{b^*\psi\psi^n\left(\sum_{i=1}^{n_0}\alpha_i b_i\right)}{s^{n+1}s_0} = \sum_{b\in\mathcal{B}} \frac{b^*\psi\left(\sum_{i=1}^{n_0}\alpha_i \psi^n(b_i)\right)}{s^{n+1}s_0} =$$

First we will briefly show that $(X_0, X_1, \dots)$ are well defined by inductively showing that for all $n \in \mathbb{N}$ there holds $\sum\limits_{b\in\mathcal{B}} \mathbb{P}(X_n = b) = 1$.
For $n = 0$ it is straight from definition of s_0:

$$\sum_{b\in\mathcal{B}} \mathbb{P}(X_0 = b) = \sum_{b\in\mathcal{B}} \frac{b^*\psi^0(x_0)}{s^0 s_0} = \sum_{b\in\mathcal{B}} \frac{b^*(x_0)}{s_0} = \frac{1}{s_0}\sum_{b\in\mathcal{B}} b^*(x_0) = \frac{s_0}{s_0} = 1.$$

Suppose that for n there holds $\sum\limits_{b\in\mathcal{B}} \mathbb{P}(X_n = b) = 1$, then:

$$\sum_{b\in\mathcal{B}} \mathbb{P}(X_{n+1} = b) = \sum_{b\in\mathcal{B}} \frac{b^*\psi^{n+1}(x_0)}{s^{n+1}s_0} = \sum_{b\in\mathcal{B}} \frac{b^*\psi\psi^n(x_0)}{ss^n s_0} = \sum_{b\in\mathcal{B}} \frac{b^*\psi\left(\frac{\psi^n(x_0)}{s^n s_0}\right)}{s}.$$

Let $\beta_1, \dots, \beta_m$ be such that $\frac{\psi^n(x_0)}{s^n s_0} = \sum\limits_{i=1}^{m} \beta_i b_i$. Then:

$$\sum_{i=1}^{m}\beta_i = \sum_{b\in\mathcal{B}} b^*\left(\sum_{i=0}^{m}\beta_i b_i\right) = \sum_{b\in\mathcal{B}} b^*\left(\frac{\psi^n(x_0)}{s^n s_0}\right) = \sum_{b\in\mathcal{B}} \frac{b^*\psi^n(x_0)}{s^n s_0} = \sum_{b\in\mathbb{B}} \mathbb{P}(X_n = b) \overset{\text{ind.}}{=} 1$$

so:

$$\sum_{b\in\mathcal{B}} \frac{b^*\psi\left(\frac{\psi^n(x_0)}{s^n s_0}\right)}{s} = \sum_{b\in\mathcal{B}} \frac{b^*\psi\left(\sum_{i=1}^{m}\beta_i b_i\right)}{s} = \frac{1}{s}\sum_{b\in\mathcal{B}} b^*\psi\left(\sum_{i=1}^{m}\beta_i b_i\right) =$$
$$\frac{1}{s}\sum_{i=1}^{m}\beta_i\sum_{b\in\mathcal{B}} b^*\psi(b_i) = \frac{1}{s}\sum_{i=1}^{m}\beta_i s = \frac{s}{s}\sum_{i=0}^{m}\beta_i = \frac{s}{s}\cdot 1 = 1.$$

$$=$$

$$\frac{b_0^*(x_0)}{s_0}\prod_{i=0}^{n-1}\frac{b_{i+1}^*\psi(b_i)}{\sum}\frac{b^*\psi(b_n)}{s} = \mathbb{P}(x_0 = b_0, X_1 = b_1, \dots, x_n = b_n)\sum$$

TO DO:

- dopisać commutative i cocommutative
- dopisać przykład polinomial i ciała do coalgebry
- dopisąć grupowy do Hopfa
- pokazać jak wygląda coproduct w noncommuting - ważne
- wprowadzić dualną do noncommuting - ważne
- w rozdziale 3 wyjaśnić
- non-commuting - Hopf square zachowuje skończone podprzestrzenie.
- wprowadzić te podprzestrzenie
- do łańcuch ów markowa dopisacć dokładniejsy opis gilbert shannon reeds - jakie sa prawdopodobieństwa oraz że forward można rozumieć na dwa równoważne sposoby. To, że są do siebie odwrotne (dualne) będzie wyprowadzone przy uzyciu algebry).
- finer grading
- dodać oznaczenia
- nie dowodzić dualności!, będzie w 3.

Now we will recall two theorems from [**?**] describing eigenbases.

Theorem 1. *bla bla eigenbasismn*

Theorem 2. *bla bla dual eigen basis*

To prove them we will need a simmetrization lemma
bla bla

TO DO: Remark of no primitive generators in other algebras

Now we can make some further observations about shuffling.
ble ble

Free assocciative, cocommutative Hopf algebra of non-commuting variables (from 2.7.1) is a model for inverse riffle shuffling and its graded dual: commutative Hopf algebra of non-cocommutative variables (from 2.7.3) is a model for forward riffle shuffling. It goes as follows. Let $\mathcal{X}$ be the finite set of all possible types of cards. Let ν be a tuple of elements from $\mathcal{X}$ that represents our actual deck of cards (the same type of card can occur multiple times, the order in ν

should be the order in which we think cards are ordered). Now we can take a look at a subspace $\mathcal{H}_\nu$ of vector space $\mathcal{H}$ built over $\mathcal{X}$ as described in 2.6.2. $\mathcal{H}_\nu$ will be the subspaces spanned by words that for every type of cards consists of exactly the same number of cards of that type as ν. So the basis of $\mathcal{H}_\nu$ will be the set of words for every arrangement of our deck of cards. Let's name this basis $\mathcal{B}_\nu$. Note that then $\mathcal{H}_\nu$ is finite dimensional. As was previously described, there are two ways of equipping $\mathcal{H}$ with Hopf algebra structure. One will correspond to inverse version of riffle shuffle and another one to the forward one. With given arrangement of cards $s \in \mathcal{B}_\nu$ applying $m\Delta$ to s yields the sum of possible outcomes after one inverse riffle shuffle while applying $\Delta^* m^*$ yields the same for forward riffle shuffle. In both cases coefficients (after normalization) are probabilities of corresponding outcomes.

Here will come an intuition why tensor products and Hopf algebras suits for describing probabilistic issues.

Let $\mathcal{X} = \{x_1, \dots x_N\}$ be our set of all possible types of cards.
We will denote a stack of k cards containing (from top to bottom) $x_{i_1}, \dots, x_{i_k}$ simply as $x_{i_1} \dots x_{i_k}$.
Imagine, that you have stack of cards $x_{i_1} \dots x_{i_k}$. After shuffling it you can get one of finitely many stack of cards each with certain probability. We want to have some representation of it in our structure. For that reason we span a vector space $\mathcal{H}$, over $\mathbb{Q}$ (but can be $\mathbb{R}$ if someone likes), with basis $\mathcal{X}^*$ (finite words over $\mathcal{X}$, which means "all possible stacks of cards of types from $\mathcal{X}$ including an empty stack").
For all $s_1, \dots, s_n \in \mathcal{X}^*$, all $0 \leqslant q_1, \dots, q_n \in K$ a non-zero vector $\sum_{i=1}^{n} q_i s_i$ is for all $i \in \{1, \dots, n\}$ interpreted as a state where we have a stack s_i with probability $\frac{q_i}{\sum_{i=1}^{n} q_i}$ or equivalently as a probabilistic measure on $\mathcal{X}^*$ with value $\frac{q_i}{\sum_{i=1}^{n} q_i}$ on s_i for every $i \in \{1, \dots, n\}$ and 0 elsewhere.
In that understanding the "+" can be read as "or".
We want also desribe a situation when: we have multiple stacks of cards on a table (some of them maybe empty), there are only finitely many options how these stacks can exactly look like and we know a probability of every option.
It is very natural situation during shuffling as when we for example split a stack of cards at some random point (with known probabilities of where the split can be) we for sure have two stacks of cards (as soon as we agree that one of them can be empty), there are only finetely many options how exactly arrangement looks like and we know a probability of each one.
We will now focus on case when we have two decks on a table.
We want to deal with that matter in similar way as we done for setting "probabilistic options" to one deck of cards. We will span a vector space with all possible arrangements of two decks as a basis. That vector space will be $\mathcal{H} \otimes \mathcal{H}$.
Now we will try to give some explanation why in fact this is quite intuitive.
For $s_1, s_2 \in \mathcal{X}^*$ let's denote (s_1, s_2) as having s_1 on the left stack and s_2 on the right stack.
Let's make an observation that for all $s, s_1, s_2 \in \mathcal{X}^*$ situation of having arrangement (s_1, s) with probability p and having arrangement (s_2, s) with probability $1 - p$ is the same situation as having s_1 with probability p or having

s_2 with probability $1-p$ on the left stack and for sure having s on the right stack. Making connection with our previously introduced notation so we want to $p(s_1, s) + (1-p)(s_2, s) = (ps_1 + (1-p)s_2, s)$ (and analogously to the second coordinate).
What is more, having for sure s_1 on the left and s_2 on the right with probability p (and with probability $(1-p)$ some else arragement, let's call it (z_1, z_2)) gives the same probability distribution on possible arrangments of two decks as, having s_1 on the left and having s_2 on the right, with probability p and with probability $1-p$ having (z_1, z_2)).
This leads us to conclusion, that we also want to $p(s_1, s_2) = (ps_1, s_2)$ (and analogously to the second coordinaate).
Ta- daaaam!

But where are that Markov chain? Where are these "subspaces preserved by Ψ"?

Let us denote $\mathcal{H}$ as a dual to $\mathcal{H}$ what we want to do is to see how induced multiplication and comultiplication look like.
"here it come".
It is forward fiffle sfufle, we can check it that corresponds to that , that to that fold product is exactly that and that, so the coeficient matches and that is ok. And then it is exactly an riffle shuffle as it is bla bla.

Now let us consider a vector space that is dual to $\mathcal{H}$. It is vector space of linear functions on $\mathcal{H}$ whith basis bla bla $1s^* : \mathcal{H} \to K$ such that for all $s \in \mathcal{X}$ We can define multiplication and comultiplication of $\mathcal{H}^*$ in the natural way. bla bla ble ble

We ckech, and what? That are exactly m_F and Δ_F matrixes of $(F_i)_{i \geqslant 0}$ and $(I_i)_{i \geqslant 0}$ are transpositions of each other.

TO DO: Do poprawki, bo żal

In the Gilbert-Shannon-Reeds model of inverse riffle shuffling there are two steps. First we are decomposing the deck by take cards from the top of deck - one after another and putting them to the left or to the right each with probability $\frac{1}{2}$. Secondly putting left stack on the right stack.
That pulling apart causes a split into two stacks, each of them can be any subset of original stack (with preservation of order) with equal probability of each option.
For $s_1, s \in \mathcal{X}^*$ let denote that s_1 is subsequence of s (a subset with preservation of order) as $s_1 \prec s_2$. Let's denote a stack arisen from removing form s its subsequence s_1 as s/s_1.
Let denote that pulling apart as a $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, then for all $s \in \mathcal{X}^*$ it will give

$$\Delta(s) = \sum_{\substack{s_1 \prec s \\ \wedge s_2 = s/s_1}} s_1 \otimes s_2.$$

For putting two piles back together by placing left on the top let us write a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that is concatenation, which means, that for all

$s_1, s_2 \in \mathcal{X}^*$

$$m(s_1 \otimes s_2) = s_1 s_2.$$

What we just define here is exactly an algebra of non-commuting variables from example 2.3.2.
Facts about its algebraic nature are presented in that section.
We can observe now that Hopf-square map $\Psi^{[2]} = m\Delta$ for Δ, m defined as above describes one iteration of the inverse riffle shuffle. For every $s \in \mathcal{X}^*$, $\Psi^{[2]}(s)$ is a sum of possible arrangements of stack with, after normalisation, corresponding probabilities.
For a fixed deck of n cards $\nu = (\nu_1, \ldots, \nu_n) \in \mathcal{X}^n$ the Markov chain of shuffling that deck is set by $\Psi^{[2]}$ restricted to the subspace spanned by S_ν = "all $s \in \mathcal{X}^*$ that are some rearangement of ν", more formally: spanned by S_ν, where:

$$S_\nu = \{s = x_{i_1} \ldots x_{i_n} \in \mathcal{X}^* \mid \exists_{\sigma \in S_n} x_{\sigma(i_1)} \ldots x_{\sigma(i_n)} = \nu_1 \ldots \nu_n.\}.$$

($\sigma \in S_n$ is a permutation, S_n is a symmetric group of n (group of all permutations of n elements)). Its equivalent to that $S_\nu = [\nu]_{\simeq_{\mathrm{sym}}}$.
Then the state space of that chain is S_ν. The transition matrix of that chain is exactly a matrix of $\Psi^{[2]}$ truncated to $\mathcal{H}_\nu := \mathrm{Lin}(S_\nu)$ (which, as we can observe is finite-dimensional and preserved by $\Psi^{[2]}$).

For forward riffle shuffle we will be working with the same space $\mathcal{H}$ (as we still are dealing with the same set of types of cards) but with differnt actions (as operations of "pulling apart" and "putting together" look now different).
We will proove that indeed forward riffle shuffle $(F_i)_{i \geqslant 0}$ and inversed riffle shuffle $(I_i)_{i \geqslant 0}$ are the same shuffling method but once aplicated "forward" and once "backward". What we mean is that for fixed deck of n cards $\nu = (\nu_1, \ldots, \nu_n) \in \mathcal{X}^n$, for all $s_1, s_2 \in \mathcal{H}_\nu$, all $n \geqslant 0$:

$$\mathbb{P}\{F_{n+1} = s_2 \mid F_n = s_1\} = \mathbb{P}\{I_{n+1} = s_1 \mid I_n = s_2\}.$$

Which means that probability of going from state s_1 to state s_2 in one step in forward riffle shuffle is qual to probability of going form s_2 to s_1 in one step in inverse riffle shuffle. (which was showed in Section 1.)
As remarked in Section 1. forward riffle shuffle can be defined as cutting the deck at some point with uniform distribution on "where" ($n + 1$ options for a deck of size n) and then putting back two piles together in the way that *everyone-had-seen-at-some-point-in-the-life* (*trrrrrrrr*) with the same probability of every possible "*trrrrrrrr*".

Let $\Delta_F : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be an linear map for decomposition of the deck for forward riffle shuffle, then for all $s \in \mathcal{X}^*$

$$\Delta_F(s) = \sum_{s_1 s_2 = s} s_1 \otimes s_2.$$

Let $m_F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ be a linear map coresponding to *trrrrrrrr*. Then for all $s_1, s_2 \in \mathcal{X}^*$:

$$m_F(s_1 \otimes s_2) = \sum_{\substack{s_1 \prec s \\ s_2 = s/s_1}} s$$

which is sum of all possible entanglements of s_1 and s_2.

TO DO: MOŻE DAĆ JEDNAK TEN DOWÓD.

b This is the end of our algebraic definitions pfuuuu...

2.1 Examples

2.1.1 Graded, connected Hopf algebra of polinomials

Let P be a vector space of polynomials of one variable over the field K with natural grading by degree. Note that the standard polynomial multiplication is compatible with that grading as for polynomials with degrees i, j, their product has degree $i+j$. Connection comes from that the identity of multiplication is a polynomial of degree 0 ($1_p = X^0$).
P can be enriched with coalgebra structure with comultiplication Δ such that for all $n \in \mathbb{N}$:

$$\Delta(X^n) = \sum_{i=0}^{n} X^i \otimes X^{n-i}.$$

it extends linearly to the rest of P.
Counit is then 0 for all elements with positive degree (degree > 0). Here comes the proof:
Since for all $n \in \mathcal{N}$

$$\begin{aligned}
(1_P \otimes \varepsilon)\Delta(X^n) &= X^n \otimes 1_K && \text{and} \\
(1_P \otimes \varepsilon)\Delta(X^n) &= \sum_{i=0}^{n} X^i \otimes \varepsilon(X^{n-i}) && \text{and} \\
\sum_{i=0}^{n} X^i \otimes \varepsilon(X^{n-i}) &= X^n \otimes \varepsilon(1_P) && + \sum_{i=0}^{n-1} X^i \otimes \varepsilon(X^{n-i}) \\
&= X^n \otimes 1_K && + \sum_{i=0}^{n-1} X^i \otimes \varepsilon(X^{n-i})
\end{aligned}$$

we have that for all $n \in \mathbb{N}$

$$\sum_{i=0}^{n-1} X^i \otimes \varepsilon(X^{n-i}) = 0$$

but we also have that

$$\sum_{i=0}^{n-1} X^i \otimes \varepsilon(X^{n-i}) = \sum_{i=0}^{n-1} \varepsilon(X^{n-i}) X^i \otimes 1_K = \left(\sum_{i=0}^{n-1} \varepsilon(X^{n-i}) X^i \right) \otimes 1_K$$

Because $X_0, \ldots, X_{n-1}$ are linearly independent we have that $\forall_{0 \leqslant i \leqslant n-1}\ \varepsilon(X^{n-i}) = 0$.
Keeping in mind that n was arbitrary, we have that for all $n \geqslant 1$ $\varepsilon(X^n) = 0$ and

then by linearity of ε, that for every polynomial $p \in P$ with a positive degree we have that $\varepsilon(p) = 0$.
We can now check that P with that structure is a graded, connected Hopf algebra that is both commutative and cocommutative.
It is a bialebra, because:

$$a$$

TO DO: BAM! DO ROBOTY!

TO DO:

- finish

Cośtam cośtam stationary distirution.
To do reasons of finding eigenbasis

If the reader feels lost in this section, it is recommended to read it in parallel to the section **??** where examples are provided or treat it just as a reference when formal definition will be needed. Another reason of arranging text like that (and possibility of treating this section just as a reference), is that for most of the time we will not be using full structure of a Hopf algebra. Nevertheless it is good to see the full shape of what we are dealing with.
So now comes the full definition but we will try to explain it piece by piece.

As well we can consider forward and inverse a-shuffles. An a-shuffle is a shuffle where decomposition is made in a-decks, so standard shuffle is a 2-shuffle. In inverse case we are putting cards with probability $1/a$ on one of a places and then putting piles together. This gives that we are making a subsets, each configuration of them with probability $1/a^n$. Similar analisis as in $a = 2$ case gives that there is a probability n^{a-1}/a^n for obtaining an identity permutation and a probability $1/a^n$ for obtaining any other possible.
For forward case

TO DO: How does it set?

Let's take a non-commuting variables algebra from its example. Let's take $\mathcal{H}_\nu$ for some $\nu \in \mathcal{X}^*$. Then $\Psi^{[2]}$ sets the Markov chain of inverse riffle shuffle the deck of cards containing cards labeled by xs appearing in ν. Chains state space is then the basis of $\mathcal{H}_\nu$. Let's call it $\mathcal{B}_\nu$ Chains transition matrix is equal to transition matrix of $\Psi^{[2]}$ written in $\mathcal{B}_\nu$.

It is grading with smallest components preserved by Ψ.

It will be put more precise in the section 3. where we will present connection between Markov chains and Hopf algebras.

$\sum$, $\boldsymbol{\hbar}$

To denote it, let's write $s_1 \prec s$ for "s_1 is a subsequence of s" (a subsequence doesn't have to be a contiguous fragment) and for s_1, s such that $s_1 \prec s$, let $s_2 = s/s_1$ denote $s_2 \prec s$ created by removing s_1 from s.

2.2 Reference to [?]

3 Summation

We had shown how to describe certain Markov chain using Hopf algebras. There are other examples of doing so. Polinomial algebra gives model for rock-bracking.

4 Summation

We had shown how to describe certain Markov chain using Hopf algebras. There are other examples of doing so. Polynomial algebra gives model for rock-breaking. Although with this approach there is some apparatus to build, it gives benefits of possibility to derive from knowledge of the well-known mathematical structures, when they appear in the study of some probabilistic aspects of behaviour of some combinatorial objects. Which is a beautiful example of how various branch

Remark. This is clear that such $s_{\mathcal{B}}$ is linear.

For a linear space V over field K, with basis $\mathcal{B}$ and its dual basis $\mathcal{B}^*$ we define a linear function $s_{\mathcal{B}} : V \to K$ as:

$$s_{\mathcal{B}}(v) = \sum_{b_i^* \in \mathcal{B}^*} b_i^*(v)$$

$s_{\mathcal{B}}(v)$ is then a sum of coefficients

$V \times W$. Note that we are taking entire $V \times W$ as a basis of Z, not just a basis of $V \times W$. Consequently, every non-zero element of Z has unique representation in the form $\sum_{i=1}^{n} \alpha_i(v_i, w_i)$. Let $\simeq$ be the smallest equivalence relation on Z satisfying:
For all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $k \in K$

$$\begin{aligned}
(v, w_1) + (v, w_2) &\simeq (v, w_1 + w_2), \\
(v_1, w) + (v_2, w) &\simeq (v_1 + v_2, w), \\
k(v, w) &\simeq (kv, w), \\
k(v, w) &\simeq (v, kw).
\end{aligned}$$

Since for all $z_1, z_2, z_3, z_4 \in Z$, all $k \in K$

$$\begin{aligned}
z_1 \simeq z_2 \wedge z_3 \simeq z_4 &\implies z_1 + z_3 \simeq z_2 + z_4 \text{ and} \\
z_1 \simeq z_2 &\implies kz_1 \simeq kz_2,
\end{aligned}$$

we treat $Z/_{\simeq}$ as a vector space with operations

$$\begin{aligned}
[z_1]_{\simeq} + [z_2]_{\simeq} &\coloneqq [z_1 + z_2]_{\simeq}, \\
k[z_1]_{\simeq} &\coloneqq [kz_1]_{\simeq}.
\end{aligned}$$

We denote equivalence class $[(v,w)]_{\simeq}$ as $v \otimes w$. The tensor product $V \otimes W := Z/_{\simeq}$. Note that in $V \otimes W$ there are vectors that cannot be written as $v \otimes w$ for any v, w. However, every $z \in V \otimes W$ can be written in as $z = \sum_{i=1}^{n} v_i \otimes w_i$ for some $v_1, \ldots, v_n \in V$, $w_1, \ldots, w_n \in W$. (More detailed explanation of this fact and the following example will come in the Observation 1..)
For example take $v_1, \ldots, v_n, w_1, \ldots, w_n$ such that they are linearly independent in corresponding spaces. Then take $\sum_{i=1}^{n}(v_i, w_i)$. There are no v, w such that $[(v,w)]_{\simeq} = \left[\sum_{i=1}^{n}(v_i, w_i)\right]_{\simeq}$. Thus, for the element $\left[\sum_{i=1}^{n}(v_i, w_i)\right]_{\simeq}$ of $V \otimes W$ there are no v, w such that $v \otimes w = \left[\sum_{i=1}^{n}(v_i, w_i)\right]_{\simeq}$. However, since $\left[\sum_{i=1}^{n}(v_i, w_i)\right]_{\simeq} = \sum_{i=1}^{n}[(v_i, w_i)]_{\simeq}$ it can be written as $\sum_{i=1}^{n} v_i \otimes w_i$.
Now we will make some further observations on how $V \otimes W$ looks like.

Observation 1. *If $\{b_i\}_{i \in I}$, $\{c_j\}_{j \in J}$ are bases of, respectively, V and W, then $\{b_i \otimes c_j : i \in I, j \in J\}$ is the basis of $V \otimes W$.*

Proof. Let $z = \sum_{i=1}^{n} \alpha_i (v_i, w_i)$ be an arbitrary non-zero element of Z. We will show that $[z]_{\simeq}$ has representation as $\sum_{i=1}^{m} \beta_i [(b_i, c_i)]_{\simeq} \left(= \sum_{i=1}^{m} \beta_i (b_i \otimes c_i)\right)$.

$$
\begin{aligned}
[z]_{\simeq} = \left[\sum_{i=1}^{n} \alpha_i (v_i, w_i)\right]_{\simeq} &= \sum_{i=1}^{n} \alpha_i [(v_i, w_i)]_{\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k}\right)\right)\right]_{\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{\simeq} \\
&= \sum_{i=1}^{n} \alpha_i \left(\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{\simeq}\right) \\
&= \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{\simeq}
\end{aligned}
$$

Thus $\{b_i \otimes c_j : i \in I, j \in J\}$ spans $V \otimes W$. To prove linear independence we can observe that if $\sum_{i=1}^{m} \alpha_i [(v_i, w_i)]_{\simeq} = 0$, then either $v_1, \ldots v_n$ or $w_1, \ldots, w_n$ have to be linearly dependent. It can't occur if $v_1, \ldots v_n$ and $w_1, \ldots, w_n$ are from the bases of V and W.
This observation also justifies recently cited fact and the example. □

Observation 2. *If V and W are finite dimensional and* $\dim(V) = n$, $\dim(W) = m$, *then* $\dim(V \otimes W) = nm$.

Proof. The proof is immediate from the Observation 1.. Since if $\{b_i\}_{i \in I}$, $\{c_j\}_{j \in J}$ are bases of, respectively, $V$and W and $\dim(V) = n$ and $\dim(W) = m$, then $| \{b_i \otimes c_j : i \in I, j \in J\} |= nm$ □

Observation 3. *$V \otimes W$ is a vector space of elements in the shape of $\sum_{i=1}^{n} v_i \otimes w_i$ with operations on them defined such that for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $k \in K$ there hold*

$$
\begin{gathered}
v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w, \\
v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2), \\
k(v \otimes w) = (kv) \otimes w = v \otimes (kw).
\end{gathered}
$$

Proof. This observation is just a recollection of the definition. $\square$

Observation 4. *For vector spaces U, V, W over the field K there is a natural isomorphism between $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ therefore there is no ambiguity in writing $U \otimes V \otimes W$ or a product of any greater number of vector spaces in that way. (We will also write "$u \otimes v \otimes w$" for some of their elements.) Form of elements, operations on them and structure of that vector spaces are fully analogous to the described above (with respect to all "coordinates" in terms like $u \otimes v \otimes w$ and so on). So the space $U \otimes V \otimes W$ has elements of shape $\sum_{i=1}^{n} u_i \otimes v_i \otimes w_i$ (each for some $u_1, \ldots, u_n \in U$, $v_1, \ldots v_n \in V$, $w_1, \ldots, w_n \in W$) and for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $k \in K$ there hold*

$$u_1 \otimes v \otimes w + u_2 \otimes v \otimes w = (u_1 + u_2) \otimes v \otimes w,$$
$$u \otimes v_1 \otimes w + u \otimes v_2 \otimes w = u \otimes (v_1 + v_2) \otimes w,$$
$$u \otimes v \otimes w_1 + u \otimes v \otimes w_2 = u \otimes v \otimes (w_1 + w_2),$$
$$k(u \otimes v \otimes w) = (ku) \otimes v \otimes w = u \otimes (kv) \otimes w = u \otimes v \otimes (kw).$$

Proof. Left to the reader. $\square$

Observation 5. *If V is a vector space over K, then all elements of $K \otimes V$ ($V \otimes K$) can be expressed in form $1 \otimes v$ ($v \otimes 1$) and there are natural isomorphisms ${}^{L}m : K \otimes V \to V$, (${}^{R}m : V \otimes K \to V$) given by*

$${}^{L}m(k \otimes v) = kv,$$
$${}^{R}m(v \otimes k) = kv.$$

Proof. An arbitrary element of $K \otimes V$ has form $\sum_{i=1}^{n} k_i \otimes v_i$ but

$$\sum_{i=1}^{n} k_i \otimes v_i = \sum_{i=1}^{n} 1 \otimes k_i v_i = 1 \otimes \sum_{i=1}^{n} k_i v_i.$$

${}^{L}m$ is linear (left for the reader) and is bijective because for all $v, v_1, v_2 \in V$

$$\varphi(1 \otimes v) = v$$

and

$$1 \otimes v_1 = 1 \otimes v_2 \iff 1 \otimes v_1 - 1 \otimes v_2 = 0 \iff$$
$$1 \otimes (v_1 - v_2) = 0 \iff v_1 - v_2 = 0 \iff v_1 = v_2.$$

The proof for $V \otimes K$ and ${}^{R}m$ is analogous. In the later sections we will use notations of ${}^{L}m$ and ${}^{R}m$ for those isomorphism for any space. $\square$

Remark. In a special case when $V = W = K$ the natural isomorphisms described above take form of $m_K : K \otimes K \to K$ that for all $k_1, k_2 \in K$ $m_K(k_1 \otimes k_2) = k_1 k_2$. This isomorphism of $K \otimes K$ and K is just a field multiplication from K.

Remark. Thanks to Observation 3. there is no ambiguity in writing $kv \otimes w$. I hope that this third observation will also help us understand what the tensor product is and what it is not. It will be good to keep it in mind when we are intensively dealing with it in a combinatorical way in the following sections.