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Two dimentional orbifolds' volumes' spectrum

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Contents

Abstract

Orbifoldy

Chapter 1

Introduction

Chapter 2

Different definitions of an orbifold

Jak ma wyglądać: jedna definicja (Thurstona) notacja może byc od Conwaya
skleić drugi i trzeci rozdział
zwięźle własności
dobre i złe w trzecim rozdziale
orbikompleksy może nie
This chapter the next will be a technical chapters. Later on we will evoke some terms and definitions without explicitly saying what they mean instead we will put a reference to this chapter with explicit saying to what definition it refers.
For example in the later chapters there will be phrases like "adding a defect of order ..." or "gluing orbifolds by boundaries" and they are explained in this and the next chapter.
We will explore various definitions of an orbifold, partially proving they are equivalent, partially linking to the sources.
Some of these definitions apply only to the special cases. Some of them contain constructions with which not all orbifolds can be made (at least some of them can't be derived as such a priori) .

2.1 Hyperbolic plane tilling

2.2 Manifolds with defects

2.2.1 Disk and sphere with defects

2.2.2 Conway notation

[2]
When it is necessary to avoid a confusion, on parts such as $*abcd$, we will be writing $*^*a^*b^*c$ instead.
We will propose some extension to a notation from [2]. We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature.

We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$ which have less capitalistic vibes than "cost". For example $\Delta(*^*2) = \frac{1}{4}$.
We will denote by $*$ an operation of cutting out a disk and by ${}^{\beta *}n$ an operation of adding a kaleidoscopic point of period n on the boundary component β. Last operation is defined only on orbifolds with boundaries.

2.3 Quotients of planes

2.4 Generalised manifolds

This approach is very similar to the previous one. It differs slightly where we put the definition burden.

Chapter 3

Characteristics, classification and properties of the orbifolds

3.1 Euler orbicharacteristic

3.1.1 Classification of orbifolds with non-negative Euler orbicharacteristic

3.1.2 Extended Euler orbicharacteristic

3.2 Uniformisation theorem (formulation)

3.3 Surgeries, modifications and constructions on orbifolds

(Some preserve an area)

Chapter 4

Order structure

Order type with zanurzenie w R
1537/137
In this chapter we will discuss order type of the set of all possible Euler orbicharacteristics of two dimensional orbifolds.
For now, until Chapter 6 Counting occurrences, we will not pay attention to how many orbifolds have the same Euler orbicharacteristic.
Because of that and since Euler orbicharacteristic does not depend on the cyclic order of points on the components of the boundary we introduce an extension of a notation from [2].
We will write $*\{a, b, c, d, \dots\}$ to denote a type of a boundary (of an orbifold) that have kaleidoscopic points of periods $a, b, c, d, \dots$, but in any order.

From what we wrote above (that Euler orbicharacteristic does not depend on the cyclic order of points on the components of the boundary), we can see that Euler orbicharacteristic is well defined when we specify only such a type of the components of the boundary of an orbifold and not a particular cyclic order.

4.1 Reductions of cases

Now we want to make some reductions to limit number of cases that we will be dealing with.
For this chapter we will consider orbifolds according to a definition from (2.2.1).
Let us observe, that:

$$\begin{aligned}
\Delta(\circ) = & \quad -2 & = \Delta(*({}^*2)^4) \\
\Delta(*) = & \quad -1 & = \Delta(({}^*2)^4) \\
\Delta(n) = & \quad \frac{n-1}{n} & = \Delta(({}^*n)^2)
\end{aligned}$$

From this we can conclude, that every Euler orbicharacteristic can be obtained by an orbifold of signature of a type (n and m are arbitrary):

$$I_1 I_2 \dots I_n \text{or}$$
$$*b_1 b_2 \dots b_m.$$

Let us denote the set of all possible Euler orbicharacteristics of orbifolds of the form $I_1 I_2 \dots I_n$ by $\sigma^I(S^2)$ and the set of all possible Euler orbicharacteristics of orbifolds of the form $*b_1 b_2 \dots b_m$ as $\sigma^b(D^2)$.
Let us denote the set of all possible Euler orbicharacteristics of two dimentional orbifolds as σ.
Let us observe that the topological structure of $\sigma^I(S^2)$ and $\sigma^b(D^2)$ are the same since

$$2\sigma^b(D^2) = \sigma^I(S^2)$$

So multiplying by 2 is the homeomorphism.

4.2 Determining the order structure

In this chapter we will justify, that the order type of all possible Euler orbicharacteristics of two dimensional orbifolds is ω^ω. We will also describe precisely where accumulation points lie and of which order (see below 4.2.1) they are.

4.2.1 Definitions regarding order of accumulation points

We start with one technical definition of "transitive order" that will be almost what we want and then, there will be the the definition of "order", which is the definition that we need.

Definition 4.2.1.1. *(Inductive). We say that the point is an acccumulation point of a transitive order* 0*, when it is an isolated point. We say that the point is an acccumulation point of a transitive order* $n+1$*, when it is an acccumulation point (in the usual sense) of the accumulation points of the transitive order* n*.*

The only issue of the definition is that the point of the transitive order n is also a point of the transitive order k, for all $0 < k \leqslant n$. We want a definition of order such that for any point, there is at most one integer that is its order. So we define:

Definition 4.2.1.2. *We say that the point is an acccumulation point of order* n *iff it is an acccumulation point of the transitive order* n *and it is not an acccumulation point of the transitive order* $n+1$*. If the point is an acccumulation point of the transitive order for an arbitrary large* n *we say that the point is an acccumulation point of order* ω*.*

4.2.2 Order structure of $\sigma^b(D^2)$

Some preliminary observations.

Let us observe, that $\lim_{n\to\infty} \Delta(^*n) = -\frac{1}{2}$. From that, we see, that for every point $x \in \sigma^b(D^2)$, the point $x - \frac{1}{2}$ is an acccumulation point. Let us observe, that also, for every point $x \in \sigma^b(D^2)$, we have that $x - \frac{1}{2} \in \sigma^b(D^2)$, because $\Delta(^*\infty) = -\frac{1}{2}$.

Now we will show that the order type of $\sigma^b(D^2)$ is ω^ω and where exactly are its accumulation points of which orders. For this we will use a handful of lemmas.

Lemma 4.2.2.1. *If x is an acccumulation point of the set $\sigma^b(D^2)$ of order n, then $x - \frac{1}{2}$ is a accumulation point of the set $\sigma^b(D^2)$ of order at least $n+1$.*

Proof.
Inductive.
• $n = 0$: If x is an isolated point of the set $\sigma^b(D^2)$, then $x \in \sigma^b(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma^b(D^2)$, from that, that $x - \frac{1}{2}$ is a accumulation point of $\sigma^b(D^2)$.
• inductive step: Let x be an acccumulation point of the set $\sigma^b(D^2)$ of an order $n > 0$. Let a_k be a sequence of accumulation points of order $n-1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of accumulation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+1$. □

Lemma 4.2.2.2. *If x is an acccumulation point of the set $\sigma^b(D^2)$ of order n, then $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n-1$.*

Proof.
Inductive
• $n = 1$: We assume, that x is an acccumulation point of isolated points of the set $\sigma^b(D^2)$. Let us observe, that for all m there are only finitely many Euler orbicharacteristics in the interval $[1, x]$ of orbifolds that have cone points of period equal at most m.
From that, for arbitrary small neighborhood $U \ni x$ and arbitrary large m there exist an orbifold that has a cone point of period grater than m, whose Euler orbicharacteristic lies in U. Let us take a sequence of such Euler orbicharacteristics a_k convergent to x, such that we can choose a sequence divergent to infinity of periods of cone points b_k of orbifolds of Euler orbicharacteristics equal a_k.
To do: picture
Let us observe, that for all k, the number $a_k + \frac{b_k-1}{2b_k}$ is in $\sigma^b(D^2)$. It is so, because a_k is an Euler orbicharacteristicof an orbifold that have a cone point of period b_k, so identical orbifold, only without this cone point has an Euler orbicharacteristicequal to $a_k + \frac{b_k-1}{2b_k}$. The sequence $a_k + \frac{b_k-1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least 0.
• inductive step: Let x be an acccumulation point of the set $\sigma^b(D^2)$ of order $n > 1$.

Let a_k be a sequence of accumulation points of the set $\sigma^b(D^2)$ of order $n-1$ convergent to x. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of an acccumulation points of the set $\sigma^b(D^2)$ of order $n-2$ convergent to $x+\frac{1}{2}$. From that $x+\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n-1$. □

Lemma 4.2.2.3. *If x is an acccumulation point of the set $\sigma^b(D^2)$ of order $n+1$, then*
$x-\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $n+2$ and
$x+\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order n.

Proof.
Let x be an acccumulation point of the set $\sigma^b(D^2)$ of order $n+1$. From the lemma 4.2.2.1 we know, that $x-\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$. Now let us assume (for a contradiction), that $x-\frac{1}{2}$ is an accumulation point of the set$\sigma^b(D^2)$ of order $k > n+2$. But then from the lemma 4.2.2.2 we have that x is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction.
Analogously, from the lemma 4.2.2.2 we know, that $x+\frac{1}{2}$ is a accumulation point of the set $\sigma^b(D^2)$ of order at least n. Let us assume (for a contradiction), that $x+\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $k > n$. But then from the lemma 4.2.2.1 we have that x is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction. □

Lemma 4.2.2.4. *For all $n \in \mathbb{N}$ all accumulation points of the set $\sigma^b(D^2)$ of order n are in $\sigma^b(D^2)$.*

Proof.
Inductive
• $n = 0$: Clear, as they are isolated points of $\sigma^b(D^2)$.
• inductive step: Let x be a accumulation point of the set$\sigma^b(D^2)$ of order $n > 0$. From the lemma 4.2.2.3 point $x+\frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $n-1$. From the inductive assumption $x+\frac{1}{2} \in \sigma^b(D^2)$. Then $x \in \sigma^b(D^2)$. □

Lemma 4.2.2.5. *If $A, B \subseteqq \mathbb{R}$ have no infinite ascending sequences, then set $A+B := \{a+b \mid a \in A, b \in B\}$ also have no infinite ascending sequences.*

Proof.
Let A, B have no infinite ascending sequences. Let $c_n \in A+B$ are elements of some sequence. With a sequence c_n there are two associated sequences a_n, b_n, such that, for all n, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradiction), that c_n is an infinite ascending sequence. Then $\forall_n\ a_{n+1} > a_n \ \vee \ b_{n+1} > b_n$. From the assumption a_n has no infinite ascending sequence, so a_n has a weakly decreasing subsequence a_{n_k}. But then subsequence b_{n_k} must be strictly increasing, what gives us a contradiction. ↯□

Lemma 4.2.2.6. *In $\sigma^b(D^2)$ there are no infinite ascending sequences.*

Proof.
Let us denote by A_n the set of all possible Euler orbicharacteristics realised by orbifolds of type $*b_1, \ldots, b_n$. Then $A_0 = \{1\}$ and $A_{n+1} = A_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$. From that, from the lemma 4.2.2.5, for all n, we have that A_n do not have infinite ascending sequence. $\sigma^b(D^2) = \bigcup_{n=0}^{\infty} A_n$. Let us also observe, that for all n, we have $A_n \subsetneqq [1-\frac{n}{4}, 1-\frac{n}{2}]$. From that we have $\sigma^b(D^2)$ do not have infinite ascending sequences. □

Theorem 4.2.2.7. *The biggest accumulation point of the set $\sigma^b(D^2)$ of order n is $1-\frac{n}{2}$.*

Proof.
Inductive
- $n = 0$: $1 \in \sigma^b(D^2)$ and 1 is the biggest element of $\sigma^b(D^2)$.
- an inductive step: From the inductive assumption we know that $1-\frac{n}{2}$ is the biggest accumulation point of the set$\sigma^b(D^2)$ of order n. From the lemma 4.2.2.3 we have then that $1-\frac{n+1}{2}$ is a accumulation point of the set$\sigma^b(D^2)$ of order $n+1$. Let us assume (for a contradiction), that there exist a bigger accumulation point of order $n+1$ equal to $y > 1-\frac{n+1}{2}$. But then, from lemma 4.2.2.3, point $y+\frac{1}{2}$ would be an acccumulation point of order n, what gives a contradiction, because $y+\frac{1}{2} > 1-\frac{n}{2}$. □

4.2.3 Order structure of the set of all possible Euler orbicharacteristics σ

Theorem 4.2.3.1. *The order type of the set of possible Euler orbicharacteristics of two dimensional orbifolds σ is ω^ω.*

Proof.
From the lemma 4.2.2.6 we know, that $\sigma^b(D^2)$ is well ordered. From this and from the theorem 4.2.2.7 we know, that for the point $1-\frac{n}{2}$ there exist a neighborhood $U = (1-\frac{n}{2}-\varepsilon, 1-\frac{n}{2}+\varepsilon)$ such that $U \cap \sigma^b(D^2)$ is homeomorphic to ω^n. From this, and again from theorem 4.2.2.7 we have that $\sigma^b(D^2) \cap [1, 1-\frac{n}{2})$ is homeomorphic with ω^n. From this $\sigma^b(D^2)$ is homeomorphic with ω^ω. From this $\sigma^I(S^2)$ is homeomorphic with ω^ω.
$\sigma^I(S^2) = 2\sigma^b(D^2)$, so for all $n \in \mathbb{N}$ set $\sigma^I(S^2) \cap [2, n)$ has a lower order type then $\sigma^b(D^2) \cap [2, n)$. From this, we have that $\sigma^I(S^2) \cup \sigma^b(D^2) \cong \omega^\omega$. □

Theorem 4.2.3.2. *The first (biggest) negative accumulation point of the setof all possible Euler orbicharacteristicof two dimensional orbifolds is $\frac{1}{12}$. It is the accumulation point of order* 1*.*

Proof.
We will show, that $-\frac{1}{12}$ is the biggest negative accumulation point of the set $\sigma^b(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two dimensional orbifolds is equal to $\sigma^I(S^2) \cup \sigma^b(D^2)$ and $\sigma^I(S^2) = 2\sigma^b(D^2)$, so the biggest negative point of the set $\sigma^I(S^2)$ is smaller than the biggest negative

accumulation point of the set $\sigma^b(D^2)$.
• $-\frac{1}{12} = \chi^{orb}((2,3)) - \frac{1}{2}$, from this we have that $-\frac{1}{12}$ an acccumulation point of the set $\sigma^b(D^2)$ of order at least 1.
• Let us assume (for the contradiction), that there exist bigger, negative accumulation point of the set $\sigma^b(D^2)$ of order at least 1. Let us denote it by x.
However, then, from the lemma 4.2.2.3 point $x + \frac{1}{2}$ is the accumulation point of the set $\sigma^b(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12}$. From the lemma 4.2.2.4 we have that x is in $\sigma^b(D^2)$. But orbifolds of the type $*b_1$ can have Euler orbicharacteristiconly greater or equal $\frac{1}{2}$. Orbifolds of the type $*b_1b_2$ can only have Euler orbicharacteristic$\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*b_1b_2b_3\dots$ can have Euler orbicharacteristiconly lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma^b(D^2) = \emptyset$ and to the contradiction.
• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma^b(D^2)$, from this $-\frac{1}{12}$ is an acccumulation point of order 1 of the set $\sigma^b(D^2)$. □

From the above discussion we can conclude following:

Corollary 4.2.3.3. *Let $x \in \sigma$. Then there exeists $n \in \mathbb{N}$ such that $x + \frac{n}{2} \in \sigma$ but $x + \frac{n+1}{2} \notin \sigma$. For such n we have that x is an accumulation point of the set σ of order n.*

Chapter 5

Algorithms for searching the spectrum

5.1 Decidability

Here we will show the proof that the problem of "deciding whether a given rational number is in an Euler orbicharacteristic's spectrum or not" is decidable by showing algorithm for doing this. Later, our algorithm will have a bonus property of determining of which order of condensation is given point if it is in fact in σ.
First stated algorithm is also very inefficient and is presented, because the idea is the most clear in it. Right after it there is stated an algorithm with two enhancements:

- determining an accumulation point of which order is a given point, if it is in fact in the spectrum (this enhancement gives also an performance boost)
- faster searching, because some cases do not need to be checked.

We start with $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$.

We want to determine whether there exist $b_1, b_2, \ldots, b_k$, such that $\chi^{orb}(*b_1 \ldots b_k) = \frac{p}{q}$.

In the case that $\frac{p}{q}$ is of the form $l * \frac{1}{4}$, for some whole l we can give the answer right away. For $l > 4$ we have that $l * \frac{1}{4}$ is not in the set and for $l \leqslant 4$ it is. Moreover for an even l it is a condensation point of order $\frac{l-4}{2}$ (see 4.2.2.7) and for an odd l it is a condensation point of order $\frac{l-3}{2}$ (see 4.2.3.3).

Now we will consider only cases when $\frac{p}{q}$ is not of such form.

The first approach of the searching algorithm is of this form:

We use:

- $\mathbb{N}_{>0}$ counters $b_1 b_2 \ldots$ with values ranging from 1, through all natural numbers, to infinity (with infinity included). Each counter correspond to one cone point on the boundry of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial cone point - namely a none cone point, a normal point).

- a pivot pointing to some counter at any time
- a flag that can be set to "greater", "smaller" or "equal" corresponding to what was the outcome of comparing Euler orbicharacteristicof the orbifold corresponding to counters' state and $\frac{p}{q}$.

We start with:

- all counters set to 1.
- pivot pointing at the first counter
- flag set to "greater"

We will do our computation such that:

- every state of the counters during runtime of the algorithm will have only finitely many counters with value non-1.
- every state in the rutime of the algorithm will have values on consequtive counters ordered in weakly decreasing order.

From now we will consider only such states.
The state of the counters $b_1 b_2 \dots$ correspond to the orbifold of Euler orbicharacteristicequal $\chi^{orb}(*b_1 b_2 \dots)$ (where the trailing 1 are trunkated).
When the algorithm is in the state:

- counters: $b_1 b_2 \dots$
- pivot: on the counter c
- flag: set to the value $flag\_value$

we procced as follows:

```
In the case, the counter is set to:
if the flag is set to "equal" then
{
    We found an orbifold and we are ending the whole
    algorithm with answer "yes, *b_1 b_2 ..."
}

if the flag is set to "smaller" then
{
    We increase the counter c by one (b_c := b_c + 1)
    and set the value of all counters on the left to b_c
    if χ^orb(*b_1 b_2 b_3 ...) < p/q then
        We set the flag to "smaller"
```

```
        We put pivot to the $c+1$ counter

}

if the flag is set to "greater" then
{
    if $\chi^{orb}(*\infty b_2 b_3 \dots) > \frac{p}{q}$ then
    {
        We search for value $@b_1$ of the first counter
        such that $\chi^{orb}(*@b_1 b_2 b_3 \dots) \geqslant \frac{p}{q}$
        and $\chi^{orb}(*(@b_1 - 1) b_2 b_3 \dots) < \frac{p}{q}$.
        More on how we search for it will be told later, for now
        we can think that we search one by one starting
        from $b_1$ and going up till $@b_1$.
        We set $b_1$ to $@b_1$.
        if $\chi^{orb}(*b_1 b_2 b_3 \dots) = \frac{p}{q}$ then
        {
            we found an orbifold and we are ending the whole
            algorithm with answer "yes, $*b_1 b_2 \dots$"
        }
        if $\chi^{orb}(*b_1 b_2 b_3 \dots) > \frac{p}{q}$
        {
            we set flag to "greater"
            and move the pivot to the second column
        }
    }
    if $\chi^{orb}(*\infty b_2 b_3 \dots) < \frac{p}{q}$ then
    {
        We set $b_1$ to $\infty$.
        We set flag to "smaller".
        We move pivot to the second column.
    }
}
```

Let $m \in \mathbb{N}$ be such that $\frac{p}{q} \in (1 - \frac{m}{2}, 1 - \frac{m+1}{2})$ Let us denote by $r \coloneqq \frac{p}{q} - (1 - \frac{m}{2})$.

We will searching in σ as such:

If $\frac{p}{q} \in \sigma$, then, from the corollary 4.2.3.3 we know, that there exist some $n \in \mathbb{N}$, such that $\frac{p}{q} + \frac{n}{2} \in \sigma$ but $\frac{p}{q} + \frac{n}{2} \notin \sigma$.

We will be consequently checking points from $1 + r$, through $1 + r - \frac{l}{2}$, for $0 \leqslant l \leqslant m$, to the $\frac{p}{q}$. We stop at the first found point. If one of these point is in the spectrum, then all smaller (so also $\frac{p}{q}$) are in the spectrum and $\frac{p}{q}$ is the accumulation point of the spectrum of order $m - l$ (from this, we can see some heuristic, that the points that have smaller order will be generally harder to find in some sense). If none of this points are in in the spectrum, then $\frac{p}{q}$ is not.

Chapter 6

Counting occurrences

abcd

6.1 Deformations on orbifolds?

Chapter 7

Power series and generating functions

Chapter 8

Conclusions

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