Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny
specjalność: teoretyczna

Bartosz Sójka

Two dimentional orbifolds' volumes' spectrum

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prof. dr hab. Tadeusza Januszkiewicza

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Abstract

Orbifoldy

Chapter 1

Introduction

Chapter 2

Different definitions of an orbifold

This chapter the next will be a technical chapters. Later on we will evoke some terms and definitions without explicitly saying what they mean instead we will put a reference to this chapter with explicit saying to what definition it refers.
For example in the later chapters there will be phrases like "adding a defect of order ..." or "gluying orbifolds by boundries" and they are explained in this and the next chapter.

We will explore various definitions of an orbifold, partially proving they are equivalent, partially linking to the sources.
Some of these definitions apply only to the special cases. Some of them contain constructions with which not all orbifolds can be made (at least some of them can't be derived as such a priori) .

2.1 Hiperbolic plane tilling

2.2 Manifolds with defects

2.2.1 Disk and sphere with defects

2.2.2 Conway notation

[2]

When it is nesseccery to avoid a confusion, on parts such as $*abcd$, we will be writting $*^*a^*b^*c$ instead.
We will propose some extension to a notation from [2]. We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature.
We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$ which have less capitalistic vibes than "cost". For example $\Delta(*^*2) = \frac{1}{4}$.
We will denote by $*$ an operation of cutting out a disk and by ${}^{\beta *}n$ an operation of

adding a kaleidoscopic point of period n on the boundry component β. Last operation is defined only on orbifolds with boundries.

2.3 Quationts of planes

2.4 Generalised manifolds

This approuch is very simmilar to the previous one. It differs slightly where we put the difinition burden.

2.5 Complexes?

Chapter 3

Order structure

In this chapter we will discuss order type of the set of all possible Euler orbicharacteristics of two dimentional orbifolds.
For now, until Chapter 4 Counting occurrences, we will not pay attension to how many orbifolds have the same Euler orbicharacteristic.
Because of that and since Euler orbicharacteristic does not depend on the cyclic order of points on the conponents of the boundry we introduce an extension of a notation from [2].

We will write $*\{a, b, c, d, \dots\}$ to denote a type of a boundry (of an orbifold) that have kaleidoscopic points of periods $a, b, c, d, \dots$, but in any order.

From what we wrote above (that Euler orbicharacteristic does not depend on the cyclic order of points on the conponents of the boundry), we can see that Euler orbicharacteristic is well defined when we specify only such a type of the components of the boundry of an orbifold and not a particular cyclic order.

3.1 Reductions of cases

Now we want to make some reductions to limit number of cases that we will be dealing with.
For this chapter we will consider orbifolds according to a definition from (2.2.1).
Let us observe, that:

$$\begin{aligned}
\Delta(\circ) &= & -2 & = \Delta(*({}^*2)^4) \\
\Delta(*) &= & -1 & = \Delta(({}^*2)^4) \\
\Delta(n) &= & \frac{n-1}{n} & = \Delta(({}^*n)^2)
\end{aligned}$$

From this we can conclude, that every Euler orbicharacteristic can be obtained by an orbifold of signature of a type (n and m are arbitrary):

$$I_1I_2\dots I_n\text{or}$$
$$*b_1b_2\dots b_m.$$

Let us denote the set of all possible Euler orbicharacteristics of orbifolds of the form $I_1I_2\dots I_n$ by $\sigma^I(S^2)$ and the set of all possible Euler orbicharacteristics of orbifolds of the form $*b_1b_2\dots b_m$ as $\sigma^b(D^2)$

Let us observe that the topological structure of $\sigma^I(S^2)$ and $\sigma^b(D^2)$ are the same since

$$2\sigma^b(D^2) = \sigma^I(S^2)$$

So multiplying by 2 is the homeomorphism.

3.2 Determining the order structure

In this chapter we will justify, that the order type of all possible Euler orbicharacteristics of two dimentional orbifolds is ω^ω. We will also describe precisely where condensation points lie and of which order (see below 3.2.1) they are.

3.2.1 Definitions regarding order of condensation points

We start with one technical definition of "transitive order" that will be almost what we want and then, there will be the the definition of "order", which is the definition that we need.

Definition 3.2.1.1. *(Inductive). We say that the point is a condensation point of a transitive order* 0*, when it is an isolated point. We say that the point is a condensation point of a transitive order* $n+1$*, when it is a condensation point (in the usual sense) of the condensation points of the transitive order* n*.*

The only issue of the definition is that the point of the transitive order n is also a point of the transitive order k, for all $0 < k \leqslant n$. We want a definition of order such that for any point, there is at most one integer that is its order. So we define:

Definition 3.2.1.2. *We say that the point is a condensation point of order* n *iff it is a condensation point of the transitive order* n *and it is not a condensation point of the transitive order* $n+1$*. If the point is a condensation point of the transitive order for an arbitraly large* n *we say that the point is a condensation point of order* ω*.*

3.2.2 $\sigma^b(D^2)$

Some preliminary observations.

Let us observe, that $\lim\limits_{n\to\infty}\Delta(^*n) = -\frac{1}{2}$. From that, we see, that for every point $x\in\sigma^b(D^2)$, the point $x-\frac{1}{2}$ is a condensation point. Let us observe, that also, for every point $x \in \sigma^b(D^2)$, we have that $x - \frac{1}{2} \in \sigma^b(D^2)$, because $\Delta((^*2)^2) = -\frac{1}{2}$.

Now we will show that the order type of $\sigma^b(D^2)$ is ω^ω and where exactly are its condensation points of which orders. For this we will use a handfull of lemmas.

Lemma 3.2.2.1. *If x is a condensation point of the set $\sigma^b(D^2)$ of order n, then $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+1$.*

Proof.
Inductive.
• $n = 0$: If x is an isolated point of the set $\sigma^b(D^2)$, then $x \in \sigma^b(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma^b(D^2)$, from that, that $x - \frac{1}{2}$ is a condensation point of $\sigma^b(D^2)$.
• inductive step: Let x be a condensation point of the set $\sigma^b(D^2)$ of an order $n > 0$. Let a_k be a sequence od condensation points of order $n-1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of condensation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+1$. □

Lemma 3.2.2.2. *If x is a condensation point of the set $\sigma^b(D^2)$ of order $n+1$, then $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least n.*

Proof.
Inductive
• n = 0:

Chapter 4

Counting occurrences

abcd

Chapter 5

Decidability

5.1 Algorithm

Here we will show the proof that the problem of decideng whether a given rational number is in an Euler orbicharacteristic's spectrum or not is decidable by showing algorithm for doing this.
We start with $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Chapter 6

Conclusions

Bibliography

[1] John Conway and Daniel Huson. The orbifold notation for two-dimensional groups. *Structural Chemistry*, 13, 08 2002.

[2] Chaim Goodman-Strauss John H. Conway, Heidi Burgiel. *The Symmetries of Things*. A K Peters, 1 edition, 2008.

[3] William P Thurston. *The geometry and topology of three-manifolds*. s.n, 1979.