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Areas of two dimensional hyperbolic orbifolds

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Abstract

This thesis aims to describe the spectrum of all possible areas of two dimensional orbifold, in particular those of negative Euler orbicharacteristic. We will analyse the spectrum both as a set and study its order type and topology as well as as a preimage of χ^{orb} – Euler orbicharacteristic and count orbifolds corresponding to a particular points to a spectrum.

Chapter 1

Introduction

1.1 Motivations

Orbifolds are geometrical spaces that encodes some of group action properties.

They played a crucial role in Thurston's geometrisation program ([Thu79]). They are also correlated and find direct use or indirect use (be unintended emergent occurrence) in different subjects, from geometry, to number theory – e.g. as spaces correlated with modular forms, physics – e.g. they are used in modeling the string theory, signal analysis – e.g. aliasing can be modeled with D_∞, an infinite dihedral group and corresponding , and art – they are visible e.g. on the prints of M.C.Esher. **TO DO: referencje**

They are also beautiful symmetrical structures on they own, providing nice and uniform language to describe platonic solids, tilings of the euclidean and hyperbolic plane, as well as general notion of symmetry

The focus of this thesis is on possible areas of two-dimensional orbifolds. Simmilar analysis was performed in dimension three for manifolds, where ([Thu79], [Gro81]) it was proved, that the spectrum of possible volumes of three dimensional manifolds form a closed, non-discrete set on the real line, that this set is well ordered, its ordinal type is ω^ω and there are only finitely many manifolds with a given volume. What is interesting, in this thesis simmilar results are obtained for two-dimensional orbifolds using more elementary methods (chapter 3).

1.2 Scope

In this thesis we would like to give some description of the spectrum of the volumes of two dimensional orbifolds, in particular those of negative Euler orbicharacteristic. We will examine order type and topology of the spectrum (chapter 3) as well as the structure of the spectrum based on spectra corresponding to different manifolds (section 1.9), We will also provide tools for determining whether a given number is in spectrum of areas (chapter 2). (chapter 4) as well as tools to compute how many orbifolds correspond to a given area in the spectrum (chapter 5).

1.3 Orbifolds

The definition of the orbifold is taken from Thurston [Thu79] (chapter 13), with slight modification described in 1.3.3. We briefly recall the concept, but for full discussion we refer to [Thu79].

An orbifold is a generalisation of a manifold. As manifold, it consists of a Hausdorff space (which we will call a base space) with some additional structure. Compared to manifolds, one allows more variety of local behaviour. On a manifold a map is a homeomorphism between $\mathbb{R}^n$ and some open set on a manifold. On an orbifold a map is a homeomorphism between a quotient of $\mathbb{R}^n$ by some finite group and some open set on an orbifold. In addition to that, the orbifold structure consist the information about that finite group and a quotient map for any such open set.

We can make an observation, that since in dimension 2, quotient of $\mathbb{R}^2$ by a finite group is topologically always either $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$, we have that in dimension 2, the underlying Hausdorff space of any orbifold is a topological manifold (possibly with a boundary).

For an orbifold O we will call this underlying manifold M a base manifold of O and denote it by $|O|$, and we will call O an M-orbifold.

In dimension two, only possible groups acting on the map sets are:

- cyclic groups $\mathbb{Z}_n$ acting by rotations around certain point
- dihedral groups $\mathbb{D}_n$ acting by reflections about n different lines crossing at the certain point
- group $\mathbb{Z}_2$ acting as reflection about certain line

Manifolds without boundary can be treated as orbifolds with trivial group for every map. Manifolds with boundary can be treated as orbifolds with trivial group on all maps from the interior and with group $\mathbb{Z}_2$ on the boundary, as described in [Thu79] (example 13.2.2.).

1.3.1 Terminology

We differ from Thurston in the terms of naming points with every map with non-trivial groups. We will call them orbipoints. If a cyclic group acts by rotations around such point, we will call it rotational point. If a dihedral group acts by reflections about n different lines crossing at such point, we will call them dihedral points. And if $\mathbb{Z}_2$ group acts by reflection about certain line consisting of such point, will call it a reflection point and such line we will call a reflection line.

If a group associated to the orbipoint has degree n, we will say that the orbipoint is of degree n.

Through the whole thesis we will consider only two dimensional manifolds and orbifolds, for this reason, for brevity, words "two-dimensional" will be sometimes omitted, nevertheless we will always mean only two dimensional manifolds and orbifolds if not stated otherwise.

1.3.2 Finite number of orbipoints

In this thesis we will consider only orbifolds with finitely many orbipoints and all orbifolds mentioned from this point are meant as such without further notice. Reason for this choice will be described in 1.6.2.

1.3.3 Compactness

Orbifold as a topological space is the same as its base manifold. We would like to restrict our interest only to compact orbifolds. However, noncompact orbifolds, such as ones from [CBG16] (chapter 18) which are quotients of a group action of an infinite group acting on $\mathbb{R}^2$ (that will be also frequently interpreted in this setting as a hyperbolic plane $\mathbb{H}^2$), also interests us. We would like to accommodate some of noncompact orbifolds that will satisfy the condition similar to the 1.3.2. To do this we will slightly expand our definition of an orbifold.

Let us start with a following construction.

For an noncompact orbifold O, let us take it's one point compactification. Let the compactification point be named x_O. For some set X, let $\#(X)$ be the number of connected components of X. Now let us consider some connected open set $U \ni x_O$. The set $U \setminus \{x_O\}$ is not necessarily connected.

Let

$$C(O) \coloneqq \sup_{\substack{U \ni x_O \\ \text{connected,} \\ \text{open}}} \#(U \setminus \{x_O\}). \tag{1.3.3.0.1}$$

We will be interested only in the case, when $C(O)$ is finite. If $C(O)$ is finite, we take some U that realise the supremum and compactify each of connected components of $U \setminus \{x_O\}$ with a separate point. We will call these points "cusps". If for some cusp x, in every $U \ni x$ there are points from the boundary of O, we will call it a cusp on a boundary. In the other case, we will call it a cusp in the interior.

The result is a compact topological space. We will treat it as an orbifold, with cusps as orbipoints in extended definition. We extend the definition as such, that the map from the compactification of some open subset consisting point x can go to the quotient of compactification of $\mathbb{H}^2$ by S^1. The group, we will take to act on $\mathbb{H}^2$ will be:

- in the case of x being in the interior – infinite cyclic group $\mathbb{Z}$, where the generator acts as translation by 1 in the half-plane model of hyperbolic plane
- in the case of x being on the boundary – infinite dihedral group D_∞, where generators will be reflections with respect to vertical lines spaced by 1 in a half-plane model on a hyperbolic plane.

Note that in both cases, the is exactly one point in the compactification of $\mathbb{H}$, that is fixed for every element of the acting group – point from the compactification at the infinity on the top of the plane. This point is will be always mapped to the orbipoint x. We will call x an, respectively, rotational, or dihedral orbipoint of infinite degree.

1.4 Classification of two dimensional manifolds

In this thesis we aim to better recognise possible two dimensional orbifolds. The foundation that we are relaying on is the classification of two dimensional manifolds.

It is phrased as a well known theorem:

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Theorem 1.4.0.1. *Any two dimensional manifold can be obtained from S^2 by taking some number of connected sums with $\mathbb{R}P^2$ and $\mathbb{T}^2$, and by cutting out some nuber of D^2 without boundary.*

After [CBG16] we will treat two dimensional manifold as a S^2 with a collection of features made by:

- taking a connected sum with $\mathbb{R}P^2$, the corresponding feature, that we will call, after [CBG16], a cross-cap
- taking connected sum with $\mathbb{T}^2$, the corresponding feature, that we will call, after Conway, a handle
- cutting out D^2 without boundary, creating the boundary component.

1.5 Presentations

1.5.1 Manifold presentation

From 1.4 we know, two dimensional manifold can be defined, by specifying:

- how many boundary components it has,
- how many handles it has,
- how many cross-caps it has.

From this, adapting the notation from [CBG16], we will write a two dimensional manifold as a list:

$$\circ^h *^b \times^c = \underbrace{\circ \cdots \circ}_{h \text{ times}} \underbrace{* \cdots *}_{b \text{ times}} \underbrace{\times \cdots \times}_{c \text{ times}}, \tag{1.5.1.0.1}$$

where each $\times$ represents the cross-cap, each $\circ$ represent one handle and each $*$ represents a boundary component.

This presentation is not necessarily unique, as there is one relation $\circ^a \times^b = \times^{2a+b}$. For a more detailed description we refer to [CBG16] (page 101-102).

1.5.2 Orbifold presentation

For the rest of this thesis we will use slightly modified notation from [CBG16], presented below.

As stated in [CBG16] (chapter 18):

Theorem 1.5.2.1. *two two-dimensional orbifolds are isomorphic iff they have:*

- *the same base manifold,*
- *the same number of orbipoints for each type and degree*
- *orbipoints on the boundaries in the pairwise same cyclic orders – up to orientation of each alone if the base manifold is non-orientable and up to simultaniously reversing orientation of every if the base manifold is orientable.*

From this, adapting the notation from [CBG16], we will write a two dimensional orbifold as a list:

$$h^c r_1 \cdots r_n * d_1^1 \cdots d_{m_1}^1 \cdots * d_1^b \cdots d_{m_b}^b \times^c \qquad (1.5.2.1.1)$$

where each $\times$ represents the cross-cap, each $\circ$ represent one handle, $r_1 \cdots r_n$ are degrees of orbipoints in increasing order and $*d_1^1 \cdots d_{m_1}^1 \cdots * d_1^b \cdots d_{m_b}^b$ are b boundary components with dihedral orbipoints of degrees $d_1^k \cdots d_{m_k}^k$ ordered with the preservation of the cyclic order on the boundary component. In this notation, the sphere S^2 will be denoted as ε – an empty word. This presentation is not necessarily unique for a given orbifold. We can freely permute boundary components and change dihedral points inside each boundary component by cyclic permutations.

1.5.3 Feature notation

We will view any two dimensional orbifold as a S^2 with added features:

- cross-caps
- handles
- boundary components
- rotational orbipoints
- dihedral orbipoints.

For all these features except dihedral orbipoints the notation for a feature treated as an object alone (it can be view as a map from orbifolds to orbifolds, that takes one orbifold O_1 defined by some list and returns the other orbifold O_2 defined by the list of O_1 with added feature), will be the same as the notation of the feature in the list. For dihedral points, when they will appear alone, we will write $^*d_i$, instead of d_i, to signify, that they are dihedral orbipoints and we will write expressions like ^{*2} or ^{*3}.

For writing a sequence of numbers – $\underbrace{{}^*d\cdots{}^*d}_{n \text{ times}}$ in the power notation, we will use parenthesis and write $({}^*d)^n$ instead ${}^*d^n$, for example $({}^*4)^5 =^* 4^*4^*4^*4^*4$ for dihedral orbipoints, or $(4)^5 = 4\ 4\ 4\ 4\ 4$ for rotational orbipoints, to avoid confusion with raising the numbers themselves to some power.

We will usually (but not always) separate numbers by spaces not commas in this context. We will sometimes enclose whole orbifold presentation in the parentesis e.g. (2 3 7) instead of writing 2 3 7 for readability.

1.6 Euler (orbi)characteristic

1.6.1 Euler characteristic

On CW-complexes we can define Euler characteristic as additive topological invariant normalised on simplex.

On a CW-complex it is then an alternating sum of numbers of cells in a consecutive dimensions:

$$\chi(C) = \sum_{d=0}^{n} (-1)^d k_d \tag{1.6.1.0.1}$$

Among other properties we have that if a compact manifold N is a quotient of a compact manifold M, by the action of a finite group G, that acts properly discontinuously and freely on M, then

$$\chi(N) = \frac{\chi(M)}{|G|} \tag{1.6.1.0.2}$$

1.6.2 Euler orbicharacteristic

We would like to extend the definition of Euler characteristic to orbifolds in a way that will reflect on their structure. We will call the resulting additive topological invariant the Euler orbicharacteristic and denote it by χ^{orb}.

Following [Thu79] (definition 13.3.3.), but extending definition to orbifolds with cusps, we will define it as follows:

Definition 1.6.2.1. *When an orbifold O has a cell-division of the base space X, such that for each open cell the group associated to the interior points of a cell is constant, then the Euler number $\chi^{orb}(O)$ is defined by the formula:*

$$\chi^{orb}(O) := \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|}, \tag{1.6.2.1.1}$$

where c_i ranges over all cells and $|\Gamma(c_i)|$ is the order of the group $\Gamma(c_i)$ associated to each cell, if c_i is a cusp and $|\Gamma(c_i)| = \infty$ we put $\frac{1}{|\Gamma(c_i)|} = 0$. We will call $\frac{1}{|\Gamma(c_i)|}$ a weight of a cell c_i.

This definition results in the property, that if an orbifold O_2 is a quotient of a orbifold O_1, by the action of the finite group G, that acts properly discontinuously, but not necessarily freely on O_1, then:

$$\chi^{orb}(O_2) = \frac{\chi(O_1)}{|G|}. \tag{1.6.2.1.2}$$

For a two dimensional orbifolds, the possible cells with weights different than 1 are only in dimensions 0 and 1. In dimension 0 they are rotational and dihedral orbipoints. In dimension 1 they are fragments of the boundary that stabilises reflections. The weights of these cells are (with the convention that $\frac{1}{\infty} = 0$):

- for a rotational point of degree n, the weight is $\frac{1}{n}$,
- for a dihedral point of degree n, the weight is $\frac{1}{2n}$,
- for a reflection line, the weight is $\frac{1}{2}$.

We can see that weights of rotational and dihedral orbipoints are monotonously decreasing and converges to 0, as degree diverges to infinity. Moreover, the cusps – orbipoints of an infinite degree, that are stabilised by a groups of infinite degree, has weight 0.

From this we will obtain the formula for an Euler orbicharacteristic of a two dimensional orbifold with rotational points of degrees $r_1, r_2, \cdots, r_n$ and dihedral points of degrees $d_1, d_2, \cdots, d_m$:

$$\chi^{orb}(O) = \chi(M) - n + \sum_{i=1}^{n} \frac{1}{r_i} - \frac{m}{2} + \sum_{j=1}^{m} \frac{1}{2d_j} \tag{1.6.2.1.3}$$

$$= \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \tag{1.6.2.1.4}$$

For O with only rotational orbipoints:

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i}. \tag{1.6.2.1.5}$$

For O with only dihedral orbipoints:

$$\chi^{orb}(O) = \chi(M) - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j}. \tag{1.6.2.1.6}$$

From these formulas we can see, that as number of orbipoints diverges to infinity, the Euler orbicharacteristic diverges to minus infinity. For this reason, we restrict ourselves only to orbifolds with finitely many orbipoints.

Observation 1.6.2.2. *A M-orbifold that is different than M always have strictly smaller Euler orbicharacteristic than M.*

1.7 Metric structures on the orbifolds and areas of the orbifolds

1.7.1 Good and bad orbifolds

Definition presented in 1.3 says that an orbifold is locally homeomorphic do the quotient of $\mathbb{R}^n$ by some finite group.

When an orbifold as a whole is quotient of some finite group acting on a manifold we say, that it is 'good'. Otherwise we say, that it is 'bad'.

In two dimensions there are only four types of bad orbifolds, namely :

- (n) – S^2 with only one rotational orbipoint
- $(*n)$ – D^2 with only one dihedral orbipoint
- $(n_1 n_2)$ for $n_1 < n_2$ – S^2 with two rotational orbipoints of different degrees
- $(*n_1 n_2)$ for $n_1 < n_2$. – D^2 with two dihedral orbipoints of differen degrees

All other orbifolds are good – [Thu79] (theorem 13.3.6).

1.7.2 Metric structures

As described and proved in [Thu79] (13.3.6.) orbifold O is good iff have either :

- if $\chi^{orb}(O) > 0$ – an elliptic structure, i.e some metric with constant sectional curvature equal 1,
- if $\chi^{orb}(O) = 0$ – a parabolic structure, i.e some metric with constant sectional curvature equal 0,
- if $\chi^{orb}(O) < 0$ – a hyperbolic structure. i.e some metric with constant sectional curvature equal -1,

With an elliptic or a hyperbolic metric structure, we can measure the area of the orbifold as

$$A(O) = |\int_O K dA| \tag{1.7.2.0.1}$$

Also, as stated in [Thu79] (13.3.5.), the Gauss-Bonnet theorem works also for orbifolds and we have that integral over curvature is independent of the particular metric choosen and is proportional to χ^{orb}:

$$\int_O K dA = 2\pi|\chi^{orb}(O)|, \tag{1.7.2.0.2}$$

thus having that the area is:

$$A(O) = 2\pi|\chi^{orb}(O)|. \tag{1.7.2.0.3}$$

As mentioned in [Thu79] (Theorem 13.3.6), there are only 3 families of elliptical orbifolds (grouped with respect to base manifolds):

- S^2: (ε), (n), (nn), $(2\ 2\ n)$, $(2\ 3\ 3)$, $(2\ 3\ 4)$, $(2\ 3\ 5)$,
- D^2: $(*)$, $(*nn)$, $(*2\ 2\ n)$, $(*2\ 3\ 3)$, $(*2\ 3\ 4)$, $(*2\ 3\ 5)$, $n*$, $2*m$, $3*2$,
- $\mathbb{R}P^2$: $(\times)$, $(n\times)$.

only 17 parabolic orbifolds total (with 7 different families, with respect to possible base manifolds):

- S^2: $(2\ 3\ 6)$, $(2\ 4\ 4)$, $(3\ 3\ 3)$, $(2\ 2\ 2\ 2)$,
- D^2: $(*2\ 3\ 6)$, $(*2\ 4\ 4)$, $(*3\ 3\ 3)$, $(*2\ 2\ 2\ 2)$, $(2*2\ 2)$, $(3*3)$, $(4*2)$, $(2\ 2*)$,
- $\mathbb{R}P^2$: $(2\ 2\times)$,
- $\mathbb{T}^2$: $(\circ)$,
- Klein bottle: $(\times\times)$,
- annulus: $(**)$,
- Möbius band: $(*\times)$,

and infinitely many families of hyperbolic orbifolds.

Our main goal in this thesis is to describe possible areas of two dimensional orbifolds, especially those with hyperbolic structure. As from 1.7.2.0.3 we have direct correspondence between areas and Euler orbicharacteristics, and with Euler orbicharacteristic, we can restrict ourselves to rational numbers, we will try to describe possible Euler orbicharacteristic of two dimensional orbifolds.

1.8 Notation and terminology

1.8.1 Difference in Euler orbicharacteristic

We will denote the difference in Euler orbicharacteristic which is made by modifying an orbifold by a feature α as

$$\Delta(\alpha). \tag{1.8.1.0.1}$$

Let us observe, that this is well defined, since the difference of Euler orbicharacteristic between orbifolds O_1 and O_2, defined by the lists differing by one feature is independent of O_1, nor O_2, nor the choice of the list describing them at depends only on the feature.

1.8.2 Expressions involving ∞

If not stated otherwise, in the expressions containing ∞ symbol, their value is understood as $\varphi(\infty) \coloneqq \lim_{n\to\infty} \varphi(n)$. Only expressions where such limits exists will occur without further notice.

1.8.3 Sets of numbers

For $A, B \subseteq \mathbb{R}$, we define

$$A + B \coloneqq \{a + b \mid a \in A, b \in B\}, \tag{1.8.3.0.1}$$
$$AB \coloneqq \{ab \mid a \in A, b \in B\}. \tag{1.8.3.0.2}$$

For $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we define:

$$r + A = A + r \coloneqq \{a + r \mid a \in A\}, \tag{1.8.3.0.3}$$
$$rA = Ar \coloneqq \{ar \mid a \in A\}. \tag{1.8.3.0.4}$$

1.9 Spectra

We will call the set of all possible Euler orbicharacteristic of a M-orbifolds, the spectrum of M and we will denote it by $\sigma(M)$. We will denote the set of all possible Euler orbicharacteristic of a M-orbifolds that have only rotational orbipoints by $\sigma^r(M)$. We would denote the set of all possible Euler orbicharacteristic of a M-orbifolds that have only dihedral orbipoints by $\sigma^d(M)$, but from this section, it follows that we always have $\sigma^d(M) = \sigma(M)$.

We will also denote the sum of spectra of all two dimensional manifolds by:

$$\sigma \coloneqq \bigcup_M \sigma(M), \tag{1.9.0.0.1}$$

where sum is taken other all compact two dimensional manifolds possibly with boundary. This will be the main interest of this thesis.

Now we want to derive the form of the $\sigma(M)$. For a manifold M with h handles, c cross-cups and b boundary components, it's Euler characteristic is given by:

$$\chi(M) = 2 - 2h - c - b. \tag{1.9.0.0.2}$$

The sets of Δ for possible orbifold features are:

- for $b \neq 0$:

$$\{-\frac{n-1}{2n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\}\} \tag{1.9.0.0.3}$$

- for $b = 0$:

$$\{-\frac{n-1}{n} \mid n \in \mathbb{N}_{>0} \cup \{\infty\}\}. \tag{1.9.0.0.4}$$

Thus, we have that:

Observation 1.9.0.1. *The form of the spectrum of two dimensional manifold M is:*

- *for $b \neq 0$:*

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \mid n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.1.1}$$

- *for $b = 0$:*

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \tag{1.9.0.1.2}$$

Observation 1.9.0.2. *We have that $\sigma(S^2) = 2\sigma(D^2)$.*

Proof.

Indeed, since:

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.2.1}$$

and

$$\sigma(D^2) = 1 - \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \;\middle|\; m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \square \tag{1.9.0.2.2}$$

Observation 1.9.0.3. *For every two dimensional manifold M, we have that $\sigma(M)$ is homeomorphic to $\sigma(D^2)$. This homeomorphism is:*
- *for $b \neq 0$:*

$$\sigma(M) = \sigma(D^2) + \chi(M) - 1, \tag{1.9.0.3.1}$$

- *for $b = 0$:*

$$\sigma(M) = 2\sigma(D^2) + \chi(M) - 2. \tag{1.9.0.3.2}$$

Proof.

For a manifold M with h handles, c cross-cups and b boundary components, it's $\sigma(M)$ is given by:
- for $b \neq 0$:

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \;\middle|\; n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.3.3}$$

- for $b = 0$:

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \tag{1.9.0.3.4}$$

On the other hand, we have that:

$$\sigma(D^2) = 1 - \left\{ \sum_{i=1}^{n} \frac{d_i - 1}{2d_i} \;\middle|\; n \in \mathbb{N}_0,\ d_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.3.5}$$

$$\sigma(S^2) = 2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} \tag{1.9.0.3.6}$$

and

$$\sigma(S^2) = 2\sigma(D^2). \tag{1.9.0.3.7}$$

From this, the observation follows immediately. $\square$

Observation 1.9.0.4. *For every manifold M, for every $x \in \sigma(M)$, we have that $x \leqslant \chi(M)$.*

1.10 Egyptian fractions

Egyptian fraction is a finite sum of fractions, all with numerators one and positive denominators. Most of the time it is also required, that the fractions in the sum have pairwise distinct denominators. We will however take less usual convention and will drop that requirement, calling an Egyptian fraction any sum of unitary fractions.

As a side remark, we can say, that even in case of the usual, more strict definition, every positive rational number can be expressed as an Egyptian fraction. One of the methods to do so was described in [Eng13].

In our less strict definition, we can always have $\frac{m}{n} = \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}$.

1.10.1 Connection between spectra and Egyptian fractions

The terms $-\frac{r_i - 1}{r_i}$ in the sum 1.6.2.1.5 can be expressed as $-1 + \frac{1}{r_i}$ and the term $-\frac{d_j - 1}{2d_j}$ in the sum 1.6.2.1.6 can be expressed as $-\frac{1}{2} + \frac{1}{2d_j}$. Then the sums become:

$$\chi(M) - n + \underbrace{\sum_{i=1}^{n} \frac{1}{r_i}}_{\substack{\text{Egyptian}\\ \text{fraction}}} \tag{1.10.1.0.1}$$

and

$$\chi(M) - \frac{m}{2} + \frac{1}{2} \underbrace{\sum_{j=1}^{m} \frac{1}{d_j}}_{\substack{\text{Egyptian}\\ \text{fraction}}} . \tag{1.10.1.0.2}$$

In this form, the Egyptian fractions are explicitly present in expressions of points in $\sigma(M)$.

The $-n$ and $-\frac{m}{2}$ terms provide constraints on the number of fractions that can appear in the sum.

We will now translate the questions of being in the spectrum to the questions of being expressible as Egyptian fraction with the particular number of summands. It will be used in 3.3.1.1, 5.1.0.4 and 5.2.1.1.

We will now state two corollaries that follows immediately from the form of expressions 1.10.1.0.1 and 1.10.1.0.2, and from 1.9.0.1.

Corollary 1.10.1.1. *If x can be expressed as an Egyptian fraction with n summands, then for any two dimensional manifold M we have:*

$$\chi(M) - n + x \in \sigma(M) \tag{1.10.1.1.1}$$

and, if M has at least one boundary component also:

$$\chi(M) - \frac{n}{2} + \frac{1}{2}x \in \sigma(M). \tag{1.10.1.1.2}$$

Corollary 1.10.1.2. *If for some two dimensional manifold M we have that $y \in \sigma(M)$ as an Euler orbicharacteristic of an orbifold which has n rotational orbipoints and not any other, then*

$$y + n - \chi(M) \tag{1.10.1.2.1}$$

can be expressed as an Egyptian fraction with n (not necessarily distinct) summands.

If for some two dimensional manifold M with at least one boundary component we have that $y \in \sigma(M)$ as an Euler orbicharacteristic of an orbifold which has m dihedral orbipoints and not any other, then

$$2y + \frac{m}{2} - 2\chi(M) \tag{1.10.1.2.2}$$

can be expressed as an Egyptian fraction with m (not necessarily distinct) summands.

Theorem 1.10.1.3. *For any $q \in \mathbb{Q}$ and any $k \in \mathbb{N}_0$, there are only finitely many Egyptian fractions equal to q with k summands.*

Proof.

Let $\frac{1}{n_1} + \cdots + \frac{1}{n_k}$ be the Egyptian fraction of q, such that $\frac{1}{n_1} \geqslant \cdots \geqslant \frac{1}{n_k}$.

Let us observe, that then $\frac{1}{n_1} \geqslant \frac{q}{k}$. From, this, we have that $n_1 \leqslant \frac{k}{q}$, so there are only finitely many valuse, that n_1 can take. Let n_1' be the smallest value n_1 can take, such that $\frac{1}{n_1'} < \frac{1}{q}$. Then, we have that $\frac{1}{n_2} \geqslant \cdots \geqslant \frac{1}{n_k} \geqslant q - \frac{1}{n_1'}$.

We can make an inductive argument, that if $n_1', n_2', \cdots, n_{i-1}'$ are such smallest values for $n_1, n_2, \cdots, n_{i-1}$, then we have, that $\frac{1}{n_i} \geqslant \cdots \geqslant \frac{1}{n_k} \geqslant q - \frac{1}{n_1'} - \frac{1}{n_2'} - \cdots - \frac{1}{n_{i-1}}$. As such, we have, that

$$\frac{1}{n_i} \geqslant \frac{1}{k-i+1}\left(q - \frac{1}{n_1'} - \frac{1}{n_2'} - \cdots - \frac{1}{n_{i-1}}\right), \tag{1.10.1.3.1}$$

so we have that

$$n_i \leqslant \frac{k-i+1}{q - \frac{1}{n_1'} - \frac{1}{n_2'} - \cdots - \frac{1}{n_{i-1}}}, \tag{1.10.1.3.2}$$

so for every $1 \leqslant i \leqslant k$. there only finitely many values, that n_i can take. As such, there are only finitely many Egyptian fractions of q of k summands. □

□

1.11 Operations on orbifolds

As stated in 1.5 we will often see two dimensional orbifold as a sphere S^2 with a collection of features. Throughout the thesis we will frequently refer to performing the "operation" on an orbifold consisting of removing and adding those features. What we mean by this is giving as a result of an operation on one orbifold, defined by some list of features on a sphere, another one with a modified list of features according to the described operation.

When we will be talking about "adding" or "removing" a feature from an orbifold, we will mean adding or removing this feature from a list defining this orbifold and taking the orbifold defined by resulting list as a result of the operation.

As our main interest is to determine, for a given rational number $\frac{p}{q}$ which orbifolds have $\frac{p}{q}$ as their Euler orbicharacteristic and for a given orbifold O, which orbifolds have the same Euler orbicharacteristic as O, we will be particularly interested in such operations that do not change Euler orbicharacteristic, which will be used in 2.2.

1.12 Questions asked

There will be two main parts of question:

• Ones regarding σ as a set, where we will be asking of its order type and topology and relation to other sets such as $\sigma(D^2)$ and $\sigma(S^2)$. We will focus on these questions in chapter 3.

• Ones regarding σ as an image of a χ^{orb}, sending orbifolds to their Euler orbicharacteristics. There, we will ask for example how many orbifolds have particular Euler orbicharacteristic and related questions. We will focus on these questions in the chapter 5.

Chapter 2

Reduction to arithmetical questions

Reductions presented in this chapter will be more in the spirit of chapter 3, in the sense that for now, until chapter 5 , we will not pay attention to how many orbifolds have the same Euler orbicharacteristic, only whether a particular number is an Euler orbicharacteristic for at least one orbifold or not.

In chapter 5 we will explain how these reductions will be relevant to the discussion held there.

2.1 Reductions of cases

The aim of following reductions is to make it easier to answer the question of which points lie in σ and which not.

The first aspect of the structure of σ that we would like to simplify is that it is the sum of $\sigma(M)$, for every two dimensional manifold M.

$$\sigma = \bigcup_{M} \sigma(M), \tag{2.1.0.0.1}$$

where the sum is taken other all compact, two dimensional manifolds, possibly with boundary.

We aim to find a minimal set $\mathcal{M}$ of base manifolds such that:

$$\sigma = \bigcup_{M \in \mathcal{M}} \sigma(M). \tag{2.1.0.0.2}$$

It will turn out that $\mathcal{M} = \{S^2, D^2\}$ satisfies 2.1.0.0.2 and that both S^2 and D^2 are necessarily.

2.2 Sufficiency of S^2 and D^2

Given an orbifold O_1, we want to perform some operations from 1.11 on it, such that the resulting orbifold O_2 will have the same Euler orbicharacteristic, but the base manifold of O_2 would be S^2 or D^2. We would then say, that O_1 got reduced to O_2. In following subsection, we allow only such operations, that do not change Euler orbicharacteristic. When writing that we "can" do something we mean that there is possible one of the operations from 1.11.

The Euler characteristic of base manifold depends only on the number of handles, cross caps and boundary components. And, as stated in 1.6 it is:

$$2 - 2h - c - b, \tag{2.2.0.0.1}$$

for h - number of handles, c - number of cross-caps, b - number of boundary components.

For every such a manifold feature we want to find an orbifold features with the same Euler orbicharacteristic delta.

We will take two approaches, depending on whether the orbifold in question has a boundary or not.

2.2.1 Orbifold without boundary

We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(2^4) \tag{2.2.1.0.1}$$
$$\Delta(\times) = \quad -1 \quad = \Delta(2^2) \tag{2.2.1.0.2}$$

From this we can see that we can remove handles and cross-caps from any orbifold without the boundary. After such reductions we are left with a S^2 orbifold with all orbipoints being rotational in the interior.

2.2.2 Orbifold with boundary

We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta((^*2)^8) \tag{2.2.2.0.1}$$
$$\Delta(*) = \quad -1 \quad = \Delta((^*2)^4) \tag{2.2.2.0.2}$$
$$\Delta(\times) = \quad -1 \quad = \Delta((^*2)^4) \tag{2.2.2.0.3}$$

From this we can see that we can remove handles and cross-caps from any orbifold with a boundary. We can also remove all boundary components except one. We can further observe that:

$$\Delta(n) = \frac{n-1}{n} = 2\frac{n-1}{2n} = \Delta((^*n)^2) \tag{2.2.2.0.4}$$

From this we see that we can remove all the rotational orbipoints in favor for dihedral orbipoints. After such reductions we are left with a D^2 orbifold with all orbipoints being dihedral on the boundary or being reflectional on the boundary.

As a fact not necessary for our reductions, but interesting on its own, we can furthermore, observe that:

Observation 2.2.2.1. *If O_1 has not S^2 as its base manifold it can be reduced to a D^2-orbifold.*

Proof.

If O_1 has not S^2 as its base manifold M, then M has at least one handle or a cross-cup. We can observe that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(*({}^*2)^4) \qquad (2.2.2.1.1)$$
$$\Delta(\times) = \quad -1 \quad = \Delta(*). \qquad (2.2.2.1.2)$$

From this we have that the handle or the cross-cap can be replaced by a boundary component and some number of boundary orbipoints. After this reduction, we can proceed with all the other reductions from the 2.2.2 and obtain an D^2-orbifold with the same Euler orbicharacteristicas the original one. □

The results of our reductions, can be summarised as:

Observation 2.2.2.2. *If two-dimensional manifold M has no boundary, then*

$$\sigma(M) \subseteq \sigma(S^2) \qquad (2.2.2.2.1)$$

If, in addition, $M \neq S^2$, then

$$\sigma(M) \subseteq \sigma(D^2). \qquad (2.2.2.2.2)$$

Observation 2.2.2.3. *If two-dimensional manifold M has a boundary, then*

$$\sigma(M) \subseteq \sigma(D^2) \qquad (2.2.2.3.1)$$

Corollary 2.2.2.4. *We have that $\sigma = \sigma(S^2) \cup \sigma(D^2)$.*

Observation 2.2.2.5. *If a two-dimensional manifold M has a boundary, then:*

$$\sigma(M) = \sigma^d(M). \qquad (2.2.2.5.1)$$

We will postpone our discussion of necessity of both S^2 and D^2 to 2.5, after the section 2.3 which will provide us with more convenient language.

2.3 Reduction to arithmetical questions

As written in 1.9, we can express an Euler orbicharacteristicof a M-orbifold O as:

$$\chi^{orb}(O) = \chi(M) - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j}, \qquad (2.3.0.0.1)$$

where r_i and d_j are degrees of the, respectively, rotational and dihedral orbipoints of O.

From this we can express $\sigma(M)$ as:

$$\sigma(M) = \chi(M) - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \;\middle|\; n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} + \qquad (2.3.0.0.2)$$
$$- \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \;\middle|\; m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\}. \qquad (2.3.0.0.3)$$

As from 2.2 we know that $\sigma = \sigma(S^2) \cup \sigma(D^2)$, and that $\chi(S^2) = 2$ and $\chi(D^2) = 1$, we can express σ as a sum ($\cup$) of two sets:

$$2 - \left\{ \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \mid n \in \mathbb{N}_0,\ r_i \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(S^2) \tag{2.3.0.0.4}$$

and

$$1 - \left\{ \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \mid m \in \mathbb{N}_0,\ d_j \in \mathbb{N}_{>0} \cup \{\infty\} \right\} = \sigma(D^2). \tag{2.3.0.0.5}$$

From this we see, that the core of understanding σ through arithmetical viewpoint is to understand possible values of expression:

$$2 - \sum_{i=1}^{n} \frac{r_i - 1}{r_i} \tag{2.3.0.0.6}$$

and

$$1 - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j}, \tag{2.3.0.0.7}$$

with r_i and d_j ranging over $\mathbb{N}_{>0} \cup \{\infty\}$.

As $\Delta(\infty) = 1 = \Delta(2^2)$ and $\Delta(^*\infty) = \frac{1}{2} = \Delta((^*2)^2)$, we could perform further reductions to have an orbifold with particular orbicharacteristic without cusps (if needed) and then (after these reductions) we can analyse only expressions with r_i and d_j ranging over $\mathbb{N}_{>0}$ and they will still give us full spectrum. However, as stated later, it will be more convenient to us to include orbifolds with cusps so we are stating this observation only as a side remark.

2.4 Hurwitz theorem

One of the well known facts about two-dimensional orbifolds comes from [Hur93] (page 424):

Theorem 2.4.0.1. *The number of automorphisms which a smooth, connected Riemann surface of genus $g > 1$ can have, amounts at maximum to $84(g-1)$.*

Since all hiperbolic orbifolds are good, the quotient of a smooth hyperbolic manifold by some automorphism is a hyperbolic orbifold and 1.6.2.1.2, above theorem is equivalent to the one, that the orbifolds with hyperbolic structure can have an Euler orbicharacteristic at most $-\frac{1}{84}$. Here we will present the proof of this formulation of this result.

Theorem 2.4.0.2. *If a two dimensional orbifold admits hyperbolic structure, then the maximal Euler orbicharacteristic it can have is $-\frac{1}{84}$ and the only orbifold that realises this Euler orbicharacteristic is $*2\ 3\ 7$.*

Proof.

From 2.2 we know, that to check whether $-\frac{1}{84}$ is maximal negative Euler orbicharacteristic a two dimensional orbifold can have, we only need to check possible Euler orbicharacteristics of S^2 orbifolds and D^2 orbifolds.

Firstly, we will show, that $*2\ 3\ 7$ has biggest negative Euler orbicharacteristic from all D^2 orbifolds and its the only one with this Euler orbicharacteristic from D^2 orbifolds.

We have that

$$\chi^{orb}(*2\ 3\ 7) = 1 - \frac{1}{4} - \frac{2}{6} - \frac{6}{14} = -\frac{1}{84}. \tag{2.4.0.2.1}$$

From 1.9 we know, that $\sigma(D^2) = \sigma^d(D^2)$, so we can consider only dihedral orbipoints. Let us observe, that since $\max\{\Delta(^*n) \mid n \in \mathbb{N}_{>0} \cup \{\infty\}\} = \frac{1}{2}$, we have, that to have Euler orbicharacteristic < 0, D^2 orbifold has to have at least 3 orbipoints.

Let us observe, that for any D^2 orbifold that have 5 orbipoints or more, it has an Euler orbicharacteristicequal at most $1 - 5\frac{1}{4} = -\frac{1}{4} < -\frac{1}{84}$. So we can restrict our search only to orbifolds with at most 4 orbipoints.

Let us observe, that $*2\ 2\ 2\ 3$ and $*3\ 3\ 4$ are D^2 orbifolds with the biggest negative Euler orbicharacteristic among orbifolds, respectively, with four orbipoints, and, without any point of degree 2. The proof of this observation is, that any other orbifold of these respective kinds, would need to have all degrees of orbipoints (when ordered in increasing manner) pairwise $\geqslant$ than $*2\ 2\ 2\ 3$ or $*3\ 3\ 4$.

Let us observe, that

$$\chi^{orb}(*2\ 2\ 2\ 3) = 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{2}{6} = -\frac{1}{12} < -\frac{1}{84} \tag{2.4.0.2.2}$$

and

$$\chi^{orb}(*3\ 3\ 4) = 1 - \frac{2}{6} - \frac{2}{6} - \frac{3}{8} = -\frac{1}{24} < -\frac{1}{84}, \tag{2.4.0.2.3}$$

From this, we can conclude, that we can restrict our search only to orbifolds with exactly 3 orbipoints, where at least one of them is equal to 2.

Let us observe, that for such orbifold to have negative Euler orbicharacteristic, it needs to have at two orbipoints of order at least 3, otherwise it has Euler orbicharacteristic at most

$$\chi^{orb}(*2\ 2\ \infty) = 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} = 0. \tag{2.4.0.2.4}$$

Further, let us observe, that $*2\ 4\ 5$ has the greatest negative Euler orbicharacteristic among orbifolds that have no orbipoint of degree 3. The proof of this observation is similar to the proof of the previous one – any other orbifold of this kind, would need to have all degrees of orbipoints (when ordered in increasing manner) pairwise $\geqslant$ than $*2\ 4\ 5$.

Let us observe, that

$$\chi^{orb}(*2\ 4\ 5) = 1 - \frac{1}{4} - \frac{3}{8} - \frac{4}{10} = -\frac{1}{40} < -\frac{1}{84}. \tag{2.4.0.2.5}$$

From this, we conclude, that we can restrict our search to the orbifolds of the form $*2\ 3\ n$. We can also observe, that all orbifolds of the form $*2\ 3\ n$ have unique Euler orbicharacteristic among this group. The one with the biggest Euler orbicharacteristic among them is $*2\ 3\ 7$.

Let us observe that no D^2 orbifold with rotational orbipoints can't have such Euler orbicharacteristic. For the sake of contradiction, let us assume that there is some orbifold $r_1 \cdots r_n * d_1 \cdots d_m$, with $n \neq 0$, with Euler orbicharacteristic equal to $-\frac{1}{84}$. However, then also $*r_1 r_1 \cdots r_n r_n d_1 \cdots d_m$, with only dihedral orbipoints, would have Euler orbicharacteristic equal to $-\frac{1}{84}$. However, $*2\ 3\ 7$ is unique one with this Euler orbicharacteristic and it have no repeated degree, so it can't be expressed in the form $*r_1 r_1 \cdots r_n r_n d_1 \cdots d_m$ with $n \neq 0$.

Let us also observe, that the same argument shows that $\mathbb{R}P^2$ orbifolds can't have Euler orbicharacteristic equal to $-\frac{1}{84}$, since $\chi \mathbb{R}P^2 = \chi D^2$ and $\mathbb{R}P^1$ has no boundary, so $\mathbb{R}P^2$ orbifold can have only rotational orbipoints.

Now, we will prove, that no S^2 orbifold has Euler orbicharacteristic $-\frac{1}{84}$. Since we have 1.9.0.2 We can perform following argument:

For the sake of contradiction let us assume, that $-\frac{1}{84} \in \sigma(S^2)$, then, from 1.9.0.2 we know, that $\frac{1}{2}\left(-\frac{1}{84}\right) \in \sigma(D^2)$. This is a contradiction as $0 > \frac{1}{2}\left(-\frac{1}{84}\right) > -\frac{1}{84}$. As such, we ruled out all manifolds with Euler characteristic > 0.

For two dimensional manifolds with Euler characteristic $\leqslant 0$, we have that orbifolds having them as a base manifolds have Euler orbicharacteristicat most $-\frac{1}{4} < -\frac{1}{84}$. □

2.5 Necessity of S^2 and D^2

As we know from 1.11 adding an orbipoint to a manifold decreases it's orbicharacteristic. As S^2 has the highest Euler characteristic: 2 of all two dimensional manifolds, there is no other orbifold with Euler orbicharacteristic equal to 2. S^2 is then necessary to include 2.

As known from 2.4, the number $-\frac{1}{84} \in \sigma(D^2)$, it is the greatest negative Euler orbicharacteristicany two dimensional orbifold can have and $-\frac{1}{84} \notin \sigma(S^2)$. □

Further examination of connections between $\sigma(D^2)$ and $\sigma(S^2)$ is performed in 3.3.2.

Chapter 3

Order type and topology

In this chapter we will discuss that both the order type and the topology inherited from $\mathbb{R}$ of the set σ of all possible Euler orbicharacteristics of two-dimensional orbifolds are that of ω^ω.

This ω^ω will lie in $\mathbb{R}$ in reverse order, i.e. for $x, y \in \sigma$, such that such that $x <_{\mathbb{R}} y$, we will have $x >_{\omega^\omega} y$. As such, set σ will be treated as having the order type ω^ω in the sence of having order type ω^ω inherited from the reversed order in $\mathbb{R}$. However, when refering to a particular elements of σ as "greater" or "smaller" with respect to each other, we will use the usual order from $\mathbb{R}$.

To determine order type and topology of σ we will first study how $\sigma(D^2)$ looks like. Then, remembering that $\sigma = \sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$ we will make an argument for σ.

3.1 Order type and topology of $\sigma(D^2)$

In this section we will also describe precisely where accumulation points of $\sigma(D^2)$ lie and of which order (see below 3.1.1 or A.1) they are. Analysis of locations of those accumulation points, as interesting as it is alone will also be necessary for providing our argument about order type and topology of $\sigma(D^2)$.

3.1.1 Definition of order of accumulation points

These definitions are exactly the same as from appendix A.1 and are repeated here only for the readers convenience.

We start with definition of being "at least of order n" that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.
For a given set we define as follows:

Definition 3.1.1.1. *(Inductive). We say that the point x is an accumulation point of a set X of order at least 0, when it belongs to the set X. We say that the point x is an accumulation point of a set of order at least $n+1$, when it is an accumulation point (in the usual sense) of the accumulation points each of order at least n i.e. in* *every neighbourhood of x there is at least one accumulation point of a set X of order at least n, distinct from x.*

Definition 3.1.1.2. *We say that the point is an accumulation point of order n iff it is an accumulation point of order at least n and it is not an accumulation point of order at least $n+1$. If the point is an accumulation point of order at least n for an arbitrary large n we say that the point is an accumulation point of order ω.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the usual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

3.1.2 Analysis of locations of accumulation points of $\sigma(D^2)$ with respect to their order

We want to determine where exactly are accumulation points of the set $\sigma(D^2)$ with respect to their orders.

For this we will use a handful of observations and lemmas.

Observation 3.1.2.1. *Let us observe, that $\lim_{n\to\infty} \Delta(^*n) = -\frac{1}{2}$. From that, we see, that for every point $x \in \sigma(D^2)$, the point $x - \frac{1}{2}$ is an accumulation point. Let us observe, that also, for every point $x \in \sigma(D^2)$, we have that $x - \frac{1}{2} \in \sigma(D^2)$, because $\Delta(^*\infty) = -\frac{1}{2}$.*

Lemma 3.1.2.2. *For all $n \in \mathbb{N}_{\geqslant 2}$ and $x \in (-\infty, 1]$ there are only finitely many Euler orbicharacteristics in the interval $[x, 1] \cap \sigma(D^2)$ of orbifolds that have points of order equal at most n.*

Proof.

Let $x \in (-\infty, 1]$. There can be at most $\lfloor 4(1-x) \rfloor$ orbipoints on the D^2 orbifold with an Euler orbicharacteristic $y \in [x, 1]$ since each orbipoint decreases an Euler orbicharacteristic by at least $\frac{1}{4}$ and the Euler characteristic of D^2 is 1.

There are only $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible sets of $\lfloor 4(1-x) \rfloor$ orbipoints' orders that are less or equal than n. Hence, there are only at most $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible Euler orbicharacteristics.

Lemma 3.1.2.3. *If x is an accumulation point of the set $\sigma(D^2)$ of order n, then $x - \frac{1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order at least $n+1$.*

Proof.

Inductive.

• $n = 0$: If x is an isolated point of the set $\sigma(D^2)$, then $x \in \sigma(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma(D^2)$ for all $k \geqslant 1$, from that, that $x - \frac{1}{2}$ is a accumulation point of $\sigma(D^2)$.

• inductive step: Let x be an accumulation point of the set $\sigma(D^2)$ of an order $n > 0$. Let a_k be a sequence of accumulation points of order $n-1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of accumulation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order at least $n+1$. □

Lemma 3.1.2.4. *If x is an accumulation point of the set $\sigma(D^2)$ of order n, then $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order at least $n-1$.*

Proof.

Inductive

• $n = 1$: We assume, that x is an accumulation point of isolated points of the set $\sigma(D^2)$. From 3.1.2.2 we know, that for all m there are only finitely many Euler orbicharacteristics in the interval $[x, 1]$ of orbifolds that have dihedral points of order equal at most m.

From that, for arbitrary small neighborhood $U \ni x$ and arbitrary large m there exist an orbifold that has a dihedral point of period grater than m, whose Euler orbicharacteristic lies in U. Let us take a sequence of such Euler orbicharacteristics a_k convergent to x, that we can choose a divergent to infinity sequence of degrees of dihedral points b_k of orbifolds of Euler orbicharacteristics equal a_k.

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Let us observe, that for all k, the number $a_k + \frac{b_k - 1}{2b_k}$ is in $\sigma(D^2)$. It is so, because a_k is an Euler orbicharacteristic of an orbifold that have a dihedral point of period b_k, so identical orbifold, only without this dihedral point, has an Euler orbicharacteristic equal to $a_k + \frac{b_k - 1}{2b_k}$. The sequence $a_k + \frac{b_k - 1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order at least 0.

• inductive step: Let x be an accumulation point of the set $\sigma(D^2)$ of order $n > 1$. Let a_k be a sequence of accumulation points of the set $\sigma(D^2)$ of order $n-1$ convergent to x. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of an accumulation points of the set $\sigma(D^2)$ of order $n-2$ convergent to $x + \frac{1}{2}$. From that $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order at least $n-1$. □

Lemma 3.1.2.5. *If x is an accumulation point of the set $\sigma(D^2)$ of order $n+1$, then $x - \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order $n+2$ and $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order n.*

Proof.

Let x be an accumulation point of the set $\sigma(D^2)$ of order $n+1$. From the lemma 3.1.2.3 we know, that $x - \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order at least $n+2$. Now let us assume (for a contradiction), that $x - \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order $k > n+2$. But then from the lemma 3.1.2.4 we have that x is an accumulation point of the set $\sigma(D^2)$ of order at least $n+2$ and that is a contradiction.

Analogously, from the lemma 3.1.2.4 we know, that $x + \frac{1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order at least n. Let us assume (for a contradiction), that $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order $k > n$. But then from the lemma 3.1.2.3 we have that x is an accumulation point of the set $\sigma(D^2)$ of order at least $n + 2$ and that is a contradiction. □

Lemma 3.1.2.6. *For all $n \in \mathbb{N}$ all accumulation points of the set $\sigma(D^2)$ of order n are in $\sigma(D^2)$.*

Proof.

Inductive

- $n = 0$: Clear, as they are isolated points of $\sigma(D^2)$.
- inductive step: Let x be a accumulation point of the set $\sigma(D^2)$ of order $n > 0$.

From the lemma 3.1.2.5 point $x + \frac{1}{2}$ is an accumulation point of the set $\sigma(D^2)$ of order $n - 1$. From the inductive assumption $x + \frac{1}{2} \in \sigma(D^2)$. Then, from 3.1.2.1, we have that $x \in \sigma(D^2)$. □

Corollary 3.1.2.7. *Set $\sigma(D^2)$ is closed in $\mathbb{R}$.*

Theorem 3.1.2.8. *The greatest accumulation point of the set $\sigma(D^2)$ of order n is $1 - \frac{n}{2}$.*

Proof.

Inductive

- $n = 0$: We know, that $1 \in \sigma(D^2)$ and 1 is the greatest element of $\sigma(D^2)$.
- an inductive step: From the inductive assumption we know that $1 - \frac{n}{2}$ is the

greatest accumulation point of the set $\sigma(D^2)$ of order n. From the lemma 3.1.2.5 we have then that $1 - \frac{n+1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order $n + 1$. Let us assume (for a contradiction), that there exist a bigger accumulation point of order $n + 1$ equal to $y > 1 - \frac{n+1}{2}$. But then, from lemma 3.1.2.5, point $y + \frac{1}{2}$ would be an accumulation point of order n, what gives a contradiction, because $y + \frac{1}{2} > 1 - \frac{n}{2}$. □

From the above discussion we can also formulate following corollary that will be useful later:

Corollary 3.1.2.9. *Let $x \in \sigma(D^2)$. Then:*

- *there exists $n_1 \in \mathbb{N}$ such that $x + \frac{n_1}{2} \in \sigma(D^2)$ but $x + \frac{n_1+1}{2} \notin \sigma(D^2)$.*

  In other words, there exist $y \in \sigma(D^2)$ and $n_1 \in \mathbb{N}$ such that $y + \frac{1}{2} \notin \sigma(D^2)$ and such that $x = y - \frac{n_1}{2}$;

- *there exists $n_2 \in \mathbb{N}$ such that x is an accumulation point of the set $\sigma(D^2)$ of order n_2*

and $n_1 = n_2$.

3.1.3 Proof that $\sigma(D^2)$ is well ordered

Definition 3.1.3.1. *Let $B_0 = \{1\}$. For an $n \in \mathbb{N}_{>0}$, let B_n be the set of all possible Euler orbicharacteristic realised by orbifolds of type $*^*b_1, \cdots, ^* b_n$. For a given n these are D^2 orbifolds with precisely n non trivial orbipoints on their boundary.*

Observation 3.1.3.2. *There is a recursive relation, that $B_{n+1} = B_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$*

Proof.

It is so, because every orbifold with $n+1$ orbipoints can be obtained by adding one point to an orbifold with n orbipoints and the set $\{-\frac{n-1}{2n} \mid n \geqslant 2\} = \{\Delta(^*b) \mid b \geqslant 2\}$. □

Observation 3.1.3.3. *Observe that, as any orbifold has only finitely many orbipoints, we have that $\sigma(D^2) \subseteq \bigcup_{n=0}^{\infty} B_n$. We defined $\sigma(D^2)$ as a set of all possible Euler orbicharacteristic of disk orbifolds, so $\sigma(D^2) \supseteq \bigcup_{n=0}^{\infty} B_n$. From this we have that $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.*

Lemma 3.1.3.4. *For any given $n \in \mathbb{N}$ the set B_n is a subset of the interval $[1 - \frac{n}{2}, 1 - \frac{n}{4}]$.*

Proof.

Take $x \in B_n$. There exists an orbifold O with signature $*b_1, \cdots, b_n$, such that $\chi^{orb}(O) = x$. We have that $\forall_i -\frac{1}{2} \leqslant \Delta(^*b_i) \leqslant -\frac{1}{4}$. From this $-\frac{n}{2} \leqslant \Delta(^*b_1, \cdots, ^* b_n) \leqslant -\frac{n}{4}$, so $\chi^{orb}(O) \in [1 - \frac{n}{2}, 1 - \frac{n}{4}]$. □

Observation 3.1.3.5. *From 3.1.3.2 and A.2.0.1, we have that B_n do not have infinite ascending sequence for all n.*

Further, from A.2.0.2 we conclude, that $\bigcup_{n=0}^{N} B_n$ do not have infinite ascending sequence for all N.

Theorem 3.1.3.6. *In $\sigma(D^2)$ there are no infinite strictly ascending sequences, hence, it is well ordered.*

Proof.

For the sake of contradiction lets assume that c_n is an infinite strictly ascending sequence in $\sigma(D^2)$. As c_n is bounded from below by c_0 and whole $\sigma(D^2)$ is bounded from above by 1, all elements of c_n are in the interval $[c_0, 1]$. From 3.1.3.3 we have, that $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.

Lemma 3.1.3.4 says that for all n we have $B_n \subset [1 - \frac{n}{2}, 1 - \frac{n}{4}]$. From this, we know, that for any n such that $1 - \frac{n}{4} < c_0$ we have, that $B_n \cap [c_0, 1] = \varnothing$. Let n_0 be the smallest such that $1 - \frac{n_0}{4} < c_0$ (so $n_0 > 4(1 - c_0)$). Then for all $n > n_0$ we have

$1-\frac{n}{4}<c_0$, meaning, that for all $n>n_0$ we have $B_n\cap[c_0,1]=\varnothing$, so all elements of c_n are in $\bigcup_{n=0}^{n_0} B_n$. But this contradicts 3.1.3.5. □

3.1.4 Proof that order structure and topology of $\sigma(D^2)$ are those of ω^ω

Theorem 3.1.4.1. *Order type and topology inheritef from $\mathbb{R}$ of $\sigma(D^2)$ are ω^ω.*

Proof.

We will first prove, that the order type of $\sigma(D^2)$ is ω^ω.

• Order type of $\sigma(D^2)$ is at least ω^ω.

From 3.1.2.8 we know, that for every $n\in\mathbb{N}$, in $\sigma(D^2)$ there are accumulation points of order n. From this, and from A.2.0.4 we know that $\sigma(D^2)$ has an order type at least ω^n, for all $n\in\mathbb{N}$. The smallest ordinal number equal at least ω^n, for all $n\in\mathbb{N}$ is ω^ω. Thus, the order type of $\sigma(D^2)$ is at least ω^ω.

• Order type of $\sigma(D^2)$ is at most ω^ω.

For the sake of contradiction, let us suppose, that the order type η of $\sigma(D^2)$ is strictly greater than ω^ω. Then, $\sigma(D^2)$ has a set A of an order type ω^ω as it's prefix. The set A is bounded, as the $\omega^\omega+1$st element of $\sigma(D^2)$ is greater than any element of A. Let n, be such that $1-\frac{n}{2}$ is smaller than any element of A. As A is of order type ω^ω it has a prefix B of order type ω^n. From A.2.0.4 we know, that B has an accumulation point b of order n. This gives us a contradiction, as $b>1-\frac{n}{2}$, and from 3.1.2.8 we know, that $1-\frac{n}{2}$ is the greatest accumulation point of order n in $\sigma(D^2)$.

Now, we will prove, that the topology inherited from $\mathbb{R}$ on $\sigma(D^2)$ is that of ω^ω.

From 3.1.2.6 we know that every accumulation point of $\sigma(D^2)$ is in $\sigma(D^2)$. Thus, $\sigma(D^2)$ satisfies the assumptions of the lemma A.2.0.3 and we have that the topology of $\sigma(D^2)$ inherited from $\mathbb{R}$ is ω^ω.

3.2 Order type and topology of σ

Theorem 3.2.0.1. *The order type of the set and topologi inherited from $\mathbb{R}$ of the set of possible Euler orbicharacteristics of two-dimensional orbifolds σ is ω^ω.*

Proof.

From 2.2 we know, that $\sigma=\sigma(D^2)\cup\sigma(S^2)$.

From 3.1.4.1 and 1.9.0.2, we have that order types and topologies of $\sigma(D^2)$ and $\sigma(S^2)$ both are ω^ω and that $\sigma(S^2)=2\sigma(D^2)$.

We will now prove that the order type of σ is ω^ω.

From 3.1.2.8 we know, that the largest accumulation point of the set $\sigma(D^2)$ of order n is $1-\frac{n}{2}$. From, this and from the fact that $\sigma(S^2)=2\sigma(D^2)$ we know that that the largest accumulation point of the set $\sigma(S^2)$ of order n is $2-n$.

From this, we have, that for every $m \in \mathbb{N}_{>0}$, order type of $(-m, \infty) \cap \sigma(D^2)$ is ω^{2m+2} and that order type of $(-m, \infty) \cap \sigma(S^2)$ is ω^{m+2} (if $-m = 1 - \frac{n}{2}$, then $n = 2m + 2$ and if $-m = 2 - n$, then $n = m + 2$).

Thus, for every $m \in \mathbb{N}_{>0}$, we have that $(-m, \infty) \cap \sigma(D^2)$ and $(-m, \infty) \cap \sigma(S^2)$ satisfies assumptions of A.2.0.8, thus, we have that $(-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2))$ have an order type ω^{2m+2}.

From this we have that

$$\sigma = \sigma(D^2) \cup \sigma(S^2) = \bigcup_{m=1}^{\infty} \Big((-m, \infty) \cap (\sigma(D^2) \cup \sigma(S^2)) \Big) \tag{3.2.0.1.1}$$

have an order type ω^{ω}.

Now we will prove, that the topology inherited from $\mathbb{R}$ on σ is that of ω^{ω}.

We have that for $\sigma(D^2)$ $\left[\sigma(S^2)\right]$ every accumulation point of $\sigma(D^2)$ $\left[\sigma(S^2)\right]$ is in $\sigma(D^2)$ $\left[\sigma(S^2)\right]$. From this and from A.2.0.7 we have, that all accumulations points of σ are in σ. From this, from lemma A.2.0.3 we have that the topology of σ is ω^{ω}. $\square$

3.3 More about how this σ, $\sigma(S^2)$ and $\sigma(D^2)$ lie in $\mathbb{R}$

This section consists of rather loose assembly of remarks and observations about some relations between σ, $\sigma(S^2)$, $\sigma(D^2)$ and how they all lie in $\mathbb{R}$.

Observation 3.3.0.1. *The first (greatest) negative accumulation point of the set of σ is $-\frac{1}{12}$. It is the accumulation point of order* 1.

Proof.

We will show, that $-\frac{1}{12}$ is the greatest negative accumulation point of the set $\sigma(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two-dimensional orbifolds is equal to $\sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$, so the greatest negative point of the set $\sigma(S^2)$ is smaller than the greatest negative accumulation point of the set $\sigma(D^2)$.

• $-\frac{1}{12} = \chi^{orb}((2,3)) - \frac{1}{2}$, from this, from 3.1.2.5, we have that $-\frac{1}{12}$ an accumulation point of the set $\sigma(D^2)$ of order at least 1.

• Let us assume for a contradiction, that there exist bigger, negative accumulation point of the set $\sigma(D^2)$ of order at least 1. Let us denote it by x.

However, then, from the lemma 3.1.2.5 point $x + \frac{1}{2}$ is the accumulation point of the set $\sigma(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12})$. From the lemma 3.1.2.6 we have that x is in $\sigma(D^2)$. But orbifolds of the type $*d_1$ can have Euler orbicharacteristic only greater or equal $\frac{1}{2}$. Orbifolds of the type $*d_1d_2$ can only have Euler orbicharacteristic $\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*d_1d_2d_3\cdots$ can have Euler orbicharacteristic only lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma(D^2) = \emptyset$ and to the contradiction.

• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma(D^2)$, from this $-\frac{1}{12}$ is an accumulation point of order 1 of the set $\sigma(D^2)$. $\square$

3.3.1 Saturation theorem

Theorem 3.3.1.1. *For any rational number $\frac{p}{q}$, for any two dimensional manifold M there exists $N \in \mathbb{Z}$ such that for all $n \geqslant N$, we have that $\frac{p}{q} - n \in \sigma(M)$.*

Proof.

Let us take $\frac{p}{q} \in \mathbb{Q}$. From 1.10.1, we know that every rational number is expressible as an Egyptian fraction. Let us name the number of summands in some Egyptian fraction of $\frac{p}{q}$ as k. From 1.10.1.1 we know that then $\chi(M) - k + \frac{p}{q} \in \sigma(M)$. From 3.1.2.1 and 1.9.0.3 we also know, that if $x \in \sigma(M)$, then $x - l \in \sigma(M)$, for any $l \in \mathbb{N}_0$. From this, we have, that $k - \chi(M)$ is our N. □

Corollary 3.3.1.2. *For any finite set of rational numbers $\{(\frac{p}{q})_i\}_{i=1}^k$, for any finite set of two dimensional manifolds $\{M_j\}_{j=0}^l$, there exists a $N \in \mathbb{N}_0$ such that for all $n \geqslant N$, for all $i \in \{1 \ldots k\}$, for all $j \in \{1 \ldots l\}$ we have that $(\frac{p}{q})_i - n \in \sigma(M_j)$.*

Proof.

For each pair of $(\frac{p}{q})_i$ and M_j we apply 3.3.1.1, obtaining $N_{i,j}$. we take N as a minimal from $\{N_{i,j}\}_{i \in \{1 \ldots k\}, j \in \{1 \ldots l\}}$. □

3.3.2 Connections between $\sigma(S^2)$ and $\sigma(D^2)$

In this section we would like to answer some questions about relations between $\sigma(S^2)$ and $\sigma(D^2)$.

From 2.5 we know that both $\sigma(S^2)$ and $\sigma(D^2)$ are necessarily in expressing $\sigma = \sigma(S^2) \cup \sigma(D^2)$. It is shown by giving examples of two points one from $\sigma(S^2) \setminus \sigma(D^2) \ni 2$ and one from $\sigma(D^2) \setminus \sigma(S^2) \ni -\frac{1}{84}$. We found it interesting to ask further questions about the sets $\sigma(S^2) \setminus \sigma(D^2)$ and $\sigma(D^2) \setminus \sigma(S^2)$ such as what points exactly lie in one of $\sigma(S^2)$ and $\sigma(D^2)$ and not in the other, does it have any connection to the previously described order and topological structure or if the $\sigma(S^2)$ and $\sigma(D^2)$ overlap from some sufficiently distant point. This subsection is a meager attempt to answer some of these questions.

Accumulation points of the $\sigma(S^2)$

We will first state some observations that will be useful in this subsubsection.

Observation 3.3.2.1. *If an Euler orbicharacteristic x is an accumulation point of order n in $\sigma(D^2)$ $\left[\text{respectively } \sigma(S^2)\right]$, then there exist an D^2 $\left[\text{resp. } S^2\right]$ orbifold with n dihedral $\left[\text{resp. rotational}\right]$ points with that Euler orbicharacteristic.*

Proof.

From 3.1.2.9, we know, that then $x + \frac{n}{2} \in \sigma(D^2)$ $\left[\text{resp. } x + n \in \sigma(S^2)\right]$. Let O, be an orbifold with Euler orbicharacteristic equal to $x + \frac{n}{2}$ $\left[\text{resp. } x + n\right]$. Then O with

n dihedral $\Big[$resp. rotational$\Big]$ orbipoints of degree ∞ added is the orbifold we are looking for.

Observation 3.3.2.2. *If $x \in \sigma(D^2)$ $\Big[$respectively $\sigma(S^2)\Big]$, then $1 - x$ $\Big[$resp. $2 - x\Big]$ is a difference in Euler orbicharacteristic resulting from some set of dihedral $\Big[$resp. rotational$\Big]$ points. From that $1 - n(1 - x) \in \sigma(D^2)$ $\Big[$resp. $2 - n(2 - x) \in \sigma(S^2)\Big]$ for all $n \in \mathbb{N}$.*

Theorem 3.3.2.3. *All accumulation points of the $\sigma(S^2)$ are in $\sigma(D^2)$.*

There are two proofs of this theorem showing nice correspondence – one arithmetical and one geometrical.

Proof I. Arithmetical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.

By 1.9.0.2 we have, that $\frac{x}{2} \in \sigma(D^2)$ is an accumulation point of the set $\sigma(D^2)$. From 3.1.2.5 we have that $\frac{x}{2} + \frac{1}{2} \in \sigma(D^2)$. From that, from 3.3.2.2 we have, that

$$1 - \overbrace{2}^{\substack{\text{"}n\text{" from} \\ 3.3.2.2}} \Big(\overbrace{1 - (\frac{x}{2} + \frac{1}{2})}^{\substack{\text{"}1-x\text{" from} \\ 3.3.2.2}} \Big) \in \sigma(D^2). \tag{3.3.2.3.1}$$

But $1 - 2(1 - (\frac{x}{2} + \frac{1}{2})) = x$, so $x \in \sigma(D^2)$. □

Proof II. Geometrical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.

From 3.1.2.9 and 1.9.0.2 we know, that x can be expressed as $y - 1$ for some $y \in \sigma(S^2)$. Let $\mathcal{O}$ be an orbifold with the base manifold S^2, such that $\chi^{orb}(\mathcal{O}) = y$. Let $\mathcal{O}_c$ be the orbifold created from $\mathcal{O}$ by adding one cusp. Then $\chi^{orb}(\mathcal{O}_c) = y - 1 = x$. Topologically $\mathcal{O}_c$ with the cusp point removed is $\mathbb{R}^2$. We can compactify it with S^1. This operation of removing cusp point and replacing it by S^1 will not change an Euler orbicharacteristic since $\chi^{orb}(S^1) = 0$, Euler orbicharacteristic is additive and $\Delta(\infty) = \Delta(*) = 1$.

What we get is an orbifold $\mathcal{O}_D$ with the base manifold D^2 and the same orbipoints as $\mathcal{O}$. Since orbipoints of $\mathcal{O}$ create a difference in Euler orbicharacteristic equal to $2 - y$, we have that $\chi^{orb}(\mathcal{O}_D) = 1 - (2 - y) = y - 1 = x$. We can then replace all orbipoints from the interior of $\mathcal{O}_D$ by twice as many of the same degrees on its boundary 1.11, so $x \in \sigma(D^2)$. □

Chapter 4

Algorithm for searching for the spectrum

In the previous chapter we answered the questions about how σ looks like – in particular what is it's order type and topology. In this chapter we would like to develop a methods for answering the following question:

"For a given rational number, is it in σ?"

We have some sort of answer to this question – an algorithm.

It is not an ideal answer as it gives little insight of what is a general structure of the spectrum. Nevertheless it is a constructive and computable answer.

The exact question we will provide algorithm to answer here is:

For a given rational number r and manifold M, is there at least one M orbifold with r as its Euler orbicharacteristic?

We start with $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$ and a manifold M.

4.1 Reduction from arbitrary M to D^2

This reduction is based on 1.9.0.3. Note, that this is a different reduction than the one in chapter 2. In chapter 2 we are saying that for any M, we have $\sigma(M) \subseteq \sigma(S^2) \cup \sigma(D^2)$. In 1.9.0.3 on the other hand we have, that for a manifold M with h handles, c crosscaps and b boundary components:
for $b \neq 0$:

$$\sigma(M) = \sigma(D^2) + \chi(M) - 1 = \sigma(D^2) - 2h - c - (b - 1) \tag{4.1.0.0.1}$$

and for $b = 0$:

$$\sigma(M) = 2\sigma(D^2) + \chi(M) - 2 = 2\sigma(D^2) - 2h - c. \tag{4.1.0.0.2}$$

We conclude that the problem of deciding whether $\frac{p}{q}$ is in $\sigma(M)$ is equivalent to deciding:
for $b \neq 0$ if:

$$\frac{p}{q} - \chi(M) + 1 = \frac{p}{q} + 2h + c + (b - 1) \tag{4.1.0.0.3}$$

is in $\sigma^d(D^2)$;
for $b = 0$ if:

$$\frac{1}{2}\frac{p}{q} + \chi(M) + 2 = \frac{1}{2}\frac{p}{q} + h + \frac{c}{2} \tag{4.1.0.0.4}$$

is in $\sigma^d(D^2)$.

Considering this fact, from this point, WLOG we will assume that $M = D^2$ and, following 2.2.2.5, we will be concerned only with dihedral orbipoints.

4.2 Special cases

In the case that $\frac{p}{q}$ is of the form $l\frac{1}{4}$, for some $l \in \mathbb{Z}$ we can give the answer right away. For $l > 4$ we have that $l\frac{1}{4}$ is not in the set and for $l \leqslant 4$ it is (see 3.1.2.8).

Moreover for an even l we have that $l\frac{1}{4}$ is an accumultion point of order $\frac{4-l}{2}$ and for an odd l it is an accumulation point of order $\frac{3-l}{2}$ (see 3.1.2.8 and 3.1.2.9).

In the case, where $\frac{p}{q} > 1$, we also can give answer right away and this answer is "no".

Now we will consider only cases when $\frac{p}{q}$ is not of the form $l\frac{1}{4}$ and is $\leqslant 1$.

4.3 Regular cases

First we will describe what we use in the algorithm, giving the brief semantics. The detailed semantics are given in 4.4.

4.3.1 What we use

We use:

- $\mathbb{N}_{>0}$ counters $c_1, c_2, \cdots$ with values ranging on $\mathbb{N}_{>0} \cup \{\infty\}$. Each counter correspond to one dihedral point on the boundary of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial dihedral point - namely a non-orbi point, a normal point).

  We will write the state of the counters without commas, using the letter d. Note that with this convention, c_i will refer to the i-th counter and d_i will refer to the value of the i-th counter.

  So the state of the counters $d_1 d_2 \cdots$ correspond to the orbifold $*d_1 d_2 \cdots$ (where the trailing 1's are truncated).

  We will refer to the counters being "to the left" or "to the right" of each other, as the numbering would go from left to right.

- a pivot pointing at some counter

- a flag that can be set to: "Greater", "Searching" or "Less" corresponding to what was the outcome of comparing Euler orbicharacteristic of the orbifold corresponding to counters' state and $\frac{p}{q}$ or to the fact, that there is a need for a search of the next state of counters to compare with $\frac{p}{q}$.

4.3.2 What state are we starting our algorithm with

We start with:

- all counters set to 1.
- pivot pointing at the c_1
- flag set to "Greater"

4.3.3 Invariants claims

Now we will state the claims of what properties the state of the counters will maintain during all the execution of the algorithm. The proof, that this is indeed the case will be performed in 4.5

Claim 4.3.3.1. *We will do our computation such that:*

- *every state of the counters during runtime of the algorithm will have only finitely many counters with value non-*1*.*
- *every state in the runtime of the algorithm will have values on consecutive counters ordered in weakly decreasing order.*

From now we will consider only such states.

4.3.4 The algorithm for searching for a spectrum

When the algorithm is in the state:

- counters with values: $d_1 d_2 \cdots$
- pivot: at the counter c_p
- flag: set to the value $flag\_value$,

we proceed as follows :

In the case, the $flag\_value$ is equal to:
{
    "Greater", then
    {
        If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) = \frac{p}{q}$ then
        {
            We found an orbifold and we are ending the whole
            algorithm with answer "yes, $*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots$".

        }

```
13          If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) > \frac{p}{q}$ then
14          {
15              We set $d_p$ to $\infty$.
16              We set the flag to "Greater".
17              We put the pivot at the $c_{p+1}$.
18              We go to the 1st line.
19          }
20          If $\chi^{orb}(*d_1 \cdots d_{p-1} \infty d_{p+1} \cdots) < \frac{p}{q}$ then
21          {
22              We set the flag to "Searching".
23              We go to the 1st line.
24          }
25      }
26
27      "Searching", then
28      {
29          We search one by one
30          for the value $d'_p$ of the $c_p$ such that
31          $\chi^{orb}(*d_1 \cdots d_{p-1} d'_p d_{p+1} \cdots) \leqslant \frac{p}{q}$ and
32          $\chi^{orb}(*d_1 \cdots d_{p-1} (d'_p - 1) d_{p+1} \cdots) > \frac{p}{q}$.
33          We set $c_p$ and all of the counters
34          to the left of $c_p$ to the value $d'_p$.
35          if $\chi^{orb}(*d_1 d_2 d_3 \cdots) = \frac{p}{q}$ then
36          {
37              We found an orbifold and we are ending the whole
38              algorithm with answer "yes, $*d_1 d_2 \cdots$".
39
40
41
42          }
43          If $\chi^{orb}(*d_1 d_2 d_3 \cdots) < \frac{p}{q}$ then
44          {
45              We set the flag to "Less".
46              We put the pivot at the $c_{p+1}$.
47              We go to the 1st line.
48          }
49          If $\chi^{orb}(*d_1 d_2 d_3 \cdots) > \frac{p}{q}$ then
50          {
51              We set the flag to "Greater".
52              We put the pivot at the $c_1$.
53              We go to the 1st line.
54          }
55      }
56
```

```
    "Less", then
    {
        If $d_p = 1$ and the values of all the counters
        on the left of $c_p$ are equal to 2 then
        {
            We end the whole algorithm with the answer "no".
        }
        We increase $c_p$ by one ($d_p := d_p + 1$) and
        we set the value of all counters on the left of $c_p$ to $d_p$.
        If $\chi^{orb}(*d_1d_2d_3\cdots) = \frac{p}{q}$ then
        {
            We found an orbifold and we are ending the whole
             algorithm with answer "yes, $*d_1d_2\cdots$".
        }
        If $\chi^{orb}(*d_1d_2d_3\cdots) > \frac{p}{q}$ then
        {
            We set the flag to "Greater".
            We put the pivot at the $c_1$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1d_2d_3\cdots) < \frac{p}{q}$ then
        {
            We set the flag to "Less".
            We put the pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
    }
}
```

4.4 The idea of the algorithm

We will now present in more detail what the algorithm is indented to do. To do this and for the later sections, we will first introduce an order on the states of counters satisfying 4.3.3.1 (as mentioned in 4.3.3.1 we will consider only such states) and prove several lemmas about it.

4.4.1 Order on the space of states of the counters

Definition 4.4.1.1. *We define a linear order $\preceq$ on the states of counters as follows:*

Let D_1 be a state of counters equal to $d_1^1 d_2^1 \cdots$ and D_2 be a state of counters equal to $d_1^2 d_2^2 \cdots$. Let i be the greatest index where D_1 and D_2 differ, then:

bullet If $d_i^1 \leqslant d_i^2$ then $D_1 \preceq D_2$.

This is a suborder of the lexicographical order of states of counters after truncation of trailing 1's with the counters to the right being more significant.

Observation 4.4.1.2. *In general it is not true that if $D_1 \preceq D_2$ then $\chi^{orb}(*D_1) \leqslant \chi^{orb}(*D_2)$ nor that if $D_1 \preceq D_2$ then $\chi^{orb}(*D_1) \geqslant \chi^{orb}(*D_2)$.*

Observation 4.4.1.3. *Since $\preceq$ is a suborder of a lexicographical order it is a good order.*

Let us use $S(a)$ for a successor of a. We can explicitly write the form of the successor of any state $d_1 d_2 d_3 \cdots$ in $\preceq$:

Observation 4.4.1.4. *The successor of the state $d_1 d_2 d_3 \cdots$, of the form*

$$\underbrace{\infty\infty\cdots\infty}_{k-1 \text{ times}} d_k d_{k+1} d_{k+2} \cdots , \tag{4.4.1.4.1}$$

where k is such that c_k is the first counter from the left that is not set to ∞, is

$$\underbrace{(d_k+1)(d_k+1)\cdots(d_k+1)}_{k-1 \text{ times}}(d_k+1) d_{k+1} d_{k+2} \cdots , \tag{4.4.1.4.2}$$

Definition 4.4.1.5. *We will call the state $d_1 d_2 d_3 \cdots$, such that no d_k is equal to ∞ a **finite** state.*

*We will call the state $d_1 d_2 d_3 \cdots$, such that at least one of d_k is equal to ∞ an **infinite** state.*

Observation 4.4.1.6. *Using 4.3.3.1 we have that for the state $d_1 d_2 d_3 \cdots$ to be finite (resp. infinite), it is equivalent to d_1 being different from (resp. being equal to) ∞.*

Observation 4.4.1.7. *For any state D, we have that $S(D)$ is a finite state.*

Definition 4.4.1.8. *We will call the ascending sequence $\{D_n\}$ in $\preceq$, such that for all n, we have that $S(D_n) = D_{n+1}$, a **connected** sequence in $\preceq$.*

Observation 4.4.1.9. *Every connected sequence of the finite states is of the form $\{(d_1+n)d_2 d_3 \cdots\}$, where all d_n are different from ∞.*

Lemma 4.4.1.10. *Let D_1 and D_2 be finite states and let $S(D_1) = D_2$ in $\preceq$. Then $\chi^{orb}(*D_1) > \chi^{orb}(*D_2)$.*

Proof.

From 4.4.1.4 we know, that taking the successor of the finite state always changes only first counter and it is changing it by increasing it by 1. Increasing the order of the orbipoint decreases Euler orbicharacteristic. □

Corollary 4.4.1.11. *The sequence $\{\chi^{orb}(*D_n)\}$ is descending for every connected sequence of finite states $\{D_n\}$ in $\preceq$.*

Lemma 4.4.1.12. *Let D_1 be infinite state and let $D_2 := S(D_1)$ in $\preceq$. Then $\chi^{orb}(*D_1) \leqslant \chi^{orb}(*D_2)$. Furthermore there is only one element in $\preceq$ for which the equality holds: $\infty\ 1\ 1\ 1 \cdots$, for all the rest the inequality is strict.*

Proof.

For the state

$$\infty d_2 d_3 d_4 \cdots \tag{4.4.1.12.1}$$

and its successor

$$S(\infty d_2 d_3 d_4 \cdots) = (d_2+1)(d_2+1)d_3 \cdots , \tag{4.4.1.12.2}$$

we have that:

$$\begin{gathered}
\chi^{orb}(*\infty d_2 d_3 d_4 \cdots) - \chi^{orb}((d_2+1)(d_2+1)d_3 d_4 \cdots) = \\
1 + \Delta(\infty d_2) + \Delta(d_3 d_4 \cdots) - (1 + \Delta((d_2+1)(d_2+1)) + \Delta(d_3 d_4 \cdots)) = \\
\Delta(\infty d_2) - \Delta((d_2+1)(d_2+1)) = \\
-\frac{1}{2} - \frac{d_2-1}{2d_2} + 2\frac{(d_2+1)-1}{2(d_2+1)} = \\
\frac{-d_2(d_2+1) - (d_2-1)(d_2+1) + 2d_2^2}{2d_2(d_2+1)} = \frac{-d_2 - d_2 + d_2 + 1}{2d_2(d_2+1)} = \frac{1-d_2}{2d_2(d_2+1)}.
\end{gathered} \tag{4.4.1.12.3}$$

So the difference is not negative only for $d_2 = 1$ and for $d_2 = 1$ it is equal to 0. □

Lemma 4.4.1.13. *The supremum of the connected sequence of finite states*

$$\{(d_1+n)d_2 d_3 \cdots\} \tag{4.4.1.13.1}$$

is

$$\infty d_2 d_3 \cdots \tag{4.4.1.13.2}$$

, and the infimum of the corresponding sequence

$$\{\chi^{orb}(*(d_1+n)d_2 d_3 \cdots)\} \tag{4.4.1.13.3}$$

is

$$\chi^{orb}(*\infty d_2 d_3 \cdots). \tag{4.4.1.13.4}$$

Proof.

For every n we have that

$$(d_1+n)d_2 d_3 \cdots \preceq \infty d_2 d_3 \cdots . \tag{4.4.1.13.5}$$

Furthermore for every

$$d_1' d_2' d_3' \cdots \tag{4.4.1.13.6}$$

such that

$$d_1' d_2' d_3' \cdots \preceq \infty d_2 d_3 \cdots , \tag{4.4.1.13.7}$$

there exists n, such that

$$d_1' d_2' d_3' \cdots \preceq (d_1+n)d_2 d_3 \cdots . \tag{4.4.1.13.8}$$

Thus,

$$\infty d_2 d_3 \cdots \tag{4.4.1.13.9}$$

is the supremum of

$$\{(d_1+n)d_2d_3\cdots\}. \tag{4.4.1.13.10}$$

For every n we have that:

$$\begin{aligned}\chi^{orb}(*(d_1+n)d_2d_3\cdots) &= \chi^{orb}(*d_1d_2d_3\cdots) - \frac{(d_1+n)-1}{2(d_1+n)} + \frac{d_1-1}{2d_1} \\ &= \chi^{orb}(*d_1d_2d_3\cdots) - \frac{1}{2d_1} + \frac{1}{2(d_1+n)}.\end{aligned} \tag{4.4.1.13.11}$$

We also have that:

$$\begin{aligned}\chi^{orb}(*\infty d_2d_3\cdots) &= \chi^{orb}(*d_1d_2d_3\cdots) - \frac{1}{2} + \frac{d_1-1}{2d_1} \\ &= \chi^{orb}(*d_1d_2d_3\cdots) - \frac{1}{2d_1} + 0.\end{aligned} \tag{4.4.1.13.12}$$

Thus $\chi^{orb}(*\infty d_2d_3\cdots)$ is the infimum of $\{\chi^{orb}(*(d_1+n)d_2d_3\cdots)\}$. □

Observation 4.4.1.14. *We have that for $d_n \neq \infty$:*

$$\chi^{orb}(\infty\infty\cdots\infty d_n d_{n+1} d_{n+2}\cdots) > \chi^{orb}(\infty\infty\cdots\infty(d_n+1)d_{n+1}d_{n+2}\cdots). \tag{4.4.1.14.1}$$

As increasing the counter increases corresponding Euler orbicharacteristic.

Lemma 4.4.1.15. *The supremum of the sequence of states*

$$\{\infty\infty\cdots\infty(d_n+m)d_{n+1}d_{n+2}\cdots\}_m \tag{4.4.1.15.1}$$

is

$$\infty\infty\cdots\infty\infty d_{n+1}d_{n+2}\cdots, \tag{4.4.1.15.2}$$

and the infimum of the corresponding sequence

$$\{\chi^{orb}(*\infty\infty\cdots\infty(d_n+m)d_{n+1}d_{n+2}\cdots)\}_m \tag{4.4.1.15.3}$$

is

$$\chi^{orb}(*\infty\infty\cdots\infty\infty d_{n+1}d_{n+2}\cdots). \tag{4.4.1.15.4}$$

Proof.

The proof will be analogous to 4.4.1

For every m we have that

$$\infty\infty\cdots\infty(d_n+m)d_{n+1}d_{n+2}\cdots \preceq \infty\infty\cdots\infty\infty d_{n+1}d_{n+2}\cdots. \tag{4.4.1.15.5}$$

Furthermore for every $d'_1d'_2d'_3\cdots$ such that

$$d'_1d'_2d'_3\cdots \preceq \infty\infty\cdots\infty\infty d_{n+1}d_{n+2}\cdots, \tag{4.4.1.15.6}$$

there exists m, such that

$$d'_1 d'_2 d'_3 \cdots \preceq \infty\infty \cdots \infty(d_n + m)d_{n+1}d_{n+2} \cdots . \tag{4.4.1.15.7}$$

Thus,

$$\infty\infty \cdots \infty\infty d_{n+1}d_{n+2} \cdots \tag{4.4.1.15.8}$$

is the supremum of

$$\{\infty\infty \cdots \infty(d_n + m)d_{n+1}d_{n+2} \cdots\}_m. \tag{4.4.1.15.9}$$

For every m we have that:

$$\chi^{orb}(*\infty\infty \cdots \infty(d_n + m)d_{n+1}d_{n+2} \cdots) = \tag{4.4.1.15.10}$$

$$\chi^{orb}(*\infty\infty \cdots \infty d_n d_{n+1}d_{n+2} \cdots) - \frac{(d_n + m) - 1}{2(d_n + m)} + \frac{d_n - 1}{2d_n} =$$

$$\chi^{orb}(*\infty\infty \cdots \infty d_n d_{n+1}d_{n+2} \cdots) - \frac{1}{2d_n} + \frac{1}{2(d_n + m)} \tag{4.4.1.15.11}$$

We also have that:

$$\chi^{orb}(*\infty\infty \cdots \infty\infty d_{n+1}d_{n+2} \cdots) = \tag{4.4.1.15.12}$$

$$\chi^{orb}(*\infty\infty \cdots \infty d_n d_{n+1}d_{n+2} \cdots) - \frac{1}{2} + \frac{d_n - 1}{2d_n} =$$

$$\chi^{orb}(*\infty\infty \cdots \infty d_n d_{n+1}d_{n+2} \cdots) - \frac{1}{2d_n} + 0. \tag{4.4.1.15.13}$$

Thus

$$\chi^{orb}(*\infty\infty \cdots \infty\infty d_{n+1}d_{n+2} \cdots) \tag{4.4.1.15.14}$$

is the infimum of

$$\{\infty\infty \cdots \infty(d_n + m)d_{n+1}d_{n+2} \cdots\}_m. \square \tag{4.4.1.15.15}$$

Lemma 4.4.1.16. *The state of the counters in the algorithm is weakly increasing with respect to order $\preceq$.*

Proof.

The state of the counters is changed in lines 10-11, 46, 66-67. In each of these lines the counter with the greatest index of all changed counters increases in value, so the resulting state is bigger with respect to order $\preceq$. $\square$

4.4.2 Basic idea

The basic idea of the algorithm is to search through all the states of the counters going from the smallest (in the sense of $\preceq$) state of counters, which will be when all counters are set to 1, up to some upper limit beyond which we are sure that no configuration of counters will yield the Euler orbicharacteristic that we are looking for.

Now we will go through several obstacles of how to do so and solutions for them, answering for example the questions how we go through all the states and what can be this upper limit.

4.4.3 Checking all the states

This can't be done directly as there are infinite ascending sequences in $\preceq$. However, it can be done with some use of the properties we derived in the previous subsection.

4.4.4 Checking infinite connected sequences in finitely many steps

We will now present the method how to check any infinite connected sequence for solutions in finite number of steps.

First, we will perform a reduction from arbitrary infinite connected sequence to the infinite connected sequence of finite states.

Let us observe, that, by 4.4.1.7, there can be at most one infinite state in any connected sequence, and if it is present it must be the first one. If such state D_0 is present, we can check it whether $\chi(*D_0))$ is equal to $\frac{p}{q}$ or not (one step), and then all states that are left to be checked are finite and form infinite connected sequence of finite states, thus ending our reduction.

As this from this point we will present a method or checking for solutions any infinite connected sequence of finite states.

First, let us observe that thanks to 4.4.1.11, when we are searching through the infinite connected sequence of finite states in $\preceq$, once we get (without finding any solution) to the state D_n for which $\chi^{orb}(*D_n) < \frac{p}{q}$, we know that no state D_m with $m > n$ can have $\chi^{orb}(*D_m) = \frac{p}{q}$ and we can disregard whole sequence.

There is, however, another problem, namely, that when we are searching through the infinite connected sequence of the finite state, initially, we don't now, whether there will be any state $D_k = (d_1 + k)d_2d_3\cdots$ in it, that will have $\chi^{orb}(*(d_1 + k)d_2d_3\cdots) \leqslant \frac{p}{q}$. However, thanks to 4.4.1.13 we can check for this, by first comparing $\frac{p}{q}$ with $\chi(*\infty d_2d_3\cdots)$. Since from 4.4.1.13, we have that $\chi(*\infty d_2d_3\cdots)$ is the infimum of $\{*(d_1 + n)d_2d_3\cdots\}$, we have that if $\chi(*\infty d_2d_3\cdots) < \frac{p}{q}$, then there must be state $(d_1 + n)d_2d_3\cdots$ such that $\chi(*(d_1 + n)d_2d_3\cdots) < \frac{p}{q}$, for some n and we can proceed to look for it one by one through the sequence.

One case that is left, is when $\chi(*\infty d_2d_3\cdots) > \frac{p}{q}$, but then we can disregard the whole sequence right away, since $\chi(*\infty d_2d_3\cdots)$ is the infimum of $\{*(d_1+n)d_2d_3\cdots\}$.

4.4.5 What after we checked infinite connected sequence?

Let us suppose that we just checked the infinite connected sequence, together with its supremum.

The supremum is of the form $\infty d_2d_3d_4\cdots$. Then, trying to perform 4.4.2, we continue with the successor $S(\infty d_2d_3d_4\cdots) = (d_2 + 1)(d_2 + 1)d_3d_4\cdots$ (4.4.1.4). Provided the successor is not our solution, there are two options:

1. $\chi^{orb}(*(d_2 + 1)(d_2 + 1)d_3d_4\cdots) > \frac{p}{q}$,
2. $\chi^{orb}(*(d_2 + 1)(d_2 + 1)d_3d_4\cdots) < \frac{p}{q}$.

4.4.6 Case when $\chi^{orb}(*(d_2+1)(d_2+1)d_3d_4\cdots) > \frac{p}{q}$

We could start checking through the connected sequence starting at

$$(d_2+1)(d_2+1)d_3d_4\cdots, \tag{4.4.6.0.1}$$

however, if

$$\chi^{orb}(*\infty(d_2+1)d_3d_4\cdots) > \frac{p}{q}, \tag{4.4.6.0.2}$$

we would end up in the same place that we are now, only with

$$(d_2+2)(d_2+2)d_3d_4\cdots. \tag{4.4.6.0.3}$$

Without further changes, this will lead to possibly checking one by one of infinitely many states of the form

$$\infty(d_2+n)d_3d_4\cdots. \tag{4.4.6.0.4}$$

We can solve this problem, by checking the state

$$\infty\infty d_3d_4\cdots, \tag{4.4.6.0.5}$$

that have corresponding Euler orbicharacteristic lower than all of 4.4.6.0.4. If it will happen that

$$\chi^{orb}(*\infty\infty d_3d_4\cdots) > \frac{p}{q}, \tag{4.4.6.0.6}$$

we ruled out all states of the form

$$\infty(d_2+n)d_3d_4\cdots, \tag{4.4.6.0.7}$$

and we can continue this pattern on further coordinates, checking:

$$\infty\infty\cdots\infty d_k\cdots, \tag{4.4.6.0.8}$$

until we find some n, such that

$$\chi^{orb}(*\infty\infty\cdots\infty d_{n+1}\cdots) < \frac{p}{q}. \tag{4.4.6.0.9}$$

We always find n like that, because at all time only finitely many counters are set to non-1 value, so from some point moving to next coordinate will result in comparing to $\frac{p}{q}$ the number $\frac{1}{2}$ smaller than from previous coordinate.

Once we find such n, we need to perform actions described in 4.4.8.

4.4.7 Case when $\chi^{orb}(*(d_2+1)(d_2+1)d_3d_4\cdots) < \frac{p}{q}$

In this case, we know that every state D such that:

$$(d_2+1)(d_2+1)d_3d_4d_5\cdots \preceq D \prec (d_3+1)(d_3+1)(d_3+1)d_4d_5\cdots \tag{4.4.7.0.1}$$

have $\chi^{orb}(D) < \frac{p}{q}$, since for any such state D, we have that counter c_3 and all to the right of it are the same as in

$$(d_2+1)(d_2+1)d_3d_4d_5\cdots, \tag{4.4.7.0.2}$$

but counters c_1 and c_2 are at least equal to d_2+1. For this reason, we can go to

$$(d_3+1)(d_3+1)(d_3+1)d_4d_5\cdots, \tag{4.4.7.0.3}$$

as we ruled out all the sates smaller than 4.4.7.0.3. Then, we can continue from this state.

This behaviour can be generalised – whenever, in our algorithm we will have counters if the state

$$(d_n+1)(d_n+1)\cdots(d_n+1)d_{n+1}d_{n+2}d_{n+3}\cdots, \tag{4.4.7.0.4}$$

and we will know that

$$\chi^{orb}(*(d_n+1)(d_n+1)\cdots(d_n+1)d_{n+1}d_{n+2}d_{n+3}\cdots) < \frac{p}{q}, \tag{4.4.7.0.5}$$

we can rule out all the states up to (but not including) state:

$$(d_{n+1}+1)(d_{n+1}+1)\cdots(d_{n+1}+1)(d_{n+1}+1)d_{n+2}d_{n+3}\cdots \tag{4.4.7.0.6}$$

by the analogous reasoning as for $n=2$ and continue from state 4.4.7.0.6.

4.4.8 Searching

We are in the state, as described in 4.4.6, that we found n, such that

$$\chi^{orb}(*\infty\infty\cdots\infty d_{n+1}\cdots) < \frac{p}{q}. \tag{4.4.8.0.1}$$

The idea of algorithm at this point was, to rule out all the states that have corresponding Euler orbicharacteristicgreater than $\frac{p}{q}$. We ruled out all smaller or equal to (in the sense of $\preceq$) than

$$\infty\infty\cdots\infty 1 d_{n+1}\cdots. \tag{4.4.8.0.2}$$

At this point, we can use a procedure analogous to the one from 4.4.4, checking through the sequence (iterated with respect to m)

$$\infty\infty\cdots\infty m d_{n+1}\cdots, \tag{4.4.8.0.3}$$

that at some m_0 is guaranteed to have

$$\chi^{orb}(\infty\infty\cdots\infty m_0 d_{n+1}\cdots)) \leqslant \frac{p}{q}, \tag{4.4.8.0.4}$$

since

$$\chi^{orb}(\infty\infty\cdots\infty\infty d_{n+1}\cdots)) < \frac{p}{q} \tag{4.4.8.0.5}$$

and we have that 4.4.1.15.

This way, we know that no state smaller or equal than

$$\infty\infty\cdots\infty(m_0-1)d_{n+1}\cdots \qquad (4.4.8.0.6)$$

is the solution. We know that $m_0 \geqslant 2$, since we know from the procedure 4.4.6 that

$$\chi^{orb}(\infty\infty\cdots\infty 1 d_{n+1}\cdots)) > \frac{p}{q} \qquad (4.4.8.0.7)$$

We know proceed to check from the successor:

$$S(\infty\infty\cdots\infty(m_0-1)d_{n+1}\cdots) = m_0 m_0\cdots m_0 m_0 d_{n+1}\cdots \qquad (4.4.8.0.8)$$

and up.

This presents the idea of the algorithm.

4.4.9 Three "modes" of the algorithm

The algorithm has three distinct fragments that coincide with the description of the idea above:

- fragment in the lines 3-35. that will be called the "Greater" part, that corresponds to 4.4.6
- fragment in the lines 27-55, that will be called the "Searching" part, that corresponds to 4.4.4 and 4.4.8
- fragment in the lines 57-86, that will be called the "Less" part, that corresponds to 4.4.7.

The control flow of the parts can be seen on the diagram:

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The execution of the algorithm then goes as follows:

We start at "Greater" and proceed to do the procedure from 4.4.6. Once the procedure stops, we do procedure from 4.4.8, then dependent whether the result have corresponding Euler orbicharacteristic greater or smaller than $\frac{p}{q}$, we perform, respectively – again procedure from 4.4.6 or the procedure from **??**. We repeat **??** as long as necessarily. Once it gives the state that have Euler orbicharacteristic greater than $\frac{p}{q}$ we set the flag to "Greater" again and repeat the whole process starting from the procedure in 4.4.6. In the case that repeating the procedure from 4.4.7 won't give any state with corresponding Euler orbicharacteristic greater than $\frac{p}{q}$ the algorithm will hit its stopping condition and answer "no" as written in the algorithm 4.3.4 itself.

4.5 Proof of the correctness of the algorithm

4.5.1 Lemmas

Firstly, we will prove that our invariants indeed are conserved during the execution of the algorithm. We will also proof some other lemmas regarding the state of memory during the algorithm.

Lemma 4.5.1.1. *During any time of the execution of the algorithm, there are only finitely many counters that have non-*1 *value.*

Proof.

We start with the state that have only finitely many non-1 value. Let us observe, that all three of the places – lines: 15, 33-34, 64-65, where the counters are changed, change them in the way that preserves this state. As during any time of execution, there were only finitely many changes, we have the thesis. □

Lemma 4.5.1.2. *When control is at the line 15. and the pivot is at the counter c_p, all counters to the left of c_n are set to ∞.*

Proof.

We can get to the line 15th only in two ways: from line 18 in "Greater" section or from line 53 in "Searching" section. In this process "Searching" moves pivot to the first counter and "Greater" moves pivot one counter to the right. As long as the pivot is not on the 1st counter, control flow must have came then to the line 15th from "Greater" section and if pivot is at the 1st counter it must have came from the "Searching" section. From this we have, that for the pivot, to get to the counter c_p, it would need to go through all the counters to the left of c_p while being on the line 15th and setting them to ∞.

Lemma 4.5.1.3. *When control is at the line 33rd and the pivot is then at the counter c_p, all counters to the left of c_p are set to ∞ and the counter c_p is not set to ∞.*

Proof.

The only way to get to the line 33rd is from line 23rd in "Greater" section. From 4.5.1.2 we know, that then all the counters to the left of c_p are set to ∞. Counter c_p on the other hand can not be set to infinity, since, from the fact that control flow was at the block from line 21st, we know that

$$\chi^{orb}(*d_1d_2\cdots d_{p-1}\infty d_{p+1}\cdots) < \frac{p}{q} \tag{4.5.1.3.1}$$

and from the fact, that control flow was in the "Greater" block we know, that

$$\chi^{orb}(*d_1d_2d_3\cdots) > \frac{p}{q}.\square \tag{4.5.1.3.2}$$

Lemma 4.5.1.4. *For any state of counters during the execution of the algorithm $D = d_1d_2d_3\cdots$ we have that $d_1 \geqslant d_2 \geqslant d_3 \geqslant \cdots$.*

Proof.

We start with the state where $d_1 \geqslant d_2 \geqslant d_3 \geqslant \cdots$. Let us observe, that, by 4.5.1.2, changing at line 15 preserves this state. Changing at lines 33-34 or 64-65, preserve this state as they increase the value of the counter at which pivot is by one (to some $d_p + 1$) and change all counters to the left of the pivot to $d_p + 1$. □

Lemma 4.5.1.5. *When control is at the line 66th and the pivot is then at the counter c_p, all counters to the left of c_p, are set to the same value d_{p-1} and the counter c_p is not set to the value d_{p-1}.*

Proof.

The only way for the control flow to get to the line 66th is from line 47th or 84th. In both of these cases, the counters to the left of c_p were set to the same value on, respectively lines 33-34 or 64-65. Also, on lines 33-34 or 64-65, the value of the counter c_{p-1} was increased by 1. From this and from 4.5.1.4, we know, that $d_{p-1} > d_p$.□

Lemma 4.5.1.6. *All counters strictly to the left to the pivot have the same value at any stage of the execution of the algorithm.*

Proof.

As state of the counters changes only on lines 15, 33-34 or 64-65, and the pivot is moving at most by one position to the right between the changes to the state of the counters, this is the corollary from 4.5.1.2, 4.5.1.3 and 4.5.1.5. □

Lemma 4.5.1.7. *Searching procedure from lines 29-32 always terminates.*

Proof.

Let c_p be the counter at which pivot is, when the searching procedure from lines 29-32 stars. Control flow can get to the lines 29-32 only from line 23rd. This guarantees, that when starting the searching procedure, we have that:

$$\chi^{orb}(*d_1 d_2 \cdots d_{p-1} \infty d_{p+1} \cdots) < \frac{p}{q} \tag{4.5.1.7.1}$$

and

$$\chi^{orb}(*d_1 d_2 d_3 \cdots) > \frac{p}{q}. \tag{4.5.1.7.2}$$

From this and from 4.4.1.15, we know, that there exists some $d'_p < \infty$ such that

$$\chi^{orb}(*d_1 d_2 \cdots d_{p-1} d'_p d_{p+1} \cdots) < \frac{p}{q} \tag{4.5.1.7.3}$$

As such, the searching procedure stops. □

Lemma 4.5.1.8. *There are always only finitely many steps in execution of the algorithm before it changes the state of the counters.*

Proof.

The only steps not explicitly listed in the algorithm are from the searching procedure from lines 29-32. From 4.5.1.7 we know, that this procedure always terminates. All other control flow can be check explicitly to have always only finitely many steps between the change of the counters. The change of the counters itself is also a finite procedure, as we are always only changing the counters at pivot or at and to the left of the pivot, and there are only finitely many such counters. □

Lemma 4.5.1.9. *For any state D_1 that is a state of counters at some point of the execution of the algorithm and D_2 such that D_1 is changed to D_2 during the execution, we have that $D_1 \prec D_2$.*

Proof.

Let us observe that in each instance of changing the counters – in lines 15, 33-34 and 64-65. the rightmost counter that is changed is always increased. From this, the lemma follows. □

4.5.2 Proof.

Now, we will perform the proof, that the idea of the algorithm presented above in 4.4, as well as the algorithm itself 4.3.4, works as intended.

Firstly, let us observe, that algorithm gives the answer on lines 8, 14-15, 35-36, 55-56, 62-63 and always ends immediately after giving the answer. Thus, it will always give at most one answer. Furthermore let us observe that these are the only places where the algorithm terminates, so if it terminates it will give at least one answer.

There are three things to be checked:

- That the algorithm never answers "yes" if there is no orbifold of the Euler orbicharacteristic $\frac{p}{q}$ (No false positives)
- That the algorithm never answers "no" if there is an orbifold of Euler orbicharacteristic $\frac{p}{q}$ (No false negatives)
- That the algorithm always ends in a finite number of steps (Guaranteed termination).

4.5.3 No false positives

Algorithm gives answer "yes" at lines 7-8, 37-38, 68-69. At each of these places, the answer contains the example of an orbifold with Euler orbicharacteristic equal to $\frac{p}{q}$ that was explicitly checked for correctness just before giving the answer (see lines 5, 35, 66).

4.5.4 No false negatives

Let $D = d_1 d_2 d_3 \cdots$ be such that $\chi^{orb}(*d_1 d_2 d_3 \cdots) = \frac{p}{q}$.

First, we will show that the algorithm will never go beyond $d_1d_2d_3\cdots$ counter state in $\preceq$ order.

Let us observe that the only lines where the counters are changed are lines 15, 40-41 and 71-72,

Right right before the change from 15 and right after each change from 40-41, 71-72 (lines, respectively 5, 35, 66), the new state is checked if it is a solution and if it is a solution, the algorithm stops. From this, we have that going beyond $d_1d_2d_3\cdots$ can not happen from $d_1d_2d_3\cdots$, it must happen from some state $D' \preceq d_1d_2d_3\cdots$.

Furthermore while changing, only counters at pivot and to the left of the pivot are changed.

Because of that:

Observation 4.5.4.1. *Going beyond $d_1d_2d_3\cdots$ counter state could happen only in lines 15, 33-34 or 64-65, while pivot would be at the rightmost counter that is different from $d_1d_2d_3\cdots$ or further to the right. That is, if $d'_1d'_2d'_3\cdots$ is the current state, and c_k is the rightmost counter on which $d_1d_2d_3\cdots$ and $d'_1d'_2d'_3\cdots$ differ and c_n is the counter at which the pivot is on then at the moment of change that goes beyond $d_1d_2d_3\cdots$, it must hold that $k \leqslant n$.*

We will show that if the counters are below D in $\preceq$ before the change they still be below D or at D after the change.

We will now eliminate all three options arising from lines 15, 33-34 and 64-65 case by case. Let $D' = d'_1d'_2d'_3\cdots$ be current state of counters before the change during the execution of the algorithm. Let c_k be the rightmost counter on which $d_1d_2d_3\cdots$ and $d'_1d'_2d'_3\cdots$ differ. Let c_n be the counter at which the pivot is on.

Line 15

We will show that under taken assumptions, we will not get to this line. First, let us prove, that the pivot must be exactly at the counter c_k. By 4.5.1.2 we know, that if the pivot is on the counter c_n, while execution is at line 15, all the counters to the left of c_n are set to ∞.

This however means, that

$$\underbrace{\infty\infty\cdots\infty}_{n-1 \text{ times}} d_n dn + 1 d_{n+2}\cdots \preceq D. \tag{4.5.4.1.1}$$

Together with the fact, that for k, which is $\leqslant n$, we have that for any $l > k$, we have that $d_l = d'_l$ this means that if $n \geqslant k+1$, we have that $D = D'$. This is a contradiction with the assumption of k. From this we have that $n = k$.

We have then that pivot is on the rightmost counter c_k, that differs between D and D', and that all the counters strictly to the left of c_k in D' are set to ∞.

From this, since $\chi^{orb}(*d_1d_2d_3\cdots) = \frac{p}{q}$, we have, that:

$$\chi^{orb}(*d'_1d'_2d'_3\cdots d'_{k-1}\infty d'_{k+1}\cdots) \leqslant \frac{p}{q} \tag{4.5.4.1.2}$$

From this, we have the contradiction with the assumption that we will get to the line 15.

Lines 33-34

Similarly as in the previous case, this time using 4.5.1.3 we can prove, that $n = k$, as well as that $d'_k < d_k$, that all counters strictly to the left of c_k in D' also must be ∞.

This leads us to the conclusion that searching procedure from lines 29-32 will find some $d''_k \leqslant d_k$ as a result. Then, the resulting state still will be $\preceq D$.

Lines 64-65

We will show the contradiction by showing that pivot must be on the counter strictly to the left of c_k. By 4.5.1.5 we know, that all the counters to the left of c_n are set to d'_{n-1}. From this, from the fact that $d'_k < d_k$ and from the fact that $\chi^{orb}(*d_1d_2d_3\cdots) = \frac{p}{q}$, we have that, unless $n < k$, we have that

$$\chi^{orb}(*d'_1d'_2d'_3\cdots) \geqslant \frac{p}{q}. \tag{4.5.4.1.3}$$

However, this is a contradiction with being at lines 64-65, as the only way to get there with the execution, is either from lines 45-47 or the lines 82-84, reaching either require for the state of counters to have corresponding Euler orbicharacteristic $< \frac{p}{q}$.

4.5.5 Guaranteed termination

First let us observe, that Let

$${}_\infty D = \underbrace{\infty\ \infty\ \infty\cdots\infty}_{\lfloor 2(1-\frac{p}{q})\rfloor \text{ times}}\ 1\ 1\ 1\cdots \tag{4.5.5.0.1}$$

and let

$${}_2 D = \underbrace{2\ 2\ 2\cdots 2}_{\lceil 4(1-\frac{p}{q})\rceil \text{ times}}\ 1\ 1\ 1\cdots\ . \tag{4.5.5.0.2}$$

These are the state consisting of only, respectively, ∞ and 2 as a non-1 counters values, that have the property, that they have the most non-1 counters from all states of such form, having the corresponding Euler orbicharacteristic $\geqslant \frac{p}{q}$.

Let us assume that for some input M and $\frac{p}{q}$ the algorithm does not answer "yes". We will show, that then it will answer "no" in finite number of steps. By 4.5.1.8 we can show this, by showing, that it will answer "no" after finitely many of counters state changes.

Going beyond every state smaller or equal than ${}_2D$

We will show it by firstly showing, that for any state $D = d_1d_2d_3\cdots$ smaller or equal to D_2 the algorithm will go to it or beyond it.

Proof will be inductive, with respect to order $\preceq$. Our inductive assumption will be, that for a given state D, that is at most D_2, there is some state D', such that $D \preceq D''$, and that D' was the state of the counters after a finite number of steps of execution of the algorithm.

• For the fist state we have that this is true since it is the D' for itself.

• Let us suppose that for a state $D = d_1 d_2 d_3 \cdots \preceq D_2$, for all the states $\prec D$ our inductive assumption holds. We will show that it holds for D.

We will have two cases.

- D is a successor of some D_1. There are two options – either D_1 was a state of counters at some point of the execution, or it wasn't.

  If it wasn't, from the inductive assumption we know that some state equal at least D was the state of counters. From this we have the induction thesis for D.

  In the case D_1 was a state of counters, from 4.5.1.9 we have, that the next state of counters will be equal at least D.

- D is not a successor of any $D' \prec D$. From this, we have that D has at least one ∞ on its counters. Let c_l be the counter that has rightmost infinity in D. From **??** we have, that all the counters to the left of c_l also are set to ∞.

  Let us consider the state ${}^0D ={}^0 d_1^0 d_2^0 d_3' \cdots$, such that for any $i > l$, we have that ${}^0d_i = d_i$. From the induction assumption, there exists state $D' = d_1' d_2' d_3' \cdots$ such that ${}^0D \preceq D'$ and such that D' was a state of the counters after finitely many steps of the algorithm. If $D' \succeq D$, then we have induction thesis for D. Let us consider the case, where $D' \prec D$. Then, since ${}^0D \preceq D' \prec D$, we have that for every $i > l$, we have $d_i' = d_i$. From this, we have, that D' differs from D only on the counters at, or to the left of c_l. We know, that all the counters to the left of, or at c_l are set to ∞ in D and that there is at least one counter set to ∞. Moreover, we know, that D' differ from D on at least one counter.

  There are two cases:

  – $\chi^{orb}(D) \geqslant \frac{p}{q}$ From this, we conclude that $\chi^{orb}(D') > \frac{p}{q}$.
    Regardless of at which line state D' was produced, the execution will then follow, through line or to the line . From this point, since $\chi^{orb}(D) \geqslant \frac{p}{q}$, we will have consecutive sequence length at most l, of states of counters, where values of consecutive counters from D' will be replaced by ∞, up to the counter c_l at which state D will be reached. This sequence will be generated by repeatedly going through line 15 in the algorithm, since at every step, the corresponding Euler orbicharacteristicof the state of counters will be $> \frac{p}{q}$.

  – $\chi^{orb}(D) < \frac{p}{q}$ From 4.4.1.15, we know, that there must exists state

$$ {}^0D ={}^0 d_1^0 d_2^0 d_3' \cdots , \tag{4.5.5.0.3} $$

    such that all ${}^0d_i$ are less than ∞ and are the same for any $i \leqslant l$, and that $\chi^{orb}({}^0D) < \frac{p}{q}$.
    From this, we conclude, that $\chi^{orb}(D') < \frac{p}{q}$. Regardless of at which line state D' was produced, the execution will then follow, through line or to the line . From this point, since $\chi^{orb}(D) \geqslant \frac{p}{q}$, we will have consecutive

sequence length at most l, of states of counters, where values of consecutive counters from D' will be increased by one and all the counters to the left of them will be set to the value of the counter which was just increased. This sequence will be generated by repeatedly going through line 66 in the algorithm, since at every step, the corresponding Euler orbicharacteristicof the state of counters will be $< \frac{p}{q}$.
This sequence will end by going with the pivot to the counter c_{l+1}, which we know from the assumption that has value smaller than ∞, increasing it by one and setting all the counters to the left of if to value $d_{l+1}+1$, this way obtaining the state larger in $\preceq$ than D.

Reaching ${}_2D$ at finite number of steps

Now we will show that at some point, algorithm will have as a state of counters ${}_2D$.

Lemma 4.5.5.1. *Every counter with number greater than* $\lfloor 2(1-\frac{p}{q})\rfloor$ times *has pivot on it for the first time while the execution is in line*

Proof.

From this, we have, that every time there is a state of counters that for the first time involves counter with number greater than $\lfloor 2(1-\frac{p}{q})\rfloor$ times, it is the state of the form $2\ 2\ 2 \cdots\ 2\ 1\ 1\ 1 \cdots$. Since from 4.5.5 we know that for every state $D \preceq_2 D$, we will have some bigger state D', that at some point was a state of counters, we have that at some point counter $c_{\lceil 4(1-\frac{p}{q})\rceil}$ will obtain non-zero value (if the predecessor of ${}_2D$ will be the state of counters at some point, then $c_{\lceil 4(1-\frac{p}{q})\rceil}$ will have non-1 value at the next change of counters.). From 4.5.5.1, we know, that then there must have been state ${}_2D$ at some point as a state of counters. □

4.6 Another questions the algorithm can answer

4.6.1 Deciding the order of accumulation

Using above algorithm, for a point $\frac{p}{q}$ and a manifold M, we can check what order of accumulation point $\frac{p}{q}$ is in $\sigma(M)$.

Analogously to 4.1, we will wlog answer the question fr $M = D^2$.

First, we check with the algorithm, whether $\frac{p}{q} \in \sigma(D^2)$. If not, it is not an accumulation point of $\sigma(D^2)$, since 3.1.2.6.

Let us now consider the case, where $\frac{p}{q} \in \sigma(D^2)$.

From 3.1.2.9, we know, that for a point $\frac{p}{q}$ being an accumulation point of the set $\sigma(D^2)$, of order at least n is equivalent to the fact that $\frac{p}{q}+\frac{n}{2} \in \sigma(D^2)$. From 1.9.0.4 on the other hand, we know, that for any $x > 1$, we have $x \notin \sigma(D^2)$.

As such, for a given $\frac{p}{q}$, we can, using our algorithm, check one by one every number of the form $\frac{p}{q}+\frac{n}{2}$, such that $n \in \mathbb{N}_0$ and $\frac{p}{q}+\frac{n}{2} < 1$. There are at most $\lfloor 2\left(1-\frac{p}{q}\right)\rfloor$ such numbers. Let n_0, be the biggest of the checked numbers, for which

$\frac{p}{q} + \frac{n}{2} \in \sigma(D^2)$. Then, from 3.1.2.9 we know, that $\frac{p}{q}$ is an accumulation point of order n_0 in $\sigma(D^2)$.

4.7 Implementation

This algorithm is a part of the algorithm from chapter 6 where the implementation of the whole will be discussed in 6.3.

Chapter 5

Counting orbifolds

The central question of this section is: "given a rational number, how many orbifolds have that Euler orbicharacteristic?".

In 5.1, we will show that for any number, there are only finitely many orbifolds with that Euler orbicharacteristic. In 5.3,

5.1 Finiteness

In this section, we will prove, that for any $x \in \sigma$ there are only finitely many orbifolds with the Euler orbicharacteristic equal to x. We will give two proofs of this fact.

Lemma 5.1.0.1. *The function $\rho : \mathbb{N}^m \to \mathbb{Q}$, defined as:*

$$\rho(n_1, \cdots, n_m) \coloneqq \sum_{i=0}^{m} \frac{1}{n_i}, \tag{5.1.0.1.1}$$

is finite-to-1.

Proof.

Observation 5.1.0.2. *For any $x \in \sigma$ and $n \in \mathbb{N}$ there are only finitely many orbifolds with the Euler orbicharacteristic greater or equal to x and all orbipoints of order at most n.*

Proof.

For a given x, there are only finitely many manifolds with an Euler characteristic $y \geqslant x$. Only them can be base manifolds for an orbifold with an Euler orbicharacteristic $y' \geqslant x$, as adding orbipoints always decreases an Euler orbicharacteristic.

It remains to prove then, that for any base manifold M, there are only finitely many orbifolds, with M as a base manifold, that have an Euler orbicharacteristic $y \geqslant x$, and all orbipoints of order at most n.

We proceed now similarly to the proof of 3.1.2.2 – on the orbifold with an Euler orbicharacteristic $y \in [x, 2]$, there can be at most $\max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}$ orbipoints. Thus, for a given manifold M and a given x and n, there can be at most $(n-1)^{\max\{\lfloor 4(1-x)\rfloor, \lfloor 2(2-x)\rfloor\}}$ orbifolds with an Euler orbicharacteristic $y \geqslant x$, all orbipoints of order at most n and M as a base manifold.$\square$

Observation 5.1.0.3. *To prove that for any $x \in \sigma$ there are only finitely many orbifolds with the Euler orbicharacteristic equal to x, it is sufficient, to prove that for any $x \in \sigma$, for any two dimensional manifold M, there are only finitely many M-orbifolds with only one corresponding type of orbipoints (dihedral in the case M has a boundary or rotational in the case M dos not have a boundary), with the Euler orbicharacteristic equal to x.*

Proof.

Let x be a rational number. Let $\mathcal{O}$ be the set of all orbifolds with an Euler orbicharacteristic equal to x. Those orbifolds can have different base manifolds. However, the set of base manifolds of orbifolds from $\mathcal{O}$ is finite, as there are only finitely many two dimensional manifolds with an Euler characteristic greater or equal to x and an orbifold always has an Euler orbicharacteristic less or equal to the Euler characteristic of its underlying manifold.

From this, it is sufficient to prove, that for any base manifold M, the number of M orbifolds with Euler orbicharacteristic equal to x is finite.

Let M be a two dimensional manifold. If M has no boundary it can't have dihedral orbipoints and we have the the thesis for manifolds without boundary.

If M has a boundary, M-orbifolds can have both rotational and dihedral orbipoints. Let us observe, that every rotational orbipoint can be replaced by two dihedral orbipoints of the same order without changing the Euler orbicharacteristic. Thus, if there would be infinitely many M-orbifolds having both rotational and dihedral orbipoints, there would be also infinitely many M-orbifolds having only dihedral orbipoints. Thus in this case it is sufficient to prove that there are finitely many M-orbifolds that have only dihedral orbipoints. $\square$

Theorem 5.1.0.4. *For any $x \in \sigma$ there are only finitely many orbifolds with the Euler orbicharacteristic equal to x.*

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Proof I.

Let us take $x \in \sigma$. By using observation 5.1.0.3 it is only needed to prove, that for any M, there are only finitely many M-orbifolds with only one corresponding type of orbipoints (rotational in the case M has no boundary, dihedral in the case M has a boundary), with Euler orbicharacteristic equal to x.

For the sake of contradiction, assume, that the set $\mathcal{O}_M$ of M-orbifolds that have only one type of orbipoints and have Euler orbicharacteristic equal to x is infinite .

Let $\mathcal{O}_M = \{O_i\}_{i \in I}$. For each i, let $s_i = (o_i^0, \cdots, o_i^{l_i})$ be the list of degrees of the orbipoints of O_i ordered in a decreasing manner. So for each i we have, that o_i^0 is the order of the orbipoint with the highest order of all the dihedral orbipoints of O_i. By 5.1.0.2 we know that if the set $\{o_i^0\}_{i\in I}$ would be bounded by some $n \in \mathbb{N}$, by 1.5.2.1 it would mean, that $\mathcal{O}_M^d$ would be finite. As from our assumption for the contradiction, we have that $\mathcal{O}_M$ is not finite, we know that the set $\{o_i^0\}_{i\in I}$ is unbounded. Let $\{i_n\}_{n\in\mathbb{N}} \subseteq I$ be a sequence of indices such that $\{o_{i_n}^0\}_{n\in\mathbb{N}}$ is strictly increasing.

Let $\{a_n\}$ be the sequence such that $a_n = \Delta(o_{i_n}^0)$. Let $\{b_n\}$ be the sequence such that $b_n = \Delta(o_{i_n}^1, \cdots, o_{i_n}^{l_{i_n}})$. So for every n we know that $\chi^{orb}(O_{i_n}) = \chi(M) + a_n + b_n$. As $\{o_{i_n}^0\}$ is strictly increasing, we know that a_n is strictly decreasing, so b_n must be strictly increasing (we have that $\chi^{orb}(O_{i_n})$ is constant for all n, since all O_{i_n} are from the family with Euler orbicharacteristic equal to x).

But $\{b_n\} \subseteq \sigma(M) - \chi(M)$. From 3.1.3.6 and 1.9.0.3 we know that $\sigma(M)$ has no infinite strictly increasing sequences, so $\sigma(M) - \chi(M)$ has no infinite strongly increasing sequences. That gives us a contradiction. □

Proof II.

Let us take $x \in \sigma$. By using observation 5.1.0.3 it is only needed to prove, that for any M, there are only finitely many M-orbifolds with only one corresponfding type of orbipoints (rotational in the case M has no boundary, dihedral in the case M has a boundary), with Euler orbicharacteristic equal to x.

Similarly as in 3.1.2.2 and 5.1.0.2, we can deduce, that since:

- $\Delta({}^*2) = -\frac{1}{4}$,
- $\Delta(2) = -\frac{1}{2}$,
- every M with boundary has $\chi(M) \leqslant 1$,
- every M without boundary has $\chi(M) \leqslant 2$,

we have, that there can be at most $n := \max\{\lfloor 4(1-x) \rfloor, \lfloor 2(2-x) \rfloor\}$ orbipoints at any orbifold with an Euler orbicharacteristic equal to x.

Let us take $k \leqslant n$, we will show that there is only finitely many M-orbifolds O, with exactly k orbipoints of the corresponding type having Euler orbicharacteristic equal to x.

Let O be an M-orbifold with exactly k orbipoints of the corresponding type. Let $s = (o^0, \cdots, o^k)$ be the list of degrees of the orbipoints of O ordered in a decreasing manner. From 1.5.2.1 we know, that only finitely many orbifolds can have the same such list. With list s, we can associate a sum:

$$S := \sum_{i=0}^{k} \frac{1}{o^i}. \tag{5.1.0.4.1}$$

We have that $\chi^{orb}(O) = \chi(M) - \alpha k + \alpha S$, where α is equal to $\frac{1}{2}$ or 1, when, respectively M has a boundary or not.

As such $\chi^{orb}(O) = x$ iff $S = \frac{1}{\alpha}x - \frac{1}{\alpha}\chi(M) + k$.

From 1.10.1.3 we know, that there are only finitely many sums of the form of S, equal to any given number. □

5.2 Infinitness

5.2.1 Local unboundness

We know, that for any x, there are only finitely many orbifolds with x as an Euler orbicharacteristic. However, we can ask about some boundness of number of these orbipoints. In particular, we could ask, whether near any accumulation point, there will be x with an arbitrary large number of orbifolds corresponding to it. The answer will be yes, and it can be formulated as such:

Theorem 5.2.1.1. *For any neighbourhood U of any accumulation point x of $\sigma(D^2)$ of order at least 2, for any $n \in \mathbb{N}$, there exists an $y \in U$ such that there are at least n orbifolds with y as their Euler orbicharacteristic.*

Proof.

This will follow from the theorem about the sums of Egyptian fractions from [BE11] (Theorem 1. page 1). It states that:

Theorem 5.2.1.2. *For a counting function*

$$f_k(p, q) \coloneqq \# \left\{ (n_1, \cdots, n_k) \in \mathbb{N}^k_{>0} \;\middle|\; n_1 \leqslant \cdots \leqslant n_k \wedge \frac{p}{q} = \frac{1}{n_1} + \cdots + \frac{1}{n_k} \right\}, \tag{5.2.1.2.1}$$

we have that for any fixed $p \in \mathbb{N}_{>0}$, there are infinitely many values of $q \in \mathbb{N}_{>0}$ for which

$$f_2(p, q) > \exp\left((\log 3 + o(1)) \frac{\log(q)}{\log(\log(q))} \right). \tag{5.2.1.2.2}$$

From 3.1.2.5 we know, that for point x to be an accumulation points of order at least 2 of the set $\sigma(D^2)$ means, that $x + 1 \in \sigma(D^2)$. This also means, that all points of the form

$$x + 1 - \frac{d_1 - 1}{2d_1} - \frac{d_2 - 1}{2d_2} = x + \frac{1}{2d_1} + \frac{1}{2d_2} \tag{5.2.1.2.3}$$

are in $\sigma(D^2)$.

Let x be an accumulation point of the set $\sigma(D^2)$ of order at least 2, let U be some neighborhood of x and let $n \in \mathbb{N}$. Let us take $p = 1$ and q, such that:

1. 5.2.1.2.3 holds,
2. $\exp\left((\log 3 + o(1)) \frac{\log(q)}{\log(\log(q))} \right) > n$,
3. $x + \frac{1}{q} \in U$.

We can always find such one, since:

- from [BE11] we know, that there exists infinitely many q, for $p = 1$, such that 5.2.1.2.3 holds,
- $\lim_{q \to \infty} \exp\left((\log 3 + o(1)) \frac{\log(q)}{\log(\log(q))} \right) = \infty$,

- conditions 2. and 3. have the property, that if some q_1 satisfies them, then any $q_2 > q_1$ also satisfies them.

From this, we have, that $x + \frac{1}{q} \in U$ and there exists at least n different pairs of numbers $(n_1^1, n_2^1) \cdots (n_1^n, n_2^2)$, such that for any $1 \leqslant i \leqslant n$, we have $\frac{1}{n_1^i} + \frac{1}{n_2^i} = \frac{1}{q}$.

This means also that $x + \frac{1}{2q} \in U$ and that for any $1 \leqslant i \leqslant n$, we have $\frac{1}{2n_1^i} + \frac{1}{2n_2^i} = \frac{1}{2q}$.

As for any $1 \leqslant i \leqslant n$, we have that $x + \frac{1}{2n_1^i} + \frac{1}{2n_2^i}$ is of the form 5.2.1.2.3, we have, that for any $1 \leqslant i \leqslant n$, we have $x + \frac{1}{2n_1^i} + \frac{1}{2n_2^i} \in \sigma(D^2)$. Let O, be a D^2 orbifold with Euler orbicharacteristic equal to $x + 1$. All of n different orbifolds, created by adding to O two dihedral orbipoints, respectively to i, of orders, n_1^i and n_2^i have an Euler orbicharacteristicequal to

$$x + 1 - \frac{n_1^i - 1}{2n_1^i} - \frac{n_2^i - 1}{2n_2^i} = x + \frac{1}{2n_1^i} + \frac{1}{2n_2^i} = x + \frac{1}{2q}. \tag{5.2.1.2.4}$$

As such, we found $y = x + \frac{1}{2q}$, such that $y \in U$ and found n different orbifolds, with an Euler orbicharacteristic equal y. □

5.3 Dividing the problem into an arithmetical and combinatorical parts

Here will divide the question "Given the number x, how many orbifolds have x as an Euler orbicharacteristic?" into two parts. The answers to these partial questions will be given in 6 and 7.

5.4 Arithmetical part

The first part is to answer the following question:

"How many sums of the form:

$$1 - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \tag{5.4.0.0.1}$$

with $m \in \mathbb{N}$ and $\forall_j \; d_j \in \mathbb{N} \cup \{\infty\}$, are equal to x?"

It is a matter of convention (and then coherently translating this convention to the final result) what sums are we treating as "the same". The convention we will take, is that a sum is determined uniquely by the tuple $(d_1, \ldots, d_n)$ of orders of orbipoints, ordered in decreasing order, appearing in the sum.

This part describes how adding rotational orbipoints to a sphere and dihedral points to the disk changes their Euler orbicharacteristic. In the second part we will use the answer from this part.

5.5 Combinatorical part

We will take following steps:

1. First we divide the question "Given number x, how many orbifolds have x as their Euler orbicharacteristic?" into the series of questions for each two-dimensional manifolds M: "Given number x, and the manifold M, how many M-orbifolds have x as their Euler orbicharacteristic?". At the end we will sum up the answers from all these questions.

   Note, that for M such that $\chi(M) < x$, the answer is always 0, since orbifolds have smaller Euler orbicharacteristic than their base manifolds (1.6.2.2).

2. Then for each manifold M, we answer one of the following questions:

   • if M has a boundary (4.1.0.0.3):

   "How many sums of the form
$$1 - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \qquad (5.5.0.0.1)$$
   are equal to
$$\frac{p}{q} - \chi(M) - 1 \text{ ?''}, \qquad (5.5.0.0.2)$$

   • if M has no boundary (4.1.0.0.4):

   "How many sums of the form
$$1 - \sum_{j=1}^{m} \frac{d_j - 1}{2d_j} \qquad (5.5.0.0.3)$$
   are equal to
$$\frac{1}{2}\frac{p}{q} - \frac{1}{2}\chi(M) - 1 \text{ ?''}. \qquad (5.5.0.0.4)$$

   Here we consider the sums to be "the same" in the same way as in 5.4. This we can do, since these questions are equivalent to asking respective questions from arithmetical part 5.4 for $x + \chi(M)$. In the case where M has no boundary this gives us our result, since 1.5.2.1.

3. Finally we take into account two remaining things concerning the case where M has a boundary:

   3.1. For now we only considered sums corresponding to orbifolds with either rotational or dihedral orbipoints. When M has a boundary, M-orbifold can have both of these types of points. Fortunately, to take this into account, be don't have to answer the arithmetical question concerning the sums simultaneously corresponding to both types of orbipoints. It is possible to reduce (in the sense that will be described in 7) all sums that contain two types to orbipoints to sums with only dihedral orbipoints. In doing so, we will ascribe "weights" to the sums of how many other sums got reduced to it.

3.2. When the orbipoints lie on the boundary components, their order of placement around the boundary component matters as orbifolds with orbipoints on boundary components with different order are not necessary the same (see 1.5.2.1). We will take this fact into account, by affecting the aforementioned "weights" with which we will sum the number of sums. The resulting weights will be the amount of orbifolds corresponding (possibly also via reduction from 3.1) to the given sum.

This are the two phenomena causing that in the case where M has a boundary, multiple orbifolds correspond to the same sum. We will calculate the total number of orbifolds by calculating the number of the sums corresponding to dihedral points, but taking the sums with the proper "weight" – of how many orbifolds correspond to this sum.

Chapter 6

Counting orbifolds – arithmetical part

6.1 The idea of the algorithm

This is an extension of the algorithm from chapter 4. It only differs by lines after finding the solution – 7-11, 37-41 and 68-72. They all send the control flow to line 57th. Instead terminating the algorithm the solution is appended to the initially empty list and the algorithm proceeds to search through the states as if the the state that the solution was changed to at lines 64-65 during continuation of the execution was the starting configuration, together with the pointer placement and flag value.

```
1  In the case, the flag_value is equal to:
2  {
3      "Greater", then
4      {
5          If χ^orb(*d_1 ... d_{p-1}∞d_{p+1} ...) = p/q then
6          {
7              We found an orbifold, we add it to a list
8              and increase the occurrence counter by 1.
9              We set the flag to "Less".
10             We put pivot to the c_{p+1} counter.
11             We go to the 1st line.
12         }
13         If χ^orb(*d_1 ... d_{p-1}∞d_{p+1} ...) > p/q then
14         {
15             We set d_p to ∞.
16             We set the flag to "Greater".
17             We put the pivot at the c_{p+1}.
18             We go to the 1st line.
19         }
20         If χ^orb(*d_1 ... d_{p-1}∞d_{p+1} ...) < p/q then
21         {
22             We set the flag to "Searching".
23             We go to the 1st line.
```

```
        }
    }

    "Searching", then
    {
        We search one by one
        for the value $d'_p$ of the $c_p$ such that
        $\chi^{orb}(*d_1 \dots d_{p-1} d'_p d_{p+1} \dots) \leqslant \frac{p}{q}$ and
        $\chi^{orb}(*d_1 \dots d_{p-1} (d'_p - 1) d_{p+1} \dots) > \frac{p}{q}$.
        We set $c_p$ and all of the counters
        to the left of $c_p$ to the value $d'_p$.
        if $\chi^{orb}(*d_1 d_2 d_3 \dots) = \frac{p}{q}$ then
        {
            We found an orbifold, we add it to a list
            and increase the occurrence counter by 1.
            We set the flag to "Less".
            We put the pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1 d_2 d_3 \dots) < \frac{p}{q}$ then
        {
            We set the flag to "Less".
            We put the pivot at the $c_{p+1}$.
            We go to the 1st line.
        }
        If $\chi^{orb}(*d_1 d_2 d_3 \dots) > \frac{p}{q}$ then
        {
            We set the flag to "Greater".
            We put the pivot at the $c_1$.
            We go to the 1st line.
        }
    }

    "Less", then
    {
        If $d_p = 1$ and the values of all the counters
        on the left of $c_p$ are equal to 2 then
        {
            We end the whole algorithm with the answer "no".
        }
        We increase $c_p$ by one ($d_p \coloneqq d_p + 1$) and
        we set the value of all counters on the left of $c_p$ to $d_p$.
        If $\chi^{orb}(*d_1 d_2 d_3 \dots) = \frac{p}{q}$ then
        {
```

```
68                  We found an orbifold, we add it to a list
69                  and increase the occurrence counter by 1.
70                  We set the flag to "Less".
71                  We put pivot at the c_{p+1}.
72                  We go to the line 1..
73              }
74              If χ^{orb}(*d_1d_2d_3...) > p/q then
75              {
76                  We set the flag to "Greater".
77                  We put the pivot at the c_1.
78                  We go to the 1st line.
79              }
80              If χ^{orb}(*d_1d_2d_3...) < p/q then
81              {
82                  We set the flag to "Less".
83                  We put pivot at the c_{p+1}.
84                  We go to the 1st line.
85              }
86          }
87  }
```

6.2 Proof of the correctness of the algorithm

Let us observe, that whole proof from the chapter 4 was independent from the choice of the starting configuration – state of counters, flag value and pivot placement, as long as they would hold the invariants that were proved in 4.5.1 and were used in 4.5.2 and the fact, that flag value will correspond to the relation between Euler orbicharacteristic corresponding to the current state and $\frac{p}{q}$. We know, that the found solution was satisfying all the lemmas – as it was the state of the counters at some point of the execution. The only thing left to see, is that the flag value will be appropriate.

Let $D = d_1d_2d_3\cdots$ be the solution. Let c_p be the counter at which the pointer was when the solution was found. Then, since 4.5.1.6 and the fact that after each change the value of the counter that pivot is at is the same as value of the counters to the left of it and 4.5.1.4, we conclude that all states that have value of c_p greater than d_p can not be solutions. As such, we can proceed from the state

$$D' = (d_{p+1}+1)(d_{p+1}+1)(d_{p+1}+1)\cdots(d_{p+1}+1)(d_{p+1}+1)d_{p+2}d_{p+3}\cdots. \quad (6.2.0.0.1)$$

Setting flag to "Less" after finding the solution, will result in producing exactly this state. Then, flag will be set accordingly to the comparison on line 76 or 82 or another solution will be found on line 68. From 5.1.0.4 we know, that there will be only finitely many solution. Once after, finding the solution $d_1d_2d_3\cdots$ and going to "Less" won't immediately produce another solution all the invariants will be satisfied and the algorithm will proceed until it finds another solution or it stops. □

6.3 Implementation

The source of the program with implementation of this algorithm, written in Rust can be found on GitHub along with the LaTeX source of this thesis.

Chapter 7

Counting orbifolds – combinatorical part

We will go through the steps described in 5.5.

We will form our answer in a form of answering the question for a given number $\frac{p}{q}$ and a given manifold M. As stated in 5.5 at the end all answers for all M needs to be summed, and we know that for M such that $\chi(M) < \frac{p}{q}$, the answer is 0, so there are always only finitely many answers to be summed.

7.1 Manifolds without boundary

In this case we ask our algorithm from 6: "How many sums of the form

$$1 - \sum_{i=1}^{m} \frac{d_j - 1}{2d_j} \tag{7.1.0.0.1}$$

are equal to

$$\frac{1}{2}\frac{p}{q} - \frac{1}{2}\chi(M) - 1 \text{ ?''}, \tag{7.1.0.0.2}$$

and the result is our final answer, as (1.5.2.1) the list of degrees of rotational orbipoints ordered in descending order uniquely defines a two dimensional M-orbifold without boundary.

7.2 Manifolds with boundary

7.2.1 Using chapter 6

We ask our algorithm from chapter 6 for the list of all possible sums of the form

$$1 - \sum_{i=1}^{m} \frac{d_j - 1}{2d_j} \tag{7.2.1.0.1}$$

that are equal to

$$\frac{p}{q} - \chi(M) - 1. \tag{7.2.1.0.2}$$

As we know from:

- 5.1.0.4
- the fact, that to each sum corresponds at least one M-orbifold
- the fact, that to different sums corresponds different M-orbifolds

this list of sums will be finite.

7.2.2 Reduction to only dihedral orbipoints

As stated in 33.1., we need to first take into account that an orbifold can have both dihedral and rotational orbipoints.

We have complete list of sums corresponding to degrees of dihedral orbipoints that result in the $\frac{p}{q}$ orbicharacteristic on M-orbifold.

We are interested in having complete list of sums corresponding to degrees of both dihedral and rotational orbipoints that result in the $\frac{p}{q}$ orbicharacteristic on M-orbifold. We will now propose a unique reduction $(*)$ of every list corresponding to both dihedral and rotational orbipoints to the list of only dihedral orbipoints.

The reduction $(*)$ goes as follows: for a list of degrees of orbipoints consisting of $r_1 r_2 \cdots r_n$ for rotational orbipoints and $d_1 d_2 \cdots d_m$ for dihedral orbipoints, we replace each r_i from the list of rotational orbipoints by to entries of the same value on the list of dihedral orbipoints. This does not change the corresponding Euler orbicharacteristic, since $\Delta(n) = \Delta({}^*n^*n)$ (as stated in **??**).

This procedure is unambiguous and gives only one possible list of dihedral orbipoints degrees for every list of both rotational and dihedral orbipoints degrees.

Based on this, for a given sum $d_1 d_2 \cdots d_n$, we can perform the transformation of replacing n by ${}^*n^*n$ in another direction to produce all possible sums consisting of both rotational and dihedral degrees, that would be reduced to sum $d_1 d_2 \cdots d_n$. The uniqueness of reduction $(*)$ guarantees, that we won't arrive to the same unreducted sum from different starting lists of only dihedral degrees.

Based on this, after getting the list of sums from 7.2.1, we need to add to this list all possible sums that could be reducted by $(*)$ to some of the sums that we got from 7.2.1.

Sums on the new extended list have also rotational orbipoints taken into account. At this point we have full list of sums resulting in Euler orbicharacteristic equal to $\frac{p}{q}$ on a M-orbifolds. In further considerations we will not explicitly rotational orbipoints degrees as they play no role in combinatorics.

7.2.3 Fixed sum

Let us now consider a case with a manifold M, and a sum $d_1 d_2 \cdots d_n$ (as we wrote at the end of the previous section we are not writing rotational degrees as they will play no role from this point, however, they are possibly present in some sums). Then the last step of the procedure will be to sum over all sums produced in 7.2.2.

Let b be the number of boundary components of M. We know from the assumption, that $b > 0$.

We need to partition $d_1 d_2 \cdots d_n$ among boundary components of M.

At this moment we will treat boundary components as distinguishable.

Let us consider some partition of $d_1 d_2 \cdots d_n$ among the distinguishable boundary components. After this, results from all partitions need to be summed together. The fact that boundary components are not distinguishable will be taken into account in the next subsection by assigning the proper weights in the summation.

7.2.4 Fixed partition

Let us now consider some fixed partition. Suppose that in this partition for every $1 \leqslant j \leqslant b$ boundary component B_j have orbipoints of degrees: ${}^j d_1 \, {}^j d_2 \cdots {}^j d_{n_j}$. We want to know how many possible sets of cyclic orders there are on a boundary components with ${}^j d_1 \, {}^j d_2 \cdots {}^j d_{n_j}$ on B_i.

We can count sets of cyclic orders un untistinguishable boundary components by iterating through tuples of linear orders on distinguishable boundary components and summing them with proper weights.

Given the tuple of linear orders $\mathcal{L} = (L_1, \cdots, L_b)$ on distinguishable components, to calculate the weight $W(\mathcal{L})$, we will first set some weights $W(L_j)$, for every $1 \leqslant j \leqslant b$.

For a linear order L_j we set the weight $W(L_j)$ to be :

- in case M is orientable – $\frac{|\mathbb{Z}_k|}{|\mathbb{Z}_{n_j}|} = \frac{k}{n_j}$, where $\mathbb{Z}_k$ is the biggest cyclic subgroup of $\mathbb{Z}_{n_j}$, under which L_j is invariant as a linear order,
- in case M is not orientable – $\frac{|\mathbb{D}_k|}{|\mathbb{D}_{n_j}|} = \frac{2k}{2n_j} = \frac{k}{n_j}$, where $\mathbb{D}_k$ is the biggest dihedral subgroup of $\mathbb{D}_{n_j}$, under which L_j is invariant as a linear order.

Then, we put $W(L) \coloneqq SR \prod_{j=1}^{b} L_j$, where:

- $S \coloneqq \frac{|G|}{|S_b|}$, where G is the biggest subgroup of permutations S_b under which tuple of orders $\mathcal{L}$ is invariant as a tuple of cyclic orders,
- $R = \frac{1}{2}$ if M is orientable and at least one of linear orders L_j is different as a cyclic order than the reverse of L_j as a cyclic order; otherwise $R = 1$

7.2.5 Comment about possibility of a single equation

Although, given enough effort, results from this section could be summarise in one equation consisting only of b and $k_1, k_2, \cdots, k_n$ for a given sum, we feel that it would be long enough not to give any new insight into the structure of the problem. We are stopping thus at giving the above procedure.

Chapter 8

Conclusions

8.1 What was done

In 2 we proved that $\sigma(D^2) \not\subseteq \sigma(S^2)$ and $\sigma(S^2) \not\subseteq \sigma(D^2)$.

In chapter 3, among other things, we described the spectrum of possible Euler orbicharacteristics of two dimensional orbifolds and, as the result, the spectrum of all possible areas of two dimensional hyperbolic orbifolds in a ordinal and topological manner. It has order type and topology (induced from $\mathbb{R}$) of ω^ω. We also proved, that every accumulation point of $\sigma(S^2)$ is in $\sigma(D^2)$.

In chapter 4 we provided algorithm for deciding for a given number x, whether there exists an orbifold O, such that $\chi^{orb}(O) = x$ and proved its correctness.

In chapter 5 we provided some finiteness results, such as that there are always only finitely many orbifolds for a given Euler orbicharacteristic. We also proved that for every n, in every neighbourhood of every accumulation point of σ of order at least 2, there is at least one number x, such that there are at least n orbifolds such that $\chi^{orb}(O) = x$.

In chapter 6 and chapter 7 we provided an algorithm for counting for a given number x number of orbifold such that $\chi^{orb}(O) = x$, and proved its correctness.

We also discussed that its complexity is low enough for actual implementation and practical usage on a reasonably small denominators and reasonably close to zero.

8.2 Further directions

It remains unclear how Disk spectrum and Sphere spectrum lies relative to each other. In particular we still don't know, whether they coincide from a sufficiently distant point.

We don't really know why there is exactly "this" many orbifolds for a given Euler orbicharacteristic? We would like to know, whether there is some underlying geometrical reason for that?

We would like to somehow characterise points $x \in \sigma$ that has "the most" orbifolds corresponding to them. With reasonable normalisation of what it means for a number to have "more" orbifolds as we go to lesser values of Euler orbicharacteristic.

Bibliography

[Hur93] A. Hurwitz. "On Riemann surfaces with monogenic transformations into themselves." German. In: *Math. Ann.* 41 (1893), pp. 403–442. ISSN: 0025-5831. DOI: 10.1007/BF01443420.

[Eng13] F. Engel. "Entwicklung der Zahlen nach Stammbruechen". In: *Verhandlungen der 52. Versammlung deutscher Philologen und Schulmaenner in Marburg* (1913), pp. 190–191.

[Thu79] William P Thurston. *The geometry and topology of three-manifolds*. s.n, 1979.

[Gro81] Michael Gromov. "Hyperbolic manifolds according to Thurston and Jørgensen". In: (1981), pp. 40–53.

[BE11] T. D. Browning and C. Elsholtz. "The number of representations of rationals as a sum of unit fractions". In: *Illinois J. Math.* 55.2 (2011), 685–696 (2012). ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1359762408.

[CBG16] J.H. Conway, H. Burgiel, and C. Goodman-Strauss. *The Symmetries of Things*. AK Peters/CRC Recreational Mathematics Series. CRC Press, 2016. ISBN: 9781439864890. URL: https://books.google.fr/books?id=Drj1CwAAQBAJ.

Appendix A

Appendix about good orders and accumulation points

A.1 Definition of order of accumulation points

This definitions will be useful for us in chapter 3, the exact same copy of it is included there 3.1.1 as well for a readers convenience .

We start with definition of being "at least of order n" that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.
For a given set we define as follows:

Definition A.1.0.1. *(Inductive). We say that the point x is an accumulation point of a set X of order at least 0, when it belongs to the set X. We say that the point x is an accumulation point of a set of order at least $n+1$, when it is an accumulation point (in the usual sense) of the accumulation points each of order at least n i.e. in every neighbourhood of x there is at least one accumulation point of a set X of order at least n, distinct from x.*

Definition A.1.0.2. *We say that the point is an accumulation point of order n iff it is an accumulation point of order at least n and it is not an accumulation point of order at least $n+1$. If the point is an accumulation point of order at least n for an arbitrary large n we say that the point is an accumulation point of order ω.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the usual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

A.2 Lemmas

Lemma A.2.0.1. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A + B := \{a + b \mid a \in A, b \in B\}$ also have no infinite strictly ascending sequences.*

Proof.

Let A, B have no infinite strictly ascending sequences. Let $c_n \in A + B$ are elements of some sequence. With a sequence c_n there are two associated sequences a_n, b_n, such that, for all n, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradiction), that c_n is an infinite strictly ascending sequence. Then $\forall_n \; a_{n+1} > a_n \;\vee\; b_{n+1} > b_n$. From the assumption a_n has no infinite ascending sequence, so a_n has a weakly decreasing subsequence a_{n_k}. But then subsequence b_{n_k} must be strictly increasing, as c_{n_k} is strictly increasing, what gives us a contradiction. □

Lemma A.2.0.2. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A \cup B$ also have no infinite strictly ascending sequences.*

Proof.

Let A, B have no infinite strictly ascending sequences. For the sake of contradiction, lets assume, that $A \cup B$ has an infinite strictly ascending sequence c_n. Let c_{n_k}, c_{n_l} be subsequences of c_n consisting of elements from, respectively A and B. At least one of them must be infinite and strictly increasing, which gives us a contradiction. □

Concerning accumulation points, we will use the terminology, that we introduced in A.1

Lemma A.2.0.3. *Let $A \subseteq \mathbb{R}$ has an order type α. Let A be such that every accumulation point of A belong to A. Then A has not only an order type α but is also homeomorphic to α.*

Proof.

Without loss of generality, let us assume, that A has no infinite descending sequence (case with A having no infinite ascending sequence is completely analogous).

As A has an order type α we have that there is an order preserving bijection $f : \alpha \to A$.
We will prove the theorem by showing that f is a homeomorphism.

For the continuity of f and f^{-1} it is sufficient to show, that for every open $U \subseteq A$ and $V \subseteq \alpha$ from prebases of respective topologies, $f^{-1}[U]$ and $f[V]$ are open $(*)$.
Prebase open sets in A are the ones inherited from the order topology on $\mathbb{R}$, for all $s \in \mathbb{R}$:

$$\{r \mid r < s\} \cap A$$
$$\{r \mid s < r\} \cap A.$$

Prebase open sets in α are from order topology, for all $\nu \in \alpha$:

$$\{\eta \mid \eta < \nu\}$$
$$\{\eta \mid \nu < \eta\}.$$

Now, we will prove $(*)$ case by case:

• Prebase set – $\{r \mid r < s\} \cap A$:
Let $\nu \in \alpha$ be the smallest, that $s \leqslant f(\nu)$, then:

$$f^{-1}[\{r \mid r < s\} \cap A] = \{\eta \mid \eta < \nu\},$$

which is open.

• Prebase set – $\{r \mid s < r\} \cap A$:
Let $s < f(\mu)$. We have two cases:
– $s \in A$: then let ν be such that $f(\nu) = s$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.
– $s \notin A$: then, by the assumption of the theorem we know that s is not an accumulation point of A. From this we conclude, that $\exists_{t \in A}(t < s \wedge \neg\exists_{t' \in A} t < t' < s)$. Let ν be such that $f(\nu) = t$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A] = \{\eta \mid \nu < \eta\},$$

which is open.

• Prebase set – $\{\eta \mid \eta < \nu\}$:

$$f[\{\eta \mid \eta < \nu\}] = \{r \mid r < f(\nu)\} \cap A,$$

which is open.

• Prebase set – $\{\eta \mid \nu < \eta\}$:

$$f[\{\eta \mid \nu < \eta\}] = \{r \mid f(\nu) < r\} \cap A,$$

which is open.□

Remark. The reverse is also true: If $A \subsetneqq \mathbb{R}$ is homeomorphic to α, then every accumulation point of A belongs to A.

Lemma A.2.0.4. *Let $A \subseteq \mathbb{R}$ be a bounded, well ordered set. Then A has an accumulation point a of order $n \in \mathbb{N}$ (it may be that $a \notin A$) iff order type of A is at least ω^n.*

Proof.

Inductive, with respect to n in ω^n.

• $n = 0$ Let us suppose, that A has an accumulation point of order 0. Having an accumulation point of order 0 means that A is non-empty. As that it has an order type of at least $\omega^0 = 1$.

Let us suppose, that A has order type at least $\omega^0 = 1$. Then it is non-empty, so it has at least one accumulation point of order 0.

• Induction step

Let us suppose that A has an accumulation point a of order $n+1$. This means that every neighbourhood of a we can find infinitely many accumulation points of A of order n. Let take one such neighbourhood and one such family $\{b_i\}_{i\in\mathbb{N}}$ of accumulation points of order n. Let us then take family of pairwise disjoint neighbourhoods $\{U_i\}_{i\in\mathbb{N}}$ of $\{b_i\}_{i\in\mathbb{N}}$. Let $A_i \coloneqq U_i \cap A$.

From the induction assumption for all i, we have that A_i is of order type at least ω^n. As that, we managed to show an pairwise disjoint inclusions of countably many sets of order type at least ω^n into A. As that we have the order preserving inclusion of ω^{n+1} into A, so A is of order type at least ω^{n+1}.

Let us now suppose that A has the order type of at least ω^{n+1}. Then, we can find a family $\{A_i\}_{i\in\mathbb{A}}$ of pairwise disjoint subsets of A, each of order type ω^n, with the property $(*)$, that $\forall_{i,j\in\mathbb{N}} i < j \implies \forall_{x\in A_i, y\in A_j} x < y$.

From the inductive assumption, for all i, we have that A_i has an accumulation point of order n. Let $\{b_i\}_{i\in\mathbb{N}}$ be the set of those accumulation points. Because of the property $(*)$, those accumulation points are pairwise distinct, between A_i, A_j, with $i \neq j$. Since A is bounded, we have that, the set $\{b_i\}_{i\in\mathbb{N}}$ is bounded, so it has an accumulation point a. As an accumulation point of the accumulation points of order n, it is an accumulation point of order $n+1$. $\square$

Corollary A.2.0.5. *Let $A \subseteq \mathbb{R}$ be a bounded, well ordered set of the order type ω^n. Then it has exactly one accumulation point a' of order n. This point has the property that $\forall_{a\in A}\ a < a'$.*

Proof.

From A.2.0.4 we know that A has at least one accumulation point a' of order n.

For the sake of contradiction, let us assume, that there exists an accumulation point $\bar{a}$ of order n such that $\exists_{a\in A}\ a \geqslant \bar{a}$. We have that A has the order type ω^n, which means that $\forall_{a_1\in A}\exists_{a_2\in A}\ a_1 < a_2$. From this, we have, that $\exists_{a_0} a_0 > \bar{a}$. But then, we would have that the prefix $(-\infty, \bar{a}] \cup A$ of A has an accumulation point $\bar{a}$ of order n. From this, from A.2.0.4 we would conclude, that $(-\infty, \bar{a}] \cup A$ is of order type at least ω^n, which leads to the contradiction, as $(-\infty, \bar{a}] \cup A$ is a proper subset of A. Thus, we have, that for all accumulation points $\bar{a}$ of A of order n we have that $\forall_{a\in A}\ a < \bar{a}$.

It remains to show that there is only one such accumulation point - a'. For the sake of contradiction, let us assume, that there exists an accumulation point of A of order n, named $\bar{a}$, such that $\bar{a} \neq a'$. Let us assume that $\bar{a} < a'$. Then, as in every neighbourhood of a' there is a point from A, we have that $\exists_{a_0} a_0 > \bar{a}$. The absurdity of this statement is shown above. Case where $\bar{a} > a'$ is completely analogous. $\square$

Lemma A.2.0.6. *For $A, B \subseteq \mathbb{R}$, if $r \in \mathbb{R}$ is an accumulation point of order m for A and n for B and $m \leqslant n$, then r is an accumulation point of order at most n for $A \cup B$.*

Proof.

Inductive.

• $n = 0$. Then r is an isolated point of B and either r is isolated point of A or $r \notin A$. From this we have that there exists U_1, U_2 such that $B \cap U_1 = \{r\}$ and $A \cap U_2 \subseteq \{r\}$. From this we have that $(A \cup B) \cap (U_1 \cap U_2) = \{r\}$. So r is an isolated point of $A \cup B$.

• Inductive step. Let us suppose that for all $k < n$, the statement holds. Let r be an accumulation point of order n of B and order m of A, where $m \leqslant n$. From this we have that there exists $U_1, U_2 \ni r$ such that in $B \cap U_1$ there are only accumulation points of B of order at most $n-1$ and in $A \cup U_2$ there are only accumulation points of A of order at most $m-1$. From this, from the inductive assumption we have that in $(A \cup B) \cap (U_1 \cap U_2)$ there are only accumulation points of order at most $n-1$ of $A \cup B$. This means that r is an accumulation point of order at most n of $A \cup B$.

We also know that, in every $U_1, U_2 \ni r$, there are accumulation points of order exactly $n-1$ of B and exactly $m-1$ for A. From the inductive assumption we have then, that in $(A \cup B) \cap (U_1 \cap U_2)$ there are accumulation points of order $n-1$ of $A \cup B$. This means that r is an accumulation point of order exactly n of $A \cup B$. $\square$

Corollary A.2.0.7. *Let $A^{(n)}$ be the set of all accumulations point of order n of A. Then for every $n \in \mathbb{N}$ we have that $(A \cup B)^{(n)} = A^{(n)} \cup B^{(n)}$.*

Proof.

Every accumulation point of either A or B is also an accumulation point of $A \cup B$, so $(A \cup B)^{(n)} \supseteq A^{(n)} \cup B^{(n)}$.

From A.2.0.6 we know, that for any point $r \in \mathbb{R}$, if $r \in (A \cup B)^{(n)}$, then $r \in A^{(n)} \cup B^{(n)}$. $\square$

Lemma A.2.0.8. *For two bounded, well ordered sets $A, B \subseteq \mathbb{R}$, with order types, respectively ω^m and ω^n, such that $m < n$, and that $\forall_{x \in A \cup B} \exists_{b \in B} x < b$, we have that order type of $A \cup B$ is well defined and equal to ω^n.*

Proof.

From A.2.0.2, we know, that $A \cup B$ is well ordered. As such its order type is well defined and equal to some ordeal number γ.

We will show that $\gamma \leqslant \omega^n$ and $\gamma \geqslant \omega^n$, thus showing that $\gamma = \omega^n$.

Let $f : \omega^n \to B$ and $g : A \cup B \to \gamma$ be order preserving bijections.

• $\omega^n \leqslant \gamma$:

We have that $g \circ f : \omega^n \to \gamma$ is an order preserving injection, thus, $\omega^n \leqslant \gamma$.

• $\omega^n \geqslant \gamma$:

From A.2.0.5 we know, that B has exactly one accumulation point b' of order n. This point has the property that $\forall_{b \in B}\ b < b'$. As b' is the only accumulation point of order n for B and from A.2.0.5 we know also that A has no accumulation points of order n, from A.2.0.6 we know, that $A \cup B$ has exactly one accumulation point of order n, namely b'.

For the sake of contradiction, let us assume that $\omega^n < \gamma$. But then, there is some proper prefix of $A \cup B$ with order type ω^n. Let us name that prefix as P. From A.2.0.4 we know, that P has an accumulation point p' of order n. Let $b_1 \in B$ be such that $\forall_{p \in P}\ p < b_1$. Such b_1 exists, because P is a proper prefix of $A \cup B$, so $\exists_{x \in A \cup B} \forall p \in P p < x$, and from the assumptions of the lemma we have that $\forall_{x \in A \cup B} \exists_{b \in B} x < b$. We have that $p' \leqslant b_1$. But we have also that $b_1 < b'$, so $p' \neq b'$. This gives us the contradiction, as b' is the only accumulation point of order n in $A \cup B$. $\square$