$$[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i(v_i, w_i)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i[(v_i, w_i)]_{/\simeq} = \tag{1}$$

$$\sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq} = \tag{2}$$

$$\sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k}\right)\right)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1\leqslant j\leqslant l_1 \\ 1\leqslant k\leqslant l_2}}^{l_1} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{/\simeq} \tag{3}$$

$$\sum_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant l_1 \\ 1\leqslant k\leqslant l_2}}^{n} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq})_{/\simeq} \tag{4}$$

Then take $\sum_{i=1}^{n}(v_i, w_i)$. There are no v, w such that $[(v,w)]_{/\simeq} = [(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$. Thus for the element $[(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$ of $V \otimes W$ there are no v, w such that $v \otimes w = [(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq}$. However, since $[(v_1, w_1) + \cdots + (v_n, w_n)]_{/\simeq} = [(v_1, w_1)]_{/\simeq} + \cdots + [(v_n, w_n)]_{/\simeq}$ it can be writen as $v_1 \otimes w_1 + \cdots + v_n \otimes w_n$.

Now we will make some futher observations on how $V \otimes W$ looks like.

is a vector space created from of elemets of form $\{v \otimes w : v \in V, w \in W\}$ such that for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $k \in K$ there hold

$$v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$$
$$v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$$
$$k(v \otimes w) = (kv) \otimes w = v \otimes (kw)$$

with basises $\{v_i\}_{i \in I}$, $\{w_j\}_{j \in J}$ with basis and operations defined

As algebra we will understand a vector space $\mathcal{H}$ with additional linear operation $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ called multiplication. Natural example are polinomials. Trivial example is multimlication within a field. Now we will introduce an important example of algebra of noncommutative variables. Later it will rise to Hopf algebra we will be using for describing inverse riffle shuffle.

In fact this is the purpose of map u to insert the copy of the field K into a K-algebra.

("$1a$" means a from $\mathcal{H}$ multiplicated by 1 from the field K.)

means exactly what was in the definition of u. Which mean that "counit does the same thing to comultiplication as unit to the multiplication" ;)

Since

$$(\Delta \otimes I)\Delta(c) = (\Delta \otimes I)\left(\sum c_1 \otimes c_2\right) = \sum \Delta(c_1) \otimes c_2,$$
$$(I \otimes \Delta)\Delta(c) = (I \otimes \Delta)\left(\sum c_1 \otimes c_2\right) = \sum c_1 \otimes \Delta(c_2).$$

$(\mathcal{H}, m, u)$

$(\mathcal{H}, \Delta, \varepsilon)$

The canonical isomorphism between $K \otimes K$ and K for all $k_1, k_2 \in K$ taking a form $k_1 \otimes k_2 \xmapsto{\cong} k_1 k_2$ will be writen as φ_K.

and because of that we will omit it and identify $K \otimes K$ with K (and, because of associativity of a field multiplication we will identify any power $K^{\otimes n}$ with K).

Note, that later, when there will be no risk of confusion, we will be still using "$\cdot$" as multiplication on algebra setted by "m" from that algebra.

Explanation. In fact for a given vector space $\mathcal{H}$ with both an algebra structure $(\mathcal{H}, m, u)$ and a coalgebra structure $(\mathcal{H}, \Delta, \varepsilon)$, m and u are morfisms of coalgebras iff Δ and ε are morfisms of algebras and these are equivalent to conjuction of contditions that

preceisly.

where I^m means $\overbrace{I \otimes \cdots \otimes I}^{m \text{ times}}$.

$$[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i (v_i, w_i) \right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i [(v_i, w_i)]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k} \right) \right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k} \right) \right]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} \left(b_{i,j}, c_{i,k} \right) \right) \right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k}) \right]_{/\simeq} =$$

$$\sum_{i=1}^{n} \alpha_i \left(\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq} \right) = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}$$

$$[z]_{/\simeq} = \left[\sum_{i=1}^{n} \alpha_i(v_i, w_i)\right]_{/\simeq} = \sum_{i=1}^{n} \alpha_i[(v_i, w_i)]_{/\simeq}$$

$$= \sum_{i=1}^{n} \alpha_i \left[\left(\sum_{j=1}^{l_1} \gamma_{i,j} b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq}$$

$$= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(b_{i,j}, \sum_{k=1}^{l_2} \gamma_{i,k} c_{i,k}\right)\right]_{/\simeq}$$

$$= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{l_1} \gamma_{i,j} \left(\sum_{k=1}^{l_2} \gamma_{i,k} (b_{i,j}, c_{i,k})\right)\right]_{/\simeq}$$

$$= \sum_{i=1}^{n} \alpha_i \left[\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} (b_{i,j}, c_{i,k})\right]_{/\simeq}$$

$$= \sum_{i=1}^{n} \alpha_i \left(\sum_{\substack{1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}\right)$$

$$= \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l_1 \\ 1 \leqslant k \leqslant l_2}} \alpha_i \gamma_{i,j} \gamma_{i,k} [(b_{i,j}, c_{i,k})]_{/\simeq}$$

$$(u\varepsilon * f)(c) = \sum u\varepsilon(c_1) \cdot f(c_2)$$
$$= \sum \varepsilon(c_1) 1_A \cdot f(c_2)$$
$$= \sum 1_A \cdot \varepsilon(c_1) f(c_2)$$
$$= 1_A \cdot \sum \varepsilon(c_1) f(c_2)$$
$$= 1_A \cdot f(c) = f(c)$$

$$(f * u\varepsilon)(c) = \sum f(c_1) \cdot u\varepsilon(c_2)$$
$$= \sum f(c_1) \cdot \varepsilon(c_2) 1_A$$
$$= \sum f(c_1) \varepsilon(c_2) \cdot 1_A$$
$$= \left(\sum f(c_1) \varepsilon(c_2)\right) \cdot 1_A$$
$$= f(c) \cdot 1_A = f(c)$$

Hence the name.

"with respect to coordinates"

in next steps

That means that for bialgebra $\mathcal{H}$, a linear map $S \in Hom(\mathcal{H}^C, \mathcal{H}^A)$ is an antipode iff it satisfies

$$
\begin{aligned}
\Delta(x_{i_0} \ldots x_{i_k}) &= \Delta m^{[k]}(x_{i_0} \otimes \cdots \otimes x_{i_k}) \\
&= (m^{[k]} \otimes m^{[k]}) \left(\sum (x_{i_0})_1 \otimes \cdots \otimes (x_{i_k})_1 \otimes (x_{i_0})_2 \otimes \cdots \otimes (x_{i_k})_2 \right) \\
&= \sum (x_{i_0})_1 \ldots (x_{i_k})_1 \otimes (x_{i_0})_2 \ldots (x_{i_k})_2 \\
&= \sum_{S \subseteq \{i_0, \ldots i_k\}} \prod_{j \in S} x_j \otimes \prod_{j \notin S} x_j.
\end{aligned}
$$

Let $\nu = (\nu_1, \ldots \nu_N) \in \mathbb{N}^N$ be our deck of N cards (we just pick N cards from $\mathcal{X}$ in gene

Let $\nu = (\nu_1, \ldots \nu_N) \in \mathbb{N}^N$ be our deck of N cards (we just pick N cards from $\mathcal{X}$ in general case they can repeat

comes with certain probability.

we dont know how exactly these stacks looks like but

if we have two stacks (we will refer to them as (L)eft and (R)ight).

and we have with probability p s_1 on L and s on R and with probability $1 - p$ s_2 on L and s on R is exactly the same situation as having

Note that for all $k_1, \ldots, k_n \in K$, such that at least one of them is non-zero, an expression $\sum_{i=0}^{n} k_i ($

anallogly

Let denote that pulling apart as a $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, which for all $x_{i_0}, \ldots, x_{i_n}$ gives

$$
\Delta(x_{i_0} \cdots x_{i_n}) = \sum_{\substack{S \subseteq \{i_0, \ldots i_n\} \\ S = \{i_{j_1}, \ldots, i_{j_l}\} \\ S^c = \{i_{k_1}, \cdots, i_{k_{n-l}}\}}} x_{i_{j_1}} \cdots x_{i_{j_l}} \otimes x_{i_{k_1}} \cdots x_{i_{k_{n-l}}}.
$$

For putting two piles back together by placing left on the top let us write a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that is concatenation, which means, that for all $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_l} \in \mathcal{X}$

$$
m(x_{i_1} \ldots x_{i_k} \otimes x_{j_1} \ldots x_{j_l}) = x_{i_1} \ldots x_{i_k} x_{j_1} \ldots x_{j_l}.
$$

Let denote that pulling apart as a $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, then for all $s \in \mathcal{X}^*$ it will give

$$
\Delta(s) = \sum_{\substack{s_1 \prec s \\ s_2 = s/s_1}} s_1 \otimes s_2.
$$

For putting two piles back together by placing left on the top let us write a linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ that is concatenation, which means, that for all $s_1, s_2 \in \mathcal{X}^*$

$$
m(s_1 \otimes s_2) = s_1 s_2.
$$

all $s = x_{i_1} \ldots x_{i_n} \in \mathcal{X}^*$ that meet the condition:

$$\mathbb{P}F_{n+1} = s_2 \mid F_n = s_1 = \mathbb{I}_{\ltimes + \Join} = \sim_{\Join} \mid \mathbb{I}_{\ltimes} = \sim_{\nJoin}.$$

, we percive it as a certain amount of stacks
\\ Indeed it turned out that a vector space of
At this point we can start to think that it is convinient to interprete a space of possible arragements of two stacks of cards as a $\mathcal{H} \otimes \mathcal{H}$.

So we need a vector space with basis made of pairs of basis vectors from $\mathcal{H}$ with actions on them that are linear to both coordinates. A tensor product $\mathcal{H} \otimes \mathcal{H}$ is the less-degenerated vector space with that properties.

We will now show, that what we just had defined indeed give the same results as standard model of inverse riffle shuffling. In inverse riffle shuffling
Which also equvalent to that matrixes of $(F_i)_{i \geqslant 0}$ and $(I_i)_{i \geqslant 0}$ are transpositions of each other.
a \ big! \ huge! \ giant!
We will be working of an examples of riffle shuffle and inverse riffle shuffle cards shuffling as our Markov chains.
How we will put them in the algebraic way?
First we will do this with inverse riffle shuffle, the forward riffle shuffle will then apear in a natural way.

Earlier it was said that comultiplication can be sometimes viewed as a sum of possible divisions into smaller objects. It happends naturally when we are working with graded bialgebras. Like in example of polinomials, where natural grading is by degree.

0.0.1 Graded, connected Hopf algebra of non-commuting variables

This is a main example of our interest.
Let K be a field with characteristic 0. Let $\mathcal{X} = \{x_1, \ldots, x_N\}$ be a finite set. For every $n \in \mathbb{N}$ let H_n be a vector space having as a basis all words of length n made of elements of $\mathcal{X}$. Let $\mathcal{H} := \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. Let $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ be concatenation of words, that is, for all $k, l \in \mathbb{N}$ for all $x_{i_0}, \ldots, x_{i_k}, x_{j_0}, \ldots, x_{j_l} \in \mathcal{X}$

$$m(x_{i_0} \ldots x_{i_k} \otimes x_{j_0} \ldots x_{j_l}) := x_{i_0} \ldots x_{i_k} x_{j_0} \ldots x_{j_l}.$$

Let $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be defined for all elements from $\mathcal{X}$ as

$$\Delta(x_i) = x_i \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_i.$$

and extends lineary and multiplically
Lemma. Then $\mathcal{H}$ is the a graded, connected Hopf algebra that is cocomutative.

Proof. Associativity of m and coassociativity of Δ are obvious. Actions fit together, because we define them so. Algebra is graded and connnected because it is. $\square$

We can see $\Delta(x_{i_0} \ldots x_{i_k})$ how looks like:

$$
\begin{aligned}
\Delta(x_{i_0} \ldots x_{i_k}) &= \Delta m^{[k]}(x_{i_0} \otimes \cdots \otimes x_{i_k}) \\
&= (m^{[k]} \otimes m^{[k]}) \left(\sum (x_{i_0})_1 \otimes \cdots \otimes (x_{i_k})_1 \otimes (x_{i_0})_2 \otimes \cdots \otimes (x_{i_k})_2 \right) \\
&= \sum (x_{i_0})_1 \ldots (x_{i_k})_1 \otimes (x_{i_0})_2 \ldots (x_{i_k})_2 \\
&= \sum_{S \subseteq \{i_0, \ldots i_k\}} \prod_{j \in S} x_j \otimes \prod_{j \notin S} x_j.
\end{aligned}
$$

where S is a multiset, because some of the $i_0, \dot{i}_k$ can be the same. Form of that coproduct is like that because when for all $x_i \in \mathcal{X}$ we have $\Delta(x_i) = x_i \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_i$ then

a ! ! ! ! !

1 Summarize

$\sum \bigoplus + \oplus$

2 Productise

$\prod \otimes \odot \times *$

To jest źle:

Please note, that at first it can be not clear that $h^*m \in \mathcal{H}^* \otimes \mathcal{H}^*$. For sure $h^* \in (\mathcal{H} \otimes \mathcal{H})^*$ but in general case we have $\mathcal{H}^* \otimes \mathcal{H}^* \subseteq (\mathcal{H} \otimes \mathcal{H})^*$ (with inclusion given by canonical injection) but not necessarily $\mathcal{H}^* \otimes \mathcal{H}^* = (\mathcal{H} \otimes \mathcal{H})^*$. Although we are in graded bialgebra $\mathcal{H}$ and because of that we know that $h^*m \in \left(\bigoplus_{n=0}^{\infty} \mathcal{H}_n \otimes \mathcal{H}_n \right)^* = \bigoplus_{n=0}^{\infty} (\mathcal{H}_n \otimes \mathcal{H}_n)^*$. Futher, for all $n \in \mathbb{N}$ since $\mathcal{H}_n$ is finite-dimentional we have that $(\mathcal{H}_n \otimes \mathcal{H}_n)^* = \mathcal{H}_n^* \otimes \mathcal{H}_n^*$. Hence

$$
h^*m \in \bigoplus_{n=0}^{\infty} \mathcal{H}_n^* \otimes \mathcal{H}_n^* = \mathcal{H}^* \otimes \mathcal{H}^*.
$$

Now we will describe how a dual algebra to $\mathcal{H}$ looks like.

Let $\mathcal{H}^*$ be the dual vector space to $\mathcal{H}$. (A space of linear functions from $\mathcal{H}$ to K). We define multiplication $\Delta^* : \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H}^*$ and comultiplication $m^* : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*$ as (for all $h_1^*, h_2^*, h^* \in \mathcal{H}^*$):

$$
\begin{aligned}
\Delta^*(h_1^* \otimes h_2^*) &= (h_1^* \otimes h_2^*)\Delta, \\
m^*(h^*) &= h^*m.
\end{aligned}
$$

Structure detail described in this paragraph will be important in chapter about eigenbasises. [GR89] shows that symmetrized sums of certain primitive elements form basis of a free associative algebra. It will turn out that this will be left eigenbasis of $m\Delta$. Now we will introduce methods for construction of that

basis. Explanation why this is an eigenbasis will came in Chapter 4. This whole paragraph is exactly the same as in [**?**] I put it here because it is quite short and

$$\Psi^{[i]}(h) = m^{[i]}\Delta^{[i]}(h) =$$

$$m^{[i]}\big(\sum h_1 \otimes \cdots \otimes h_i\big) =$$

$$m^{[i]}\big(\overbrace{h \otimes 1_{\mathcal{H}} \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}} + \overbrace{1_{\mathcal{H}} \otimes h \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}} + \cdots + \overbrace{1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes h}^{i \text{ factors}}\big) =$$
$$\underbrace{\phantom{m^{[i]}\big(h \otimes 1_{\mathcal{H}} \otimes \cdots \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes h \otimes \cdots \otimes 1_{\mathcal{H}} + \cdots + 1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes h\big)}}_{i \text{ summands}}$$

$$\underbrace{m^{[i]}\big(\overbrace{h \otimes 1_{\mathcal{H}} \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}}\big) + m^{[i]}\big(\overbrace{1_{\mathcal{H}} \otimes h \otimes \cdots \otimes 1_{\mathcal{H}}}^{i \text{ factors}}\big) + \cdots + m^{[i]}\big(\overbrace{1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes h}^{i \text{ factors}}\big)}_{i \text{ summands}} =$$

For simplifying the notation we will write a symbol for algebra multiplication also for componentwise multiplication, so for all $^1h, \ldots, {}^nh \in \mathcal{H}$:

$$\Delta^{[m]}m^{[n]}\big(^1h \otimes \cdots \otimes {}^nh\big) ==: \sum \tag{5}$$

Proof.

$$\Psi^{[a]}\big(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)}\big) =$$

$$m^{[a]}\Delta^{[a]}\big(\sum_{\sigma \in S_k} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)}\big) =$$

$$m^{[a]}\big(\sum_{\sigma \in S_k} \Delta^{[a]}\big(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)}\big)\big) =$$

$$m^{[a]}\big(\sum_{\sigma \in S_k} \big(x_{\sigma(1)} \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_{\sigma(1)}\big) \cdot \ldots \cdot \big(x_{\sigma(k)} \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x_{\sigma(k)}\big) =$$

$$\square$$

$$\Delta([x,y]) = \Delta(x \cdot y - y \cdot x) = \Delta(x \cdot y) - \Delta(y \cdot x) = \Delta m(x \otimes y) - \Delta m(y \otimes x) = \sum x_1 \cdot y_1 \otimes x_2 \cdot y_2 - \sum y_1 \cdot$$