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## Two dimensional orbifolds' volumes' spectrum

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*dla Wujka*



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## Abstract

Orbifoldy

# Chapter 1

## Introduction

### 1.1 Motivations

Quotients by a groups

Praca o ujemnym ale z najwyższą orbicharakterystyką

### 1.2 Definitions

## TO DO: jak sie juz wszystko zbierze co ma tu być, to to dopisać

The definition of the orbifold is taken from Thurston [8] (chapter 13). We briefly recall the concept, but for full discussion we refer to [8].

An orbifold is a generalisation of a manifold. One allows more variety of local behaviour. On a manifold a map is a homeomorphism between  $\mathbb{R}^n$  and some open set on a manifold. On an orbifold a map is a homeomorphism between a quotient of  $\mathbb{R}^n$  by some finite group and some open set on an orbifold. In addition to that, the orbifold structure consist the informations about that finite group and a quotient map for any such open set.

Above definition says that an orbifold is locally homeomorphic do the quotient of  $\mathbb{R}^n$  by some finite group.

When an orbifold as a whole is quotient of some finite group acting on a manifold we say, that it is 'good'. Otherwise we say, that it is 'bad'.

We are also adopting notation from [8].

In two dimentions there are only four types of bad orbifolds, namely:

- $S^2(n)$
- $D^2(; n)$
- $S^2(n_1, n_2)$  for  $n_1 < n_2$
- $D^2(; n_1, n_2)$  for  $n_1 < n_2$ .



All other orbifolds are good. As manifolds are special case of orbifolds with all ... We differ from Thurston in the terms of naming points with maps with non-trivial groups. We call them orbipoints. If the group acts as the group of rotations (so a cyclic group) we call them rotational points. If the group is a dihedral group we call them dihedral points. And if it is point on the boundry that stabilises reflection it is a reflection point.

quotients of groups

Map between orbifolds can be defined as follows: [6]

### 1.2.1 Sameness

From this we can treat that and that orbifolds as the same.

## 1.3 Euler (orbi)characteristic

# TO DO: dać cytowanie do charakterystyki

quotients

### 1.3.1 Euler characteristic

We can define Euler characteristic as additive topological invariant defined normalised on simplex.

we will treat as we will treat manifolds as orbifolds we will always refer we will from of Euler orbicharacteristic on two dim orbifolds

### 1.3.2 Classification of orbifolds with non-negative Euler orbicharacteristic

The list of all orbifolds with non-negative Euler orbicharacteristic Powiedzieć coś o tym, że orbicharakterystyka odpowiada polom (Gauss Bonnet itd.)

### 1.3.3 Extended Euler orbicharacteristic

(with cusps) Write about cusp as a limit.

Write about isomorphism of all spectra

$M$  - orbifold

### 1.3.4 Spectra

This will be the main interest of this thesis.  $\sigma$ . dadada

### 1.3.5 Definition and properties of order of accumulation points

Correspondence between accumulation points and cantor bendixon derivative are describen in appendix

**To do: link to appendix**

. We start with definition of being "at least of order  $n$ " that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.

For a given set we define as follows:

**Definition 1.3.5.1.** (*Inductive*). We say that the point  $x$  is an accumulation point of a set  $X$  of order at least 0, when it belongs to the set  $X$ . We say that the point  $x$  is an accumulation point of a set of order at least  $n + 1$ , when it is an accumulation point (in the usual sense) of the accumulation points each of order at least  $n$  i.e. in every neighbourhood of  $x$  there is at least one accumulation point of a set  $X$  of order at least  $n$ , distincs from  $x$ .

**Definition 1.3.5.2.** We say that the point is an accumulation point of order  $n$  iff it is an accumulation point of order at least  $n$  and it is not an accumulation point of order at least  $n + 1$ . If the point is an accumulation point of order at least  $n$  for an arbitrary large  $n$  we say that the point is an accumulation point of order  $\omega$ .

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the usual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

## 1.4 Uniformisation theorem (formulation)

# TO DO: twierdzenie o klasyfikacji powierzchni

## 1.5 Operations and constructions on orbifolds

Write about the general operations we are interested in i.e. taking any number of features (handles cross caps, parts of boundry components with orbipoints on it, orbipoints in the interior) and replacing it by any other feauters (Some preserve the area) Write about operations nesseserie for reduction of cases write that every operation reduces Euler orbicharacteristic.

## 1.6 Notation

"feature"

We treat manifolds and orbifolds as a sphere with some features added by the operations.

dać na sferę  $\varepsilon$  słowo puste. We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature.

We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as  $\Delta(modification)$ .

## TO DO: rozwinać

dopisać, że w Conwayowej  $\geq 2$

If not stated otherwise, in the expressions containing  $\infty$  symbol, their value is understood as  $\varphi(\infty) := \lim_{n \rightarrow \infty} \varphi(n)$ .

## TO DO:

Addition of sets and numbers.

Warning throught the whole thesis we will consider only two dimensional manifolds and orbifolds, because of that words "two-dimensional" will be usually omitted

delta h c b

## 1.7 Questions asked

There will be two main parts of question:

- Ones regarding  $\sigma$  as a set, where we will be asking of its order type and topology and relation to other sets such as  $\sigma(D^2)$  and  $\sigma(S^2)$ . We will focus on these questions in 3.

- Ones regarding  $\sigma$  as an image of a  $\chi^{orb}$ , sending orbifolds to their Euler orbicharacteristics. There, we will ask for example how many orbifolds have particular Euler orbicharacteristic and related questions. We will focus on these questions in the chapter 4.

Reduction presented in 2 are with .

# Chapter 2

## Reduction to arithmetical questions

Reductions presented in this chapter will be more in the spirit of chapter 3, in the sense that for now, until chapter 4, we will not pay attention to how many orbifolds have the same Euler orbicharacteristic, only whether a particular number is an Euler orbicharacteristic for at least one orbifold or not.

In chapter 4 we will explain how these reductions will be relevant to the discussion holded there.

### 2.1 Reductions of cases

The aim of following reductions is to make it easier to answer the question of which points lie in  $\sigma$  and which not.

The first problem with the structure of  $\sigma$  is that it is the sum of  $\sigma(M)$ , for every two dimensional manifold  $M$ .

$$\sigma = \bigcup_{M : 2d \text{ manifold}} \sigma(M). \quad (2.1.0.0.1)$$

We aim to find a minimal set  $\mathcal{M}$  of base manifolds such that:

$$\sigma = \bigcup_{M \in \mathcal{M}} \sigma(M). \quad (2.1.0.0.2)$$

It will turn out that  $\mathcal{M} = \{S^2, D^2\}$  satisfies 2.1.0.0.2 and that both  $S^2$  and  $D^2$  are necessary.

### 2.2 Sufficiency of $S^2$ and $D^2$

Given an orbifold  $O_1$ , we want to perform some operations from 1.5 on it, such that the resulting orbifold  $O_2$  will have the same Euler orbicharacteristic, but the base manifold of  $O_2$  would be  $S^2$  or  $D^2$ . We would then say, that  $O_1$  got reduced to  $O_2$ . In following subsection, we allow only such operations, that do not change Euler orbicharacteristic. When writing that we "can" do something we mean that there is possible one of the operations from 1.5.

The Euler characteristic of base manifold depends only on the number of handles, cross caps and boundry components. And, as stated in 1.3 it is:

$$2 - 2h - c - b, \quad (2.2.0.0.1)$$

for  $h$  - number of handles,  $c$  - number of cross-caps,  $b$  - number of boundary components.

For every such a manifold feature we want to find an orbifold features with the same Euler orbicharacteristic delta.

We will take to approuches, depending on whether the orbifold in question has a boundary or not.

### 2.2.1 Orbifold without boundary

We can observe that:

$$\Delta(\circ) = -2 = \Delta(2^4) \quad (2.2.1.0.1)$$

$$\Delta(\times) = -1 = \Delta(2^2) \quad (2.2.1.0.2)$$

From this we can see that we can remove handles and cross-caps from any orbifold without the boundary. After such reductions we are left with a  $S^2$  orbifold with all orbipoints being rotational in the interior.

### 2.2.2 Orbifold with boundary

7 We can observe that:

$$\Delta(\circ) = -2 = \Delta((^*2)^8) \quad (2.2.2.0.1)$$

$$\Delta(*) = -1 = \Delta((^*2)^4) \quad (2.2.2.0.2)$$

$$\Delta(\times) = -1 = \Delta((^*2)^4) \quad (2.2.2.0.3)$$

From this we can see that we can remove handles and cross-caps from any orbifold with a boundary. We can also remove all boundary components exept one. We can further observe that:

$$\Delta(n) = \frac{n-1}{n} = 2\frac{n-1}{2n} = \Delta((^*n)^2) \quad (2.2.2.0.4)$$

From this we see that we can remove all the rotational orbipoints in favor for dihedral orbipoints. After such reductions we are left with a  $D^2$  orbifold with all orbipoints being dihedral on the boundary or being reflectional on the boundary.

As a fact not necessary for our reductions, but interestung on its own, we can furthermore, observe that:

**Observation 2.2.2.1.** *If  $O_1$  has not  $S^2$  as its base manifold it can be reduced to a  $D^2$ -orbifold.*

**Proof.**

If  $O_1$  has not  $S^2$  as its base manifold  $M$ , then  $M$  has at least one handle or a cross-cup. We can observe that:

$$\Delta(\circ) = -2 = \Delta(*2^4) \quad (2.2.2.1.1)$$

$$\Delta(\times) = -1 = \Delta(*). \quad (2.2.2.1.2)$$

From this we have that the handle or the cross-cap can be replaced by a boundary component and some number of boundary orbipoints. After this reduction, we can proceed with all the other reductions from the 2.2.2 and obtain an  $D^2$ -orbifold with the same Euler orbicharacteristicas the original one.  $\square$

The results of our reductions, can be summarised as:

**Observation 2.2.2.2.** *If two-dimentional manifold  $M$  has no boundry, then*

$$\sigma(M) \subseteq \sigma(S^2) \quad (2.2.2.2.1)$$

*If, in addition,  $M \neq S^2$ , then*

$$\sigma(M) \subseteq \sigma(D^2). \quad (2.2.2.2.2)$$

**Observation 2.2.2.3.** *If two-dimentional manifold  $M$  has a boundry, then*

$$\sigma(M) \subseteq \sigma(D^2) \quad (2.2.2.3.1)$$

**Corollary 2.2.2.4.** *We have that  $\sigma = \sigma(S^2) \cup \sigma(D^2)$ .*

We will postpone our discussion of neccessity of both  $S^2$  and  $D^2$  to 2.4, after the section 2.3 which will provide us with more convenient language.

## 2.3 Reduction to arithmetical questions

As written in 1.3.1, we can express an Euler orbicharacteristic of a  $M$ -orbifold  $O$  as:

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^n \frac{r_i - 1}{r_i} - \sum_{j=1}^m \frac{d_j - 1}{2d_j}, \quad (2.3.0.0.1)$$

where  $r_i$  and  $d_j$  are degrees of the, respectively, rotational and diheadral orbipoints of  $O$ .

From this we can express  $\sigma(M)$  as:

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^n \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0, r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} + \quad (2.3.0.0.2)$$

$$- \left\{ \sum_{j=1}^m \frac{d_j - 1}{2d_j} \mid m \in \mathbb{N}_0, d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \quad (2.3.0.0.3)$$

As from 2.2 we know that  $\sigma = \sigma(S^2) \cup \sigma(D^2)$ , and that  $\chi(S^2) = 2$  and  $\chi(D^2) = 1$ , we can express  $\sigma$  as a sum ( $\cup$ ) of two sets:

$$2 - \left\{ \sum_{i=1}^n \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0, r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(S^2) \quad (2.3.0.0.4)$$

and

$$1 - \left\{ \sum_{j=1}^m \frac{d_j - 1}{2d_j} \mid m \in \mathbb{N}_0, d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(D^2). \quad (2.3.0.0.5)$$

From this we see, that the core of understanding  $\sigma$  through arithmetical viewpoint is to understand possible values of expression:

$$2 - \sum_{i=1}^n \frac{r_i - 1}{r_i} \quad (2.3.0.0.6)$$

and

$$1 - \sum_{j=1}^m \frac{d_j - 1}{2d_j}, \quad (2.3.0.0.7)$$

with  $r_i$  and  $d_j$  ranging over  $\mathbb{N}_{>0} \cup \{\infty\}$ .

As stated in ?? we can perform further reductions to have an orbifold with particular orbicharacteristic without cusps (if needed) and then (after these reductions) we can analyse only expressions with  $r_i$  and  $d_j$  ranging over  $\mathbb{N}_{>0}$  and they will still give us full spectrum. However, as stated later, it will be more convenient to us to include orbifolds with cusps so we are stating this observation only as a side remark. The fact that this agrees with the definition of the Euler orbicharacteristic on the geometrical terms was addressed in 1.3.3.

**Observation 2.3.0.1.** *We have that  $\sigma(S^2) = 2\sigma(D^2)$ .*

**Proof.**

Indeed, since:

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^n \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0, r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \quad (2.3.0.1.1)$$

and

$$\sigma(D^2) = 1 - \left\{ \sum_{j=1}^m \frac{d_j - 1}{2d_j} \mid m \in \mathbb{N}_0, d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \square \quad (2.3.0.1.2)$$

## 2.4 Necessity of $S^2$ and $D^2$

As we know from 1.5 adding an orbipoint to a manifold decreases its orbicharacteristic. As  $S^2$  has the highest Euler characteristic - 2 of all two dimensional manifolds, there is no other orbifold with Euler orbicharacteristic equal to 2.  $S^2$  is then necessary to include 2.

As known from [?najwiÅŻkszyorbifold], the number  $-\frac{1}{84} \in \sigma(D^2)$  and it is the greatest negative Euler orbicharacteristic any two dimensional orbifold can have. We will now show, that  $-\frac{1}{84} \notin \sigma(S^2)$ . For the sake of contradiction let us assume, that  $-\frac{1}{84} \in \sigma(S^2)$ , then, from 2.3.0.1 we know, that  $\frac{1}{2} \left(-\frac{1}{84}\right) \in \sigma(D^2)$ . This is a contradiction as  $0 > \frac{1}{2} \left(-\frac{1}{84}\right) > -\frac{1}{84}$ .  $\square$

Futher examination of connections between  $\sigma(D^2)$  and  $\sigma(S^2)$  is performed in 3.5.1.

## 2.5 Translating questions to ones about Egyptian fractions

The term in the sums can be expressed as  $1 - \frac{1}{I_i}$ , then the sums become:  $2 - n + \sum_{i=1}^n \frac{1}{I_i}$ . This is very simmilar in notion to the egyptian fraction, which is a rational number that can be expressed as a sum of a fractions with numerator 1 and different denominators. Sometimes the "distinc denominators" condition is dropped and we will follow that convention here. The particular interest is in translating questions related to spectra to the questions of egyptian fractions. What the data of the spectra imposes is the number of fractions to be summed.



# Chapter 3

## Order type and topology

In this chapter we will discuss that both the order type and the topology of the set  $\sigma$  of all possible Euler orbicharacteristics of two-dimensional orbifolds are that of  $\omega^\omega$ .

To determine order type and topology of  $\sigma$  we will first study how  $\sigma(D^2)$  looks like. Then, remembering that  $\sigma = \sigma(S^2) \cup \sigma(D^2)$  and  $\sigma(S^2) = 2\sigma(D^2)$  we will make an argument for  $\sigma$ .

### 3.1 Order type and topology of $\sigma(D^2)$

In this section we will also describe precisely where accumulation points of  $\sigma(D^2)$  lie and of which order (see below 3.1.1 or 1.3.5) they are. Analysis of locations of those accumulation points, as interesting as it is alone will also be necessary for providing our argument about order type and topology of  $\sigma(D^2)$ .

#### 3.1.1 Definition and properties of order of accumulation points

These definitions are exactly the same as from 1.3.5 and are repeated here only for the readers convenience.

We start with definition of being "at least of order  $n$ " that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.

For a given set we define as follows:

**Definition 3.1.1.1.** (*Inductive*). We say that the point  $x$  is an accumulation point of a set  $X$  of order at least 0, when it belongs to the set  $X$ . We say that the point  $x$  is an accumulation point of a set of order at least  $n+1$ , when it is an accumulation point (in the usual sense) of the accumulation points each of order at least  $n$  i.e. in every neighbourhood of  $x$  there is at least one accumulation point of a set  $X$  of order at least  $n$ , distinct from  $x$ .

**Definition 3.1.1.2.** We say that the point is an accumulation point of order  $n$  iff it is an accumulation point of order at least  $n$  and it is not an accumulation point of order at least  $n+1$ . If the point is an accumulation point of order at least  $n$  for an arbitrary large  $n$  we say that the point is an accumulation point of order  $\omega$ .

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the usual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

### 3.1.2 Analysis of locations of accumulation points of $\sigma(D^2)$ with respect to their order

We want to determine where exactly are accumulation points of the set  $\sigma(D^2)$  with respect to their orders.

For this we will use a handful of observations and lemmas.

**Observation 3.1.2.1.** *Let us observe, that  $\lim_{n \rightarrow \infty} \Delta(*n) = -\frac{1}{2}$ . From that, we see, that for every point  $x \in \sigma(D^2)$ , the point  $x - \frac{1}{2}$  is an accumulation point. Let us observe, that also, for every point  $x \in \sigma(D^2)$ , we have that  $x - \frac{1}{2} \in \sigma(D^2)$ , because  $\Delta(*\infty) = -\frac{1}{2}$ .*

**Lemma 3.1.2.2.** *For all  $n \in \mathbb{N}_{\geq 2}$  and  $x \in (-\infty, 1]$  there are only finitely many Euler orbicharacteristics in the interval  $[x, 1] \cap \sigma(D^2)$  of orbifolds that have points of order equal at most  $n$ .*

**Proof.**

Let  $x \in (-\infty, 1]$ . There can be at most  $\lfloor 4(1-x) \rfloor$  orbipoints on the  $D^2$  orbifold with an Euler orbicharacteristic  $y \in [x, 1]$  since each orbipoint decreases an Euler orbicharacteristic by at least  $\frac{1}{4}$  and the Euler characteristic of  $D^2$  is 1.

There are only  $(n-1)^{\lfloor 4(1-x) \rfloor}$  possible sets of  $\lfloor 4(1-x) \rfloor$  orbipoints' orders that are less or equal than  $n$ . Hence, there are only at most  $(n-1)^{\lfloor 4(1-x) \rfloor}$  possible Euler orbicharacteristics.

**Lemma 3.1.2.3.** *If  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n$ , then  $x - \frac{1}{2}$  is a accumulation point of the set  $\sigma(D^2)$  of order at least  $n+1$ .*

**Proof.**

Inductive.

- $n = 0$ : If  $x$  is an isolated point of the set  $\sigma(D^2)$ , then  $x \in \sigma(D^2)$ . From that, we have, that points  $x - \frac{k-1}{2k}$  are in  $\sigma(D^2)$  for all  $k \geq 1$ , from that, that  $x - \frac{1}{2}$  is a accumulation point of  $\sigma(D^2)$ .

- inductive step: Let  $x$  be an accumulation point of the set  $\sigma(D^2)$  of an order  $n > 0$ . Let  $a_k$  be a sequence of accumulation points of order  $n-1$  convergent to  $x$ . From the inductive assumption, we have, that  $a_k - \frac{1}{2}$  is a sequence of accumulation points of order at least  $n$ . From the basic sequence arithmetic it is convergent to  $x - \frac{1}{2}$ . From that, we have that  $x - \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n+1$ .  $\square$

**Lemma 3.1.2.4.** *If  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n$ , then  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n-1$ .*

**Proof.**

Inductive

- $n = 1$ : We assume, that  $x$  is an accumulation point of isolated points of the set  $\sigma(D^2)$ . From 3.1.2.2 we know, that for all  $m$  there are only finitely many Euler orbicharacteristics in the interval  $[x, 1]$  of orbifolds that have dihedral points of order equal at most  $m$ .

From that, for arbitrary small neighborhood  $U \ni x$  and arbitrary large  $m$  there exist an orbifold that has a dihedral point of period greater than  $m$ , whose Euler orbicharacteristic lies in  $U$ . Let us take a sequence of such Euler orbicharacteristics  $a_k$  convergent to  $x$ , such that we can choose a sequence divergent to infinity of periods of dihedral points  $b_k$  of orbifolds of Euler orbicharacteristics equal  $a_k$ .

**To do: picture**

Let us observe, that for all  $k$ , the number  $a_k + \frac{b_k-1}{2b_k}$  is in  $\sigma(D^2)$ . It is so, because  $a_k$  is an Euler orbicharacteristic of an orbifold that have a dihedral point of period  $b_k$ , so identical orbifold, only without this dihedral point, has an Euler orbicharacteristic equal to  $a_k + \frac{b_k-1}{2b_k}$ . The sequence  $a_k + \frac{b_k-1}{2b_k}$  converge to  $x + \frac{1}{2}$ . From that we have, that  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order at least 0.

- inductive step: Let  $x$  be an accumulation point of the set  $\sigma(D^2)$  of order  $n > 1$ . Let  $a_k$  be a sequence of accumulation points of the set  $\sigma(D^2)$  of order  $n-1$  convergent to  $x$ . From the inductive assumption the sequence  $a_k + \frac{1}{2}$  is a sequence of an accumulation points of the set  $\sigma(D^2)$  of order  $n-2$  convergent to  $x + \frac{1}{2}$ . From that  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n-1$ .  $\square$

**Lemma 3.1.2.5.** *If  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n+1$ , then*

*$x - \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n+2$  and  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n$ .*

**Proof.**

Let  $x$  be an accumulation point of the set  $\sigma(D^2)$  of order  $n+1$ . From the lemma 3.1.2.3 we know, that  $x - \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n+2$ . Now let us assume (for a contradiction), that  $x - \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order  $k > n+2$ . But then from the lemma 3.1.2.4 we have that  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n+2$  and that is a contradiction.

Analogously, from the lemma 3.1.2.4 we know, that  $x + \frac{1}{2}$  is a accumulation point of the set  $\sigma(D^2)$  of order at least  $n$ . Let us assume (for a contradiction), that  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order  $k > n$ . But then from the lemma 3.1.2.3 we have that  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order at least  $n+2$  and that is a contradiction.  $\square$

**Lemma 3.1.2.6.** *For all  $n \in \mathbb{N}$  all accumulation points of the set  $\sigma(D^2)$  of order  $n$  are in  $\sigma(D^2)$ .*

**Proof.**

Inductive

- $n = 0$ : Clear, as they are isolated points of  $\sigma(D^2)$ .

- inductive step: Let  $x$  be a accumulation point of the set  $\sigma(D^2)$  of order  $n > 0$ . From the lemma 3.1.2.5 point  $x + \frac{1}{2}$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n - 1$ . From the inductive assumption  $x + \frac{1}{2} \in \sigma(D^2)$ . Then, from 3.1.2.1, we have that  $x \in \sigma(D^2)$ .  $\square$

**Theorem 3.1.2.7.** *The greatest accumulation point of the set  $\sigma(D^2)$  of order  $n$  is  $1 - \frac{n}{2}$ .*

**Proof.**

Inductive

- $n = 0$ : We know, that  $1 \in \sigma(D^2)$  and 1 is the greatest element of  $\sigma(D^2)$ .
- an inductive step: From the inductive assumption we know that  $1 - \frac{n}{2}$  is the greatest accumulation point of the set  $\sigma(D^2)$  of order  $n$ . From the lemma 3.1.2.5 we have then that  $1 - \frac{n+1}{2}$  is a accumulation point of the set  $\sigma(D^2)$  of order  $n + 1$ . Let us assume (for a contradiction), that there exist a bigger accumulation point of order  $n + 1$  equal to  $y > 1 - \frac{n+1}{2}$ . But then, from lemma 3.1.2.5, point  $y + \frac{1}{2}$  would be an accumulation point of order  $n$ , what gives a contradiction, because  $y + \frac{1}{2} > 1 - \frac{n}{2}$ .  $\square$

From the above discussion we can also formulate following corollary that will be useful later:

**Corollary 3.1.2.8.** *Let  $x \in \sigma(D^2)$ . Then:*

- there exists  $n_1 \in \mathbb{N}$  such that  $x + \frac{n_1}{2} \in \sigma(D^2)$  but  $x + \frac{n_1+1}{2} \notin \sigma(D^2)$ .  
In other words, there exist  $y \in \sigma(D^2)$  and  $n_1 \in \mathbb{N}$  such that  $y + \frac{1}{2} \notin \sigma(D^2)$  and such that  $x = y - \frac{n_1}{2}$ ;
- there exists  $n_2 \in \mathbb{N}$  such that  $x$  is an accumulation point of the set  $\sigma(D^2)$  of order  $n_2$

and  $n_1 = n_2$ .

### 3.1.3 Proof that $\sigma(D^2)$ is well ordered

**Definition 3.1.3.1.** *Let  $B_0 = \{1\}$ . For an  $n \in \mathbb{N}_{>0}$ , let  $B_n$  be the set of all possible Euler orbicharacteristic realised by orbifolds of type  $*b_1, \dots, *b_n$ . For a given  $n$  these are  $D^2$  orbifolds with precisely  $n$  non trivial orbipoits on their boundry.*

**Observation 3.1.3.2.** *There is a recursive relation, that  $B_{n+1} = B_n + \{-\frac{n-1}{2n} \mid n \geq 2\}$*

**Proof.**

It is so, because every orbifold with  $n + 1$  orbipoits can be obtained by adding one point to an orbifold with  $n$  orbipoits and the set  $\{-\frac{n-1}{2n} \mid n \geq 2\} = \{\Delta(*b) \mid b \geq 2\}$ .

$\square$

**Observation 3.1.3.3.** *Observe that, as any orbifold has only finitely many orbipoits, we have that  $\sigma(D^2) \subseteq \bigcup_{n=0}^{\infty} B_n$ . We defined  $\sigma(D^2)$  as a set of all possible Euler orbicharacteristic of disk orbifolds, so  $\sigma(D^2) \supseteq \bigcup_{n=0}^{\infty} B_n$ . From this we have that  $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$ .*

**Lemma 3.1.3.4.** *For any given  $n \in \mathbb{N}$  the set  $B_n$  is a subset of the interval  $[1 - \frac{n}{2}, 1 - \frac{n}{4}]$ .*

**Proof.**

Take  $x \in B_n$ . There exists an orbifold  $O$  with signature  $*b_1, \dots, *b_n$ , such that  $\chi^{orb}(O) = x$ . We have that  $\forall_i -\frac{1}{2} \leq \Delta(*b_i) \leq -\frac{1}{4}$ . From this  $-\frac{n}{2} \leq \Delta(*b_1, \dots, *b_n) \leq -\frac{n}{4}$ , so  $\chi^{orb}(O) \in [1 - \frac{n}{2}, 1 - \frac{n}{4}]$ .  $\square$

**Observation 3.1.3.5.** *From 3.1.3.2 and A.1.0.1, we have that  $B_n$  do not have infinite ascending sequence for all  $n$ .*

*Further, from A.1.0.2 we conclude, that  $\bigcup_{n=0}^N B_n$  do not have infinite ascending sequence for all  $N$ .*

**Theorem 3.1.3.6.** *In  $\sigma(D^2)$  there are no infinite strictly ascending sequences, hence, it is well ordered.*

**Proof.**

For the sake of contradiction let's assume that  $c_n$  is an infinite strictly ascending sequence in  $\sigma(D^2)$ . As  $c_n$  is bounded from below by  $c_0$  and whole  $\sigma(D^2)$  is bounded from above by 1, all elements of  $c_n$  are in the interval  $[c_0, 1]$ . From 3.1.3.3 we have, that  $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$ .

Lemma 3.1.3.4 says that for all  $n$  we have  $B_n \subset [1 - \frac{n}{2}, 1 - \frac{n}{4}]$ . From this, we know, that for any  $n$  such that  $1 - \frac{n}{4} < c_0$  we have, that  $B_n \cap [c_0, 1] = \emptyset$ . Let  $n_0$  be the smallest such that  $1 - \frac{n_0}{4} < c_0$  (so  $n_0 > 4(1 - c_0)$ ). Then for all  $n > n_0$  we have  $1 - \frac{n}{4} < c_0$ , meaning, that for all  $n > n_0$  we have  $B_n \cap [c_0, 1] = \emptyset$ , so all elements of  $c_n$  are in  $\bigcup_{n=0}^{n_0} B_n$ . But this contradicts 3.1.3.5.  $\square$

### 3.1.4 Proof that order structure and topology of $\sigma(D^2)$ are those of $\omega^\omega$

**Theorem 3.1.4.1.** *Order type and topology of  $\sigma(D^2)$  is  $\omega^\omega$ .*

**Proof.**

We will first prove, that the order type of  $\sigma(D^2)$  is  $\omega^\omega$ .

- Order type of  $\sigma(D^2)$  is at least  $\omega^\omega$ .

From 3.1.2.7 we know, that for every  $n \in \mathbb{N}$ , in  $\sigma(D^2)$  there are accumulation points of order  $n$ . From this, and from A.1.0.4 we know that  $\sigma(D^2)$  has an order type at least  $\omega^n$ , for all  $n \in \mathbb{N}$ . The smallest ordinal number equal at least  $\omega^n$ , for all  $n \in \mathbb{N}$  is  $\omega^\omega$ . Thus, the order type of  $\sigma(D^2)$  is at least  $\omega^\omega$ .

- Order type of  $\sigma(D^2)$  is at most  $\omega^\omega$ .

For the sake of contradiction, let us suppose, that the order type  $\eta$  of  $\sigma(D^2)$  is strictly greater than  $\omega^\omega$ . Then,  $\sigma(D^2)$  has a set  $A$  of an order type  $\omega^\omega$  as it's prefix. The set  $A$  is bounded, as the  $\omega^\omega + 1$ st element of  $\sigma(D^2)$  is greater than any element of  $A$ . Let  $n$ , be such that  $1 - \frac{n}{2}$  is smaller than any element of  $A$ . As  $A$  is of order

type  $\omega^\omega$  it has a prefix  $B$  of order type  $\omega^n$ . From A.1.0.4 we know, that  $B$  has an accumulation point  $b$  of order  $n$ . This gives us a contradiction, as  $b > 1 - \frac{n}{2}$ , and from 3.1.2.7 we know, that  $1 - \frac{n}{2}$  is the greatest accumulation point of order  $n$  in  $\sigma(D^2)$ .

Now, we will prove, that the topology of  $\sigma(D^2)$  is that of  $\omega^\omega$ .

From 3.1.2.6 we know that every accumulation point of  $\sigma(D^2)$  is in  $\sigma(D^2)$ . Thus,  $\sigma(D^2)$  satisfies the assumptions of the lemma A.1.0.3 and we have that the topology of  $\sigma(D^2)$  is  $\omega^\omega$ .

## 3.2 Order type and topology of $\sigma(M)$

**Observation 3.2.0.1.** *For every two dimensional manifold  $M$ , we have that  $\sigma(M)$  is homeomorphic to  $\sigma(S^2)$ . For  $M$  with  $h$  handles,  $c$  cross-cups and  $b$  boundary components, this homeomorphism is:*

- for  $b \neq 0$ :

$$\sigma(M) = \sigma(D^2) - 2h - c - (b - 1), \quad (3.2.0.1.1)$$

- for  $b = 0$ :

$$\sigma(M) = 2\sigma(D^2) - 2h - c. \quad (3.2.0.1.2)$$

**Proof.**

For a manifold  $M$  with  $h$  handles,  $c$  cross-cups and  $b$  boundary components, it's Euler characteristic is given by:

$$\chi(M) = 2 - 2h - c - b. \quad (3.2.0.1.3)$$

The possible  $\Delta$  for possible orbifold features are:

- for  $b \neq 0$ :

$$\left\{ -\frac{n-1}{2n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \quad (3.2.0.1.4)$$

- for  $b = 0$ :

$$\left\{ -\frac{n-1}{n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \quad (3.2.0.1.5)$$

Thus, we have that:

- for  $b \neq 0$ :

$$\sigma(M) = 2 - 2h - c - b - \left\{ \sum_{i=1}^n \frac{d_i - 1}{2d_i} \mid n \in \mathbb{N}_0, d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \quad (3.2.0.1.6)$$

- for  $b = 0$ :

$$\sigma(M) = 2 - 2h - c - \left\{ \sum_{i=1}^n \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0, r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \quad (3.2.0.1.7)$$

On the other hand, we have that:

$$\sigma(D^2) = 1 - \left\{ \sum_{i=1}^n \frac{d_i - 1}{2d_i} \mid n \in \mathbb{N}_0, d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \quad (3.2.0.1.8)$$

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^n \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0, r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \quad (3.2.0.1.9)$$

and

$$\sigma(S^2) = 2\sigma(D^2). \quad (3.2.0.1.10)$$

From this, the observation follows immedietly.  $\square$

### 3.3 Order type and topology of $\sigma$

**Theorem 3.3.0.1.** *The order type of the set of possible Euler orbicharacteristics of two-dimensional orbifolds  $\sigma$  is  $\omega^\omega$ .*

**Proof.**

From 2.2.2.2 we know, that  $\sigma = \sigma(D^2) \cup \sigma(S^2)$ .

From 3.1.4.1 and ??, we have that order types and topologies of  $\sigma(D^2)$  and  $\sigma(S^2)$  both are  $\omega^\omega$  and that  $\sigma(S^2) = 2\sigma(D^2)$ .

We will now prove that the order type of  $\sigma$  is  $\omega^\omega$ .

From 3.1.2.7 we know, that the largest accumulation point of the set  $\sigma(D^2)$  of order  $n$  is  $1 - \frac{n}{2}$ . From, this and from the fact that  $\sigma(S^2) = 2\sigma(D^2)$  we know that that the largest accumulation point of the set  $\sigma(S^2)$  of order  $n$  is  $2 - n$ . From this, we have, that for every  $m \in \mathbb{N}_{>0}$ , order type of  $(-m, \infty) \cap \sigma(D^2)$  is  $\omega^{2m+2}$  and that order type of  $(-m, \infty) \cap \sigma(S^2)$  is  $\omega^{m+2}$ . Thus, for every  $m \in \mathbb{N}_{>0}$ , we have that  $(-m, \infty) \cap \sigma(D^2)$  and  $(-m, \infty) \cap \sigma(S^2)$  satisfies assumptions of A.1.0.8, thus, we have that  $(-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2))$  have an order type  $\omega^{2m+2}$ .

From this we have that

$$\sigma = \sigma(D^2) \cup \sigma(S^2) = \bigcup_{m=1}^{\infty} \left( (-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2)) \right) \quad (3.3.0.1.1)$$

have an order type  $\omega^\omega$ .

Now we will prove, that the topology of  $\sigma$  is that of  $\omega^\omega$  too.

We have that for  $\sigma(D^2) \left[ \sigma(S^2) \right]$  that every accumulation point of  $\sigma(D^2) \left[ \sigma(S^2) \right]$  is in  $\sigma(D^2) \left[ \sigma(S^2) \right]$ . From this and from A.1.0.7 we have, that all accumulations points of  $\sigma$  are in  $\sigma$ . From this, from lemma A.1.0.3 we have that the topology of  $\sigma$  is  $\omega^\omega$ .  $\square$

### 3.4 Order type and topology of n-th cantor bendixon derivative of $\sigma$

$\sigma_n$

taking limit points is the order type of  $\omega^\omega$  but not homeomorphic anymore.

**TO DO: lemat o braku nowych punktów**

### 3.5 More about how this $\sigma$ , $\sigma(D^2)$ and $\sigma(S^2)$ lie in $\mathbb{R}$

**Observation 3.5.0.1.** *The first (greatest) negative accumulation point of the set of  $\sigma$  is  $-\frac{1}{12}$ . It is the accumulation point of order 1.*

**Proof.**

We will show, that  $-\frac{1}{12}$  is the greatest negative accumulation point of the set  $\sigma(D^2)$ . From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two-dimensional orbifolds is equal to  $\sigma(S^2) \cup \sigma(D^2)$  and  $\sigma(S^2) = 2\sigma(D^2)$ , so the greatest negative point of the set  $\sigma(S^2)$  is smaller than the greatest negative accumulation point of the set  $\sigma(D^2)$ .

- $-\frac{1}{12} = \chi^{orb}((2, 3)) - \frac{1}{2}$ , from this we have that  $-\frac{1}{12}$  an accumulation point of the set  $\sigma(D^2)$  of order at least 1.

- Let us assume (for a contradiction), that there exist bigger, negative accumulation point of the set  $\sigma(D^2)$  of order at least 1. Let us denote it by  $x$ .

However, then, from the lemma 3.1.2.5 point  $x + \frac{1}{2}$  is the accumulation point of the set  $\sigma(D^2)$ . What is more, since  $x \in (0, -\frac{1}{12})$ , then  $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12})$ . From the lemma 3.1.2.6 we have that  $x$  is in  $\sigma(D^2)$ . But orbifolds of the type  $*b_1$  can have Euler orbicharacteristic only greater or equal  $\frac{1}{2}$ . Orbifolds of the type  $*b_1b_2$  can only have Euler orbicharacteristic  $\frac{1}{2}, \frac{5}{12}$  and some smaller. Orbifolds of the type  $*b_1b_2b_3 \dots$  can have Euler orbicharacteristic only lower than  $\frac{1}{4}$ . This analysis of the cases leads us to the conclusion, that  $(\frac{1}{2}, \frac{5}{12}) \cap \sigma(D^2) = \emptyset$  and to the contradiction.

- Above analysis of the cases leads us also to the conclusion, that  $\frac{5}{12}$  is an isolated point of the set  $\sigma(D^2)$ , from this  $-\frac{1}{12}$  is an accumulation point of order 1 of the set  $\sigma(D^2)$ .  $\square$

#### 3.5.1 Connections between $\sigma(D^2)$ and $\sigma(S^2)$

In this section we would like to answer some questions about relations between  $\sigma(D^2)$  and  $\sigma(S^2)$ .

## TO DO: dopisać trochę inną motywację

We now state some observations that will be usefull in this section.

**Observation 3.5.1.1.** *If an Euler orbicharacteristic is an accumulation point of order  $n$  in  $\sigma(D^2)$  [respectively  $\sigma(S^2)$ ], there exist an  $D^2$  [resp.  $S^2$ ] orbifold with  $n$  dihedral [resp. rotational] points of that Euler orbicharacteristic.*

prrof. from chapter 3. (todo: dopisać)

**Observation 3.5.1.2.** *If  $x \in \sigma(D^2)$  [respectively  $\sigma(S^2)$ ], then  $1 - x$  [resp.  $2 - x$ ] is a difference in Euler orbicharacteristic resulting from some set of dihedral [resp.*



rotational] points. From that  $1 - n(1 - x) \in \sigma(D^2)$  [resp.  $2 - n(2 - x) \in \sigma(S^2)$ ] for all  $n \in \mathbb{N}$ .

$$-\frac{1}{84} \text{ and } -\frac{1}{42}$$

//Why it is how it is//

### Accumulation points of the $\sigma(S^2)$

**Theorem 3.5.1.3.** *All accumulation points of the  $\sigma(S^2)$  are in  $\sigma(D^2)$ .*

There are two proofs of this theorem showing nice correspondence – one arithmetical and one geometrical.

#### Proof I. Arithmetical reason

We assume that  $x \in \sigma(S^2)$  is an accumulation point of the set  $\sigma(S^2)$ .

By ?? we have, that  $\frac{x}{2} \in \sigma(D^2)$  is an accumulation point of the set  $\sigma(D^2)$ . From 3.1.2.5 we have that  $\frac{x}{2} + \frac{1}{2} \in \sigma(D^2)$ . From that, from 3.5.1.2 we have, that

$$1 - \underbrace{\frac{n}{2}}_{\substack{\text{"n" from} \\ 3.5.1.2}} \left( 1 - \underbrace{\left( \frac{x}{2} + \frac{1}{2} \right)}_{\substack{\text{"1-x" from} \\ 3.5.1.2}} \right) \in \sigma(D^2). \quad (3.5.1.3.1)$$

But  $1 - 2(1 - (\frac{x}{2} + \frac{1}{2})) = x$ , so  $x \in \sigma(D^2)$ .  $\square$

#### Proof II. Geometrical reason

We assume that  $x \in \sigma(S^2)$  is an accumulation point of the set  $\sigma(S^2)$ .

From 3.1.2.8 we know, that  $x$  can be expressed as  $y - 1$  for some  $y \in \sigma(S^2)$ .

Let  $\mathcal{O}$  be an orbifold with the base manifold  $S^2$ , such that  $\chi^{orb}(\mathcal{O}) = y$ .

Let  $\mathcal{O}_c$  be the orbifold created from  $\mathcal{O}$  by adding one cusp. Then  $\chi^{orb}(\mathcal{O}_c) = y - 1 = x$ .

Topologically  $\mathcal{O}_c$  with the cusp point removed (which do not change an orbicharacteristic) is  $\mathbb{R}^2$ . We can compactify it with  $S^1$ . This will not change an Euler orbicharacteristic since  $\chi^{orb}(S^1) = 0$  and Euler orbicharacteristic is additive.

What we get is an orbifold  $\mathcal{O}_D$  with the base manifold  $D^2$  and the same orbipoints as  $\mathcal{O}$ . Since orbipoints of  $\mathcal{O}$  create a difference in Euler orbicharacteristic equal to  $2 - y$ , we have that  $\chi^{orb}(\mathcal{O}_D) = 1 - (2 - y) = y - 1 = x$ . We can then move all orbipoints from the interior of  $\mathcal{O}_D$  to its boundry by doubling them, so  $x \in \sigma(D^2)$ .  $\square$

Do  $\sigma(D^2)$  and  $\sigma(S^2)$  coincide? It is easy to answer that  $\sigma(D^2) \neq \sigma(S^2)$  (and we will do that along some harder questions in the moment), but do they coincide starting from a sufficiently distant point? Or maybe, for every denominator, do they coincide from a sufficiently distatn point? (Yes.)

# Chapter 4

## Counting occurrences

The central question of this section is: "given a rational number, how many orbifolds have that Euler orbicharacteristic?". The warning for this chapter is, that this question will remain unanswered, however, we can provide some glimpse into the partial answer.

In the first section, we will show that for any number, there are only finitely many orbifolds with that Euler orbicharacteristic. In subsequent section, we will separate the problem of finding the exact number into more arithmetical part and one more combinatorial.

### 4.1 Finiteness

**Observation 4.1.0.1.** *For any  $x \in \sigma$  and  $n \in \mathbb{N}$  there are only finitely many orbifolds with the Euler orbicharacteristic greater or equal to  $x$  and all orbipoints of order at most  $n$ .*

**Proof:**

For a given  $x$ , there are only finitely many manifolds with an Euler characteristic  $y \geq x$ . Only them can be base manifolds for an orbifold with an Euler orbicharacteristic  $y' \geq x$ , as adding orbipoints always decreases an Euler orbicharacteristic.

It remains to prove then, that for any base manifold  $M$ , there are only finitely many orbifolds, with  $M$  as a base manifold, that have an Euler orbicharacteristic  $y \geq x$ , and all orbipoints of order at most  $n$ .

We proceed now similarly to the proof of 3.1.2.2 – on the orbifold with an Euler orbicharacteristic  $y \in [x, 2]$ , there can be at most  $\max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}$  orbipoints. Thus, for a given manifold  $M$  and a given  $x$  and  $n$ , there can be at most  $(n-1)^{\max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}}$  orbifolds with an Euler orbicharacteristic  $y \geq x$ , all orbipoints of order at most  $n$  and  $M$  as a base manifold.  $\square$

**Theorem 4.1.0.2.** *For any  $x \in \sigma$  there are only finitely many orbifolds with the Euler orbicharacteristic equal to  $x$ .*

**Proof:**

Let  $x$  be a rational number. Let  $\mathcal{O}$  be the set of all orbifolds with an Euler orbicharacteristic equal to  $x$ . Those orbifolds can have different base manifolds.

However, the set of base manifolds of orbifolds from  $\mathcal{O}$  is finite, as there are only finitely many two dimensional manifolds with an Euler characteristic greater or equal to  $x$  and an orbifold always has an Euler orbicharacteristic less or equal to the Euler characteristic of its underlying manifold.

It remains to proof, that for any base manifold  $M$ , the number of  $M$  orbifolds with Euler orbicharacteristic equal to  $x$  is finite:

Let  $M$  be a two dimensional manifold.

For the sake of contradiction, assume, that there exists an infinite set  $\mathcal{O}_M$  of  $M$ -orbifolds such that  $\mathcal{O}_M \subseteq \mathcal{O} (*)$ .

If  $M$  has some boundary, orbifolds in  $\mathcal{O}_M$  can have both rotational and dihedral orbipoints. For the simplicity of the following part of the proof we want now to reduce this case to a case where only one type of the orbipoints is present.

Let us observe, that every rotational orbipoint can be replaced by two dihedral orbipoints of the same order without changing the Euler orbicharacteristic. Thus, if there would be infinitely many orbifolds in  $\mathcal{O}_M$  having both rotational and dihedral orbipoints, there would be also infinitely many orbifolds in  $\mathcal{O}_M$  having only dihedral orbipoints. Thus it is sufficient to prove that there are finitely many orbifolds in  $\mathcal{O}_M$  that have only one type of the orbipoints.

We will now perform the proof of above statement in the case of dihedral orbipoints. The proof for rotational orbipoints is completely analogous.

Let us call the subset of  $\mathcal{O}_M$  that consists only of orbifolds with only dihedral orbipoints by  $\mathcal{O}_M^d$ .

Let  $\mathcal{O}_M = \{O_i\}_{i \in I}$ . For each  $i$ , let  $s_i = (b_i^0, \dots, b_i^{l_i})$  be the signature of  $O_i$  written with decreasing orders of dihedral points. So for each  $i$  we have, that  $b_i^0$  is the order of the orbipoint with the highest order of all the dihedral orbipoints of  $O_i$ . By 4.1.0.1 we know that if the set  $\{b_i^0\}_{i \in I}$  would be bounded by some  $n \in \mathbb{N}$  it would mean, that  $\mathcal{O}_M^d$  would be finite. As from  $(*)$  this is not the case, we know that the set  $\{b_i^0\}_{i \in I}$  is unbounded. Let  $\{i_n\}_{n \in \mathbb{N}} \subseteq I$  be a sequence of indices such that  $\{b_{i_n}^0\}_{n \in \mathbb{N}}$  is strictly increasing.

Let  $\{x_n\}$  be the sequence such that  $x_n = \Delta(b_{i_n}^0)$ . Let  $\{y_n\}$  be the sequence such that  $y_n = \Delta(b_{i_n}^1, \dots, b_{i_n}^{l_{i_n}})$ . So for every  $n$  we know that  $\chi^{orb}(O_{i_n}) = \chi(M) + a_n + b_n$ . As  $\{b_{i_n}^0\}$  is strictly increasing, we know that  $a_n$  is strictly decreasing, so  $b_n$  must be strictly increasing (we have that  $\chi^{orb}(O_{i_n})$  is constant for all  $n$ , since all  $O_{i_n}$  are from the family with Euler orbicharacteristic equal to  $x$ ).

But  $\{b_n\} \subseteq \sigma(M) - \chi(M)$ . From 3.1.3.6 and ?? we know that  $\sigma(M)$  has no infinite strongly increasing sequences, so  $\sigma(M) - \chi(M)$  has no infinite strongly increasing sequences. That gives us a contradiction.  $\square$

## 4.2 Infinitness

### 4.2.1 Unboundeness of some number of occurences

We know, that for any  $x$ , there are only finitely many orbifolds with  $x$  as an Euler orbicharacteristic. However, we can ask about some boundness of number of these orbipoints. In particular, we could ask, whether near any accumulation point, there

will be  $x$  with an arbitrary large number of orbifolds corresponding to it. The answer will be yes, and it can be formulated as such:

**Theorem 4.2.1.1.** *For any neighbourhood  $U$  of any accumulation point of  $\sigma(D^2)$  of order at least 2, for any  $n \in \mathbb{N}$ , there exists an  $x \in U$  such that there are at least  $n$  orbifolds with  $x$  as their Euler orbicharacteristic.*

**Proof.**

This will follow from the theorem about the sums of egyptian fractions from [2]. It states that for ...

### 4.3 Dividing the problem into an arithmetical and combinatorial parts

Given a number  $x$ , the question about how many orbifolds have  $x$  as an Euler orbicharacteristic can be partially expressed by asking the question how many of sums of the form:

$$1 - \sum_{i=1}^n \frac{d_i - 1}{2d_i} \quad (4.3.0.0.1)$$

and

$$2 - \sum_{i=1}^n \frac{r_i - 1}{r_i} \quad (4.3.0.0.2)$$

with  $n \in \mathbb{N}$  and  $\forall_i d_i, r_i \in \mathbb{N} \cup \{\infty\}$ , are equal to  $x$ .

It is a matter of convention (and then coherently translating this convention to the final result) what sums are we treating as "the same". The convention we will take, is that a sum is determined uniquely by the tuple  $(d_1, \dots, d_n)$  [or  $(r_1, \dots, r_n)$ ] of orders of orbipoints, ordered in deacreasing order, appearing in the sum.

This covers the arithmetics and combinatorics of the "orbifold" part of the problem. In this sums there is a structure of what Euler orbicharacteristic orbipoints can produce. The question "How many different sums (understood by above convention) are equal to a given  $x$ ?" will be the first part of the problem – the arithmetical part, adressed in 4.5. This will be the hard and unanswered part.

This however, does not give us the full information. For once, without changing Euler orbicharacteristic some orbipoints can be replaced by a features on a manifold such as handles, cross-cups and boundry components, giving orbifold of the same Euler orbicharacteristic, but different base manifold. Secondly, when the orbipoints lie on the boundry components, their order matters, and orbifolds with orbipoints on boundry components with different order are not neccesery the same.

This gives us a question – "How many different orbifolds produce the same sum?". This will be answered in the section 4.6. The other question is – "How many manifold with some sums corrspond to  $x$ ?" and this will be treated in 4.4 together with more precise notions of these questions.

## 4.4 Different manifolds

The question about different manifolds can be changed for the question about order of accumulation.

For a point  $x$ , if it is of order  $n$ , then all  $x + 1, \dots, x + n$  are also in the spectrum as such we can take differences

all that  $n$  can go into manifold features.

## 4.5 Arithmetical part

## 4.6 Combinatorial part

This case is simple in

# Chapter 5

## Algorithms for searching in the spectrum

Using results from previous chapters, we can now prove, that some computational problems related to spectra are solvable.

We will do it in a constructive way, by writing explicitly the algorithms and proving their correctness and properties.

The questions we will provide algorithms to answer here are: For a given rational number  $r$  and manifold  $M$ :

- How many  $M$  orbifolds with  $r$  as their Euler orbicharacteristic are there?
- The accumulation point of what degree is  $r$  in  $\sigma(M)$ ?

We start with  $\frac{p}{q}$ , where  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}_{>0}$  and a manifold  $M$ .

### 5.1 Reduction from arbitrary $M$ to $D^2$

Using 3.2 we conclude that the problem of deciding whether  $\frac{p}{q}$  is in  $\sigma(M)$  is equivalent to deciding whether *some expression* is in  $\sigma^b(D^2)$ .

Considering this fact, from this point, WLOG we will assume that  $M = D^2$  and we will be concerned only with dihedral orbipoints.

We want to determine whether there exists  $b_1, b_2, \dots, b_k$ , such that  $\chi^{orb}(*b_1 \dots b_k) = \frac{p}{q}$ .

### 5.2 Special cases

In the case that  $\frac{p}{q}$  is of the form  $l\frac{1}{4}$ , for some whole  $l$  we can give the answer right away. For  $l > 4$  we have that  $l\frac{1}{4}$  is not in the set and for  $l \leq 4$  it is.

Moreover for an even  $l$  it is a condensation point of order  $\frac{4-l}{2}$  (see ??) and for an odd  $l$  it is a condensation point of order  $\frac{3-l}{2}$  (see 3.1.2.8).

In the case, where  $\frac{p}{q} > 1$ , we also can give answer right away and this answer is "no".

Now we will consider only cases when  $\frac{p}{q}$  is not of the form  $l\frac{1}{4}$  and is  $\leq 1$ .

### 5.3 Simpler version of the question

To present the idea of searching the spectrum for the orbifolds with a given Euler orbicharacteristic, we will first present the algorithm that answers a little easier question, namely:

*For a given rational number  $r$  and manifold  $M$ , is there at least one  $M$  orbifold with  $r$  as their Euler orbicharacteristic?*

This algorithm will mirror what we are focused on in ??, giving us the computational tool for deciding whether a given number is in the spectrum or no.

We use:

- $\mathbb{N}_{>0}$  counters  $b_1 b_2 \dots$  with values ranging from  $\mathbb{N}_{>0} \cup \{\infty\}$ . Each counter correspond to one cone point on the boundry of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial cone point - namely a none cone point, a normal point).
- a pivot pointing at some counter
- a flag that can be set to "Greater" or "Less" corresponding to what was the outcome of comparing Euler orbicharacteristic of the orbifold corresponding to counters' state and  $\frac{p}{q}$ .

We start with:

- all counters set to 1.
- pivot pointing at the first counter
- flag set to "Greater"

We will do our computation such that:

- every state of the counters during runtime of the algorithm will have only finitely many counters with value non-1.
- every state in the rutime of the algorithm will have values on consecutive counters ordered in weakly decreasing order.

From now we will consider only such states.

The state of the counters  $b_1 b_2 \dots$  correspond to the orbifold of Euler orbicharacteristic equal  $\chi^{orb}(*b_1 b_2 \dots)$  (where the trailing 1 are trunkated).

When the algorithm is in the state:

- counters:  $b_1 b_2 \dots$
- pivot: at the counter  $b_c$
- flag: set to the value *flag\_value*,

we procced as follows :

```

1 In the case, the flag is set to:
2 {
3     "Less", then
4     {
5         We increase the pivot counter by one ( $b_c := b_c + 1$ ).
6         If  $b_c = 2$  and the values of all the counters
7         on the left are also equal 2 then
8         {
9             We end the whole algorithm with the answer "no".
10        }
11        We set the value of all counters on the left to  $b_c$ 
12        If  $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$  then
13        {
14            We found an orbifold and we are ending the whole
15            algorithm with answer "yes,  $*b_1b_2\dots$ ".
16        }
17        If  $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$  then
18        {
19            We set the flag to "Greater".
20            We put the pivot on the first counter.
21            We go to the line 1..
22        }
23        If  $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$  then
24        {
25            We set the flag to "Less".
26            We put pivot to the  $c+1$  counter.
27            We go to the line 1..
28        }
29    }
30
31    "Greater", then
32    {
33        If  $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) = \frac{p}{q}$  then
34        {
35            We found an orbifold and we are ending the whole
36            algorithm with answer "yes,  $*b_1\dots b_{c-1}\infty b_{c+1}\dots$ ".
37        }
38        If  $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) > \frac{p}{q}$  then
39        {
40            We set  $b_c$  to  $\infty$ .
41            We set the flag to "Greater".
42            We move pivot to the  $c+1$  counter.
43            We go to the line 1..
44        }

```



```

45     If  $\chi^{orb}(*b_1 \dots b_{c-1} \infty b_{c+1} \dots) < \frac{p}{q}$  then
46     {
47         We search for value  $b'_c$  of the  $c$  counter
48         such that  $\chi^{orb}(*b_1 \dots b_{c-1} b'_c b_{c+1} \dots) \leq \frac{p}{q}$ 
49         and  $\chi^{orb}(*b_1 \dots b_{c-1} (b'_c - 1) b_{c+1} \dots) > \frac{p}{q}$ .
50         // More on how we search for it will be told later ,
51         // for now we can think that we search one by one ,
52         // starting from  $b_c$  and going up till  $b'_c$ .
53         We set  $b_c$  to  $b'_c$ .
54         if  $\chi^{orb}(*b_1 b_2 b_3 \dots) = \frac{p}{q}$  then
55         {
56             We found an orbifold and we are ending the whole
57             algorithm with answer "yes ,  $*b_1 b_2 \dots$ ".
58         }
59         We set all the counters to the left to value  $b_c$ .
60         if  $\chi^{orb}(*b_1 b_2 b_3 \dots) = \frac{p}{q}$  then
61         {
62             We found an orbifold and we are ending the whole
63             algorithm with answer "yes ,  $*b_1 b_2 \dots$ ".
64         }
65         If  $\chi^{orb}(*b_1 b_2 b_3 \dots) < \frac{p}{q}$  then
66         {
67             We set flag to "Less".
68             We move the pivot to the column  $c+1$ .
69             We go to the line 1..
70         }
71         If  $\chi^{orb}(*b_1 b_2 b_3 \dots) > \frac{p}{q}$  then
72         {
73             We set the flag to "Greater".
74             We move the pivot to the first counter.
75             We go to the line 1..
76         }
77     }
78 }
79 }

```

### 5.3.1 Why this works

## 5.4 Counting occurrences algorithm

Searching for all occurrences

The difficulty here is to carefully step over an occurrence.

Compared to the previous version, we also use an occurrence counter, starting with it set to 0 and with the list of orbifolds, which is empty at the start.

```

1 In the case , the flag is set to:
2 {
3     "Less", then
4     {
5         We increase the pivot counter by one ( $b_c := b_c + 1$ ).
6         If  $b_c = 2$  and the values of all the counters
7         on the left are also equal 2 then
8         {
9             We end the whole algorithm with the answer "no".
10        }
11        We set the value of all counters on the left to  $b_c$ 
12        If  $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$  then
13        {
14            We found an orbifold , we add it to a list
15            and increase the occurrence counter by 1.
16            We set the flag to "Less".
17            We put pivot to the  $c+1$  counter.
18            We go to the line 1..
19        }
20        If  $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$  then
21        {
22            We set the flag to "Greater".
23            We put the pivot on the first counter.
24            We go to the line 1..
25        }
26        If  $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$  then
27        {
28            We set the flag to "Less".
29            We put pivot to the  $c+1$  counter.
30            We go to the line 1..
31        }
32    }
33
34    "Greater", then
35    {
36        If  $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) = \frac{p}{q}$  then
37        {
38            We found an orbifold , we add it to a list
39            and increase the occurrence counter by 1.
40            We set the flag to "Less".
41            We put pivot to the  $c+1$  counter.
42            We go to the line 1..
43        }
44        If  $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) > \frac{p}{q}$  then

```

```

45      {
46          We set  $b_c$  to  $\infty$ .
47          We set the flag to "Greater".
48          We move pivot to the  $c+1$  counter.
49          We go to the line 1..
50      }
51      If  $\chi^{orb}(*b_1 \dots b_{c-1} \infty b_{c+1} \dots) < \frac{p}{q}$  then
52      {
53          We search for value  $b'_c$  of the  $c$  counter
54          such that  $\chi^{orb}(*b_1 \dots b_{c-1} b'_c b_{c+1} \dots) \leq \frac{p}{q}$ 
55          and  $\chi^{orb}(*b_1 \dots b_{c-1} (b'_c - 1) b_{c+1} \dots) > \frac{p}{q}$ .
56          // More on how we search for it will be told later,
57          // for now we can think that we search one by one,
58          // starting from  $b_c$  and going up till  $b'_c$ .
59          We set  $b_c$  to  $b'_c$ .
60          if  $\chi^{orb}(*b_1 b_2 b_3 \dots) = \frac{p}{q}$  then
61          {
62              We found an orbifold, we add it to a list
63              and increase the occurrence counter by 1.
64              We set flag to "Less".
65              We go to the line 1..
66          }
67          We set all the counters to the left to value  $b_c$ .
68          if  $\chi^{orb}(*b_1 b_2 b_3 \dots) = \frac{p}{q}$  then
69          {
70              We found an orbifold, we add it to a list
71              and increase the occurrence counter by 1.
72              We set flag to "Less".
73              We move the pivot to the column  $c+1$ .
74              We go to the line 1..
75          }
76          If  $\chi^{orb}(*b_1 b_2 b_3 \dots) < \frac{p}{q}$  then
77          {
78              We set flag to "Less".
79              We move the pivot to the column  $c+1$ .
80              We go to the line 1..
81          }
82          If  $\chi^{orb}(*b_1 b_2 b_3 \dots) > \frac{p}{q}$  then
83          {
84              We set the flag to "Greater".
85              We move the pivot to the first counter.
86              We go to the line 1..
87          }
88      }

```

89        }  
90    }

### 5.4.1 Why this works

## 5.5 Deciding the order

Let  $m \in \mathbb{N}$  be such that  $\frac{p}{q} \in (1 - \frac{m}{2}, 1 - \frac{m+1}{2})$  Let us denote by  $r := \frac{p}{q} - (1 - \frac{m}{2})$ .

We will searching in  $\sigma$  as such:

If  $\frac{p}{q} \in \sigma$ , then, from the corollary 3.1.2.8 we know, that there exist some  $n \in \mathbb{N}$ , such that  $\frac{p}{q} + \frac{n}{2} \in \sigma$  but  $\frac{p}{q} + \frac{n}{2} \notin \sigma$ .

We will be consequently checking points from  $1 + r$ , through  $1 + r - \frac{l}{2}$ , for  $0 \leq l \leq m$ , to the  $\frac{p}{q}$ . We stop at the first found point. If one of these point is in the spectrum, then all smaller (so also  $\frac{p}{q}$ ) are in the spectrum and  $\frac{p}{q}$  is the accumulation point of the spectrum of order  $m - l$  (from this, we can see some heuristic, that the points that have smaller order will be generally harder to find in some sense). If none of this points are in in the spectrum, then  $\frac{p}{q}$  is not.

## 5.6 Implementation

As an appendix in the separate document, there is a source of a program with implementation of this algorithm with full enhancements described in this chapter. It is written in Rust. It can be also found on

**To do:** [dać ref do github](#)

along with the L<sup>A</sup>T<sub>E</sub>X source of this thesis.

# Chapter 6

## Conclusions

We described the spectrum of possible Euler orbicharacteristics of two dimensional orbifolds. It has topology of  $\omega^\omega$  and the problem, whether the given point is in the spectrum is decidable.

We also provided some finiteness results, such as that there are always only finitely many orbifolds for a given Euler orbicharacteristic. So the problem how much are for a given number is also decidable.

From **To do** we know, that there are however, blab la dowolnie dużo na Euler orbicharacteristic.

It remains unclear how Disk spectrum and Sphere spectrum lies relative to each other, but some result was, shown, namely, that every accumulation point of Sphere spectrum is also in the disk spectrum.

### 6.1 Further directions

#### 6.1.1 Asked, but unanswered questions

Our ultimate goal is to give the answer to the questions such as:

- For a given  $x \in \sigma$ , how many orbifolds have  $x$  as their Euler orbicharacteristic?
- Why? Is there some underlying geometrical reason for that?
- Can we characterise points  $x \in \sigma$  that has the most orbifolds corresponding to them?
- Is there any reasonable normalisation to counter the effect that there are 'more' points as we go to lesser values. (What we mean by 'more' was stated in)

The first question we can tackle is stemming from the chapter 3 and it is – Do  $\sigma(D^2)$  and  $\sigma(S^2)$  coincide? It is easy to answer that  $\sigma(D^2) \neq \sigma(S^2)$  (and we will do that along some harder questions in the moment), but do they coincide starting from a sufficiently distant point? Or maybe, for every denominator, do they coincide from a sufficiently distant point? (Yes.)

# TO DO: przenieść (przekopiować?) część może do futher direction

write about cyclic order

6.1.2 Unasked and unanswered questions

6.1.3 Power series and generating functions

6.1.4 Seifert manifolds

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# Appendix A

## Appendix about well orders, accumulation points and Canor-Bendixon derivatives

TO DO: Not e between equivalence of  
being accumulation point and being in  
Cantor bendixon derivative

### A.1 Lemmas

**Lemma A.1.0.1.** *If  $A, B \subseteq \mathbb{R}$  have no infinite strictly ascending sequences, then set  $A + B := \{a + b \mid a \in A, b \in B\}$  also have no infinite strictly ascending sequences.*

**Proof.**

Let  $A, B$  have no infinite strictly ascending sequences. Let  $c_n \in A + B$  are elements of some sequence. With a sequence  $c_n$  there are two associated sequences  $a_n, b_n$ , such that, for all  $n$ , we have  $a_n \in A, b_n \in B$  and  $a_n + b_n = c_n$ . Assume (for contradiction), that  $c_n$  is an infinite strictly ascending sequence. Then  $\forall_n a_{n+1} > a_n \vee b_{n+1} > b_n$ . From the assumption  $a_n$  has no infinite ascending sequence, so  $a_n$  has a weakly decreasing subsequence  $a_{n_k}$ . But then subsequence  $b_{n_k}$  must be strictly increasing, as  $c_{n_k}$  is strictly increasing, what gives us a contradiction.  $\square$

**Lemma A.1.0.2.** *If  $A, B \subseteq \mathbb{R}$  have no infinite strictly ascending sequences, then set  $A \cup B$  also have no infinite strictly ascending sequences.*

**Proof.**

Let  $A, B$  have no infinite strictly ascending sequences. For the sake of contradiction, lets assume, that  $A \cup B$  has an infinite strictly ascending sequence  $c_n$ . Let  $c_{n_k}, c_{n_l}$  be subsequences of  $c_n$  consisting of elements from, respectively  $A$  and  $B$ . At least one



of them must be infinite and strictly increasing, which gives us a contradiction.  $\square$

Concerning accumulation points, we will use the terminology, that we introduced in 1.3.5

**Lemma A.1.0.3.** *Let  $A \subseteq \mathbb{R}$  has an order type  $\alpha$ . Let  $A$  be such that every accumulation point of  $A$  belong to  $A$ . Then  $A$  has not only an order type  $\alpha$  but is also homeomorphic to  $\alpha$ .*

## TO DO: dopisać, że to równoważność

**Proof.**

Without loss of generality, let us assume, that  $A$  has no infinite descending sequence (case with  $A$  having no infinite ascending sequence is completely analogous).

As  $A$  has an order type  $\alpha$  we have that there is an order preserving bijection  $f : \alpha \rightarrow A$ .

We will prove the theorem by showing that  $f$  is a homeomorphism.

For the continuity of  $f$  and  $f^{-1}$  it is sufficient to show, that for every open  $U \subseteq A$  and  $V \subseteq \alpha$  from prebases of respective topologies,  $f^{-1}[U]$  and  $f[V]$  are open (\*). Prebase open sets in  $A$  are the ones inherited from the order topology on  $\mathbb{R}$ , for all  $s \in \mathbb{R}$ :

$$\begin{aligned} \{r \mid r < s\} \cap A \\ \{r \mid s < r\} \cap A. \end{aligned}$$

Prebase open sets in  $\alpha$  are from order topology, for all  $\nu \in \alpha$ :

$$\begin{aligned} \{\eta \mid \eta < \nu\} \\ \{\eta \mid \nu < \eta\}. \end{aligned}$$

Now, we will prove (\*) case by case:

- Prebase set –  $\{r \mid r < s\} \cap A$ :

Let  $\nu \in \alpha$  be the smallest, that  $s \leq f(\nu)$ , then:

$$f^{-1}[\{r \mid r < s\} \cap A] = \{\eta \mid \eta < \nu\},$$

which is open.

- Prebase set –  $\{r \mid s < r\} \cap A$ :

Let  $s < f(\mu)$ . We have two cases:

–  $s \in A$ : then let  $\nu$  be such that  $f(\nu) = s$ . Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.

–  $s \notin A$ : then, by the assumption of the theorem we know that  $s$  is not an accumulation point of  $A$ . From this we conclude, that  $\exists_{t \in A}(t < s \wedge \neg \exists_{t' \in A} t < t' < s)$ . Let  $\nu$  be such that  $f(\nu) = t$ . Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.

- Prebase set –  $\{\eta \mid \eta < \nu\}$ :

$$f[\{\eta \mid \eta < \nu\}] = \{r \mid r < f(\nu)\} \cap A,$$

which is open.

- Prebase set –  $\{\eta \mid \nu < \eta\}$ :

$$f[\{\eta \mid \nu < \eta\}] = \{r \mid f(\nu) < r\} \cap A,$$

which is open.  $\square$

**Lemma A.1.0.4.** *Let  $A \subseteq \mathbb{R}$  be a bounded, well ordered set. Then  $A$  has an accumulation point  $a$  of order  $n \in \mathbb{N}$  (it may be that  $a \notin A$ ) iff order type of  $A$  is at least  $\omega^n$ .*

**Proof.**

Inductive, with respect to  $n$  in  $\omega^n$ .

- $n = 0$  Let us suppose, that  $A$  has an accumulation point of order 0. Having an accumulation point of order 0 means that  $A$  is non-empty. As that it has an order type of at least  $\omega^0 = 1$ .

Let us suppose, that  $A$  has order type at least  $\omega^0 = 1$ . Then it is non-empty, so it has at least one accumulation point of order 0.

- Induction step

Let us suppose that  $A$  has an accumulation point  $a$  of order  $n + 1$ . This means that every neighbourhood of  $a$  we can find infinitely many accumulation points of  $A$  of order  $n$ . Let take one such neighbourhood and one such family  $\{b_i\}_{i \in \mathbb{N}}$  of accumulation points of order  $n$ . Let us then take family of pairwise disjoint neighbourhoods  $\{U_i\}_{i \in \mathbb{N}}$  of  $\{b_i\}_{i \in \mathbb{N}}$ . Let  $A_i := U_i \cap A$ .

From the induction assumption for all  $i$ , we have that  $A_i$  is of order type at least  $\omega^n$ . As that, we managed to show an pairwise disjoint inclusions of countably many sets of order type at least  $\omega^n$  into  $A$ . As that we have the order preserving inclusion of  $\omega^{n+1}$  into  $A$ , so  $A$  is of order type at least  $\omega^{n+1}$ .

Let us now suppose that  $A$  has the order type of at least  $\omega^{n+1}$ . Then, we can find a family  $\{A_i\}_{i \in \mathbb{A}}$  of pairwise disjoint subsets of  $A$ , each of order type  $\omega^n$ , with the property  $(*)$ , that  $\forall_{i,j \in \mathbb{N}} i < j \implies \forall_{x \in A_i, y \in A_j} x < y$ .

From the inductive assumption, for all  $i$ , we have that  $A_i$  has an accumulation point of order  $n$ . Let  $\{b_i\}_{i \in \mathbb{N}}$  be the set of those accumulation points. Because of the property (\*), those accumulation points are pairwise distinct, between  $A_i, A_j$ , with  $i \neq j$ . Since  $A$  is bounded, we have that, the set  $\{b_i\}_{i \in \mathbb{N}}$  is bounded, so it has an accumulation point  $a$ . As an accumulation point of the accumulation points of order  $n$ , it is an accumulation point of order  $n + 1$ .  $\square$

**Corollary A.1.0.5.** *Let  $A \subseteq \mathbb{R}$  be a bounded, well ordered set of the order type  $\omega^n$ . Then it has exactly one accumulation point  $a'$  of order  $n$ . This point has the property that  $\forall_{a \in A} a < a'$ .*

**Proof.**

From A.1.0.4 we know that  $A$  has at least one accumulation point  $a'$  of order  $n$ .

For the sake of contradiction, let us assume, that there exists an accumulation point  $\bar{a}$  of order  $n$  such that  $\exists_{a \in A} a \geq \bar{a}$ . We have that  $A$  has the order type  $\omega^n$ , which means that  $\forall_{a_1 \in A} \exists_{a_2 \in A} a_1 < a_2$ . From this, we have, that  $\exists_{a_0} a_0 > \bar{a}$ . But then, we would have that the prefix  $(-\infty, \bar{a}] \cup A$  of  $A$  has an accumulation point  $\bar{a}$  of order  $n$ . From this, from A.1.0.4 we would conclude, that  $(-\infty, \bar{a}] \cup A$  is of order type at least  $\omega^n$ , which leads to the contradiction, as  $(-\infty, \bar{a}] \cup A$  is a proper subset of  $A$ . Thus, we have, that for all accumulation points  $\bar{a}$  of  $A$  of order  $n$  we have that  $\forall_{a \in A} a < \bar{a}$ .

It remains to show that there is only one such accumulation point -  $a'$ . For the sake of contradiction, let us assume, that there exists an accumulation point of  $A$  of order  $n$ , named  $\bar{a}$ , such that  $\bar{a} \neq a'$ . Let us assume that  $\bar{a} < a'$ . Then, as in every neighbourhood of  $a'$  there is a point from  $A$ , we have that  $\exists_{a_0} a_0 > \bar{a}$ . The absurdity of this statement is shown above. Case where  $\bar{a} > a'$  is completely analogous.  $\square$

**Lemma A.1.0.6.** *For  $A, B \subseteq \mathbb{R}$ , if  $r \in \mathbb{R}$  is an accumulation point of order at most  $m$  for  $A$  and at most  $n$  for  $B$  and  $m \leq n$ , then  $r$  is an accumulation point of order at most  $n$  for  $A \cup B$ .*

**Proof.**

Inductive.

- $n = 0$ . Then  $r$  is an isolated point of  $B$  and either  $r$  is isolated point of  $A$  or  $r \notin A$ . From this we have that there exists  $U_1, U_2$  such that  $B \cap U_1 = \{r\}$  and  $A \cap U_2 \subseteq \{r\}$ . From this we have that  $(A \cup B) \cap (U_1 \cap U_2) = \{r\}$ . So  $r$  is an isolated point of  $A \cup B$ .

- Inductive step. Let us suppose that for all  $k < n$ , the statement holds. Let  $r$  be an accumulation point of order  $n$  of  $B$  and order  $m$  of  $A$ , where  $m \leq n$ . From this we have that there exists  $U_1, U_2$  such that in  $B \cap U_1$  there are only accumulation points of  $B$  of order at most  $n - 1$  and in  $A \cup B_2$  there are only accumulation points of  $A$  of order at most  $m - 1$ . From this, from the inductive assumption we have that in  $(A \cup B) \cap (U_1 \cap U_2)$  there are only accumulation points of order at most  $n - 1$  of  $A \cup B$ . This means that  $r$  is an accumulation point of order at most  $n$  of  $A \cup B$ .  $\square$

**Corollary A.1.0.7.** *For every  $n \in \mathbb{N}$  we have that  $(A \cup B)^{(n)} = A^{(n)} \cup B^{(n)}$ .*

**Proof.**

Every accumulation point of either  $A$  or  $B$  is also an accumulation point of  $A \cup B$ , so  $(A \cup B)^{(n)} \supseteq A^{(n)} \cup B^{(n)}$ .

From A.1.0.6 we know, that for any point  $r \in \mathbb{R}$ , if  $r \in (A \cup B)^{(n)}$ , then  $r \in A^{(n)} \cup B^{(n)}$ .  $\square$

**Lemma A.1.0.8.** *For two bounded, well ordered sets  $A, B \subseteq \mathbb{R}$ , with order types, respectively  $\omega^m$  and  $\omega^n$ , such that  $m < n$ , and that  $\forall_{x \in A \cup B} \exists_{b \in B} x < b$ , we have that order type of  $A \cup B$  is well defined and equal to  $\omega^n$ .*

**Proof.**

From A.1.0.2, we know, that  $A \cup B$  is well ordered. As such its order type is well defined and equal to some ordinal number  $\gamma$ .

We will show that  $\gamma \leq \omega^n$  and  $\gamma \geq \omega^n$ , thus showing that  $\gamma = \omega^n$ .

Let  $f : \omega^n \rightarrow B$  and  $g : A \cup B \rightarrow \gamma$  be order preserving bijections.

•  $\omega^n \leq \gamma$ :

We have that  $g \circ f : \omega^n \rightarrow \gamma$  is an order preserving injection, thus,  $\omega^n \leq \gamma$ .

•  $\omega^n \geq \gamma$ :

From A.1.0.5 we know, that  $B$  has exactly one accumulation point  $b'$  of order  $n$ . This point has the property that  $\forall_{b \in B} b < b'$ . As  $b'$  is the only accumulation point of order  $n$  for  $B$  and from A.1.0.5 we know also that  $A$  has no accumulation points of order  $n$ , from A.1.0.6 we know, that  $A \cup B$  has exactly one accumulation point of order  $n$ , namely  $b'$ .

For the sake of contradiction, let us assume that  $\omega^n < \gamma$ . But then, there is some proper prefix of  $A \cup B$  with order type  $\omega^n$ . Let us name that prefix as  $P$ . From A.1.0.4 we know, that  $P$  has an accumulation point  $p'$  of order  $n$ . Let  $b_1 \in B$  be such that  $\forall_{p \in P} p < b_1$ . Such  $b_1$  exists, because  $P$  is a proper prefix of  $A \cup B$ , so  $\exists_{x \in A \cup B} \forall p \in P p < x$ , and from the assumptions of the lemma we have that  $\forall_{x \in A \cup B} \exists_{b \in B} x < b$ . We have that  $p' \leq b_1$ . But we have also that  $b_1 < b'$ , so  $p' \neq b'$ . This gives us the contradiction, as  $b'$  is the only accumulation point of order  $n$  in  $A \cup B$ .  $\square$