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Two dimentional orbifolds' volumes' spectrum

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dla Wujka

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Abstract

Orbifoldy

Chapter 1

Introduction

1.1 Motivations

1.2 Questions asked

Chapter 2

Definition, characteristics, classification and properties of the orbifolds

2.1 Definition

TO DO: jak sie juz wszystko zbierze co ma tu być, to to dopisać

The definition of the orbifold is taken from Thurston [6] (chapter 13). We briefly recall the concept, but for full discussion we refer to [6].
An orbifold is a generalisation of a manifold. One allows more variety of local behaviour. On a manifold a map is a homeomorphism between $\mathbb{R}^n$ and some open set on a manifold. On an orbifold a map is a homeomorphism between a quotient of $\mathbb{R}^n$ by some finite group and some open set on an orbifold. In addition to that, the orbifold structure consist the informations about that finite group and a quotient map for any such open set.

Above definition says that an orbifold is locally homeomorphic do the quotient of $\mathbb{R}^n$ by some finite group.
When an orbifold as a whole is quotient of some finite group acting on a manifold we say, that it is 'good'. Otherwise we say, that it is 'bad'.

We are also adopting notation from [6].

In two dimentions there are only four types of bad orbifolds, namely:
- $S^2(n)$
- $D^2(;n)$
- $S^2(n_1, n_2)$ for $n_1 < n_2$
- $D^2(;n_1, n_2)$ for $n_1 < n_2$.

All other orbifolds are good. As manifolds are special case of orbifolds with all ...

We differ from Thurston in the terms of naming points with maps with non-trivial groups. We call them orbipoints. If the group acts as the group of rotations (so a cyclic group) we call them rotational points. If the group is a dihedreal group we call them dihedreal points. And if it is point on the boundry that stabilises relfecition it is a reflection point.

2.2 Euler orbicharacteristic

we will treat as we will treat manifolds as orbifolds we will always refer we will

2.2.1 Classification of orbifolds with non-negative Euler orbicharacteristic

The list of all orbifolds with non-negative Euler orbicharacteristic Powiedzieć coś o tym, że orbicharatkeryttyka odpowiada polom (Gauss Bonett itd.)

2.2.2 Extended Euler orbicharacteristic

(with cusps) Write about cusp as a limit.
Write about isomorphism of all spectra

2.3 Uniformisation theorem (formulation)

TO DO: twierdzenie o klasyfikacji powierzchn

2.4 Operations and constructions on orbifolds

Write about the general sugeries we are interested in i.e. taking any number of features (handles cross caps, parts of boundry components with orbipoints on it, orbipoints in the interior) and replacing it by any other feauters (Some preserve the area) Write about surgeries nesseserie for reduction of cases

2.5 Notation

We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature. We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$.

TO DO: rozwinąć

dopisać, że w Conwayowej >= 2

Chapter 3

Order type and topology

In this chapter we will discuss that both the order type and the topology of the set of all possible Euler orbicharacteristics of two-dimensional orbifolds are that of ω^ω. We will call this set σ.

Disclaimer
For now, until Chapter 5 named "Counting occurrences", we will not pay attention to how many orbifolds have the same Euler orbicharacteristic.

3.1 Reductions of cases

Now we want to make some reductions to limit number of cases that we will be dealing with.
We aim to find a minimal set B of base manifolds that "covers all the cases" i.e. B such that any for any $x \in \sigma$ there is an orbifold O with a base manifold from B such that $\chi^{orb}(O) = x$. It will turn out it can be done such that we are left with $B = \{S^2, D^2\}$ and there is no futher reduction possible.
Given an orbifold O_1, we want to perform some operations from 2.4 on it such that the resulting orbifold O_2 will have the same Euler orbicharactristic, but the base manifold of O_2 would have as big Euler characteristic as possible.
The Euler orbicharacteristic of base manifold depends only on the number of handles, cross caps and boundry components. And, as stated in 2.2 it is:
For ever such a manifold feature we want to find an orbifold features with the same Euler orbicharacteristic delta.
One of the ways to do that is by observing that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(*(^*2)^4) \qquad (3.1.0.0.1)$$
$$\Delta(*) = \quad -1 \quad = \Delta((^*2)^4) \qquad (3.1.0.0.2)$$
$$\Delta(\times) = \quad -1 \quad = \Delta((^*2)^4) \qquad (3.1.0.0.3)$$

So we see that from any orbifold we can eradicate handles ....

$$\Delta(n) = \quad \frac{n-1}{n} \quad = \Delta((^*n)^2) \qquad (3.1.0.0.4)$$

From this we can conclude that every Euler orbicharacteristic can be obtained by an orbifold with base manifold S^2 or D^2. Examples of rational numbers from $\sigma(S^2) \setminus \sigma(D^2)$ and $\sigma(D^2) \setminus \sigma(S^2)$ are: We will provide examples Futher examination of connections between $\sigma(D^2)$ and $\sigma(S^2)$ is performed in **??**
In the terms of set relations:

Observation 3.1.0.1. *If two-dimentional manifold M has no boundry, then*

$$\sigma(M) \subseteq \sigma(S^2) \tag{3.1.0.1.1}$$

If, in addition, $M \neq S^2$, then

$$\sigma(M) \subseteq \sigma(D^2). \tag{3.1.0.1.2}$$

Observation 3.1.0.2. *If two-dimentional manifold M has a boundry, then*

$$\sigma(M) \subseteq \sigma(D^2) \tag{3.1.0.2.1}$$

In the terms of arithmetical expressions:

Observation 3.1.0.3. *From above reductions we can conclued that our problem boiles down to the analysis of all the possible values of the expressions:*

$$2 - \sum_{i=1}^{n} \frac{I_i - 1}{I_i} \tag{3.1.0.3.1}$$

and

$$1 - \sum_{j=1}^{m} \frac{b_j - 1}{2b_j}, \tag{3.1.0.3.2}$$

with I_i and b_j ranging over $\mathbb{N}_{>0} \cup \{\infty\}$.

As stated in **??** we can perform futher reductions to have an orbifold with particular orbicharacteristic without cusps (if needed) and then (after these reductions) we can analyse only expressions with I_i and b_j ranging over $\mathbb{N}_{>0}$ and they will still give us full spectrum.
However, as stated later, it will be more convenient to us to include orbifolds with cusps so we are stating above remark only for readers information.
The fact that it agrees with the definition of the Euler orbicharacteristic on the geometrical terms was addressed in 2.2.2.

3.2 Order type and topology of $\sigma(D^2)$

To determine order type and topology of σ we will first study how $\sigma(D^2)$ looks like. Then, remembering that $\sigma = \sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$ we will make an argument for σ.
In this section we will also describe precisely where accumulation points of $\sigma(D^2)$ lie and of which order (see below 3.2.1) they are. Analysis of locations of those accumulation points, as interesting as it is alone will also be necessery for providing our argument about order type and topology of $\sigma(D^2)$.

3.2.1 Definition and properties of order of accumulation points

We start with definition of being "at least of order n" that will be almost what we want and then, there will be the definition of being "order", which is the definition that we need.
For a given set we define as follows:

Definition 3.2.1.1. *(Inductive). We say that the point x is an acccumulation point of a set X of order at least 0, when it belongs to the set X. We say that the point x is an acccumulation point of a set of order at least $n+1$, when it is an acccumulation point (in the usual sense) of the accumulation points each of order at least n i.e. in every neighbourhood of x there is at least one accumulation point of a set X of order at least n, distincs from x.*

Definition 3.2.1.2. *We say that the point is an acccumulation point of order n iff it is an acccumulation point of order at least n and it is not an acccumulation point of order at least $n+1$. If the point is an acccumulation point of order at least n for an arbitrary large n we say that the point is an acccumulation point of order ω.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the ussual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

Lemma 3.2.1.3.

3.2.2 Analysis of locations of accumulation points of $\sigma(D^2)$ with respect to their order

We want to determine where exactly are accumulation points of the set $\sigma(D^2)$ with respect to their orders.
For this we will use a handful of observations and lemmas.

Observation 3.2.2.1. *Let us observe, that $\lim_{n\to\infty} \Delta({}^*n) = -\frac{1}{2}$. From that, we see, that for every point $x \in \sigma(D^2)$, the point $x - \frac{1}{2}$ is an acccumulation point. Let us observe, that also, for every point $x \in \sigma(D^2)$, we have that $x - \frac{1}{2} \in \sigma(D^2)$, because $\Delta({}^*\infty) = -\frac{1}{2}$.*

Lemma 3.2.2.2. *For all $n \in \mathbb{N}_{\geqslant 2}$ and $x \in (-\infty, 1]$ there are only finitely many Euler orbicharacteristics in the interval $[x, 1] \cap \sigma(D^2)$ of orbifolds that have points of order equal at most n.*

Proof.
Let $x \in (-\infty, 1]$. There can be at most $\lfloor 4(1-x) \rfloor$ orbipoints on the D^2 orbifold with an Euler orbicharacteristic $y \in [x, 1]$ since each orbipoint decreases an Euler orbicharacteristic by at least $\frac{1}{4}$ and the Euler characteristic of D^2 is 1.
There are only $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible sets of $\lfloor 4(1-x) \rfloor$ orbipoints' orders that are less or equal than n. Hence, there are only at most $(n-1)^{\lfloor 4(1-x) \rfloor}$ possible Euler orbicharacteristics.

Lemma 3.2.2.3. *If x is an acccumulation point of the set $\sigma(D^2)$ of order n, then $x - \frac{1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order at least $n+1$.*

Proof.
Inductive.
• $n = 0$: If x is an isolated point of the set $\sigma(D^2)$, then $x \in \sigma(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma(D^2)$ for all $k \geqslant 1$, from that, that $x - \frac{1}{2}$ is a accumulation point of $\sigma(D^2)$.
• inductive step: Let x be an acccumulation point of the set $\sigma(D^2)$ of an order $n > 0$. Let a_k be a sequence of accumulation points of order $n-1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of accumulation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+1$. □

Lemma 3.2.2.4. *If x is an acccumulation point of the set $\sigma(D^2)$ of order n, then $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n-1$.*

Proof.
Inductive
• $n = 1$: We assume, that x is an acccumulation point of isolated points of the set $\sigma(D^2)$. From 3.2.2.2 we know, that for all m there are only finitely many Euler orbicharacteristics in the interval $[x, 1]$ of orbifolds that have dihedral points of order equal at most m.
From that, for arbitrary small neighborhood $U \ni x$ and arbitrary large m there exist an orbifold that has a dihedral point of period grater than m, whose Euler orbicharacteristic lies in U. Let us take a sequence of such Euler orbicharacteristics a_k convergent to x, such that we can choose a sequence divergent to infinity of periods of dihedral points b_k of orbifolds of Euler orbicharacteristics equal a_k.
To do: picture
Let us observe, that for all k, the number $a_k + \frac{b_k-1}{2b_k}$ is in $\sigma(D^2)$. It is so, because a_k is an Euler orbicharacteristic of an orbifold that have a dihedral point of period b_k, so identical orbifold, only without this dihedral point, has an Euler orbicharacteristic equal to $a_k + \frac{b_k-1}{2b_k}$. The sequence $a_k + \frac{b_k-1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least 0.
• inductive step: Let x be an acccumulation point of the set $\sigma(D^2)$ of order $n > 1$. Let a_k be a sequence of accumulation points of the set $\sigma(D^2)$ of order $n-1$ convergent to x. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of an acccumulation points of the set $\sigma(D^2)$ of order $n-2$ convergent to $x + \frac{1}{2}$. From that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n-1$. □

Lemma 3.2.2.5. *If x is an acccumulation point of the set $\sigma(D^2)$ of order $n+1$, then*
$x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $n+2$ and
$x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order n.

Proof.
Let x be an acccumulation point of the set $\sigma(D^2)$ of order $n+1$. From the lemma

3.2.2.3 we know, that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$. Now let us assume (for a contradiction), that $x - \frac{1}{2}$ is an accumulation point of the set$\sigma(D^2)$ of order $k > n+2$. But then from the lemma 3.2.2.4 we have that x is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$ and that is a contradiction.
Analogously, from the lemma 3.2.2.4 we know, that $x + \frac{1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order at least n. Let us assume (for a contradiction), that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $k > n$. But then from the lemma 3.2.2.3 we have that x is an acccumulation point of the set $\sigma(D^2)$ of order at least $n+2$ and that is a contradiction. □

Lemma 3.2.2.6. *For all $n \in \mathbb{N}$ all accumulation points of the set $\sigma(D^2)$ of order n are in $\sigma(D^2)$.*

Proof.
Inductive
• $n = 0$: Clear, as they are isolated points of $\sigma(D^2)$.
• inductive step: Let x be a accumulation point of the set$\sigma(D^2)$ of order $n > 0$. From the lemma 3.2.2.5 point $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma(D^2)$ of order $n-1$. From the inductive assumption $x + \frac{1}{2} \in \sigma(D^2)$. Then, from 3.2.2.1, we have that $x \in \sigma(D^2)$. □

Theorem 3.2.2.7. *The greatest accumulation point of the set $\sigma(D^2)$ of order n is $1 - \frac{n}{2}$.*

Proof.
Inductive
• $n = 0$: We know, that $1 \in \sigma(D^2)$ and 1 is the greatest element of $\sigma(D^2)$.
• an inductive step: From the inductive assumption we know that $1 - \frac{n}{2}$ is the greatest accumulation point of the set $\sigma(D^2)$ of order n. From the lemma 3.2.2.5 we have then that $1 - \frac{n+1}{2}$ is a accumulation point of the set $\sigma(D^2)$ of order $n+1$. Let us assume (for a contradiction), that there exist a bigger accumulation point of order $n+1$ equal to $y > 1 - \frac{n+1}{2}$. But then, from lemma 3.2.2.5, point $y + \frac{1}{2}$ would be an acccumulation point of order n, what gives a contradiction, because $y + \frac{1}{2} > 1 - \frac{n}{2}$. □

From the above discussion we can also formulate following corollary that will be useful later:

Corollary 3.2.2.8. *Let $x \in \sigma(D^2)$. Then:*

- *there exists $n_1 \in \mathbb{N}$ such that $x + \frac{n_1}{2} \in \sigma(D^2)$ but $x + \frac{n_1+1}{2} \notin \sigma(D^2)$.*
  In other words, there exist $y \in \sigma(D^2)$ and $n_1 \in \mathbb{N}$ such that $y + \frac{1}{2} \notin \sigma(D^2)$ and such that $x = y - \frac{n_1}{2}$;
- *there exists $n_2 \in \mathbb{N}$ such that x is an accumulation point of the set $\sigma(D^2)$ of order n_2*

and $n_1 = n_2$.

3.2.3 Proof that $\sigma(D^2)$ is well ordered

Definition 3.2.3.1. *Let $B_0 = \{1\}$. For an $n \in \mathbb{N}_{>0}$, let B_n be the set of all possible Euler orbicharacteristic realised by orbifolds of type $*b_1, \cdots, b_n$. For a given n these are D^2 orbifolds with precisely n non trivial orbipoits on their boundry.*

Observation 3.2.3.2. *There is a recursive relation, that $B_{n+1} = B_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$*

Proof.
It is so, because every orbifold with $n+1$ orbipoints can be obtained by adding one point to an orbifold with n orbipoints and the set $\{-\frac{n-1}{2n} \mid n \geqslant 2\} = \{\Delta(^*b) \mid b \geqslant 2\}$.
□

Observation 3.2.3.3. *Observe that, as any orbifold has only finitely many orbipoints, we have that $\sigma(D^2) \subseteq \bigcup_{n=0}^{\infty} B_n$. We defined $\sigma(D^2)$ as a set of all possible Euler orbicharacteristic of disk orbifolds, so $\sigma(D^2) \supseteq \bigcup_{n=0}^{\infty} B_n$. From this we have that $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.*

Lemma 3.2.3.4. *For any given $n \in \mathbb{N}$ the set B_n is a subset of the interval $[1 - \frac{n}{2}, 1 - \frac{n}{4}]$.*

Proof.

Observation 3.2.3.5. *From 3.2.3.2 and A.1.0.1, we have that B_n do not have infinite ascending sequence for all n.*
Further, from A.1.0.2 we conclude, that $\bigcup_{n=0}^{N} B_n$ do not have infinite ascending sequence for all N.

Theorem 3.2.3.6. *In $\sigma(D^2)$ there are no infinite strictly ascending sequences, hence, it is well ordered.*

Proof.
For the sake of contradiction lets assume that c_n is an infinite strictly ascending sequence in $\sigma(D^2)$. As c_n is bounded from below by c_0 and whole $\sigma(D^2)$ is bounded from above by 1, all elements of c_n are in the interval $[c_0, 1]$. From 3.2.3.3 we have, that $\sigma(D^2) = \bigcup_{n=0}^{\infty} B_n$.
Lemma 3.2.3.4 says that for all n we have $B_n \subset [1 - \frac{n}{2}, 1 - \frac{n}{4}]$. From this, we know, that for any n such that $1 - \frac{n}{4} < c_0$ we have, that $B_n \cap [c_0, 1] = \varnothing$. Let n_0 be the smallest such that $1 - \frac{n_0}{4} < c_0$ (so $n_0 > 4(1 - c_0)$). Then for all $n > n_0$ we have $1 - \frac{n}{4} < c_0$, meaning, that for all $n > n_0$ we have $B_n \cap [c_0, 1] = \varnothing$, so all elements of c_n are in $\bigcup_{n=0}^{n_0} B_n$. But this contradicts 3.2.3.5. □

3.2.4 Proof that order structure and topology of $\sigma(D^2)$ are those of ω^ω

To do: rozbić to na dwa?

Theorem 3.2.4.1. *Order type of $\sigma(D^2)$ is ω^ω.*

Proof.
- proof that it is at least ω^ω
suppouse that it is less than ω^ω
then it is smaller than ω^n for some n.
but for sufficiently distant have accumulation point.
done
- proof that it is at most ω^ω
suppouse it is bigger
then there is a point before which there is omega omega order
wooo, but it cant be since first with accumulation is at something spmething
done
ok, so how to give convincing correspondence between accumulation points and ω^n??
Lemma
there cant be order of type
i wish I can cite this from somewhere

From the theorem A.2.0.1 and ... we have that the topology of $\sigma(D^2)$ is ω^ω.

3.3 Order type and topology of $\sigma(M)$

Ok, all isomorphic with $\sigma(D^2)$
here write about it
Tell me about it!

3.4 Order type and topology of σ

Theorem 3.4.0.1. *The order type of the set of possible Euler orbicharacteristics of two-dimensional orbifolds σ is ω^ω.*

TO DO: tutaj też dopisać dowód

dopisać w apendiksie, że jak wysumuję dwa o różnym typie porządkowym, to wychodzi większy równy

3.5 Order type and topology of some subsets of σ and $\sigma(M)$

σ_n

taking limit points is the order type of ω^ω but not homeomorphic anymore.

3.6 More about how this ω^ω lies in $\mathbb{R}$

Observation 3.6.0.1. *The first (greatest) negative accumulation point of the set of σ is $-\frac{1}{12}$. It is the accumulation point of order* 1.

Proof.

We will show, that $-\frac{1}{12}$ is the greatest negative accumulation point of the set $\sigma(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two-dimensional orbifolds is equal to $\sigma(S^2) \cup \sigma(D^2)$ and $\sigma(S^2) = 2\sigma(D^2)$, so the greatest negative point of the set $\sigma(S^2)$ is smaller than the greatest negative accumulation point of the set $\sigma(D^2)$.

• $-\frac{1}{12} = \chi^{orb}((2,3)) - \frac{1}{2}$, from this we have that $-\frac{1}{12}$ an acccumulation point of the set $\sigma(D^2)$ of order at least 1.

• Let us assume (for a contradiction), that there exist bigger, negative accumulation point of the set $\sigma(D^2)$ of order at least 1. Let us denote it by x.

However, then, from the lemma 3.2.2.5 point $x + \frac{1}{2}$ is the accumulation point of the set $\sigma(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12}$. From the lemma 3.2.2.6 we have that x is in $\sigma(D^2)$. But orbifolds of the type $*b_1$ can have Euler orbicharacteristiconly greater or equal $\frac{1}{2}$. Orbifolds of the type $*b_1b_2$ can only have Euler orbicharacteristic$\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*b_1b_2b_3\cdots$ can have Euler orbicharacteristiconly lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma(D^2) = \emptyset$ and to the contradiction.

• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma(D^2)$, from this $-\frac{1}{12}$ is an acccumulation point of order 1 of the set $\sigma(D^2)$. □

Chapter 4

Algorithms for searching the spectrum

4.1 Decidability

TO DO: oj dokończyć

Here we will show the proof that the problem of "deciding whether a given rational number is in an Euler orbicharacteristic's spectrum or not" is decidable by showing algorithm for doing this. Later, our algorithm will have a bonus property of determining of which order of condensation is given point if it is in fact in σ.

To do: Może od razu postawić pełny problem

First stated algorithm is also very inefficient and is presented, because the idea is the most clear in it. Right after it there is stated an algorithm with two enhancements:

- determining an accumulation point of which order is a given point, if it is in fact in the spectrum (this enhancement gives also a performance boost)
- faster searching, because some cases do not need to be checked.

We start with $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$.

We want to determine whether there exists $b_1, b_2, \ldots, b_k$, such that $\chi^{orb}(*b_1 \ldots b_k) = \frac{p}{q}$.

In the case that $\frac{p}{q}$ is of the form $l\frac{1}{4}$, for some whole l we can give the answer right away. For $l > 4$ we have that $l\frac{1}{4}$ is not in the set and for $l \leqslant 4$ it is. Moreover for an even l it is a condensation point of order $\frac{4-l}{2}$ (see **??**) and for an odd l it is a condensation point of order $\frac{3-l}{2}$ (see 3.2.2.8).

In the case, where $\frac{p}{q} > 2$, we also can give answer rigth away and this answr is "no".

Now we will consider only cases when $\frac{p}{q}$ is not of the form $l\frac{1}{4}$ and is $\leqslant 2$.

4.1.1 The first approach to the searching algorithm

We use:

- $\mathbb{N}_{>0}$ counters $b_1 b_2 \ldots$ with values ranging from $\mathbb{N}_{>0} \cup \{\infty\}$. Each counter correspond to one cone point on the boundry of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial cone point - namely a none cone point, a normal point).
- a pivot pointing to some counter at any time
- a flag that can be set to "Greater" or "Less" corresponding to what was the outcome of comparing Euler orbicharacteristic of the orbifold corresponding to counters' state and $\frac{p}{q}$.

We start with:

- all counters set to 1.
- pivot pointing at the first counter
- flag set to "Greater"

We will do our computation such that:

- every state of the counters during runtime of the algorithm will have only finitely many counters with value non-1.
- every state in the rutime of the algorithm will have values on consequtive counters ordered in weakly decreasing order.

From now we will consider only such states.
The state of the counters $b_1 b_2 \ldots$ correspond to the orbifold of Euler orbicharacteristic equal $\chi^{orb}(*b_1 b_2 \ldots)$ (where the trailing 1 are trunkated).
When the algorithm is in the state:

- counters: $b_1 b_2 \ldots$
- pivot: on the counter c
- flag: set to the value $flag\_value$,

we procced as follows :

```
In the case, the flag is set to:
{
    "Less", then
    {
        We increase the counter c by one (b_c := b_c + 1).
        If b_c = 2 and the values of all the counters
        on the left are also equal 2 then
        {
```

```
9              We end the whole algorithm with the answer "no".
10         }
11         We set the value of all counters on the left to $b_c$
12         If $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
13         {
14             We found an orbifold and we are ending the whole
15             algorithm with answer "yes, $*b_1b_2\dots$".
16         }
17         If $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$ then
18         {
19             We set the flag to "Greater".
20             We put the pivot on the first counter.
21             We go to the line 1..
22         }
23         If $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$ then
24         {
25             We set the flag to "Less".
26             We put pivot to the $c+1$ counter.
27             We go to the line 1..
28         }
29     }
30
31     "Greater", then
32     {
33         If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) = \frac{p}{q}$ then
34         {
35             We found an orbifold and we are ending the whole
36             algorithm with answer "yes, $*b_1\dots b_{c-1}\infty b_{c+1}\dots$".
37         }
38         If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) > \frac{p}{q}$ then
39         {
40             We set $b_c$ to $\infty$.
41             We set the flag to "Greater".
42             We move pivot to the $c+1$ counter.
43             We go to the line 1..
44         }
45         If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) < \frac{p}{q}$ then
46         {
47             We search for value $b'_c$ of the $c$ counter
48             such that $\chi^{orb}(*b_1\dots b_{c-1}b'_cb_{c+1}\dots) \leqslant \frac{p}{q}$
49             and $\chi^{orb}(*b_1\dots b_{c-1}(b'_c-1)b_{c+1}\dots) > \frac{p}{q}$.
50             More on how we search for it will be told later, for now
51             we can think that we search one by one starting
52             from $b_c$ and going up till $b'_c$.
```

```
                We set $b_c$ to $b'_c$.
                if $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
                {
                    We found an orbifold and we are ending the whole
                    algorithm with answer "yes, $*b_1b_2\dots$".
                }
                We set all the counters to the left to value $b_c$.
                if $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
                {
                    We found an orbifold and we are ending the whole
                    algorithm with answer "yes, $*b_1b_2\dots$".
                }
                If $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$ then
                {
                    We set flag to "Less".
                    We move the pivot to the column $c+1$.
                    We go to the line 1..
                }
                If $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$ then
                {
                    We set the flag to "Greater".
                    We move the pivot to the first counter.
                    We go to the line 1..
                }
            }
        }
}
```

4.1.2 Why this works

4.1.3 Improvements

Let $m \in \mathbb{N}$ be such that $\frac{p}{q} \in (1 - \frac{m}{2}, 1 - \frac{m+1}{2})$ Let us denote by $r \coloneqq \frac{p}{q} - (1 - \frac{m}{2})$.

We will searching in σ as such:

If $\frac{p}{q} \in \sigma$, then, from the corollary 3.2.2.8 we know, that there exist some $n \in \mathbb{N}$, such that $\frac{p}{q} + \frac{n}{2} \in \sigma$ but $\frac{p}{q} + \frac{n}{2} \notin \sigma$.

We will be consequently checking points from $1 + r$, through $1 + r - \frac{l}{2}$, for $0 \leqslant l \leqslant m$, to the $\frac{p}{q}$. We stop at the first found point. If one of these point is in the spectrum, then all smaller (so also $\frac{p}{q}$) are in the spectrum and $\frac{p}{q}$ is the accumulation point of the spectrum of order $m - l$ (from this, we can see some heuristic, that the points that have smaller order will be generally harder to find in some sense). If none of this points are in in the spectrum, then $\frac{p}{q}$ is not.

Searching for all occurences

4.1.4 Implementation

As an appendix, there is a source of a program with implementation of this algorithm with full enhancments described in this chapter. It is written in Rust.

Chapter 5

Counting occurrences

Our ultimate goal is to give the answer to the questions such as:
- For a given $x \in \sigma$, how many orbifolds have x as their Euler orbicharacteristic?
- Why? Is there some underlying geometrical reason for that?
- Can we characterise points $x \in \sigma$ that has the most orbifolds corresponding to them?
- Is there any reasonable normalisation to counter the effect that there are 'more' points as we go to lesser values. (What we mean by 'more' was sted in)

The first equstion we can tackle is steaming from the chapter 3 and it is – Do $\sigma(D^2)$ and $\sigma(S^2)$ coincide? It is easy to answer that $\sigma(D^2) \neq \sigma(S^2)$ (and we will do that along some harder questions in the moment), but do they coincide starting from a sufficiently distant point? Or maybe, for every denominator, do they coincide from a sufficiently distatn point? (Yes.)

TO DO: przenieść (przekopiować?) część może do futher direction

write about cyclic order

5.1 Finitenes

Observation 5.1.0.1. *For any $x \in \sigma$ and $n \in \mathbb{N}$ there are only finitely many orbifolds with the Euler orbicharacteristic greater or equal to x and all orbipoints of order at most n.*

Proof:

Theorem 5.1.0.2. *For any $x \in \sigma$ there are only finitely many orbifolds with the Euler orbicharacteristic equal to x.*

Proof:
Let x be a rational number. Let $\mathcal{O}$ be the set of all orbifolds with an Euler orbicharacteristic qual to x. Those orbifolds can have different base manifolds. However, the set of base manifolds of orbifolds from $\mathcal{O}$ is finite, as there are only finitely many

two dimentional manifolds with an Euler characteristic greater or equal to x and an orbifold always has an Euler orbicharacteristic less or equal to the Euler characteristic of its underlying manifold.
It remains to proof, that for any base manifold M, the number of M orbifolds with Euler orbicharacteristic equal to x is finite. Let M be a two dimentional manifold. For the sake of contradiction, assume, that there exists an infinite set of M orbifolds $\mathcal{O}_M \subseteq \mathcal{O}$. Let $\mathcal{O}_M = \{O\}_{i \in I}$.
For each i, let $s_i = (I_i^0, \cdots, I_i^{k_i}; b_i^0, \cdots, b_i^{l_i})$ be the signature of O_i written with decreasing orders of rotational orbipoints and decreasing orders of dihedral points. So for each i we have, that I_i^0 is the order of the orbipoint with the highest order of all the rotational orbipoints of $\mathcal{O}_i$ and b_i^0 is the order of the orbipoint with the highest order of all the dihedral orbipoints of $\mathcal{O}_i$. By 5.1.0.1 we know that if the set $\{I_i^0\}_{i \in I} \cup \{b_i^0\}_{i \in I}$ would be bounded by some $n \in \mathbb{N}$ it would mean, that $\mathcal{O}_M$ would be finite. As (from the assumption) this is not a case, we know that the set $\{I_i^0\}_{i \in I} \cup \{b_i^0\}_{i \in I}$ is unbounded. Let $\{i_n\}_{n \in \mathbb{N}} \subseteq I$ be a sequence of indeces such that $\{(I_{i_n}^0, b_{i_n}^0)\}_{n \in \mathbb{N}}$ is strictly increasing on one cordinate and non-decreasing on the other. Let $\{x_n\}$ be the sequence such that $x_n = \Delta(I_{i_n}^0, b_{i_n}^0)$. Let $\{y_n\}$ be the sequence such that $y_n = \Delta(I_{i_n}^1, \cdots, I_{i_n}^{k_{i_n}}, b_{i_n}^1, \cdots, b_{i_n}^{l_{i_n}})$. So for every n we know $\chi^{orb}(O_n) = \chi(M) + a_n + b_n$. As $\{(I_{i_n}^0, b_{i_n}^0)\}$ is strictly increasing on one cordinate and non-decreasing on the other, we know that x_n is strictly decreasing, so y_n must be strictly increasing, because $\chi^{orb}(O_n)$ is constant for all n (all O_n are from the family with Euler orbicharacteristic equal to x).
But $\{y_n\} \subseteq \sigma(M) - \chi(M)$. From 3.2.3.6 and **??** we know that $\sigma(M)$ has no infinite strongly increasing sequences, so $\sigma(M) - \chi(M)$ has no infinite strongly increasing sequences. That gives us a contradiction. □

5.2 $\sigma(D^2)$ and $\sigma(S^2)$

In this section we would like to answer some questions about relations between $\sigma(D^2)$ and $\sigma(S^2)$.
The first, stated in **??** is that $2\sigma(D^2) = \sigma(S^2)$. This tells us all about simmilarities of their topological structures – namely, they are the same, but it does not directly answers questions about how they lie in $\mathbb{R}$, relative to each other.

TO DO: dopisać trochę inną motywację

We now state some observations that will be usefull in this section.

Observation 5.2.0.1. *If an Euler orbicharacteristic is an accumulation point of order n in $\sigma(D^2)$ (respectively $\sigma(S^2)$), there exist an D^2 (resp. S^2) orbifold with n dihedral (resp. rotational) points of that Euler orbicharacteristic.*

prrof. from chapter 3. (todo: dopisać)

Observation 5.2.0.2. *If $x \in \sigma(D^2)$ (respectively $\sigma(S^2)$), then $1 - x$ (resp. $2 - x$) is a difference in Euler orbicharacteristic resulting from some set of dihedral (resp. rotational) points. From that $1 - n(1 - x) \in \sigma(D^2)$ (resp. $2 - n(2 - x) \in \sigma(S^2)$) for all $n \in \mathbb{N}$.*

5.2.1 $-\frac{1}{84}$ and $-\frac{1}{42}$

//Why it is how it is//

5.2.2 Accumulation points of the $\sigma(S^2)$

Theorem 5.2.2.1. *All accumulation points of the $\sigma(S^2)$ are in $\sigma(D^2)$.*

There are two proofs of this theorem showing nice correnpondence – one arithmetical and one geometrical.

Proof I. Arithmetical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.
By **??** we have, that $\frac{x}{2} \in \sigma(D^2)$ is an accumulation point of the set $\sigma(D^2)$. From 3.2.2.5 we have that $\frac{x}{2} + \frac{1}{2} \in \sigma(D^2)$. From that, from 5.2.0.2 we have, that $1 - \overbrace{2}^{\text{"}n\text{" from 5.2.0.2}} \, (\overbrace{1 - (\frac{x}{2} + \frac{1}{2})}^{\text{"}1-x\text{" from 5.2.0.2}}) \in \sigma(D^2)$. But $1 - 2(1 - (\frac{x}{2} + \frac{1}{2})) = x$, so $x \in \sigma(D^2)$. $\square$

Proof II. Geometrical reason

We assume that $x \in \sigma(S^2)$ is an accumulation point of the set $\sigma(S^2)$.
From 3.2.2.8 we know, that x can be expressed as $y - 1$ for some $y \in \sigma(S^2)$.
Let $\mathcal{O}$ be an orbifold with the base manifold S^2, such that $\chi^{orb}(\mathcal{O}) = y$.
Let $\mathcal{O}_c$ be the orbifold created from $\mathcal{O}$ by adding one cusp. Then $\chi^{orb}(\mathcal{O}_c) = y - 1 = x$. Topologically $\mathcal{O}_c$ with the cusp point removed (which do not change an orbicharacteristic) is $\mathbb{R}^2$. We can compactify it with S^1. This will not change an Euler orbicharacteristic since $\chi^{orb}(S^1) = 0$ and Euler orbicharacteristic is additive. What we get is an orbifold $\mathcal{O}_D$ with the base manifold D^2 and the same orbipoints as $\mathcal{O}$. Since orbipoints of $\mathcal{O}$ create a difference in Euler orbicharacteristic equal to $2 - y$, we have that $\chi^{orb}(\mathcal{O}_D) = 1 - (2 - y) = y - 1 = x$. We can then move all orbipoints from the interior of $\mathcal{O}_D$ to its boundry by doubling them, so $x \in \sigma(D^2)$. $\square$

5.3 Translating questions to ones about Egyptian fractions

5.4 Estimations of the number of occurences

TO DO: dac jakieś źródła i ok

Chapter 6

Conclusions

Chapter 7

Further directions

7.1 Asked, but unanswered questions

7.2 Unasked and unanswered questions

7.3 Power series and generating functions

7.4 Seifert manifolds

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Appendix A

Appendix about well orders

A.1 Usefull lemmas

Lemma A.1.0.1. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A + B := \{a + b \mid a \in A, b \in B\}$ also have no infinite strictly ascending sequences.*

Proof.
Let A, B have no infinite strictly ascending sequences. Let $c_n \in A + B$ are elements of some sequence. With a sequence c_n there are two associated sequences a_n, b_n, such that, for all n, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradiction), that c_n is an infinite strictly ascending sequence. Then $\forall_n \ a_{n+1} > a_n \ \vee \ b_{n+1} > b_n$. From the assumption a_n has no infinite ascending sequence, so a_n has a weakly decreasing subsequence a_{n_k}. But then subsequence b_{n_k} must be strictly increasing, as c_{n_k} is strictly increasing, what gives us a contradiction. □

Lemma A.1.0.2. *If $A, B \subseteq \mathbb{R}$ have no infinite strictly ascending sequences, then set $A \cup B$ also have no infinite strictly ascending sequences.*

Proof.
Let A, B have no infinite strictly ascending sequences. For the sake of contradiction, lets assume, that $A \cup B$ has an infinite strictly ascending sequence c_n. Let c_{n_k}, c_{n_l} be subsequences of c_n consisting of elements from, respectively A and B. At least one of them must be infinite and strictly increasing, which gives us a contradiction. □

A.2 Order preserving homeomorphisms

Conserning accumulation points, we will use the terminology, that we introduced in 3.2.1

Theorem A.2.0.1. *Let $A \subseteq \mathbb{R}$ has order type α. Let A be such that every accumulation point of A belong to A. Then A has not only order type α but is also homeomorphic to α.*

Proof.
Let us observe, that α must be countable – as A is well ordered we can assign to

every element $a_1 \in A$ an non-empty open interval, between a_1 and it's succesor a_2. From every such interval we can pick a rational number. From this we have injection of A into $\mathbb{Q}$.
Without loss of generality, let us assume, that A has no infinite descending sequence (case with A having no infinite ascending sequence is completly analogus).

As A has an order type α we have that there is an order preserving bijection $f : \alpha \to A$.
For an ordinal $\mu < \alpha$, let us denote $f_\mu \coloneqq f\Big|_{\mu+1}$ so f_μ is defined on all ordinals less or equal to μ. Then we have that $f = \bigcup\limits_{\mu<\alpha} f_\mu$. Let us also denote $A_\mu \coloneqq f_\mu[\mu + 1]$ – the image of $\mu + 1$ (as a set of all ordinals less or equal to μ) in A. Let us remark, that A_μ has an order type $\mu + 1$.

We will prove the theorem by inductively showing for all $\mu < \alpha$, that f_μ is a homeomorphism, and by showing that this is sufficient for f to be a homeomorphism.

Our inductive assumption for a given μ will be that for all $\nu < \mu$ function f_ν is a homeomorphism between $\nu + 1$ (so defined on all ordinals less or equal to ν) and A_ν; and that for every $\nu_1 < \nu_2 < \mu$ we have $f_{\nu_2}\Big|_{\nu_1+1} = f_{\nu_1}$.

• $\mu = 0$:
Function f_0 is an homeomorphism between one element sets, both with discreate topology.

• $\mu < \alpha$ is a successor ordinal:
Since μ is a successor ordinal, $\mu - 1$ exists.
From inductive assumtion we know, that $f_{\mu-1} : \mu \to A_{\mu-1}$ is a homeomorphism, so it is sufficient to show that preimages of open sets containing $f_\mu(\mu)$ and images of open sets containing μ are open.
Since $f_\mu(\mu)$ is a successor (because f_μ preserves the order) and since A_μ is well ordered, we have, that $f_\mu(\mu)$ is an isolated point in A_μ.
Simmilarly μ is an isolated point in $\mu + 1$ as an successor ordinal.
From this we have, that open sets containing $f_\mu(\mu)$ (resp. μ) are of the form

$$U \cup \{f_\mu(\mu)\} \text{ (resp. } V \cup \{\mu\}) \tag{A.2.0.1.1}$$

for some U – open set in $A_{\mu-1}$. (resp. V – open set in μ).
Let U be an open set in $A_{\mu-1}$ and $V = f_{\mu-1}^{-1}[U]$ an open set in μ.
we have that

$$f_\mu^{-1}\left[U \cup \{f_\mu(\mu)\}\right] = f_\mu^{-1}\left[U\right] \cup f_\mu^{-1}\left[\{f_\mu(\mu)\}\right] = f_{\mu-1}^{-1}\left[U\right] \cup \{\mu\} = V \cup \{\mu\} \tag{A.2.0.1.2}$$

and

$$f_\mu\left[V \cup \{\mu\}\right] = f_\mu\left[V\right] \cup f_\mu\left[\{\mu\}\right] = f_{\mu-1}\left[V\right] \cup \{f_\mu(\mu)\} = U \cup \{f_\mu(\mu)\} \tag{A.2.0.1.3}$$

so by A.2.0.1.1, we have that preimages of open sets containing $f_\mu(\mu)$ and images of open sets containing μ are indeed open.

• $\mu < \alpha$ is a limit ordinal:
Let $f_{\nu<\mu} := f_\mu\Big|_{\{\nu\,|\,\nu<\mu\}}$. Then $f_{\nu<\mu}$ is a well ordered net from μ to $\mathbb{R}$ with image $A_\mu \setminus \{f(\mu)\}$. It is bounded by $f(\mu)$, so, as $\mathbb{R}$ is Hausdorff, from [5] (chapter 2, theorem 3, page 67) we know, that $f_{\nu<\mu}$ has a unique limit as a net. Let us call this limit r. This limit is as well an accumulation point of A, so by the assumption of the theorem, we have that $r \in A$.

Observation A.2.0.2. *We have that $r = f_\mu(\mu)$.*

Proof.
For the sake of contradiction, let us assume, that $r > f_\mu(\mu)$. Then, as r is an accumulation point of $f_{\nu<\mu}$, we have that $\exists_{\nu<\mu}\, f_\mu(\mu) < f_\mu(\nu)$. Which is a contradiction as f preservers the order. Hence, $r \leqslant f_\mu(\mu)$. Now, for the sake of contradiction, let us assume, that $r < f_\mu(\mu)$.
As we have, that $r \in A$, we have, that there exist some η such that $r = f(\eta)$. Since f preserves order and we assumed that $r < f_\mu(\mu)$, we have, that $\eta < \mu$. But then, as μ is a limit ordinal, we have, that $\eta + 1 < \mu$ as well. From this, we conclude that there exist some ordinal $< \mu$, namely $\eta + 1$, such that $f_\mu(\eta + 1) > r$. This however is a contradiction, as f preserves the order and r is an accumulation point. Hence, $r \geqslant f_\mu(\mu)$.
From this we conclude, that indeed $r = f_\mu(\mu)$. □
For the continuity of f and f^{-1} it is sufficient to show, that for every $U \subseteq A_\mu$ and $V \subseteq \mu + 1$ in prebases of respective topologies, $f_\mu^{-1}[U]$ and $f_\mu[V]$ are open $(*)$.
Prebase open sets in A_μ are the ones inherited from the order topology on $\mathbb{R}$, for all $s \in \mathbb{R}$:

$$\{r \mid r < s\} \cap A_\mu$$
$$\{r \mid s < r\} \cap A_\mu.$$

Prebase open sets in $\mu + 1$ are from order topology, for all $\nu \in \mu + 1$:

$$\{\eta \mid \eta < \nu\}$$
$$\{\eta \mid \nu < \eta\}.$$

From our induction assumption we have $(*)$ checked for following prebase sets:
$\{r \mid r < s\} \cap A_\mu$, for all $s < f(\mu)$.
$\{\eta \mid \eta < \nu\}$, for all $\nu < \mu$,
It was left to be shown, that $(*)$ holds for the prebase sets:
$\{r \mid r < f(\mu)\} \cap A_\mu$,
$\{r \mid s < r\} \cap A_\mu$, for all $s \leqslant f(\mu)$
$\{\eta \mid \eta < \mu\}$,
$\{\eta \mid \nu < \eta\}$, for all $\nu \leqslant \mu$,

Prebase set – $\{r \mid r < f(\mu)\} \cap A_\mu$:
We have that:

$$f^{-1}[\{r \mid s < f(\mu)\} \cap A_\mu] = \{\eta \mid \eta < \mu\},$$

which is open.
Prebase set – $\{r \mid s < r\} \cap A_\mu$:
Let $s < f(\mu)$. We have two cases:
– $s \in A$: then let ν be such that $f(\nu) = s$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A_\mu] = \{\eta \mid \nu < \eta\},$$

which is open.
– $s \notin A$: then, by the assumption of the theorem we know that s is not an accumulation point of A. From this we conclude, that $\exists_{t \in A}(t < s \wedge \neg\exists_{t' \in A} t < t' < s)$. Let ν be such that $f(\nu) = t$. Then we have that:

$$f^{-1}[\{r \mid s < r\} \cap A_\mu] = \{\eta \mid \nu < \eta\},$$

which is open.
Prebase set – : a
This concludes the inductive part of the proof, that for every $\mu < \alpha$ we have that f_μ is a homeomorphism. Now we will show, that this is sufficient for f to be a homeomorphism.

TO DO: wszystkie ograniczone otwarte działają

TO DO: wszystkie nieograniczone też