Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny
specjalność: teoretyczna

Bartosz Sójka

Two dimentional orbifolds' volumes' spectrum

Praca magisterska
napisana pod kierunkiem
prof. dr hab. Tadeusza Januszkiewicza

Wrocław 2021-2023

dla Wujka

Contents

Abstract

Orbifoldy

Chapter 1

Introduction

1.1 Motivations

Quotiens by a groups

Praca o ujemnym ale z najwyzszą orbicharakterystyką

1.2 Technical introductions

Now, we will proceed to give technical introductions about orbifolds, Euler orbicharacteristic and the technics we will use in this thesis, alongside with some definitions and notation and naming conventions.

1.3 Orbifolds

1.3.1 Definition

The definition of the orbifold is taken from Thurston [Thu79] (chapter 13), with slight modification described in 1.3.5. We briefly recall the concept, but for full discussion we refer to [Thu79].

An orbifold is a generalisation of a manifold. As manifold, it consists of a Hausdorf space with some additional structure. Compared to manifolds, one allows more variety of local behaviour. On a manifold a map is a homeomorphism between $\mathbb{R}^n$ and some open set on a manifold. On an orbifold a map is a homeomorphism between a quotient of $\mathbb{R}^n$ by some finite group and some open set on an orbifold. In addition to that, the orbifold structure consist the informations about that finite group and a quotient map for any such open set.

We can make an observation, that since in dimension 2, quatient of $\mathbb{R}^2$ by a finite group is topologically always either $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$, we have that in dimension 2, the underlying Hausdorf space of any orbifold is a topological manifold (possibly with a boundary). For an orbifold O we will call this underlying manifold M a base manifold of O and we will call O an M-orbifold.

1.3.2 Good and bad orbifolds

Above definition says that an orbifold is locally homeomorphic do the quotient of $\mathbb{R}^n$ by some finite group.

When an orbifold as a whole is quotient of some finite group acting on a manifold we say, that it is 'good'. Otherwise we say, that it is 'bad'.

In two dimentions there are only four types of bad orbifolds, namely (adopting notation from [Thu79]):

- $S^2(n)$
- $D^2(;n)$
- $S^2(n_1, n_2)$ for $n_1 < n_2$
- $D^2(;n_1, n_2)$ for $n_1 < n_2$.

All other orbifolds are good – [Thu79] (theorem 13.3.6).

Manifolds without boundary can be treated as orbifolds with trivial group for every map and will. Manifolds with boundary can be treated as orbifolds with trivial group on all maps from the interior and with group $\mathbb{Z}_2$ on the boundary, as described in [Thu79] (example 13.2.2.).

1.3.3 Therminology

We differ from Thurston in the terms of naming points with maps with non-trivial groups. We will call them orbipoints. If the group acts as the group of rotations (so a cyclic group) we will call them rotational points. If the group is a dihedreal group we will call them dihedreal points. And if the point is on the boundry that stabilises reflection we will call it a reflection point.

If a group associated to the orbipoint has degree n, we will say that tha orbipoint is of degree n.

1.3.4 Finite number of orbipoints

In this thesis we will consider only orbifolds with finitely many orbipoints and all orbifolds mentioned from this point are meant as such without further notice. Reason for this choice will be described in 1.4.2.

1.3.5 Compactness

Orbifold as a topological space is the same as its base manifold. We, would like to restrict our interest only to compact orbifolds. However, noncompact orbifolds, such as ones from [CBG16] (chapter 18) which are quotients of a group action of an infinite group acting on $\mathbb{R}^2$ (that will be also frequently interpreted in this setting as a hyperbolic plane $\mathbb{H}^2$), also interests us. We would like to accomodate some of noncompact orbifolds that will satisfy the condition simmilar to the 1.3.4. To do this we will slightly expand our definition of an orbifold.

Let us start with a following construction.

For an noncompact orbifold O, let us take it's one point compactification. Let the compactification point be named x_O. For some set X, let $\#(X)$ be the number of connected components of X. Now let us consider some connected open set $U \ni x_O$. The set $U \setminus \{x_O\}$ is not neccesserly connected.

Let

$$C(O) \coloneqq \sup_{\substack{U \ni x_O \\ \text{connected,} \\ \text{open}}} \#(U \setminus \{x_O\}). \tag{1.3.5.0.1}$$

We will be interested only in the case, when $C(O)$ is finite. If $C(O)$ is finite, we take some U that realise the supremum and compactify each of connected components of $U \setminus \{x_O\}$ with a separate point. We will call these points "cusps". If for some cusp x, in every $U \ni x$ there are points from the boundary of O, we will call it a cusp on a boundary. In the other case, we will call it a cusp in the interior.

The result is a compact topological space. We will treat it as an orbifold, with cusps as orbipoits in extended definition. We extend the definition as such, that the map from the compactification of some open subset consisting point x can go to the quationt of compactification of $\mathbb{H}^2$ by S^1. The group, we will take to act on $\mathbb{H}^2$ will be:

- in the case of x being in the interior – infinite cyclic group $\mathbb{Z}$, where the generator acts as translation by 1 in the half-plane model of hyperbolic plane

- in the case of x being on the boundary – infinite dihedral group D_∞, where generators will be reflections with respect to vertical lines spaced by 1 in a half-plane model on a hyperbolic plane.

Note that in both cases, the is exactly one point in the compactification of $\mathbb{H}$, that is fixed for every elemnt of the acting group – point from the compactification at the infinity on the top of the plane. This point is will be always mapped to the orbipoint x.

1.3.6 Equivalence

From this we can treat that and that orbifolds as the same.

Without boundary – uniquely determined by orbipoints list.

With boundary – in general not.

1.4 Euler (orbi)characteristic

1.4.1 Euler characteristic

We can define Euler characteristic as additive topological invariant defined normalised on symplexes.

we will treat as we will treat manifolds as orbifolds we will always refer we will from of Euler orbicharacteristicon two dim orbifolds

1.4.2 Euler orbicharacteristic

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \tag{1.4.2.0.1}$$

For O with only rotational orbipoints:

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \tag{1.4.2.0.2}$$

For O with only dihedral orbipoints:

$$\chi^{orb}(O) = \chi(M) - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \tag{1.4.2.0.3}$$

From these formulas we can see, that as number of orbipoints diverges to infinity, the Euler orbicharacteristic diverges to minus infinity. For this reason, we restric ourselves only to orbifolds with finitely many orbipoints.

Observation 1.4.2.1. *An M-orbifold that is different than M always have strictly smaller Euler orbicharacteristic than M.*

1.4.3 Extended Euler orbicharacteristic

(with cusps) Write about cusp as a limit.

1.5 Volumes

1.5.1 Metric structure on the orbifolds

1.6 Classification of orbifolds

The list of all orbifolds with non-negative Euler orbicharacteristic Powiedzieć coś o tym, że orbicharatkeryttyka odpowiada polom (Gauss Bonett itd.) []

1.6.1 Non-negative Euler orbicharacteristic

1.7 Uniformisation theorem (formulation)

In this thesis we aim to better recognise possible two dimensional orbifolds. The fundament that we are relaying on is the classification of two dimensional manifolds.

It is phrased as a well known theorem: []

1.8 Notation

For the rest of this thesis we will adopt notation from []. "feature"

We treat manifolds and orbifolds as a sphere with some features added by the operations.

dać na sferę ε słowo puste. We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature.
We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$.

TO DO: rozwinąć

dopisać, że w Conwayowej $>= 2$

If not stated othewise, in the expressions containing ∞ symbol, their value is understood as $\varphi(\infty) := \lim_{n\to\infty} \varphi(n)$.

TO DO:

Addition of sets and numbers.

Warning throught the whole thesis we will consider only two dimensional manifolds and orbifolds, because of that words "two-dimensional" will be usually omited

delta h c b

$\sigma^r(S^2), \sigma^d(D^2)$

1.9 Spectra

We will call the set of all possible Euler orbicharacteristic of a manifold M, the spectrum of M and we will denote it by $\sigma(M)$.

We will also denote the sum of spectra of all two dimentional manifolds by:

$$\sigma := \bigcup_{M\ :\ \text{2d manifold}} \sigma(M). \tag{1.9.0.0.1}$$

This will be the main interest of this thesis.

Now we want to derive the form of the $\sigma(M)$. For a manifold M with h handles, c cross-cups and b boundary components, it's Euler characteristic is given by:

$$\chi(M) = 2 - 2h - c - b. \tag{1.9.0.0.2}$$

The set of Δ for possible orbifold features are:
- for $b \neq 0$:

$$\{-\frac{n-1}{2n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\}\} \tag{1.9.0.0.3}$$

- for $b = 0$:

$$\{-\frac{n-1}{n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\}\}. \tag{1.9.0.0.4}$$

Thus, we have that the form of the spectrum of two dimensional manifold M is:
- for $b \neq 0$:

$$\sigma(M) = 2 - 2h - c - b - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \;\middle|\; n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.0.5}$$

• for $b = 0$:

$$\sigma(M) = 2 - 2h - c - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \qquad (1.9.0.0.6)$$

Observation 1.9.0.1. *We have that* $\sigma(S^2) = 2\sigma(D^2)$.

Proof.

Indeed, since:

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \qquad (1.9.0.1.1)$$

and

$$\sigma(D^2) = 1 - \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \;\middle|\; m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \square \qquad (1.9.0.1.2)$$

Observation 1.9.0.2. *For every two dimentional manifold M, we have that $\sigma(M)$ is homeomorphic to $\sigma(D^2)$. For M with h handles, c cross-cups and b boundary components, this homeomorphism is:*
• *for $b \neq 0$:*

$$\sigma(M) = \sigma(D^2) - 2h - c - (b - 1), \qquad (1.9.0.2.1)$$

• *for $b = 0$:*

$$\sigma(M) = 2\sigma(D^2) - 2h - c. \qquad (1.9.0.2.2)$$

Proof.
For a manifold M with h handles, c cross-cups and b boundary components, it's $\sigma(M)$ is given by:
• for $b \neq 0$:

$$\sigma(M) = 2 - 2h - c - b - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \;\middle|\; n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \qquad (1.9.0.2.3)$$

• for $b = 0$:

$$\sigma(M) = 2 - 2h - c - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \qquad (1.9.0.2.4)$$

On the other hand, we have that:

$$\sigma(D^2) = 1 - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \;\middle|\; n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \qquad (1.9.0.2.5)$$

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \qquad (1.9.0.2.6)$$

and

$$\sigma(S^2) = 2\sigma(D^2). \qquad (1.9.0.2.7)$$

From this, the observation follows immedietly. $\square$

Observation 1.9.0.3. *For every manifold M, for every $x \in \sigma(M)$, we have that $x \leqslant \chi(M)$.*

1.10 Egyptian franctions

Egyptian frantion is a finite sum of fractions, all with numerators one. Most of the time it is also required, that the fractions in the sum have pairwise distinct denominators. We will however take less usual convention and will drop that requirement, calling an egyptian fraction any sum of unitary fractions.

1.11 Connection between spectra and Egyptian fractions

The terms $-\frac{r_i-1}{r_i}$ in the sum 1.4.2.0.2 can be expressed as $-1+\frac{1}{r_i}$ and the term $-\frac{d_j-1}{2d_j}$ in the sum 1.4.2.0.3 can be expressed as $-\frac{1}{2}+\frac{1}{2d_j}$. Then the sums become:

$$\chi(M) - n + \underbrace{\sum_{i=1}^{n} \frac{1}{r_i}}_{\substack{\text{Egyptian}\\ \text{fraction}}} \tag{1.11.0.0.1}$$

and

$$\chi(M) - \frac{m}{2} + \frac{1}{2} \underbrace{\sum_{j=1}^{m} \frac{1}{d_j}}_{\substack{\text{Egyptian}\\ \text{fraction}}} . \tag{1.11.0.0.2}$$

In this form, the egyptian fractions are explicitly present in expresions of points in $\sigma(M)$.

The $-n$ and $-\frac{m}{2}$ terms provide constraints on the number of fractions that can apprear in the sum.

We will now translate the questions of being in the spectrum to the questions of being expressable as egyptian fraction with the particular number of summands. It will be used in 3.3.1.1 and 5.2.1.1.

We will now state two corrolaries that follows immediately from the form of expressions 1.11.0.0.1 and 1.11.0.0.2, and from 1.9.

Corollary 1.11.0.1. *If x can be expressed as an egyptian fraction with n summands, then for any two dimensional manifold M we have:*

$$\chi(M) - n + x \in \sigma(M) \tag{1.11.0.1.1}$$

and, if M has at least one boundary component also:

$$\chi(M) - \frac{n}{2} + \frac{1}{2}x \in \sigma(M). \tag{1.11.0.1.2}$$

Corollary 1.11.0.2. *If for some two dimensional manifold M we have that $y \in \sigma(M)$ as an Euler orbicharacteristic of an orbifold which has n rotational orbipoints and not any other, then*

$$y + n - \chi(M) \tag{1.11.0.2.1}$$

can be expressed as an egyptian fraction with n (not nessecerely distincs) summands.

If for some two dimensional manifold M with at least one boundary component we have that $y \in \sigma(M)$ as an Euler orbicharacteristicof an orbifold which has m dihedral orbipoints and not any other, then

$$2y + \frac{m}{2} - 2\chi(M) \tag{1.11.0.2.2}$$

can be expressed as an egyptian fraction with m (not nessecerely distinct) summands.

1.12 Operations and constructions on orbifolds

Write about the general operations we are interested in i.e. taking any number of features (handles cross caps, parts of boundry components with orbipoints on it, orbipoints in the interior) and replacing it by any other feauters (Some preserve the area) Write about operations nesseserie for reduction of cases write that every operation reduces Euler orbicharacteristic.

1.13 Questions asked

There will be two main parts of question:

- Ones regarding σ as a set, where we will be asking of its order type and topology and relation to other sets such as $\sigma(D^2)$ and $\sigma(S^2)$. We will focus on these questions in 3.
- Ones regarding σ as an image of a χ^{orb}, sending orbifolds to their Euler orbicharacteristics. There, we will ask for example how namy orbifolds have particular Euler orbicharacteristic and related questions. We will focus on these questions in the chapter 5.

Chapter 2

Reduction to arithmetical questions

Reductions presented in this chapter will be more in the spirit of chapter 3, in the sense that for now, until chapter 5 , we will not pay attention to how many orbifolds have the same Euler orbicharacteristic, only whether a particular number is an Euler orbicharacteristic for at least one orbifold or not.

In chapter 5 we will explain how these reductions will be relevant to the discussion holded there.

2.1 Reductions of cases

The aim of following reductions is to make it easier to answear the question of which points lie in σ and which not.

The first aspect of the structure of σ that we would like to simplify is that it is the sum of $\sigma(M)$, for every two dimentional manifold M.

$$\sigma = \bigcup_{M\,:\,\text{2d manifold}} \sigma(M). \tag{2.1.0.0.1}$$

We aim to find a minimal set $\mathcal{M}$ of base manifolds such that:

$$\sigma = \bigcup_{M\in\mathcal{M}} \sigma(M). \tag{2.1.0.0.2}$$

It will turn out that $\mathcal{M} = \{S^2, D^2\}$ satisfies 2.1.0.0.2 and that both S^2 and D^2 are neccesery.

2.2 Sufficiency of S^2 and D^2

Given an orbifold O_1, we want to perform some operations from 1.12 on it, such that the resulting orbifold O_2 will have the same Euler orbicharactristic, but the base manifold of O_2 would be S^2 or D^2. We would then say, that O_1 got reduced to O_2. In following subsection, we allow only such operations, that do not change Euler orbicharacteristic. When writing that we "can" do something we mean that there is possible one of the operations from 1.12.

The Euler characteristic of base manifold depends only on the number of handles, cross caps and boundry components. And, as stated in 1.4 it is:

$$2 - 2h - c - b, \tag{2.2.0.0.1}$$

for h - number of handles, c - number of cross-caps, b - number of boundary components.

For every such a manifold feature we want to find an orbifold features with the same Euler orbicharacteristic delta.

We will take two approuches, depending on whether the orbifold in question has a boundary or not.

2.2.1 Orbifold without boundary

We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(2^4) \tag{2.2.1.0.1}$$
$$\Delta(\times) = \quad -1 \quad = \Delta(2^2) \tag{2.2.1.0.2}$$

From this we can see that we can remove handles and cross-caps from any orbifold without the boundary. After such reductions we are left with a S^2 orbifold with all orbipoints being rotational in the interior.

2.2.2 Orbifold with boundary

We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta((^*2)^8) \tag{2.2.2.0.1}$$
$$\Delta(*) = \quad -1 \quad = \Delta((^*2)^4) \tag{2.2.2.0.2}$$
$$\Delta(\times) = \quad -1 \quad = \Delta((^*2)^4) \tag{2.2.2.0.3}$$

From this we can see that we can remove handles and cross-caps from any orbifold with a boundary. We can also remove all boundary components exept one. We can further observe that:

$$\Delta(n) = \frac{n-1}{n} = 2\frac{n-1}{2n} = \Delta((^*n)^2) \tag{2.2.2.0.4}$$

From this we see that we can remove all the rotational orbipoints in favor for dihedral orbipoints. After such reductions we are left with a D^2 orbifold with all orbipoints being dihedral on the boundary or being reflectional on the boundary.

As a fact not necessary for our reductions, but interestung on its own, we can furthermore, observe that:

Observation 2.2.2.1. *If O_1 has not S^2 as its base manifold it can be reduced to a D^2-orbifold.*

Proof.
If O_1 has not S^2 as its base manifold M, then M has at least one handle or a cross-cup. We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(*(^*2)^4) \qquad (2.2.2.1.1)$$
$$\Delta(\times) = \quad -1 \qquad = \Delta(*). \qquad (2.2.2.1.2)$$

From this we have that the handle or the cross-cap can be replaced by a boundary component and some number of boundary orbipoints. After this reduction, we can proceed with all the other reductions from the 2.2.2 and obtain an D^2-orbifold with the same Euler orbicharacteristicas the original one. □

The results of our reductions, can be summarised as:

Observation 2.2.2.2. *If two-dimentional manifold M has no boundry, then*

$$\sigma(M) \subseteq \sigma(S^2) \qquad (2.2.2.2.1)$$

If, in addition, $M \neq S^2$, then

$$\sigma(M) \subseteq \sigma(D^2). \qquad (2.2.2.2.2)$$

Observation 2.2.2.3. *If two-dimentional manifold M has a boundry, then*

$$\sigma(M) \subseteq \sigma(D^2) \qquad (2.2.2.3.1)$$

Corollary 2.2.2.4. *We have that $\sigma = \sigma(S^2) \cup \sigma(D^2)$.*

Observation 2.2.2.5. *If a two-dimensional manifold M has a boundary, then:*

$$\sigma(M) = \sigma^d(M). \qquad (2.2.2.5.1)$$

We will postpone our discussion of neccessity of both S^2 and D^2 to 2.4, after the section 2.3 which will provide us with more convenient language.

2.3 Reduction to arithmetical questions

As written in 1.4.1, we can express an Euler orbicharacteristicof a M-orbifold O as:

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j}, \qquad (2.3.0.0.1)$$

where r_i and d_j are degrees of the, respectively, rotational and diheadral orbipoints of O.

From this we can express $\sigma(M)$ as:

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} + \qquad (2.3.0.0.2)$$
$$- \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \;\middle|\; m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \qquad (2.3.0.0.3)$$

As from 2.2 we know that $\sigma = \sigma(S^2) \cup \sigma(D^2)$, and that $\chi(S^2) = 2$ and $\chi(D^2) = 1$, we can expresss σ as a sum ($\cup$) of two sets:

$$2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(S^2) \tag{2.3.0.0.4}$$

and

$$1 - \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \mid m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(D^2). \tag{2.3.0.0.5}$$

From this we see, that the core of understanding σ through arithmetical viewpoint is to understand possible values of expression:

$$2 - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \tag{2.3.0.0.6}$$

and

$$1 - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j}, \tag{2.3.0.0.7}$$

with r_i and d_j ranging over $\mathbb{N}_{>0} \cup \{\infty\}$.

As stated in **??** we can perform futher reductions to have an orbifold with particular orbicharacteristic without cusps (if needed) and then (after these reductions) we can analyse only expressions with r_i and d_j ranging over $\mathbb{N}_{>0}$ and they will still give us full spectrum. However, as stated later, it will be more convenient to us to include orbifolds with cusps so we are stating this observation only as a side remark.

2.4 Neccessity of S^2 and D^2

As we know from 1.12 adding an orbipoint to a manifold decreases it's orbicharacteristic. As S^2 has the highest Euler characteristic: 2 of all two dimentional manifolds, there is no other orbifold with Euler orbicharacteristic equal to 2. S^2 is then necesery to include 2.

As known from [**najwiĂŹkszy orbifold**], the number $-\frac{1}{84} \in \sigma(D^2)$ and it is the greatest negative Euler orbicharacteristicany two dimensional orbifold can have. We will now show, that $-\frac{1}{84} \notin \sigma(S^2)$. For the sake of contradiction let us assume, that $-\frac{1}{84} \in \sigma(S^2)$, then, from 1.9.0.1 we know, that $\frac{1}{2}\left(-\frac{1}{84}\right) \in \sigma(D^2)$. This is a contradiction as $0 > \frac{1}{2}\left(-\frac{1}{84}\right) > -\frac{1}{84}$. $\square$

Futher examination of connections between $\sigma(D^2)$ and $\sigma(S^2)$ is performed in 3.3.2.

Chapter 3

Order type and topology

In this chapter we will discuss that both the order type and the topology of the set σ of all possible Euler orbicharacteristics of two-dimensional orbifolds are that of ω^ω.

To determine order type and topology of σ we will first study how $\sigma(D^2)$ looks like. Then, remembering that $\sigma = \sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$ we will make an argument for σ.

3.1 Order type and topology of $\sigma(D^2)$

In this section we will also describe precisely where accumulation points of $\sigma(D^2)$ lie and of which order (see below 3.1.1 or A.1) they are. Analysis of locations of those accumulation points, as interesting as it is alone will also be necessery for providing our argument about order type and topology of $\sigma(D^2)$.

3.1.1 Definition of order of accumulation points

These definitions are exactly the same as from A.1 and are repeated here only for the readers convenience.

We start with definition of being "at least of order n" that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.
For a given set we define as follows:

Definition 3.1.1.1. *(Inductive). We say that the point x is an acccumulation point of a set X of order at least 0, when it belongs to the set X. We say that the point x is an acccumulation point of a set of order at least $n+1$, when it is an acccumulation point (in the usual sense) of the accumulation points each of order at least n i.e. in every neighbourhood of x there is at least one accumulation point of a set X of order at least n, distincs from x.*

Definition 3.1.1.2. *We say that the point is an acccumulation point of order n iff it is an acccumulation point of order at least n and it is not an acccumulation point of order at least $n+1$. If the point is an acccumulation point of order at least n for an arbitrary large n we say that the point is an acccumulation point of order ω.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the ussual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

3.1.2 Analysis of locations of accumulation points of $\sigma(D^2)$ with respect to their order

We want to determine where exactly are accumulation points of the set $\sigma(D^2)$ with respect to their orders.

For this we will use a handful of observations and lemmas.

Observation 3.1.2.1. *Let us observe, that* $\lim_{n\to\infty} \Delta(^*n) = -\frac{1}{2}$. *From that, we see, that for every point* $x \in \sigma(D^2)$, *the point* $x - \frac{1}{2}$ *is an acccumulation point. Let us observe, that also, for every point* $x \in \sigma(D^2)$, *we have that* $x - \frac{1}{2} \in \sigma(D^2)$, *because* $\Delta(^*\infty) = -\frac{1}{2}$.

Lemma 3.1.2.2. *For all* $n \in \mathbb{N}_{\geqslant 2}$ *and* $x \in (-\infty, 1]$ *there are only finitely many Euler orbicharacteristics in the interval* $[x, 1] \cap \sigma(D^2)$ *of orbifolds that have points of order equal at most* n.

Proof.

Let $x \in (-\infty, 1]$. There can be at most $\lfloor 4(1-x) \rfloor$ orbipoints on the D^2 orbifold with an Euler orbicharacteristic $y \in [x, 1]$ since each orbipoint decreases an Euler orbicharacteristic by at least $\frac{1}{4}$ and the Euler characteristic of D^2 is 1.

There are only $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible sets of $\lfloor 4(1-x) \rfloor$ orbipoints' orders that are less or equal than n. Hence, there are only at most $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible Euler orbicharacteristics.

Lemma 3.1.2.3. *If* x *is an acccumulation point of the set* $\sigma(D^2)$ *of order* n, *then* $x - \frac{1}{2}$ *is a accumulation point of the set* $\sigma(D^2)$ *of order at least* $n + 1$.

Proof.

Inductive.

• $n = 0$: If x is an isolated point of the set $\sigma(D^2)$, then $x \in \sigma(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma(D^2)$ for all $k \geqslant 1$, from that, that $x - \frac{1}{2}$ is a accumulation point of $\sigma(D^2)$.

• inductive step: Let x be an acccumulation point of the set $\sigma(D^2)$ of an order $n > 0$. Let a_k be a sequence of accumulation points of order $n - 1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of accumulation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n + 1$. □

Lemma 3.1.2.4. *If* x *is an acccumulation point of the set* $\sigma(D^2)$ *of order* n, *then* $x + \frac{1}{2}$ *is an acccumulation point of the set* $\sigma(D^2)$ *of order at least* $n - 1$.

Proof.

Inductive

• $n = 1$: We assume, that x is an acccumulation point of isolated points of the set $\sigma(D^2)$. From 3.1.2.2 we know, that for all m there are only finitely many Euler orbicharacteristics in the interval $[x, 1]$ of orbifolds that have dihedral points of order equal at most m.

From that, for arbitrary small neighborhood $U \ni x$ and arbitrary large m there exist an orbifold that has a dihedral point of period grater than m, whose Euler orbicharacteristic lies in U. Let us take a sequence of such Euler orbicharacteristics a_k convergent to x, that we can choose a divergent to infinity sequence of degrees of dihedral points b_k of orbifolds of Euler orbicharacteristics equal a_k.

To do: picture

Let us observe, that for all k, the number $a_k + \frac{b_k-1}{2b_k}$ is in $\sigma(D^2)$. It is so, because a_k is an Euler orbicharacteristic of an orbifold that have a dihedral point of period b_k, so identical orbifold, only without this dihedral point, has an Euler orbicharacteristic equal to $a_k + \frac{b_k-1}{2b_k}$. The sequence $a_k + \frac{b_k-1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least 0.

• inductive step: Let x be an acccumulation point of the set $\sigma(D^2)$ of order $n > 1$. Let a_k be a sequence of accumulation points of the set $\sigma(D^2)$ of order $n-1$ convergent to x. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of an acccumulation points of the set $\sigma(D^2)$ of order $n-2$ convergent to $x + \frac{1}{2}$. From that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n - 1$. □

Lemma 3.1.2.5. *If x is an acccumulation point of the set $\sigma(D^2)$ of order $n+1$, then*
$x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $n+2$ and
$x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order n.

Proof.

Let x be an acccumulation point of the set $\sigma(D^2)$ of order $n+1$. From the lemma 3.1.2.3 we know, that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$. Now let us assume (for a contradiction), that $x - \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order $k > n+2$. But then from the lemma 3.1.2.4 we have that x is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$ and that is a contradiction.

Analogously, from the lemma 3.1.2.4 we know, that $x + \frac{1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order at least n. Let us assume (for a contradiction), that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $k > n$. But then from the lemma 3.1.2.3 we have that x is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$ and that is a contradiction. □

Lemma 3.1.2.6. *For all $n \in \mathbb{N}$ all accumulation points of the set $\sigma(D^2)$ of order n are in $\sigma(D^2)$.*

Proof.

Inductive

- $n = 0$: Clear, as they are isolated points of $\sigma(D^2)$.
- inductive step: Let x be a accumulation point of the set $\sigma(D^2)$ of order $n > 0$. From the lemma 3.1.2.5 point $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $n - 1$. From the inductive assumption $x + \frac{1}{2} \in \sigma(D^2)$. Then, from 3.1.2.1, we have that $x \in \sigma(D^2)$. □

Theorem 3.1.2.7. *The greatest accumulation point of the set $\sigma(D^2)$ of order n is $1 - \frac{n}{2}$.*

Proof.

Inductive

- $n = 0$: We know, that $1 \in \sigma(D^2)$ and 1 is the greatest element of $\sigma(D^2)$.
- an inductive step: From the inductive assumption we know that $1 - \frac{n}{2}$ is the greatest accumulation point of the set $\sigma(D^2)$ of order n. From the lemma 3.1.2.5 we have then that $1 - \frac{n+1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order $n + 1$. Let us assume (for a contradiction), that there exist a bigger accumulation point of order $n + 1$ equal to $y > 1 - \frac{n+1}{2}$. But then, from lemma 3.1.2.5, point $y + \frac{1}{2}$ would be an acccumulation point of order n, what gives a contradiction, because $y + \frac{1}{2} > 1 - \frac{n}{2}$. □

From the above discussion we can also formulate following corollary that will be useful later:

Corollary 3.1.2.8. *Let $x \in \sigma(D^2)$. Then:*

- *there exists $n_1 \in \mathbb{N}$ such that $x + \frac{n_1}{2} \in \sigma(D^2)$ but $x + \frac{n_1+1}{2} \notin \sigma(D^2)$.*

  In other words, there exist $y \in \sigma(D^2)$ and $n_1 \in \mathbb{N}$ such that $y + \frac{1}{2} \notin \sigma(D^2)$ and such that $x = y - \frac{n_1}{2}$;
- *there exists $n_2 \in \mathbb{N}$ such that x is an accumulation point of the set $\sigma(D^2)$ of order n_2*

and $n_1 = n_2$.

3.1.3 Proof that $\sigma(D^2)$ is well ordered

Definition 3.1.3.1. *Let $B_0 = \{1\}$. For an $n \in \mathbb{N}_{>0}$, let B_n be the set of all possible Euler orbicharacteristic realised by orbifolds of type $*^*b_1, \cdots, ^* b_n$. For a given n these are D^2 orbifolds with precisely n non trivial orbipoits on their boundry.*

Observation 3.1.3.2. *There is a recursive relation, that $B_{n+1} = B_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$*

Proof.

It is so, because every orbifold with $n+1$ orbipoints can be obtained by adding one point to an orbifold with n orbipoints and the set $\{-\frac{n-1}{2n} \mid n \geqslant 2\} = \{\Delta(^*b) \mid b \geqslant 2\}$. $\square$

Observation 3.1.3.3. *Observe that, as any orbifold has only finitely many orbipoints, we have that* $\sigma(D^2) \subseteq \bigcup_{n=0}^{\infty} B_n$. *We defined* $\sigma(D^2)$ *as a set of all possible Euler orbicharacteristic of disk orbifolds, so* $\sigma(D^2) \supseteq \bigcup_{n=0}^{\infty} B_n$. *From this we have that* $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.

Lemma 3.1.3.4. *For any given* $n \in \mathbb{N}$ *the set* B_n *is a subset of the interval* $[1-\frac{n}{2}, 1-\frac{n}{4}]$.

Proof.

Take $x \in B_n$. There exists an orbifold O with signature $*^*b_1, \cdots, ^*b_n$, such that $\chi^{orb}(O) = x$. We have that $\forall_i -\frac{1}{2} \leqslant \Delta(^*b_i) \leqslant -\frac{1}{4}$. From this $-\frac{n}{2} \leqslant \Delta(^*b_1, \cdots, ^*b_n) \leqslant -\frac{n}{4}$, so $\chi^{orb}(O) \in [1-\frac{n}{2}, 1-\frac{n}{4}]$. $\square$

Observation 3.1.3.5. *From 3.1.3.2 and A.2.0.1, we have that* B_n *do not have infinite ascending sequence for all* n.

Further, from A.2.0.2 we conclude, that $\bigcup_{n=0}^{N} B_n$ *do not have infinite ascending sequence for all* N.

Theorem 3.1.3.6. *In* $\sigma(D^2)$ *there are no infinite strictly ascending sequences, hence, it is well ordered.*

Proof.

For the sake of contradiction lets assume that c_n is an infinite strictly ascending sequence in $\sigma(D^2)$. As c_n is bounded from below by c_0 and whole $\sigma(D^2)$ is bounded from above by 1, all elements of c_n are in the interval $[c_0, 1]$. From 3.1.3.3 we have, that $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.

Lemma 3.1.3.4 says that for all n we have $B_n \subset [1-\frac{n}{2}, 1-\frac{n}{4}]$. From this, we know, that for any n such that $1-\frac{n}{4} < c_0$ we have, that $B_n \cap [c_0, 1] = \varnothing$. Let n_0 be the smallest such that $1-\frac{n_0}{4} < c_0$ (so $n_0 > 4(1-c_0)$). Then for all $n > n_0$ we have $1-\frac{n}{4} < c_0$, meaning, that for all $n > n_0$ we have $B_n \cap [c_0, 1] = \varnothing$, so all elements of c_n are in $\bigcup_{n=0}^{n_0} B_n$. But this contradicts 3.1.3.5. $\square$

3.1.4 Proof that order structure and topology of $\sigma(D^2)$ are those of ω^ω

Theorem 3.1.4.1. *Order type and topology of* $\sigma(D^2)$ *is* ω^ω.

Proof.

We will first prove, that the order type of $\sigma(D^2)$ is ω^ω.

• Order type of $\sigma(D^2)$ is at least ω^ω.

From 3.1.2.7 we know, that for every $n \in \mathbb{N}$, in $\sigma(D^2)$ there are accumulation points of order n. From this, and from A.2.0.4 we know that $\sigma(D^2)$ has an order type at least ω^n, for all $n \in \mathbb{N}$. The smallest ordinal number qual at least ω^n, for all $n \in \mathbb{N}$ is ω^ω. Thus, the order type of $\sigma(D^2)$ is at least ω^ω.

• Order type of $\sigma(D^2)$ is at most ω^ω.

For the sake of contradiction, let us suppouse, that the order type η of $\sigma(D^2)$ is strictly greater than ω^ω. Then, $\sigma(D^2)$ has a set A of an order type ω^ω as it's prefix. The set A is bounded, as the $\omega^\omega + 1$st element of $\sigma(D^2)$ is greater than any alement of A. Let n, be such that $1 - \frac{n}{2}$ is smaller than any lement of A. As A is of order type ω^ω it has a prefix B of order type ω^n. From A.2.0.4 we know, that B has an accumulation point b of order n. This gives us a contradiction, as $b > 1 - \frac{n}{2}$, and from 3.1.2.7 we know, that $1-\frac{n}{2}$ is the greatest accumulation point of order n in $\sigma(D^2)$.

Now, we will prove, that the topology of $\sigma(D^2)$ is that of ω^ω.

From 3.1.2.6 we know that every accumulation point of $\sigma(D^2)$ is in $\sigma(D^2)$. Thus, $\sigma(D^2)$ satisfies the assumtions of the lemma A.2.0.3 and we have that the topology of $\sigma(D^2)$ is ω^ω.

3.2 Order type and topology of σ

Theorem 3.2.0.1. *The order type of the set of possible Euler orbicharacteristics of two-dimensional orbifolds σ is ω^ω.*

Proof.

From 2.2 we know, that $\sigma = \sigma(D^2) \cup \sigma(S^2)$.

From 3.1.4.1 and 1.9.0.1, we have that order types and topologies of $\sigma(D^2)$ and $\sigma(S^2)$ both are ω^ω and that $\sigma(S^2) = 2\sigma(D^2)$.

We will now prove that the order type of σ is ω^ω.

From 3.1.2.7 we know, that the largest accumulation point of the set $\sigma(D^2)$ of order n is $1 - \frac{n}{2}$. From, this and from the fact that $\sigma(S^2) = 2\sigma(D^2)$ we know that that the largest accumulation point of the set $\sigma(S^2)$ of order n is $2 - n$.

From this, we have, that for every $m \in \mathbb{N}_{>0}$, order type of $(-m, \infty) \cap \sigma(D^2)$ is ω^{2m+2} and that order type of $(-m, \infty) \cap \sigma(S^2)$ is ω^{m+2} (if $-m = 1 - \frac{n}{2}$, then $n = 2m + 2$ and if $-m = 2 - n$, then $n = m + 2$).

Thus, for every $m \in \mathbb{N}_{>0}$, we have that $(-m, \infty) \cap \sigma(D^2)$ and $(-m, \infty) \cap \sigma(S^2)$ satisfies assumtions of A.2.0.8, thus, we have that $(-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2))$ have an order type ω^{2m+2}.

From this we have that

$$\sigma = \sigma(D^2) \cup \sigma(S^2) = \bigcup_{m=1}^{\infty} \Big((-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2))\Big) \qquad (3.2.0.1.1)$$

have an order type ω^ω.

Now we will prove, that the topology of σ is that of ω^ω too.

We have that for $\sigma(D^2)$ $\Big[\sigma(S^2)\Big]$ that every accumulation point of $\sigma(D^2)$ $\Big[\sigma(S^2)\Big]$ is in $\sigma(D^2)$ $\Big[\sigma(S^2)\Big]$. From this and from A.2.0.7 we have, that all accumulations points of σ are in σ. From this, from lemma A.2.0.3 we have that the topology of σ is ω^ω. $\square$

3.3 More about how this σ, $\sigma(S^2)$ and $\sigma(D^2)$ lie in $\mathbb{R}$

This section consists of rather loose assempbly of remarks and observations about some relations between σ, $\sigma(S^2)$, $\sigma(D^2)$ and how they all lie in $\mathbb{R}$.

Observation 3.3.0.1. *The first (greatest) negative accumulation point of the set of σ is $-\frac{1}{12}$. It is the accumulation point of order* 1.

Proof.

We will show, that $-\frac{1}{12}$ is the greatest negative accumulation point of the set $\sigma(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two-dimensional orbifolds is equal to $\sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$, so the greatest negative point of the set $\sigma(S^2)$ is smaller than the greatest negative accumulation point of the set $\sigma(D^2)$.

• $-\frac{1}{12} = \chi^{orb}((2,3)) - \frac{1}{2}$, from this we have that $-\frac{1}{12}$ an acccumulation point of the set $\sigma(D^2)$ of order at least 1.

• Let us assume (for a contradiction), that there exist bigger, negative accumulation point of the set $\sigma(D^2)$ of order at least 1. Let us denote it by x.

However, then, from the lemma 3.1.2.5 point $x + \frac{1}{2}$ is the accumulation point of the set $\sigma(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12}$. From the lemma 3.1.2.6 we have that x is in $\sigma(D^2)$. But orbifolds of the type $*b_1$ can have Euler orbicharacteristiconly greater or equal $\frac{1}{2}$. Orbifolds of the type $*b_1b_2$ can only have Euler orbicharacteristic$\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*b_1b_2b_3\cdots$ can have Euler orbicharacteristiconly lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma(D^2) = \emptyset$ and to the contradiction.

• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma(D^2)$, from this $-\frac{1}{12}$ is an acccumulation point of order 1 of the set $\sigma(D^2)$. $\square$

3.3.1 Saturation theorem

Theorem 3.3.1.1. *For any rational number $\frac{p}{q}$, for any two dimensional manifold M there exists $N \in \mathbb{N}_0$ such that for all $n \geqslant N$, we have that $\frac{p}{q} - n \in \sigma(M)$.*

Proof.

Let us take $\frac{p}{q} \in \mathbb{Q}$. From 1.11, after [**every number is expressible as an egyptian fraction**] we know that every rational number is expressable as an egyptian fraction. Let us name the number of summends in some egyptian fraction of $\frac{p}{q}$ as k. From 1.11.0.1 we know that then $\chi(M) - k + \frac{p}{q} \in \sigma(M)$. □

TO DO: dokończyć

Corollary 3.3.1.2. *For any finite set of rational numbers $\{(\frac{p}{q})_i\}_{i=1}^{k}$, for any finite set of two dimensional manifolds $\{M_j\}_{j=0}^{l}$, there exists a $N \in \mathbb{N}_0$ such that for all $n \geqslant N$, for all $i \in \{1 \dots k\}$, for all $j \in \{1 \dots l\}$ we have that $(\frac{p}{q})_i - n \in \sigma(M_j)$.*

Proof.

For each pair of $(\frac{p}{q})_i$ and M_j we apply 3.3.1.1, obtaining $N_{i,j}$. we take N as a minimal from $\{N_{i,j}\}_{i\in\{1\dots k\}, j\in\{1\dots l\}}$. □

3.3.2 Connections between $\sigma(S^2)$ and $\sigma(D^2)$

In this section we would like to answer some questions about relations between $\sigma(S^2)$ and $\sigma(D^2)$.

From 2.4 we know that both $\sigma(S^2)$ and $\sigma(D^2)$ are neccessery in expressing $\sigma = \sigma(S^2) \cup \sigma(D^2)$. It is shown by giving examples of two points one from $\sigma(S^2) \setminus \sigma(D^2) \ni 2$ and one from $\sigma(D^2) \setminus \sigma(S^2) \ni -\frac{1}{84}$. We found it interesting to ask further questions about the sets $\sigma(S^2) \setminus \sigma(D^2)$ and $\sigma(D^2) \setminus \sigma(S^2)$ such as what points exactly lie in one of $\sigma(S^2)$ and $\sigma(D^2)$ and not in the other, does it have any connection to the previously described order and topological structure or if the $\sigma(S^2)$ and $\sigma(D^2)$ overlap from some sufficiently distant point. This subsection is an meager attempt to answer some of these questions.

Importance of $-\frac{1}{84}$ and $-\frac{1}{42}$

As described in [**greatest negative orbifold**] we know that $-\frac{1}{84}$ is the greatest possible negative Euler orbicharacteristicfor an two dimensional orbifolds. As described in 2.4 we know that because of that $-\frac{1}{84}$ can't be in $\sigma(S^2)$. This provides us an example of a point that is in $\sigma(D^2)$ but not in $\sigma(S^2)$, showing that including $\sigma(D^2)$ in the statement $\sigma = \sigma(S^2) \cup \sigma(D^2)$ is neccessery.

Accumulation points of the $\sigma(S^2)$

We will first state some observations that will be usefull in this subsubsection.

Observation 3.3.2.1. *If an Euler orbicharacteristic is an accumulation point of order n in $\sigma(D^2)$ $\Big[$respectively $\sigma(S^2)\Big]$, then there exist an D^2 $\Big[$resp. $S^2\Big]$ orbifold with n dihedral $\Big[$resp. rotational$\Big]$ points with that Euler orbicharacteristic.*

prrof. from chapter 3. (todo: dopisać)

Observation 3.3.2.2. *If $x \in \sigma(D^2)$ $\Big[$respectively $\sigma(S^2)\Big]$, then $1-x$ $\Big[$resp. $2-x\Big]$ is a difference in Euler orbicharacteristic resulting from some set of dihedral $\Big[$resp. rotational$\Big]$ points. From that $1-n(1-x) \in \sigma(D^2)$ $\Big[$resp. $2-n(2-x) \in \sigma(S^2)\Big]$ for all $n \in \mathbb{N}$.*

Theorem 3.3.2.3. *All accumulation points of the $\sigma(S^2)$ are in $\sigma(D^2)$.*

There are two proofs of this theorem showing nice correnpondence – one arithmetical and one geometrical.

Proof I. Arithmetical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.

By 1.9.0.1 we have, that $\frac{x}{2} \in \sigma(D^2)$ is an accumulation point of the set $\sigma(D^2)$. From 3.1.2.5 we have that $\frac{x}{2}+\frac{1}{2} \in \sigma(D^2)$. From that, from 3.3.2.2 we have, that

$$1 - \overbrace{2}^{\substack{\text{"}n\text{" from} \\ 3.3.2.2}} \left(\overbrace{1 - (\frac{x}{2} + \frac{1}{2})}^{\substack{\text{"}1-x\text{" from} \\ 3.3.2.2}} \right) \in \sigma(D^2). \tag{3.3.2.3.1}$$

But $1-2(1-(\frac{x}{2}+\frac{1}{2})) = x$, so $x \in \sigma(D^2)$. □

Proof II. Geometrical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.

From 3.1.2.8 and 1.9.0.1 we know, that x can be expressed as $y-1$ for some $y \in \sigma(S^2)$. Let $\mathcal{O}$ be an orbifold with the base manifold S^2, such that $\chi^{orb}(\mathcal{O}) = y$. Let $\mathcal{O}_c$ be the orbifold created from $\mathcal{O}$ by adding one cusp. Then $\chi^{orb}(\mathcal{O}_c) = y-1 = x$. Topologically $\mathcal{O}_c$ with the cusp point removed is $\mathbb{R}^2$. We can compactify it with S^1. This operation of removing cusp point and replacing it by S^1 will not change an Euler orbicharacteristic since $\chi^{orb}(S^1) = 0$, Euler orbicharacteristic is additive and $\Delta(\infty) = \Delta(*) = 1$.

What we get is an orbifold $\mathcal{O}_D$ with the base manifold D^2 and the same orbipoints as $\mathcal{O}$. Since orbipoints of $\mathcal{O}$ create a difference in Euler orbicharacteristic equal to $2-y$, we have that $\chi^{orb}(\mathcal{O}_D) = 1-(2-y) = y-1 = x$. We can then replace all orbipoints from the interior of $\mathcal{O}_D$ by twice as many of the same degrees on its boundry 1.12, so $x \in \sigma(D^2)$. □

Chapter 4

Algorithm for searching for the spectrum

In the previous chapter we ans△wered the questions about how σ looks like – in particular what is it's order type and topology. In this chapter we would like to develop a methods for answearing the following question:

"For a given rational number, is it in σ?"

We have some sort of answear to this question – an algorithm.

It is not an ideal answear as it gives little insight of what is a general structure of the spectrum. Nethertheless it is a constructive and computable answear.

The exact question we will provide algorithm to answear here is:

For a given rational number r and manifold M, is there at least one M orbifold with r as its Euler orbicharacteristic?

We start with $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$ and a manifold M.

4.1 Reduction from arbitrary M do D^2

This reduction is based on 1.9.0.2. Note, that this is a different reduction than the one in 2. In 2 we are saying that for any M, we have $\sigma(M) \subseteq \sigma(S^2) \cup \sigma(D^2)$. In 1.9.0.2 on the other hand we have, that for a manifold M with h handles, c crosscaps and b boundary components:
for $b \neq 0$:

$$\sigma(M) = \sigma(D^2) - 2h - c - (b - 1) \tag{4.1.0.0.1}$$

and for $b = 0$:

$$\sigma(M) = 2\sigma(D^2) - 2h - c. \tag{4.1.0.0.2}$$

Using 1.9.0.2 we conclude that the problem of deciding whether $\frac{p}{q}$ is in $\sigma(M)$ is equivalent to deciding:
for $b \neq 0$ if:

$$\frac{p}{q} + 2h + c + (b - 1) \tag{4.1.0.0.3}$$

is in $\sigma^d(D^2)$;
for $b = 0$ if:

$$\frac{1}{2}\frac{p}{q} + h + \frac{c}{2} \tag{4.1.0.0.4}$$

is in $\sigma^d(D^2)$.

Considering this fact, from this point, WLOG we will assume that $M = D^2$ and, following 2.2.2.5, we will be conserned only with dihedral orbipoints.

4.2 Special cases

In the case that $\frac{p}{q}$ is of the form $l\frac{1}{4}$, for some whole l we can give the answer right away. For $l > 4$ we have that $l\frac{1}{4}$ is not in the set and for $l \leqslant 4$ it is (see 3.1.2.7).

Moreover for an even l we have that $l\frac{1}{4}$ is a condensation point of order $\frac{4-l}{2}$ and for an odd l it is a condensation point of order $\frac{3-l}{2}$ (see 3.1.2.7 and 3.1.2.8).

In the case, where $\frac{p}{q} > 1$, we also can give answer rigth away and this answr is "no".

Now we will consider only cases when $\frac{p}{q}$ is not of the form $l\frac{1}{4}$ and is $\leqslant 1$.

4.3 Regular cases

First we will describe what we use in the algorithm, giving the brief semantics. The detailed semantics are given in **??**.

4.3.1 What we use

We use:

- $\mathbb{N}_{>0}$ counters $c_1, c_2, \cdots$ with values ranging on $\mathbb{N}_{>0} \cup \{\infty\}$. Each counter correspond to one dihedral point on the boundry of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial dihedral point - namely a non-orbi point, a normal point).

  We will write the state of the counters without commas, using the letter d. Note that with this convention, c_i will refer to the i-th counter and d_i will refet to the value of the i-th counter.

  So the state of the counters $d_1d_2\cdots$ correspond to the orbifold $*d_1d_2\cdots$ (where the trailing 1's are trunkated).

  We will refer to the counters being "to the left" or "to the right" of each other, as the numbering would go from left to right.

- a pivot pointing at some counter

- a flag that can be set to: "Greater", "Searching" or "Less" corresponding to what was the outcome of comparing Euler orbicharacteristic of the orbifold corresponding to counters' state and $\frac{p}{q}$ or to the fact, that there is a need for a search of the next state of counters to compare with $\frac{p}{q}$.

4.3.2 What state are we starting our algorithm with

We start with:

- all counters set to 1.
- pivot pointing at the c_1
- flag set to "Greater"

4.3.3 Invariants claims

Now we will state the claims of what properties the state of the counters will maintain during all the execution of the algorithm. The proof, that this is indeed the case will be performed in **??**

Claim 4.3.3.1. *We will do our computation such that:*

- *every state of the counters during runtime of the algorithm will have only finitely many counters with value non-*1*.*
- *every state in the rutime of the algorithm will have values on consequtive counters ordered in weakly decreasing order.*

From now we will consider only such states.

4.3.4 The algorithem for searching for a spectrum

When the algorithm is in the state:

- counters with values: $d_1 d_2 \cdots$
- pivot: at the counter c_p
- flag: set to the value $flag\_value$,

we procced as follows :

```
In the case, the $flag\_value$ is equal to:
{
    "Greater", then
    {
        If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) = \frac{p}{q}$ then
        {
            We found an orbifold and we are ending the whole
             algorithm with answer "yes, $*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots$".



        }
```

```
13          If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) > \frac{p}{q}$ then
14          {
15              We set $d_p$ to $\infty$.
16              We set the flag to "Greater".
17              We put the pivot at the $c_{p+1}$.
18              We go to the 1st line.
19          }
20          If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) < \frac{p}{q}$ then
21          {
22              We set the flag to "Searching".
23              We go to the 1st line.
24          }
25      }
26
27      "Searching", then
28      {
29          We search one by one
30          for the value $d'_p$ of the $c_p$ such that
31          $\chi^{orb}(*d_1 \cdots d_{p-1} d'_p d_{p+1} \cdots) \leqslant \frac{p}{q}$ and
32          $\chi^{orb}(*d_1 \cdots d_{p-1} (d'_p - 1) d_{p+1} \cdots) > \frac{p}{q}$.
33          if $\chi^{orb}(*d_1 \cdots d_{p-1} d'_p d_{p+1} \cdots) = \frac{p}{q}$ then
34          {
35              We found an orbifold and we are ending the whole
36              algorithm with answer "yes, $*d_1 \cdots d_{p-1} d'_p d_{p+1} \cdots$".
37
38
39          }
40          We set $d_p$ and values of all the counters
41          to the left of $c_p$ to the value $d'_p$.
42          if $\chi^{orb}(*d_1 d_2 d_3 \cdots) = \frac{p}{q}$ then
43          {
44              We found an orbifold and we are ending the whole
45              algorithm with answer "yes, $*d_1 d_2 \cdots$".
46
47
48
49          }
50          If $\chi^{orb}(*d_1 d_2 d_3 \cdots) < \frac{p}{q}$ then
51          {
52              We set the flag to "Less".
53              We put the pivot at the $c_{p+1}$.
54              We go to the 1st line.
55          }
56          If $\chi^{orb}(*d_1 d_2 d_3 \cdots) > \frac{p}{q}$ then
```

```
        {
            We set the flag to "Greater".
            We put the pivot at the $c_1$.
            We go to the 1st line.
        }
    }

    "Less", then
    {
        If $d_p = 1$ and the values of all the counters
        on the left of $c_p$ are equal to 2 then
        {
            We end the whole algorithm with the answer "no".
        }
        We increase $c_p$ by one $(d_p := d_p + 1)$ and
        we set the value of all counters on the left of $c_p$ to $d_p$.
        If $\chi^{orb}(*d_1 d_2 d_3 \cdots) = \frac{p}{q}$ then
        {
            We found an orbifold and we are ending the whole
             algorithm with answer "yes, $*d_1 d_2 \cdots$".



        }
        If $\chi^{orb}(*d_1 d_2 d_3 \cdots) > \frac{p}{q}$ then
        {
            We set the flag to "Greater".
            We put the pivot at the $c_1$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1 d_2 d_3 \cdots) < \frac{p}{q}$ then
        {
            We set the flag to "Less".
            We put the pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
    }
}
```

4.4 The idea of the algorithm

We will now present in more detail what the algorithm is intented to do. To do this and for the later sections, we will first introduce an order on the states of counters satisfying 4.3.3.1 (as mentioned in 4.3.3.1 we will consider only such states) and prove several lemmas about it.

4.4.1 Order on the space of states of the counters

Definition 4.4.1.1. *We define a linear order $\preceq$ on the states of counters as follows:*

Let D_1 be a state of counters equal to $d_1^1 d_2^1 \cdots$ and D_2 be a state of counters equal to $d_1^2 d_2^2 \cdots$. Let i be the greatest index where D_1 and D_2 differ, then:
bullet If $d_i^1 \leqslant d_i^2$ then $D_1 \preceq D_2$.

This is a suborder of the lexicographical order of states of counters after truncation of trailing 1's with the counters to the right being more significant.

Observation 4.4.1.2. *In general it is not true that if $D_1 \preceq D_2$ then $\chi^{orb}(*D_1) \leqslant \chi^{orb}(*D_2)$ nor that if $D_1 \preceq D_2$ then $\chi^{orb}(*D_1) \geqslant \chi^{orb}(*D_2)$.*

Observation 4.4.1.3. *Since $\preceq$ is a suborder of a lexicographical order it is a good order.*

We can explicitly write the form of the successor of any state $d_1 d_2 d_3 \cdots$ in $\preceq$:

Observation 4.4.1.4. *The successor of the state $d_1 d_2 d_3 \cdots$, of the form*

$$\underbrace{\infty\infty\cdots\infty}_{k-1 \text{ times}} d_k d_{k+1} d_{k+2} \cdots \,, \tag{4.4.1.4.1}$$

where k is such that c_k is the first counter from the left that is not set to ∞, is

$$\underbrace{(d_k+1)(d_k+1)\cdots(d_k+1)}_{k-1 \text{ times}}(d_k+1) d_{k+1} d_{k+2} \cdots \,, \tag{4.4.1.4.2}$$

Definition 4.4.1.5. *We will call the state $d_1 d_2 d_3 \cdots$, such that no d_k is equal to ∞ a* ***finite*** *state.*

We will call the state $d_1 d_2 d_3 \cdots$, such that at least one of d_k is equal to ∞ an ***infinite*** *state.*

Observation 4.4.1.6. *Using 4.3.3.1 we have that for the state $d_1 d_2 d_3 \cdots$ to be finite (resp. infinite), it is equivalent to d_1 being different from (resp. being equal to) ∞.*

Observation 4.4.1.7. *For any state D, the successor of D is a finite state.*

Definition 4.4.1.8. *We will call the ascending sequence $\{D_n\}$ in $\preceq$, such that for all n, we have that D_{n+1} is the succesor of D_n, a* ***connected*** *sequence in $\preceq$.*

Observation 4.4.1.9. *Every connected sequence of the finite states is of the form $\{(d_1+n)d_2 d_3 \cdots\}$, where all d_n are different from ∞.*

Observation 4.4.1.10. *Let D_1 and D_2 be finite states and let D_2 be the successor of D_1 in $\preceq$. Then $\chi^{orb}(*D_1) > \chi^{orb}(*D_2)$.*

Proof. From 4.4.1.4 we know, that taking the successor of the finite state always chages only first counter and it is changing it by increasing it by 1. Increasing the order of the orbipoint decreases Euler orbicharacteristic. $\square$

Corollary 4.4.1.11. *The sequence $\{\chi^{orb}(*D_n)\}$ is descending for every connected sequence of finite states $\{D_n\}$ in $\preceq$.*

Lemma 4.4.1.12. *The supremum of the connected sequence of finite states $\{(d_1 + n)d_2d_3\cdots\}$ is $\infty d_2d_3\cdots$, and the infimum of the corresponding sequence $\{\chi(*(d_1 + n)d_2d_3\cdots)\}$ is $\chi(*\infty d_2d_3\cdots)$.*

Proof.

For every n we have that $(d_1 + n)d_2d_3\cdots \preceq \infty d_2d_3\cdots$. Furthermore for every $d'_1d'_2d'_3\cdots$ such that $d'_1d'_2d'_3\cdots \preceq \infty d_2d_3\cdots$, there exists n, such that $d'_1d'_2d'_3\cdots \preceq (d_1 + n)d_2d_3\cdots$. Thus, $\infty d_2d_3\cdots$ is the supremum of $\{(d_1 + n)d_2d_3\cdots\}$.

For every n we have that:

$$\begin{aligned}\chi(*(d_1+n)d_2d_3\cdots) &= \chi(*d_1d_2d_3\cdots) - \frac{(d_1+n)-1}{2(d_1+n)} + \frac{d_1-1}{2d_1}\\ &= \chi(*d_1d_2d_3\cdots) - \frac{1}{2d_1} + \frac{1}{2(d_1+n)} \qquad (4.4.1.12.1)\end{aligned}$$

We also have that:

$$\begin{aligned}\chi(*\infty d_2d_3\cdots) &= \chi(*d_1d_2d_3\cdots) - \frac{1}{2} + \frac{d_1-1}{2d_1}\\ &= \chi(*d_1d_2d_3\cdots) - \frac{1}{2d_1} + 0. \qquad (4.4.1.12.2)\end{aligned}$$

Thus $\chi(*\infty d_2d_3\cdots)$ is the infimum of $\{\chi(*(d_1 + n)d_2d_3\cdots)\}$. □

Lemma 4.4.1.13. *The state of the counters in the algorithm is weakly increasing with respect to order $\preceq$.*

Proof.

The state of the counters is changed in lines 10-11, 46, 66-67. In each of these lines the counter with the gretest index of all changed counters increases in value, so the resulting state is bigger with respect to order $\preceq$. □

4.4.2 Basic idea

The basic idea of the algorithm is to search through all the states of the counters going from the smallest (in the sense of $\preceq$) state of counters, which will be when all counters are set to 1, up to some upper limit beyond which we are sure that no configuration of counters will yield the Euler orbicharacteristic that we are looking for.

Now we will go through several obstacles of how to do so and solutions for them, answearing for example the questions how we go through all the states and what can be this upper limit.

4.4.3 Checking all the states

This can't be done directly as there are infinite ascending sequences in $\preceq$. However, it can be done with some use of the properties we derived in the previous subsection.

Checking infinite connected sequences in finitely many steps

We will now present the method how to check any infinite connected sequence for solutions in finite number of steps.

First, we will perform a reduction from arbitrary infinite connected sequence to the inifite connected sequence of finite states.

Let us observe, that, by 4.4.1.7, there can be at most one infinite state in any connected sequence, and if it is present it must be the first one. If such state D_0 is present, we can chceck it whether $\chi(*D_0))$ is equal to $\frac{p}{q}$ or not (one step), and then all states that are left to be checked are finite and form infinite connected sequence of finite states, thus ending our reduction.

As this from this point we will present a method or checking for solutions any infinite connected sequence of finite states.

First, let us observe that thanks to 4.4.1.11, when we are searching through the infinite connected sequence of finite states in $\preceq$, once we get (without fiding any solution) to the state D_n for which $\chi^{orb}(*D_n) < \frac{p}{q}$, we know that no state D_m with $m > n$ can have $\chi^{orb}(*D_m) = \frac{p}{q}$ and we can disregard whole sequence.

There is, however, another problem, namely, that when we are searching through the infinite connected sequence of the finite state, initialy, we don't now, whether there will be any state $D_k = (d_1 + k)d_2d_3\cdots$ in it, that will have $\chi^{orb}(*(d_1 + k)d_2d_3\cdots) \leqslant \frac{p}{q}$. However, thanks to 4.4.1.12 we can check for this, by first comparing $\frac{p}{q}$ with $\chi(*\infty d_2d_3\cdots)$. Since from 4.4.1.12, we have that $\chi(*\infty d_2d_3\cdots)$ is the infimum of $\{*(d_1 + n)d_2d_3\cdots\}$, we have that if $\chi(*\infty d_2d_3\cdots) < \frac{p}{q}$, then there must be state $(d_1 + n)d_2d_3\cdots$ such that $\chi(*(d_1 + n)d_2d_3\cdots) < \frac{p}{q}$, for some n and we can proceed to look for it one by one through the sequence.

One case that is left, is when $\chi(*\infty d_2d_3\cdots) > \frac{p}{q}$, but then we can disregard the whole sequence right away, since $\chi(*\infty d_2d_3\cdots)$ is the infimum of $\{*(d_1+n)d_2d_3\cdots\}$.

What after we checked infinite connected sequence?

Let us uppouse that we just checked th

4.4.4 Three "modes" of the algorithm

The algorithm has three distinct fragments:

- fragment in the lines 3-35. that will be called the "Greater" part,
- fragment in the lines 27-62, that will be called the "Searching" part,
- fragment in the lines 64-93, that will be called the "Less" part.

4.5 Proof of the correctness of the algorithm

Firstly, let us observe, that algorithm gives the answear on lines 8, 14-15, 35-36, 55-56, 62-63 and always ends immedietly after giving the answear. Thus, it will always give at most one answear. Furthermore let us observe that these are the only places where the algorithm terminates, so if it terminates it will give at least one answear.

There are three things to be checked:

• That the algorithm never answears "yes" if there is no orbifold of the Euler orbicharacteristic $\frac{p}{q}$ (No false positives)

• That the algorithm never answears "no" if there is an orbifold of Euler orbicharacteristic $\frac{p}{q}$ (No false negatives)

• That the algorithm always ends in a finite number of steps (Guaranteed termination).

4.5.1 No false positives

Algorithm gives answear "yes" at lines 7-8, 35-36, 44-45, 75-76. At each of these places, the answear contains the example of an orbifold with Euler orbicharacteristic equal to $\frac{p}{q}$ that was explicitly checked for correctness just before giving the answear (see lines 5, 33, 42, 73). $\square$

4.5.2 No false negatives

Let $d_1 d_2 d_3 \cdots$ be such that $\chi^{orb}(*d_1 d_2 d_3 \cdots) = \frac{p}{q}$.

First, we will show that the algorithm will never go beyond $d_1 d_2 d_3 \cdots$ counter state in $\preceq$ order.

Let us observe that the only lines where the counters are changed are lines 15, 40-41 and 71-72, and while changing, only counters at pivot and to the left of the pivot are changed.

Because of that, going beyond $d_1 d_2 d_3 \cdots$ counter state could happen only in lines 15, 40-41 or 71-72, while pivot would be on the rightmost counter that is differrent from $d_1 d_2 d_3 \cdots$.

We will now eliminate all three options case by case.

Line 15

Lines 40-41

Lines 71-72

4.5.3 Guaranteed termination

Let us assume that for some input M and $\frac{p}{q}$ the algorithm does not anwear "yes". We will show, that then it will answear "no" in finite number of steps.

We will show it by firstly show, that for any state d1d2d3 smaller than 1/21/21/21/2 the algorithm will go beyond it.

4.6 Another questions the algorithm can answear

4.6.1 Deciding the order of accumulation

Let $m \in \mathbb{N}$ be such that $\frac{p}{q} \in (1 - \frac{m}{2}, 1 - \frac{m+1}{2})$ Let us denote by $r \coloneqq \frac{p}{q} - (1 - \frac{m}{2})$.

We will searching in σ as such:

If $\frac{p}{q} \in \sigma$, then, from the corollary 3.1.2.8 we know, that there exist some $n \in \mathbb{N}$, such that $\frac{p}{q} + \frac{n}{2} \in \sigma$ but $\frac{p}{q} + \frac{n}{2} \notin \sigma$.

We will be consequently checking points from $1 + r$, through $1 + r - \frac{l}{2}$, for $0 \leqslant l \leqslant m$, to the $\frac{p}{q}$. We stop at the first found point. If one of these point is in the spectrum, then all smaller (so also $\frac{p}{q}$) are in the spectrum and $\frac{p}{q}$ is the accumulation point of the spectrum of order $m - l$ (from this, we can see some heuristic, that the points that have smaller order will be generally harder to find in some sense). If none of this points are in in the spectrum, then $\frac{p}{q}$ is not.

4.7 Implementation

This algorithm is a part of the algorithm from 6 where the implementation of the whole will be discused in 6.2.

Chapter 5

Counting orbifolds

The central question of this section is: "given a rational number, how many orbifolds have that Euler orbicharacteristic?".

In 5.1, we will show that for any number, there are only finitely many orbifolds with that Euler orbicharacteristic. In 5.3,

5.1 Finitenes

Observation 5.1.0.1. *For any $x \in \sigma$ and $n \in \mathbb{N}$ there are only finitely many orbifolds with the Euler orbicharacteristic greater or equal to x and all orbipoints of order at most n.*

Proof:

For a given x, there are only finitely many manifolds with an Euler characteristic $y \geqslant x$. Only them can be base manifolds for an orbifold with an Euler orbicharacteristic $y' \geqslant x$, as adding orbipoints always decreases an Euler orbicharacteristic.

It remains to prove then, that for any base manifold M, there are only finetly many orbifolds, with M as a base manifold, that have an Euler orbicharacteristic $y \geqslant x$, and all orbipoints of order at most n.

We proceed now simmilarly to the proof of 3.1.2.2 – on the orbifold with an Euler orbicharacteristic $y \in [x, 2]$, there can be at most $\max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}$ orbipoints. Thus, for a given manifold M and a given x and n, there can be at most $(n-1)^{\max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}}$ orbifolds with an Euler orbicharacteristic $y \geqslant x$, all orbipoints of order at most n and M as a base manifold.$\square$

Theorem 5.1.0.2. *For any $x \in \sigma$ there are only finitely many orbifolds with the Euler orbicharacteristic equal to x.*

Proof:

Let x be a rational number. Let $\mathcal{O}$ be the set of all orbifolds with an Euler orbicharacteristic qual to x. Those orbifolds can have different base manifolds. However, the set of base manifolds of orbifolds from $\mathcal{O}$ is finite, as there are only finitely many two dimentional manifolds with an Euler characteristic greater or equal to x and an orbifold always has an Euler orbicharacteristic less or equal to the Euler characteristic of its underlying manifold.

It remains to proof, that for any base manifold M, the number of M orbifolds with Euler orbicharacteristic equal to x is finite:

Let M be a two dimentional manifold.

For the sake of contradiction, assume, that there exists an infinite set $\mathcal{O}_M$ of M-orbifolds such that $\mathcal{O}_M \subseteq \mathcal{O}$ $(*)$.

If M has some boundary, orbifolds in $\mathcal{O}_M$ can have both rotational and dihedral orbipoints. For the simplicity of the following part of the proof we want now to reduce this case to a case where only one type of the orbipoints is present.

Let us observe, that every rotational orbipoint can be replaced by two dihedral orbipoints of the same order without changing the Euler orbicharacteristic. Thus, if there would be infinitely many obifolds in $\mathcal{O}_M$ having both rotational and dihedral orbipoints, there would be also infinitely many orbifolds in $\mathcal{O}_M$ having only dihedral orbipoints. Thus it is sufficient to prove that there are finetly many orbifolds in $\mathcal{O}_M$ that have only one type of the orbipoints.

We will now perform the proof of above statement in the case of dihedral orbipoints. The proof for rotational orbipoints is completely analogous.

Let us call the subset of $\mathcal{O}_M$ that consists only of orbifolds with only dihedral orbipoints by $\mathcal{O}_M^d$.

Let $\mathcal{O}_M = \{O_i\}_{i\in I}$. For each i, let $s_i = (b_i^0, \cdots, b_i^{l_i})$ be the signature of O_i written with decreasing orders of dihedral points. So for each i we have, that b_i^0 is the order of the orbipoint with the highest order of all the dihedral orbipoints of O_i. By 5.1.0.1 we know that if the set $\{b_i^0\}_{i\in I}$ would be bounded by some $n \in \mathbb{N}$ it would mean, that $\mathcal{O}_M^d$ would be finite. As from $(*)$ this is not the case, we know that the set $\{b_i^0\}_{i\in I}$ is unbounded. Let $\{i_n\}_{n\in\mathbb{N}} \subseteq I$ be a sequence of indeces such that $\{b_{i_n}^0\}_{n\in\mathbb{N}}$ is strictly increasing.

Let $\{x_n\}$ be the sequence such that $x_n = \Delta(b_{i_n}^0)$. Let $\{y_n\}$ be the sequence such that $y_n = \Delta(b_{i_n}^1, \cdots, b_{i_n}^{l_{i_n}})$. So for every n we know that $\chi^{orb}(O_{i_n}) = \chi(M) + a_n + b_n$. As $\{b_{i_n}^0\}$ is strictly increasing, we know that a_n is strictly decreasing, so b_n must be strictly increasing (we have that $\chi^{orb}(O_{i_n})$ is constant for all n, since all O_{i_n} are from the family with Euler orbicharacteristic equal to x).

But $\{b_n\} \subseteq \sigma(M) - \chi(M)$. From 3.1.3.6 and **??** we know that $\sigma(M)$ has no infinite strongly increasing sequences, so $\sigma(M) - \chi(M)$ has no infinite strongly increasing sequences. That gives us a contradiction. $\square$

5.2 Infinitness

5.2.1 Unboundeness of some number of occurences

We know, that for any x, there are only finetely many orbifolds with x as an Euler orbicharacteristic. However, we can ask about some boundness of number of these orbipoints. In particular, we could ask, whether near any accumulation point, there will be x with an arbitrary large number of orbifolds corresponding to it. The answer will be yes, and it can be formulated as such:

Theorem 5.2.1.1. *For any neighbourhood U of any accumulation point of $\sigma(D^2)$ of order at least 2, for any $n \in \mathbb{N}$, there exists an $x \in U$ such that there are at least n orbifolds with x as their Euler orbicharacteristic.*

Proof.
This will follow from the theorem about the sums of egiptian fractions from [**Browning2011**]. It states that for ...

5.3 Dividing the problem into an arithmetical and combinatorical parts

Here will divide the question "Given the number x, how many orbifolds have x as an Euler orbicharacteristic?" into two parts. The answears to these partial questions will be given in **??** and **??**.

5.4 Arithmetical part

The first part is to aswear the following question:
"How many sums of the form:

$$1 - \sum_{i=1}^{m} \frac{d_j - 1}{2d_j} \tag{5.4.0.0.1}$$

and

$$2 - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \tag{5.4.0.0.2}$$

with $n \in \mathbb{N}$ and $\forall_i \; d_i, r_i \in \mathbb{N} \cup \{\infty\}$, are equal to x?"

It is a matter of convention (and then coherently translating this convention to the final result) what sums are we treating as "the same". The convention we will take, is that a sum is determined uniquly by the tuple $(d_1, \ldots, d_n)$ $\Big[$or $(r_1, \ldots, r_n)\Big]$ of orders of orbipoints, ordered in deacreasing order, appearing in the sum.

This part describes how adding rotational orbipoints to a shepre and dihedral points to the disk changes their Euler orbicharacteristic. In the second part we will use the answear from this part.

5.5 Combinatorical part

We will take following steps:

1. First we divide the question "Given number x, how namy orbifolds have x as their Euler orbicharacteristic?" into the series of questions for each two-dimentional manifolds M: "Given number x, and the manifold M, how namy M-orbifolds have x as their Euler orbicharacteristic?". At the end we will sum up the answears from all these questions.

Note, that for M such that $\chi(M) < x$, the answear is always 0, since orbifolds have smaller Euler orbicharacteristic than their base manifolds (1.4.2.1).

2. Then for each manifold M, we answer one of the following questions:

   • if M has a boundary:

   "How many sums of the ?",

   • if M has no boundary:

   "How namy sums of the form?".

   Here we consider the sums to be "the same" in the same way as in 5.4. This we can do, since these questions are equivalent to asking respective questions from arithmetical part 5.4 for $x + \chi(M)$. In the case where M has no boundary this gives us our result, since 1.3.6.

3. Finally we take into account two remaining things concerning the case where M has a boundary:

   3.1. For now we only considered sums coresponding to orbifolds with either rotational or dihedral orbipoints. When M has a boundary, M-orbifold can have both of these types of points. Fortunatelly, to take this into account, be don't have to answear the arithmetical question concerning the sums simjulatiously coresponding to both types of orbipoints. It is possible to reduce (in the sense that will be discribed in 7) all sums that contain two types to orbipoints to sums with only dihedral orbipoints. In doing so, we will ascribe "weights" to the sums of how many other sums got reduced to it.

   3.2. When the orbipoints lie on the boundry components, their order of placement around the boundary component matters as orbifolds with orbipoints on boundry components with different order are not neccesery the same (see 1.3.6). We will take this fact into account, by affecting the afformentioned "weights" with which we will sum the number of sums. The resulting weights will be the amount of orbifolds coresponding (possibly also via reduction from 3.1) to the given sum.

   This are the two phenomena causing that in the case where M has a boundary, multiple orbifolds correspond to the same sum. We will calculate the total number of orbifolds by calculting the number of the sums coresponding to dihedral points, but taking the sums with the proper "weight" – of how many orbifolds correspond to this sum.

Chapter 6

Counting orbifolds – arithmetical part

This will be an extention of the algorithm from **??**.

```
In the case, the flag_value is equal to:
{
    "Greater", then
    {
        If $\chi^{orb}(*d_1 \dots d_{p-1} \infty d_{p+1} \dots) = \frac{p}{q}$ then
        {
            We found an orbifold, we add it to a list
            and increase the occurence counter by 1.
            We set the flag to "Less".
            We put pivot to the $c_{p+1}$ counter.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1 \dots d_{p-1} \infty d_{p+1} \dots) > \frac{p}{q}$ then
        {
            We set $d_p$ to $\infty$.
            We set the flag to "Greater".
            We put the pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1 \dots d_{p-1} \infty d_{p+1} \dots) < \frac{p}{q}$ then
        {
            We set the flag to "Searching".
            We go to the 1st line.
        }
    }

    "Searching", then
    {
        We search one by one
        for the value $d'_p$ of the $c_p$ such that
        $\chi^{orb}(*d_1 \dots d_{p-1} d'_p d_{p+1} \dots) \leqslant \frac{p}{q}$ and
```

```
32          $\chi^{orb}(*d_1 \dots d_{p-1}(d'_p - 1)d_{p+1} \dots) > \frac{p}{q}$.
33          if $\chi^{orb}(*d_1 \dots d_{p-1}d'_p d_{p+1} \dots) = \frac{p}{q}$ then
34          {
35              We found an orbifold, we add it to a list
36              and increase the occurence counter by 1.
37              We set flag to "Less".
38              We go to the 1st line.
39          }
40          We set $d_p$ and values of all the counters
41          to the left of $c_p$ to the value $d'_p$.
42          if $\chi^{orb}(*d_1 d_2 d_3 \dots) = \frac{p}{q}$ then
43          {
44              We found an orbifold, we add it to a list
45              and increase the occurence counter by 1.
46              We set the flag to "Less".
47              We put the pivot at the $c_{p+1}$.
48              We go to the 1st line.
49          }
50          If $\chi^{orb}(*d_1 d_2 d_3 \dots) < \frac{p}{q}$ then
51          {
52              We set the flag to "Less".
53              We put the pivot at the $c_{p+1}$.
54              We go to the 1st line.
55          }
56          If $\chi^{orb}(*d_1 d_2 d_3 \dots) > \frac{p}{q}$ then
57          {
58              We set the flag to "Greater".
59              We put the pivot at the $c_1$.
60              We go to the 1st line.
61          }
62      }
63
64      "Less", then
65      {
66          If $d_p = 1$ and the values of all the counters
67          on the left of $c_p$ are equal to 2 then
68          {
69              We end the whole algorithm with the answer "no".
70          }
71          We increase $c_p$ by one ($d_p \coloneqq d_p + 1$) and
72          we set the value of all counters on the left of $c_p$ to $d_p$.
73          If $\chi^{orb}(*d_1 d_2 d_3 \dots) = \frac{p}{q}$ then
74          {
75              We found an orbifold, we add it to a list
```

```
            and increase the occurence counter by 1.
            We set the flag to "Less".
            We put pivot at the $c_{p+1}$.
            We go to the line 1..
        }
        If $\chi^{orb}(*d_1d_2d_3\dots) > \frac{p}{q}$ then
        {
            We set the flag to "Greater".
            We put the pivot at the $c_1$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1d_2d_3\dots) < \frac{p}{q}$ then
        {
            We set the flag to "Less".
            We put pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
    }
}
```

6.1 Why this works

6.2 Implementation

As an appendix in the separate document, there is a source of a program with implementation of this algorithm with full enhancments described in this chapter. It is written in Rust. It can be also found on
To do: dać ref do github
along with the LaTeX source of this thesis.

6.2.1 Optimisations

Binary search

6.2.2 Limitations

i64

Chapter 7

Counting orbifolds – combinatorical part

Chapter 8

Conclusions

We described the spectrum of possible Euler orbicharacteristics of two dimentional orbifolds. It has topology of ω^ω and the problem, whether the given point is in the spectrum is desidable.

We also provided some finitess results, such as that there are always only finately many orbifolds for a given Euler orbicharacteristic. So the problem how much are for a given number is also decidable.

From **To do** we know, that there are howe ver, blab la dowolnie dużo na Euler orbicharacteristic.

It remains unclear how Disk spectrum and Sphere spcectrum lies relative to each other, but some result was, shown, namely, that every accumulation point of Sphere spectum is also in the disk spectrum.

for every denominator, do they coincide from a sufficiently distatn point? (Yes.)

8.1 Further directions

8.1.1 Asked, but unanswered questions

Our ultimate goal is to give the answer to the questions such as:
- For a given $x \in \sigma$, how many orbifolds have x as their Euler orbicharacteristic?
- Why? Is there some underlying geometrical reason for that?
- Can we characterise points $x \in \sigma$ that has the most orbifolds corresponding to them?
- Is there any reasonable normalisation to counter the effect that there are 'more' points as we go to lesser values. (What we mean by 'more' was sted in)

The first equstion we can tackle is steaming from the chapter 3 and it is – Do $\sigma(D^2)$ and $\sigma(S^2)$ coincide? It is easy to answer that $\sigma(D^2) \neq \sigma(S^2)$ (and we will do that along some harder questions in the moment), but do they coincide starting from a sufficiently distant point? write about cyclic order

Bibliography

[Thu79] William P Thurston. *The geometry and topology of three-manifolds*. s.n, 1979.

[CBG16] J.H. Conway, H. Burgiel, and C. Goodman-Strauss. *The Symmetries of Things*. AK Peters/CRC Recreational Mathematics Series. CRC Press, 2016. ISBN: 9781439864890. URL: `https://books.google.fr/books?id=Drj1CwAAQBAJ`.

Appendix A

Appendix about good orders and accumulation points

A.1 Definition of order of accumulation points

This definitions is usefull for us in chapter 3, the exact same copy of it is included there 3.1.1 as well for a readers convenience .

We start with definition of being "at least of order n" that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.
For a given set we define as follows:

Definition A.1.0.1. *(Inductive). We say that the point x is an acccumulation point of a set X of order at least 0, when it belongs to the set X. We say that the point x is an acccumulation point of a set of order at least $n+1$, when it is an acccumulation point (in the usual sense) of the accumulation points each of order at least n i.e. in every neighbourhood of x there is at least one accumulation point of a set X of order at least n, distincs from x.*

Definition A.1.0.2. *We say that the point is an acccumulation point of order n iff it is an acccumulation point of order at least n and it is not an acccumulation point of order at least $n+1$. If the point is an acccumulation point of order at least n for an arbitrary large n we say that the point is an acccumulation point of order ω.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the ussual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

A.2 Lemmas

Lemma A.2.0.1. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A + B := \{a + b \mid a \in A, b \in B\}$ also have no infinite strictly ascending sequences.*

Proof.
Let A, B have no infinite strictly ascending sequences. Let $c_n \in A + B$ are elements of some sequence. With a sequence c_n there are two associated sequences a_n, b_n, such that, for all n, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradiction), that c_n is an infinite strictly ascending sequence. Then $\forall_n \; a_{n+1} > a_n \;\vee\; b_{n+1} > b_n$. From the assumption a_n has no infinite ascending sequence, so a_n has a weakly decreasing subsequence a_{n_k}. But then subsequence b_{n_k} must be strictly increasing, as c_{n_k} is strictly increasing, what gives us a contradiction. $\square$

Lemma A.2.0.2. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A \cup B$ also have no infinite strictly ascending sequences.*

Proof.
Let A, B have no infinite strictly ascending sequences. For the sake of contradiction, lets assume, that $A \cup B$ has an infinite strictly ascending sequence c_n. Let c_{n_k}, c_{n_l} be subsequences of c_n consisting of elements from, respectively A and B. At least one of them must be infinite and strictly increasing, which gives us a contradiction. $\square$

Conserning accumulation points, we will use the terminology, that we introduced in A.1

Lemma A.2.0.3. *Let $A \subseteq \mathbb{R}$ has an order type α. Let A be such that every accumulation point of A belong to A. Then A has not only an order type α but is also homeomorphic to α.*

Proof.
Without loss of generality, let us assume, that A has no infinite descending sequence (case with A having no infinite ascending sequence is completly analogus).

As A has an order type α we have that there is an order preserving bijection $f : \alpha \to A$.
We will prove the theorem by showing that f is a homeomorphism.

For the continuity of f and f^{-1} it is sufficient to show, that for every open $U \subseteq A$ and $V \subseteq \alpha$ from prebases of respective topologies, $f^{-1}[U]$ and $f[V]$ are open $(*)$.
Prebase open sets in A are the ones inherited from the order topology on $\mathbb{R}$, for all $s \in \mathbb{R}$:

$$\{r \mid r < s\} \cap A$$
$$\{r \mid s < r\} \cap A.$$

Prebase open sets in α are from order topology, for all $\nu \in \alpha$:

$$\{\eta \mid \eta < \nu\}$$
$$\{\eta \mid \nu < \eta\}.$$

Now, we will prove $(*)$ case by case:

• Prebase set – $\{r \mid r < s\} \cap A$:
Let $\nu \in \alpha$ be the smallest, that $s \leqslant f(\nu)$, then:

$$f^{-1}[\{r \mid r < s\} \cap A] = \{\eta \mid \eta < \nu\},$$

which is open.

• Prebase set – $\{r \mid s < r\} \cap A$:
Let $s < f(\mu)$. We have two cases:
– $s \in A$: then let ν be such that $f(\nu) = s$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.
– $s \notin A$: then, by the assumption of the theorem we know that s is not an accumulation point of A. From this we conclude, that $\exists_{t \in A}(t < s \wedge \neg\exists_{t' \in A} t < t' < s)$. Let ν be such that $f(\nu) = t$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.

• Prebase set – $\{\eta \mid \eta < \nu\}$:

$$f[\{\eta \mid \eta < \nu\}] = \{r \mid r < f(\nu)\} \cap A,$$

which is open.

• Prebase set – $\{\eta \mid \nu < \eta\}$:

$$f[\{\eta \mid \nu < \eta\}] = \{r \mid f(\nu) < r\} \cap A,$$

which is open.□

Remark. The reverse is also true: If $A \subsetneq \mathbb{R}$ is homeomorphic to α, then every accumulation point of A belongs to A.

Lemma A.2.0.4. *Let $A \subseteq \mathbb{R}$ be a bounded, well ordered set. Then A has an accumulation point a of order $n \in \mathbb{N}$ (it may be that $a \notin A$) iff order type of A is at least ω^n.*

Proof.
Inductive, with respect to n in ω^n.
• $n = 0$ Let us suppouse, that A has an accumulation point of order 0. Having an accumulation point of order 0 means that A is non-empty. As that it has an order type of at least $\omega^0 = 1$.

Let us suppouse, that A has order type at least $\omega^0 = 1$. Then it is non-empty, so it has at least one accumulation point of order 0.

• Induction step

Let us suppouse that A has an accumulation point a of order $n+1$. This means that every neighbourhood of a we can find infinetely many accumulation points of A of order n. Let take one such neighbourhood and one such family $\{b_i\}_{i\in\mathbb{N}}$ of accumulation points of order n. Let us then take family of pairwise disjoint neibourhoods $\{U_i\}_{i\in\mathbb{N}}$ of $\{b_i\}_{i\in\mathbb{N}}$. Let $A_i := U_i \cap A$.

From the induction assumption for all i, we have that A_i is of order type at least ω^n. As that, we managed to show an pairwise disjoint inclusions of countably many sets of order type at least ω^n into A. As that we have the order preserving inclusion of ω^{n+1} into A, so A is of order type at least ω^{n+1}.

Let us now suppouse that A has the order type of at least ω^{n+1}. Then, we can find a family $\{A_i\}_{i\in\mathbb{A}}$ of pairwise disjoint subsets of A, each of order type ω^n, with the property $(*)$, that $\forall_{i,j\in\mathbb{N}} i < j \implies \forall_{x\in A_i, y\in A_j} x < y$.

From the inductive assumption, for all i, we have that A_i has an accumulation point of order n. Let $\{b_i\}_{i\in\mathbb{N}}$ be the set of thouse accumulation points. Because of the property $(*)$, thouse accumulation points are pairwise distinc, between A_i, A_j, with $i \neq j$. Since A is bounded, we have that, the set $\{b_i\}_{i\in\mathbb{N}}$ is bounded, so it has an accumulation point a. As an accumulation point of the accumulation points of order n, it is an accumulation point of order $n+1$. □

Corollary A.2.0.5. *Let $A \subseteq \mathbb{R}$ be a bounded, well ordered set of the order type ω^n. Then it has exactly one accumulation point a' of order n. This point has the property that $\forall_{a\in A}\ a < a'$.*

Proof.
From A.2.0.4 we know that A has at least one accumulation point a' of order n.

For the sake of contradiction, let us assume, that there exists an accumulation point $\bar{a}$ of order n such that $\exists_{a\in A}\ a \geqslant \bar{a}$. We have that A has the order type ω^n, which means that $\forall_{a_1\in A}\exists_{a_2\in A}\ a_1 < a_2$. From this, we have, that $\exists_{a_0} a_0 > \bar{a}$. But then, we would have that the prefix $(-\infty, \bar{a}] \cup A$ of A has an accumulation point $\bar{a}$ of order n. From this, from A.2.0.4 we would conclude, that $(-\infty, \bar{a}] \cup A$ is of order type at least ω^n, which leads to the contradiction, as $(-\infty, \bar{a}] \cup A$ is a proper subset of A. Thus, we have, that for all accumulation points $\bar{a}$ of A of order n we have that $\forall_{a\in A}\ a < \bar{a}$.

It remains to show that there is only one such accumulation piont - a'. For the sake of contradiction, let us assume, that there exists an accumulation point of A of order n, named $\bar{a}$, such that $\bar{a} \neq a'$. Let us assume that $\bar{a} < a'$. Then, as in every neibourhood of a' there is a point from A, we have that $\exists_{a_0} a_0 > \bar{a}$. The absurdity of this statement is shown above. Case where $\bar{a} > a'$ is completly analogous. □

Lemma A.2.0.6. *For $A, B \subseteq \mathbb{R}$, if $r \in \mathbb{R}$ is an accumulation point of order m for A and n for B and $m \leqslant n$, then r is an accumulation point of order at most n for $A \cup B$.*

Proof.
Inductive.

• $n = 0$. Then r is an isolated point of B and either r is isolated point of A or $r \notin A$. From this we have that there exists U_1, U_2 such that $B \cap U_1 = \{r\}$ and $A \cap U_2 \subseteq \{r\}$. From thi we have that $(A \cup B) \cap (U_1 \cap U_2) = \{r\}$. So r is an isolated point of $A \cup B$.

• Inductive step. Let us suppouse that for all $k < n$, the statement holds. Let r be an accumulation point of order n of B and order m of A, where $m \leqslant n$. From this we have that there exists $U_1, U_2 \ni r$ such that in $B \cap U_1$ there are only accumulation points of B of order at most $n - 1$ and in $A \cup U_2$ there are only accumulation points of A of order at most $m - 1$. From this, from the inductive assumption we have that in $(A \cup B) \cap (U_1 \cap U_2)$ there are only accumulation points of order at most $n - 1$ of $A \cup B$. This means that r is an accumulation point of order at most n of $A \cup B$.

We also know that, in every $U_1, U_2 \ni r$, there are accumulation points of order exactly $n - 1$ of B and exatly $m - 1$ for A. From the inductive assumption we have then, that in $(A \cup B) \cap (U_1 \cap U_2)$ there are accumulation points of order $n - 1$ of $A \cup B$. This means that r is an accumulation point of order exactly n of $A \cup B$. □

Corollary A.2.0.7. *Let $A^{(n)}$ be the set of all accumulations point of order n of A. Then for every $n \in \mathbb{N}$ we have that $(A \cup B)^{(n)} = A^{(n)} \cup B^{(n)}$.*

Proof.
Every accumulation point of either A or B is also an accumulation point of $A \cup B$, so $(A \cup B)^{(n)} \supseteq A^{(n)} \cup B^{(n)}$.

From A.2.0.6 we know, that for any point $r \in \mathbb{R}$, if $r \in (A \cup B)^{(n)}$, then $r \in A^{(n)} \cup B^{(n)}$. □

Lemma A.2.0.8. *For two bounded, well ordered sets $A, B \subseteq \mathbb{R}$, with order types, respectively ω^m and ω^n, such that $m < n$, and that $\forall_{x \in A \cup B} \exists_{b \in B} x < b$, we have that order type of $A \cup B$ is well defined and equal to ω^n.*

Proof.
From A.2.0.2, we know, that $A \cup B$ is well ordered. As such its order type is well defined and equal to some ordeal number γ.

We will show that $\gamma \leqslant \omega^n$ and $\gamma \geqslant \omega^n$, thus showing that $\gamma = \omega^n$.

Let $f : \omega^n \to B$ and $g : A \cup B \to \gamma$ be order preserving bijections.

• $\omega^n \leqslant \gamma$:

We have that $g \circ f : \omega^n \to \gamma$ is an order preserving injection, thus, $\omega^n \leqslant \gamma$.

• $\omega^n \geqslant \gamma$:

From A.2.0.5 we know, that B has exactly one accumulation point b' of order n. This point has the propertie that $\forall_{b \in B}\ b < b'$. As b' is the only accumulation point of order n for B and from A.2.0.5 we know also that A has no accumulation points of order n, from A.2.0.6 we know, that $A \cup B$ has exactly one accumulation point of order n, namely b'.

For the sake of contradiction, let us assume that $\omega^n < \gamma$. But then, there is some proper prefix of $A \cup B$ with order type ω^n. Let us name that prefix as P. From A.2.0.4 we know, that P has an accumulation point p' of order n. Let $b_1 \in B$

be such that $\forall_{p \in P}\ p < b_1$. Such b_1 exists, because P is a proper prefix of $A \cup B$, so $\exists_{x \in A \cup B} \forall p \in P p < x$, and from the assumptions of the lemma we have that $\forall_{x \in A \cup B} \exists_{b \in B} x < b$. We have that $p' \leqslant b_1$. But we have also that $b_1 < b'$, so $p' \neq b'$. This gives us the contradiction, as b' is the only accumulation point of order n in $A \cup B$. $\square$