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Two dimentional orbifolds' volumes' spectrum

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dla Wujka

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Orbifoldy

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Introduction

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Chapter 2

Definition, characteristics, classification and properties of the orbifolds

2.1 Definition

The definition of the orbifold is taken from Thurston [4] (chapter 13). We briefly recall the concept, but for full discussion we refer to [4].
An orbifold is a generalisation of a manifold. One allows more variety of local behaviour. On a manifold a map is a homeomorphism between $\mathbb{R}^n$ and some open set on a manifold. On an orbifold a map is a homeomorphism between a quotient of $\mathbb{R}^n$ by some finite group and some open set on an orbifold. In addition to that, the orbifold structure consist the informations about that finite group and a quotient map for any such open set.

Above definition says that an orbifold is locally homeomorphic do the quotient of $\mathbb{R}^n$ by some finite group.
When an orbifold as a whole is quotient of some finite group acting on a manifold we say, that it is 'good'. Otherwise we say, that it is 'bad'.

We are also adopting notation from [4].

In two dimentions there are only four types of bad orbifolds, namely:
- $S^2(n)$
- $D^2(;n)$
- $S^2(n_1, n_2)$ for $n_1 < n_2$
- $D^2(;n_1, n_2)$ for $n_1 < n_2$.

All other orbifolds are good.

2.2 Euler orbicharacteristic

2.2.1 Classification of orbifolds with non-negative Euler orbicharacteristic

The list of all orbifolds with non-negative Euler orbicharacteristic Powiedzieć coś o tym, że orbicharatkeryttyka odpowiada polom (Gauss Bonett itd.)

2.2.2 Extended Euler orbicharacteristic

(with cusps) Write about cusp as a limit.

2.3 Uniformisation theorem (formulation)

2.4 Surgeries, modifications and constructions on orbifolds

(Some preserve the area)

2.5 Notation

We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature. We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$.

TO DO: rozwinąć

Chapter 3

Order type and topology

In this chapter we will discuss that both the order type and the topology of the set of all possible Euler orbicharacteristics of two dimensional orbifolds are that of ω^ω. We will call this set σ.
We will see (in the observation 3.1.0.1) that the problem of determining this boils down to the analysis of all the possible values of the expressions:

$$2 - \sum_{i=1}^{n} \frac{I_i - 1}{I_i} \tag{3.0.0.0.1}$$

and

$$1 - \sum_{j=1}^{m} \frac{b_j - 1}{2b_j}, \tag{3.0.0.0.2}$$

where I_i, b_j varies over $\mathbb{N}_{>0} \cup \{\infty\}$.
As

$$2 - \sum_{i=1}^{n} \frac{I_i - 1}{I_i} = 2 - n + \sum_{i=1}^{n} \frac{1}{I_i} \tag{3.0.0.0.3}$$

and

$$1 - \sum_{j=1}^{m} \frac{b_j - 1}{2b_j} = 1 - m + \sum_{j=1}^{m} \frac{1}{2b_j}, \tag{3.0.0.0.4}$$

some questions about the spectrum are equivalent to some regarding Egyptian fractions. More on this connection is discussed in 5.4.

Disclaimer
For now, until Chapter 5 named "Counting occurrences", we will not pay attention to how many orbifolds have the same Euler orbicharacteristic.

3.1 Reductions of cases

Now we want to make some reductions to limit number of cases that we will be dealing with.

Let us observe, that:

$$\Delta(\circ) = \quad -2 \quad = \Delta(*({}^*2)^4) \tag{3.1.0.0.1}$$
$$\Delta(*) = \quad -1 \quad = \Delta(({}^*2)^4) \tag{3.1.0.0.2}$$
$$\Delta(n) = \quad \frac{n-1}{n} \quad = \Delta(({}^*n)^2) \tag{3.1.0.0.3}$$

From this we can conclude, that every Euler orbicharacteristic can be obtained by an orbifold of signature of a type (n and m are arbitrary):

$$I_1 I_2 \dots I_n \text{ or}$$
$$*b_1 b_2 \dots b_m.$$

Let us denote the set of all possible Euler orbicharacteristics of orbifolds of the form $I_1 I_2 \dots I_n$ by $\sigma^I(S^2)$ and the set of all possible Euler orbicharacteristics of orbifolds of the form $*b_1 b_2 \dots b_m$ as $\sigma^b(D^2)$. So we have that $\sigma = \sigma^b(D^2) \cup \sigma^I(S^2)$.
Let us also observe that the order type and topology of $\sigma^I(S^2)$ and $\sigma^b(D^2)$ are the same since

$$2\sigma^b(D^2) = \sigma^I(S^2) \tag{3.1.0.0.4}$$

and multiplying by 2 is the order preserving homeomorphism of $\mathbb{R}$.

Now we can make aforementioned observation:

Observation 3.1.0.1. *From above reductions we can conclued that our problem boiles down to the analysis of all the possible values of the expressions:*

$$2 - \sum_{i=1}^{n} \frac{I_i - 1}{I_i} \tag{3.1.0.1.1}$$

and

$$1 - \sum_{j=1}^{m} \frac{b_j - 1}{2b_j}. \tag{3.1.0.1.2}$$

We also have shown that all possible Euler orbicharacteristics are achieved without using cusps. As such, we will use cusps, remembering, that we can always get rid of them, if needed. So above I_i and b_j are ranging over $\mathbb{N}_{>0} \cup \{\infty\}$, where expressions for infinity are defined as a limits. The fact that it agrees with the definition of the Euler orbicharacteristic on the geometrical terms was addressed in 2.2.2.
Here is the notation that will be used:
σ - spectrum of all possible Euler orbicharacteristic of two-dimentional orbifolds
$\sigma(M)$ - spectrum of all possible Euler orbicharacteristic of M orbifolds
$\sigma^I(M)$ - spectrum of all possible Euler orbicharacteristic of M orbifolds with orbipoints only in the interior of M
$\sigma^b(M)$ - spectrum of all possible Euler orbicharacteristic of M orbifolds with orbipoints only on the boundry of M (when M has no boundry $\sigma^b(M) = \{\chi(M)\}$)

Examples that will be extensively use:
$\sigma^I(S^2)$ - spectrum of all possible Euler orbicharacteristic of S^2 orbifolds
$\sigma^b(D^2)$ - spectrum of all possible Euler orbicharacteristic of D^2 orbifolds with orbipoints only on the boundry (equal to $\sigma(D^2)$ - spectrum of all possible Euler orbicharacteristic of all D^2 orbifolds)

For any two-dimentional manifold M, observe, that:
Now we state what we can already conclude.

Observation 3.1.0.2. *If two-dimentional manifold M has no boundry, then:*

$$\sigma(M) = \sigma^I(M) = \chi(M) - (\sigma^I(S^2) - 2) \tag{3.1.0.2.1}$$

and

$$\sigma(M) \subseteq \sigma^I(S^2) \tag{3.1.0.2.2}$$

If, in addition, $M \neq S^2$, then

$$\sigma(M) \subseteq \sigma^b(D^2). \tag{3.1.0.2.3}$$

Proof
Let M be a two-dimentional manifold with no boundry, then orbipoints can be only in the interior of M, so $\sigma(M) = \sigma^I(M)$. From **??** we have, that:

$$\sigma(M) = \{\chi(M) - \sum_{i=1}^{n} \frac{I_i - 1}{I_i} \mid n \in \mathbb{N} \wedge \forall_i I_i \in \mathbb{N} \cup \infty\}. \tag{3.1.0.2.4}$$

And from $\chi(M) - (\sigma^I(S^2) - 2)$

Observation 3.1.0.3. *If two-dimentional manifold M has a boundry, then*

$$\begin{gathered} \sigma(M) = \sigma^b(M) = \\ \{\chi(M) - \sum_{j=1}^{m} \frac{b_j - 1}{2b_j} \mid m \in \mathbb{N} \wedge \forall_j b_j \in \mathbb{N} \cup \infty\} = \\ \chi(M) + (\sigma^b(D^2) - 1) \end{gathered} \tag{3.1.0.3.1}$$

.

Observation 3.1.0.4. *If two-dimentional manifold M has no boundry, then $\sigma(M) \subseteq \sigma^I(S^2)$. If, in addition, $M \neq S^2$, then $\sigma(M) \subseteq \sigma^b(D^2)$.*

Observation 3.1.0.5. *If two-dimentional manifold M has a boundry, then $\sigma(M) \subseteq \sigma^b(D^2)$.*

Theorem 3.1.0.6. *For any two two-dimentional manifolds M, N spectras $\sigma(M)$ and $\sigma(N)$ have the same order type and topology.*

Proof

3.2 Determining the order type

In this section we will justify, that the order type of the spectrum all possible Euler orbicharacteristics of two dimensional orbifolds σ is ω^{ω}. We will also describe precisely where accumulation points of the spectrum lie and of which order (see below 3.2.1) they are.
To do so, remembering that $\sigma = \sigma^{I}(S^2) \cup \sigma^{b}(D^2)$ and 3.1.0.6 we will first study how $\sigma^{b}(D^2)$ looks like.

3.2.1 Definitions regarding order of accumulation points

We start with one technical definition of "transitive order" that will be almost what we want and then, there will be the the definition of "order", which is the definition that we need.

Definition 3.2.1.1. *(Inductive). We say that the point is an acccumulation point of a transitive order* 0*, when it is an isolated point. We say that the point is an acccumulation point of a transitive order* $n+1$*, when it is an acccumulation point (in the usual sense) of the accumulation points of the transitive order* n*.*

The only issue of the above definition is that the point of the transitive order n is also a point of the transitive order k, for all $0 < k \leqslant n$. We want a definition of order such that for any point, there is at most one integer that is its order. So we define:

Definition 3.2.1.2. *We say that the point is an acccumulation point of order* n *iff it is an acccumulation point of the transitive order* n *and it is not an acccumulation point of the transitive order* $n+1$*. If the point is an acccumulation point of the transitive order for an arbitrary large* n *we say that the point is an acccumulation point of order* ω*.*

When we will say that a point is an accumulation point of some set without specifying an order then we will mean being an accumulation point in the ussual sense; from the point of view of above definitions, that is, an accumulation point of order at least one.

3.2.2 Order structure of $\sigma^{b}(D^2)$

Some preliminary observations

Let us observe, that $\lim\limits_{n\to\infty} \Delta(^{*}n) = -\frac{1}{2}$. From that, we see, that for every point $x \in \sigma^{b}(D^2)$, the point $x - \frac{1}{2}$ is an acccumulation point. Let us observe, that also, for every point $x \in \sigma^{b}(D^2)$, we have that $x - \frac{1}{2} \in \sigma^{b}(D^2)$, because $\Delta(^{*}\infty) = -\frac{1}{2}$.

Now we will show that the order type of $\sigma^{b}(D^2)$ is ω^{ω} and where exactly are its accumulation points of which orders. For this we will use a handful of lemmas.

Lemma 3.2.2.1. *If* x *is an acccumulation point of the set* $\sigma^{b}(D^2)$ *of order* n*, then* $x - \frac{1}{2}$ *is a accumulation point of the set* $\sigma^{b}(D^2)$ *of order at least* $n+1$*.*

Proof.
Inductive.
• $n = 0$: If x is an isolated point of the set $\sigma^b(D^2)$, then $x \in \sigma^b(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma^b(D^2)$, from that, that $x - \frac{1}{2}$ is a accumulation point of $\sigma^b(D^2)$.
• inductive step: Let x be an acccumulation point of the set $\sigma^b(D^2)$ of an order $n > 0$. Let a_k be a sequence of accumulation points of order $n-1$ convergent to x. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of accumulation points of order at least n. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+1$. $\square$

Lemma 3.2.2.2. *If x is an acccumulation point of the set $\sigma^b(D^2)$ of order n, then $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n-1$.*

Proof.
Inductive
• $n = 1$: We assume, that x is an acccumulation point of isolated points of the set $\sigma^b(D^2)$. Let us observe, that for all m there are only finitely many Euler orbicharacteristics in the interval $[1, x]$ of orbifolds that have cone points of period equal at most m.
From that, for arbitrary small neighborhood $U \ni x$ and arbitrary large m there exist an orbifold that has a cone point of period grater than m, whose Euler orbicharacteristic lies in U. Let us take a sequence of such Euler orbicharacteristics a_k convergent to x, such that we can choose a sequence divergent to infinity of periods of cone points b_k of orbifolds of Euler orbicharacteristics equal a_k.
To do: picture
Let us observe, that for all k, the number $a_k + \frac{b_k-1}{2b_k}$ is in $\sigma^b(D^2)$. It is so, because a_k is an Euler orbicharacteristicof an orbifold that have a cone point of period b_k, so identical orbifold, only without this cone point has an Euler orbicharacteristicequal to $a_k + \frac{b_k-1}{2b_k}$. The sequence $a_k + \frac{b_k-1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least 0.
• inductive step: Let x be an acccumulation point of the set $\sigma^b(D^2)$ of order $n > 1$. Let a_k be a sequence of accumulation points of the set $\sigma^b(D^2)$ of order $n-1$ convergent to x. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of an acccumulation points of the set $\sigma^b(D^2)$ of order $n-2$ convergent to $x + \frac{1}{2}$. From that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n-1$. $\square$

Lemma 3.2.2.3. *If x is an acccumulation point of the set $\sigma^b(D^2)$ of order $n+1$, then*
$x - \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $n+2$ and
$x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order n.

Proof.
Let x be an acccumulation point of the set $\sigma^b(D^2)$ of order $n+1$. From the lemma 3.2.2.1 we know, that $x - \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$. Now let us assume (for a contradiction), that $x - \frac{1}{2}$ is an accumulation

point of the set$\sigma^b(D^2)$ of order $k > n+2$. But then from the lemma 3.2.2.2 we have that x is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction.
Analogously, from the lemma 3.2.2.2 we know, that $x + \frac{1}{2}$ is a accumulation point of the set $\sigma^b(D^2)$ of order at least n. Let us assume (for a contradiction), that $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $k > n$. But then from the lemma 3.2.2.1 we have that x is an acccumulation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction. □

Lemma 3.2.2.4. *For all $n \in \mathbb{N}$ all accumulation points of the set $\sigma^b(D^2)$ of order n are in $\sigma^b(D^2)$.*

Proof.
Inductive
• $n = 0$: Clear, as they are isolated points of $\sigma^b(D^2)$.
• inductive step: Let x be a accumulation point of the set$\sigma^b(D^2)$ of order $n > 0$. From the lemma 3.2.2.3 point $x + \frac{1}{2}$ is an acccumulation point of the set $\sigma^b(D^2)$ of order $n-1$. From the inductive assumption $x + \frac{1}{2} \in \sigma^b(D^2)$. Then $x \in \sigma^b(D^2)$. □

Lemma 3.2.2.5. *If $A, B \subseteqq \mathbb{R}$ have no infinite ascending sequences, then set $A+B := \{a + b \mid a \in A, b \in B\}$ also have no infinite ascending sequences.*

Proof.
Let A, B have no infinite ascending sequences. Let $c_n \in A + B$ are elements of some sequence. With a sequence c_n there are two associated sequences a_n, b_n, such that, for all n, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradiction), that c_n is an infinite ascending sequence. Then $\forall_n \ a_{n+1} > a_n \ \vee \ b_{n+1} > b_n$. From the assumption a_n has no infinite ascending sequence, so a_n has a weakly decreasing subsequence a_{n_k}. But then subsequence b_{n_k} must be strictly increasing, what gives us a contradiction. ↯□

Lemma 3.2.2.6. *In $\sigma^b(D^2)$ there are no infinite ascending sequences.*

Proof.
Let us denote by A_n the set of all possible Euler orbicharacteristics realised by orbifolds of type $*b_1, \ldots, b_n$. Then $A_0 = \{1\}$ and $A_{n+1} = A_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$. From that, from the lemma 3.2.2.5, for all n, we have that A_n do not have infinite ascending sequence. $\sigma^b(D^2) = \bigcup\limits_{n=0}^{\infty} A_n$. Let us also observe, that for all n, we have $A_n \subseteqq [1 - \frac{n}{4}, 1 - \frac{n}{2}]$. From that we have $\sigma^b(D^2)$ do not have infinite ascending sequences. □

Theorem 3.2.2.7. *The biggest accumulation point of the set $\sigma^b(D^2)$ of order n is $1 - \frac{n}{2}$.*

Proof.
Inductive
• $n = 0$: $1 \in \sigma^b(D^2)$ and 1 is the biggest element of $\sigma^b(D^2)$.
• an inductive step: From the inductive assumption we know that $1 - \frac{n}{2}$ is the biggest

accumulation point of the set $\sigma^b(D^2)$ of order n. From the lemma 3.2.2.3 we have then that $1-\frac{n+1}{2}$ is a accumulation point of the set$\sigma^b(D^2)$ of order $n+1$. Let us assume (for a contradiction), that there exist a bigger accumulation point of order $n+1$ equal to $y > 1-\frac{n+1}{2}$. But then, from lemma 3.2.2.3, point $y+\frac{1}{2}$ would be an acccumulation point of order n, what gives a contradiction, because $y+\frac{1}{2} > 1-\frac{n}{2}$. □

3.2.3 Order structure of the set of all possible Euler orbicharacteristics σ

Theorem 3.2.3.1. *The order type of the set of possible Euler orbicharacteristics of two dimensional orbifolds σ is ω^ω.*

Proof.
From the lemma 3.2.2.6 we know, that $\sigma^b(D^2)$ is well ordered. From this and from the theorem 3.2.2.7 we know, that for the point $1-\frac{n}{2}$ there exist a neighborhood $U = (1-\frac{n}{2}-\varepsilon, 1-\frac{n}{2}+\varepsilon)$ such that $U \cap \sigma^b(D^2)$ is homeomorphic to ω^n. From this, and again from theorem 3.2.2.7 we have that $\sigma^b(D^2) \cap [1, 1-\frac{n}{2})$ is homeomorphic with ω^n. From this $\sigma^b(D^2)$ is homeomorphic with ω^ω. From this $\sigma^I(S^2)$ is homeomorphic with ω^ω.
$\sigma^I(S^2) = 2\sigma^b(D^2)$, so for all $n \in -\mathbb{N}$ set $\sigma^I(S^2) \cap [2, n)$ has a lower order type then $\sigma^b(D^2) \cap [2, n)$. From this, we have that $\sigma^I(S^2) \cup \sigma^b(D^2) \cong \omega^\omega$. □

From the above discussion we can conclude following:

Corollary 3.2.3.2. *Let $x \in \sigma$. Then:*

- *there exists $n_1 \in \mathbb{N}$ such that $x + \frac{n_1}{2} \in \sigma$ but $x + \frac{n_1+1}{2} \notin \sigma$.*
  In other words, there exist $y \in \sigma$ and $n_1 \in \mathbb{N}$ such that $y+\frac{1}{2} \notin \sigma$ and such that $x = y - \frac{n_1}{2}$;
- *there exists $n_2 \in \mathbb{N}$ such that x is an accumulation point of the set σ of order n_2*

and $n_1 = n_2$.

3.3 Determining the topology

3.4 Which points are in the σ?

Here we will try to understand better the conditions that let us determine wether the point lie in σ or not.

3.5 More about how this ω^ω lies in $\mathbb{R}$

Theorem 3.5.0.1. *The first (biggest) negative accumulation point of the set of all possible Euler orbicharacteristicof two dimensional orbifolds is $-\frac{1}{12}$. It is the accumulation point of order 1.*

Proof.
We will show, that $-\frac{1}{12}$ is the biggest negative accumulation point of the set $\sigma^b(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two dimensional orbifolds is equal to $\sigma^I(S^2) \cup \sigma^b(D^2)$ and $\sigma^I(S^2) = 2\sigma^b(D^2)$, so the biggest negative point of the set $\sigma^I(S^2)$ is smaller than the biggest negative accumulation point of the set $\sigma^b(D^2)$.
• $-\frac{1}{12} = \chi^{orb}((2,3)) - \frac{1}{2}$, from this we have that $-\frac{1}{12}$ an acccumulation point of the set $\sigma^b(D^2)$ of order at least 1.
• Let us assume (for a contradiction), that there exist bigger, negative accumulation point of the set $\sigma^b(D^2)$ of order at least 1. Let us denote it by x.
However, then, from the lemma 3.2.2.3 point $x + \frac{1}{2}$ is the accumulation point of the set $\sigma^b(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12}$. From the lemma 3.2.2.4 we have that x is in $\sigma^b(D^2)$. But orbifolds of the type $*b_1$ can have Euler orbicharacteristiconly greater or equal $\frac{1}{2}$. Orbifolds of the type $*b_1b_2$ can only have Euler orbicharacteristic$\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*b_1b_2b_3\dots$ can have Euler orbicharacteristiconly lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma^b(D^2) = \emptyset$ and to the contradiction.
• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma^b(D^2)$, from this $-\frac{1}{12}$ is an acccumulation point of order 1 of the set $\sigma^b(D^2)$. □

Chapter 4

Algorithms for searching the spectrum

4.1 Decidability

Here we will show the proof that the problem of "deciding whether a given rational number is in an Euler orbicharacteristic's spectrum or not" is decidable by showing algorithm for doing this. Later, our algorithm will have a bonus property of determining of which order of condensation is given point if it is in fact in σ.

To do: Może od razu postawić pełny problem

First stated algorithm is also very inefficient and is presented, because the idea is the most clear in it. Right after it there is stated an algorithm with two enhancements:

- determining an accumulation point of which order is a given point, if it is in fact in the spectrum (this enhancement gives also a performance boost)
- faster searching, because some cases do not need to be checked.

We start with $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$.

We want to determine whether there exists $b_1, b_2, \ldots, b_k$, such that $\chi^{orb}(*b_1 \ldots b_k) = \frac{p}{q}$.

In the case that $\frac{p}{q}$ is of the form $l\frac{1}{4}$, for some whole l we can give the answer right away. For $l > 4$ we have that $l\frac{1}{4}$ is not in the set and for $l \leqslant 4$ it is. Moreover for an even l it is a condensation point of order $\frac{4-l}{2}$ (see 3.2.2.7) and for an odd l it is a condensation point of order $\frac{3-l}{2}$ (see 3.2.3.2).

Now we will consider only cases when $\frac{p}{q}$ is not of the form $l\frac{1}{4}$.

4.1.1 The first approach to the searching algorithm

We use:

- $\mathbb{N}_{>0}$ counters $b_1 b_2 \dots$ with values ranging from 1, through all natural numbers, to infinity (with infinity included). Each counter correspond to one cone point on the boundry of the disk of period equal to the value of the counter (with the note, that if counter is set to 1 it means a trivial cone point - namely a none cone point, a normal point).
- a pivot pointing to some counter at any time
- a flag that can be set to "Greater" or "Less" corresponding to what was the outcome of comparing Euler orbicharacteristic of the orbifold corresponding to counters' state and $\frac{p}{q}$.

We start with:

- all counters set to 1.
- pivot pointing at the first counter
- flag set to "Greater"

We will do our computation such that:

- every state of the counters during runtime of the algorithm will have only finitely many counters with value non-1.
- every state in the rutime of the algorithm will have values on consequtive counters ordered in weakly decreasing order.

From now we will consider only such states.
The state of the counters $b_1 b_2 \dots$ correspond to the orbifold of Euler orbicharacteristic equal $\chi^{orb}(*b_1 b_2 \dots)$ (where the trailing 1 are trunkated).
When the algorithm is in the state:

- counters: $b_1 b_2 \dots$
- pivot: on the counter c
- flag: set to the value $flag\_value$,

we procced as follows :

```
In the case, the flag is set to:
{
    "Less", then
    {
        We increase the counter c by one (b_c := b_c + 1).
        If b_c = 2 and the values of all the counters
        on the left are also equal 2 then
        {
            We end the whole algorithm with the answer "no".
        }
```

```
11          We set the value of all counters on the left to $b_c$
12          If $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
13          {
14              We found an orbifold and we are ending the whole
15              algorithm with answer "yes, $*b_1b_2\dots$".
16          }
17          If $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$ then
18          {
19              We set the flag to "Greater".
20              We put the pivot on the first counter.
21              We go to the line 1..
22          }
23          If $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$ then
24          {
25              We set the flag to "Less".
26              We put pivot to the $c+1$ counter.
27              We go to the line 1..
28          }
29      }
30
31      "Greater", then
32      {
33          If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) = \frac{p}{q}$ then
34          {
35              We found an orbifold and we are ending the whole
36              algorithm with answer "yes, $*b_1\dots b_{c-1}\infty b_{c+1}\dots$".
37          }
38          If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) > \frac{p}{q}$ then
39          {
40              We set $b_c$ to $\infty$.
41              We set the flag to "Greater".
42              We move pivot to the $c+1$ counter.
43              We go to the line 1..
44          }
45          If $\chi^{orb}(*b_1\dots b_{c-1}\infty b_{c+1}\dots) < \frac{p}{q}$ then
46          {
47              We search for value $b'_c$ of the $c$ counter
48              such that $\chi^{orb}(*b_1\dots b_{c-1}b'_cb_{c+1}\dots) \leqslant \frac{p}{q}$
49              and $\chi^{orb}(*b_1\dots b_{c-1}(b'_c-1)b_{c+1}\dots) > \frac{p}{q}$.
50              More on how we search for it will be told later, for now
51              we can think that we search one by one starting
52              from $b_c$ and going up till $b'_c$.
53              We set $b_c$ to $b'_c$.
54              if $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
```

```
                {
                    We found an orbifold and we are ending the whole
                     algorithm with answer "yes, $*b_1b_2\dots$".
                }
               We set all the counters to the left to value $b_c$.
                if $\chi^{orb}(*b_1b_2b_3\dots) = \frac{p}{q}$ then
                {
                    We found an orbifold and we are ending the whole
                     algorithm with answer "yes, $*b_1b_2\dots$".
                }
                If $\chi^{orb}(*b_1b_2b_3\dots) < \frac{p}{q}$ then
                {
                    We set flag to "Less".
                    We move the pivot to the column $c+1$.
                    We go to the line 1..
                }
                If $\chi^{orb}(*b_1b_2b_3\dots) > \frac{p}{q}$ then
                {
                    We set the flag to "Greater".
                    We move the pivot to the first counter.
                    We go to the line 1..
                }
            }
        }
}
```

4.1.2 Why this works

4.1.3 Improvements

Let $m \in \mathbb{N}$ be such that $\frac{p}{q} \in (1 - \frac{m}{2}, 1 - \frac{m+1}{2})$ Let us denote by $r \coloneqq \frac{p}{q} - (1 - \frac{m}{2})$.

We will searching in σ as such:

If $\frac{p}{q} \in \sigma$, then, from the corollary 3.2.3.2 we know, that there exist some $n \in \mathbb{N}$, such that $\frac{p}{q} + \frac{n}{2} \in \sigma$ but $\frac{p}{q} + \frac{n}{2} \notin \sigma$.

We will be consequently checking points from $1 + r$, through $1 + r - \frac{l}{2}$, for $0 \leqslant l \leqslant m$, to the $\frac{p}{q}$. We stop at the first found point. If one of these point is in the spectrum, then all smaller (so also $\frac{p}{q}$) are in the spectrum and $\frac{p}{q}$ is the accumulation point of the spectrum of order $m - l$ (from this, we can see some heuristic, that the points that have smaller order will be generally harder to find in some sense). If none of this points are in in the spectrum, then $\frac{p}{q}$ is not.

Searching for all occurences

4.1.4 Implementation

As an appendix, there is a sample implementation of this algorithm with full described enhancments, written in Rust. It is in the separate file, as it would take too much space in this document and wouldn't be readable.

Chapter 5

Counting occurrences

Our ultimate goal is to give the answer to the questions such as:
- For a given $x \in \sigma$, how many orbifolds have x as their Euler orbicharacteristic?
- Why? Is there some underlying geometrical reason for that?
- Can we characterise points $x \in \sigma$ that has the most orbifolds corresponding to them?
- Is there any reasonable normalisation to counter the effect that there are 'more' points as we go to lesser values. (What we mean by 'more' was sted in)

The first equstion we can tackle is steaming from the chapter 3 and it is – Do $\sigma^b(D^2)$ and $\sigma^I(S^2)$ coincide? It is easy to answer that $\sigma^b(D^2) \neq \sigma^I(S^2)$ (and we will do that along some harder questions in the moment), but do they coincide starting from a sufficiently distant point? Or maybe, for every denominator, do they coincide from a sufficiently distatn point? (Yes.)

5.1 Finitenes

First we will show that for any $x \in \sigma$ there are always only finitely many orbifolds with an Euler orbicharacteristic equal to x.
Let us observe, that we only need to show this for S^2 orbifolds. It is like that, because, as discussed in 2.4 every orbifold can be obtained be modyfying the sphere and there is only finitely many possible modifications that are not adding an orbipoint, each changing Euler orbicharacteristic by non-zero value.

Theorem 5.1.0.1. *For any $x \in \sigma$ there are always only finitely many orbifolds with an Euler orbicharacteristic equal to x.*

Proof:
According to the note above, we only need to proof this for S^2 orbifolds.
Let x be a rational number. For the sake of contradiction, assume, that there exists an infinite family of orbifolds $\{\mathcal{O}\}_{i\in I}$ with an Euler orbicharacteristic of each qual to x. For each i, tet m_i be the order of the orbipoint with the highest order of $\mathcal{O}_i$. As for every $n \in \mathbb{N}$ there are only finitely many S^2 orbifolds with all orbipoints of

order less than n, we have that the set $\{m_i\}_{i\in I}$ is unbounded. Let $\{m_n\}_{n\in\mathbb{N}}$ be some strictly increasing sequence of elements of $\{m_i\}_{i\in I}$ that diverges into infinity.
Let $\{a_n\}$ be the sequence of differences in Euler orbicharacteristic caused by points corresponding to $\{m_i\}$. Let $\{b_n\}$ be the sequence of differences in Euler orbicharacteristic caused by other points on those orbifolds. So for every n we have $\chi^{orb}(\mathcal{O}_n) = 2+a_n+b_n$. As $\{m_n\}$ is strictly increasing we have that a_n is strictly decreasing, so b_n must be strictly increasing, because $\chi^{orb}(\mathcal{O}_n)$ is constant for all n (all $\{\mathcal{O}_n\}$ are from the family with Euler orbicharacteristic equal to x).
But $\{b_n\} \subseteq \sigma^I(S^2) - 2$, so it is well ordered as $\sigma^I(S^2)$ is well ordered. From 3.2.2.6 and 3.1.0.0.4 we know that $\sigma^I(S^2)$ has no infinite strongly increasing sequences, so $\sigma^I(S^2) - 2$ has no infinite strongly increasing sequences. That gives us a contradiction. $\lightning$ $\square$

5.2 Some connections between Euler orbicharacteristic and geometry of corresponding orbifolds

Here we will state some observations and corollaries derived from previous chapters about ...

Observation 5.2.0.1. *If an Euler orbicharacteristic is an accumulation point of order n in $\sigma^b(D^2)$ $//\sigma^I(S^2)//$, there exist an orbifold of the type ... $////$ with n cone $//$gyration$//$ points of that Euler orbicharacteristic.*

prrof. from chapter 3. (todo: dopisać)

Observation 5.2.0.2. *If $x \in \sigma^b(D^2)$ $//\sigma^I(S^2)//$, then $1-x$ $//2-x//$ is a difference in Euler orbicharacteristic resulting from some set of cone $//$gyration$//$ points. From that $1 - n(1 - x) \in \sigma^b(D^2)$ $//2 - n(2 - x) \in \sigma^I(S^2)//$ for all $n \in \mathbb{N}$.*

5.3 $\sigma^b(D^2)$ and $\sigma^I(S^2)$

In this section we would like to develop the tools and answer some questions about relations between $\sigma^b(D^2)$ and $\sigma^I(S^2)$.
The first, stated in 3 is that $2\sigma^b(D^2) = \sigma^I(S^2)$. This tells us all about simmilarities of their topological structures – namely, they are the same, but it does not directly answers questions about how they lie in $\mathbb{R}$, relative to each other.

5.3.1 $-\frac{1}{84}$ and $-\frac{1}{42}$

//Why it is how it is//

5.3.2 Accumulation points of the $\sigma^I(S^2)$

Theorem 5.3.2.1. *All accumulation points of the $\sigma^I(S^2)$ are in $\sigma^b(D^2)$.*

There are two proofs of this theorem showing nice correnpondence – one arithmetical and one geometrical.

Proof I. Arithmetical reason

We assume that $x \in \sigma^I(S^2)$ is an accumulation point of the set $\sigma^I(S^2)$.
By 3.1.0.0.4 we have, that $\frac{x}{2} \in \sigma^b(D^2)$ is an accumulation point of the set $\sigma^b(D^2)$.
From 3.2.2.3 we have that $\frac{x}{2} + \frac{1}{2} \in \sigma^b(D^2)$. From that, from 5.2.0.2 we have, that

$1 - \overbrace{2}^{\text{"}n\text{" from 5.2.0.2}} \left(\overbrace{1 - (\frac{x}{2} + \frac{1}{2})}^{\text{"}1-x\text{" from 5.2.0.2}} \right) \in \sigma^b(D^2)$. But $1 - 2(1 - (\frac{x}{2} + \frac{1}{2})) = x$, so $x \in \sigma^b(D^2)$. □

Proof II. Geometrical reason

We assume that $x \in \sigma^I(S^2)$ is an accumulation point of the set $\sigma^I(S^2)$.
From 3.2.3.2 we know, that x can be expressed as $y - 1$ for some $y \in \sigma^I(S^2)$.
Let $\mathcal{O}$ be an orbifold with the base manifold S^2, such that $\chi^{orb}(\mathcal{O}) = y$.
Let $\mathcal{O}_c$ be the orbifold created from $\mathcal{O}$ by adding one cusp. Then $\chi^{orb}(\mathcal{O}_c) = y - 1 = x$. Topologically $\mathcal{O}_c$ with the cusp point removed (which do not change an orbicharacteristic) is $\mathbb{R}^2$. We can compactify it with S^1. This will not change an Euler orbicharacteristic since $\chi^{orb}(S^1) = 0$ and Euler orbicharacteristic is additive. What we get is an orbifold $\mathcal{O}_D$ with the base manifold D^2 and the same orbipoints as $\mathcal{O}$. Since orbipoints of $\mathcal{O}$ create a difference in Euler orbicharacteristic equal to $2 - y$, we have that $\chi^{orb}(\mathcal{O}_D) = 1 - (2 - y) = y - 1 = x$. We can then move all orbipoints from the interior of $\mathcal{O}_D$ to its boundry by doubling them, so $x \in \sigma^b(D^2)$. □

5.4 Translating questions to ones about Egyptian fractions

5.5 Estimations of the number of occurences

5.6 Deformations on orbifolds?

Chapter 6

Conclusions

Chapter 7

Further directions

7.1 Power series and generating functions

7.2 Seifert manifolds

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