**Uniwersytet Wrocławski**
**Wydział Matematyki i Informatyki**
**Instytut Matematyczny**
*specjalność: teoretyczna*

***Bartosz Sójka***

# Two dimentional orbifolds' volumes' spectrum

Praca magisterska
napisana pod kierunkiem
prof. dr hab. Tadeusza Januszkiewicza

Wrocław Rok 2021

# Contents

**Abstract**

Orbifoldy

# Chapter 1

# Introduction

# Chapter 2

# Different definitions of an orbifold

Jak ma wyglądać: jedna definicja (Thurstona) notacja może byc od Conwaya

skleić drugi i trzeci rozdział

zwięźle własności

dobre i złe w trzecim rozdziale

orbikompleksy może nie

This chapter the next will be a technical chapters. Later on we will evoke some terms and definitions without explicitly saying what they mean instead we will put a reference to this chapter with explicit saying to what definition it refers.
For example in the later chapters there will be phrases like "adding a defect of order ..." or "gluying orbifolds by boundries" and they are explained in this and the next chapter.

We will explore various definitions of an orbifold, partially proving they are equivalent, partially linking to the sources.
Some of these definitions apply only to the special cases. Some of them contain constructions with which not all orbifolds can be made (at least some of them can't be derived as such a priori) .

## 2.1 Hiperbolic plane tilling

## 2.2 Manifolds with defects

### 2.2.1 Disk and sphere with defects

### 2.2.2 Conway notation

[2]

When it is nesseccery to avoid a confusion, on parts such as $*abcd$, we will be writting $*^*a^*b^*c$ instead.
We will propose some extension to a notation from [2]. We will regard parts of that notation not only as features on an orbifold but also as an operations on orbifolds transforming one to another by adding particular feature.

We will denote the difference in Euler characteristic which is made by modifying an orbifold by such a feature as $\Delta(modification)$ which have less capitalistic vibes than "cost". For example $\Delta(*^{*}2) = \frac{1}{4}$.
We will denote by $*$ an operation of cutting out a disk and by ${}^{\beta *}n$ an operation of adding a kaleidoscopic point of period $n$ on the boundry component $\beta$. Last operation is defined only on orbifolds with boundries.

## 2.3 Quationts of planes

## 2.4 Generalised manifolds

This approuch is very simmilar to the previous one. It differs slightly where we put the difinition burden.

# Chapter 3

# Characteristics, classification and properties of the orbifolds

## 3.1 Euler orbicharacteristic

### 3.1.1 Classification of orbifolds with non-negative Euler orbicharacteristic

### 3.1.2 Extended Euler orbicharacteristic

## 3.2 Uniformisation theorem (formulation)

## 3.3 Surgeries, modifications and constructions on orbifolds

(Some preserve an area)

# Chapter 4

# Order structure

Order type with zanurzenie w R

1537/137

In this chapter we will discuss order type of the set of all possible Euler orbicharacteristics of two dimentional orbifolds.
For now, until Chapter 6 Counting occurrences, we will not pay attension to how many orbifolds have the same Euler orbicharacteristic.
Because of that and since Euler orbicharacteristic does not depend on the cyclic order of points on the conponents of the boundry we introduce an extension of a notation from [2].

We will write $*\{a, b, c, d, \dots\}$ to denote a type of a boundry (of an orbifold) that have kaleidoscopic points of periods $a, b, c, d, \dots$, but in any order.

From what we wrote above (that Euler orbicharacteristic does not depend on the cyclic order of points on the conponents of the boundry), we can see that Euler orbicharacteristic is well defined when we specify only such a type of the components of the boundry of an orbifold and not a particular cyclic order.

## 4.1 Reductions of cases

Now we want to make some reductions to limit number of cases that we will be dealing with.
For this chapter we will consider orbifolds according to a definition from (2.2.1).
Let us observe, that:

$$\begin{aligned} \Delta(\circ) = && -2 && = \Delta(*(^*2)^4) \\ \Delta(*) = && -1 && = \Delta((^*2)^4) \\ \Delta(n) = && \frac{n-1}{n} && = \Delta((^*n)^2) \end{aligned}$$

From this we can conclude, that every Euler orbicharacteristic can be obtained by an orbifold of signature of a type ($n$ and $m$ are arbitrary):

$$I_1 I_2 \dots I_n \text{or}$$
$$*b_1 b_2 \dots b_m.$$

Let us denote the set of all possible Euler orbicharacteristics of orbifolds of the form $I_1 I_2 \dots I_n$ by $\sigma^I(S^2)$ and the set of all possible Euler orbicharacteristics of orbifolds of the form $*b_1 b_2 \dots b_m$ as $\sigma^b(D^2)$

Let us observe that the topological structure of $\sigma^I(S^2)$ and $\sigma^b(D^2)$ are the same since

$$2\sigma^b(D^2) = \sigma^I(S^2)$$

So multiplying by 2 is the homeomorphism.

## 4.2 Determining the order structure

In this chapter we will justify, that the order type of all possible Euler orbicharacteristics of two dimentional orbifolds is $\omega^\omega$. We will also describe precisely where condensation points lie and of which order (see below 4.2.1) they are.

### 4.2.1 Definitions regarding order of condensation points

We start with one technical definition of "transitive order" that will be almost what we want and then, there will be the the definition of "order", which is the definition that we need.

**Definition 4.2.1.1.** *(Inductive). We say that the point is a condensation point of a transitive order* $0$*, when it is an isolated point. We say that the point is a condensation point of a transitive order* $n+1$*, when it is a condensation point (in the usual sense) of the condensation points of the transitive order* $n$*.*

The only issue of the definition is that the point of the transitive order $n$ is also a point of the transitive order $k$, for all $0 < k \leqslant n$. We want a definition of order such that for any point, there is at most one integer that is its order. So we define:

**Definition 4.2.1.2.** *We say that the point is a condensation point of order* $n$ *iff it is a condensation point of the transitive order* $n$ *and it is not a condensation point of the transitive order* $n+1$*. If the point is a condensation point of the transitive order for an arbitraly large* $n$ *we say that the point is a condensation point of order* $\omega$*.*

### 4.2.2 $\sigma^b(D^2)$

**Some preliminary observations.**

Let us observe, that $\lim\limits_{n\to\infty} \Delta(^*n) = -\frac{1}{2}$. From that, we see, that for every point $x \in \sigma^b(D^2)$, the point $x - \frac{1}{2}$ is a condensation point. Let us observe, that also, for every point $x \in \sigma^b(D^2)$, we have that $x - \frac{1}{2} \in \sigma^b(D^2)$, because $\Delta(^*\infty) = -\frac{1}{2}$.

Now we will show that the order type of $\sigma^b(D^2)$ is $\omega^\omega$ and where exactly are its condensation points of which orders. For this we will use a handfull of lemmas.

**Lemma 4.2.2.1.** *If $x$ is a condensation point of the set $\sigma^b(D^2)$ of order $n$, then $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+1$.*

**Proof.**
Inductive.
• $n = 0$: If $x$ is an isolated point of the set $\sigma^b(D^2)$, then $x \in \sigma^b(D^2)$. From that, we have, that points $x - \frac{k-1}{2k}$ are in $\sigma^b(D^2)$, from that, that $x - \frac{1}{2}$ is a condensation point of $\sigma^b(D^2)$.
• inductive step: Let $x$ be a condensation point of the set $\sigma^b(D^2)$ of an order $n > 0$. Let $a_k$ be a sequence od condensation points of order $n-1$ convergent to $x$. From the inductive assumption, we have, that $a_k - \frac{1}{2}$ is a sequence of condensation points of order at least $n$. From the basic sequence arithmetic it is convergent to $x - \frac{1}{2}$. From that, we have that $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+1$. □

**Lemma 4.2.2.2.** *If $x$ is a condensation point of the set $\sigma^b(D^2)$ of order $n$, then $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n-1$.*

**Proof.**
Inductive
• $n = 1$: We assume, that $x$ is a condensation point of isolated points of the set $\sigma^b(D^2)$. Let us observe, that for all $m$ there are only finitely many Euler orbicharacteristics in the interval $[1, x]$ of orbifolds that have cone points of period equal at most $m$. From that, for arbirary small neighborhood $U \ni x$ and arbitrary large $m$ there exist an orbifold that has a cone point of period grater than $m$, whose Euler orbicharacteristic lies in $U$. Let us take a sequence of such Euler orbicharacteristics $a_k$ convergent to $x$, such that we can choose a sequence divergent to infinity of periods of cone points $b_k$ of orbifolds of Euler orbicharacteristics equal $a_k$.
**To do: picture**
Let us observe, that for all $k$, the number $a_k + \frac{b_k-1}{2b_k}$ is in $\sigma^b(D^2)$. It is so, because $a_k$ is an Euler orbicharacteristicof an orbifold that have a cone point of period $b_k$, so identical orbifold, only without this cone point has an Euler orbicharacteristicequal to $a_k + \frac{b_k-1}{2b_k}$. The sequence $a_k + \frac{b_k-1}{2b_k}$ converge to $x + \frac{1}{2}$. From that we have, that $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least 0.
• inductive step: Let $x$ be a condensation point of the set $\sigma^b(D^2)$ of order $n > 1$. Let $a_k$ be a sequence of condensation points of the set $\sigma^b(D^2)$ of order $n-1$ convergent to $x$. From the inductive assumption the sequence $a_k + \frac{1}{2}$ is a sequence of a condensation points of the set $\sigma^b(D^2)$ of order $n-2$ convergent to $x + \frac{1}{2}$. From that $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n-1$. □

**Lemma 4.2.2.3.** *If $x$ is a condensation point of the set $\sigma^b(D^2)$ of order $n+1$, then $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order $n+2$ and $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order $n$.*

**Proof.**
Let $x$ be a condensation point of the set $\sigma^b(D^2)$ of order $n+1$. From the lemma 4.2.2.1 we know, that $x - \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+2$. Now tet us assume (for a contradiction), that $x - \frac{1}{2}$ is an condensation point of the set$\sigma^b(D^2)$ of order $k > n+2$. But then from the lemma 4.2.2.2 we have that $x$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction.
Analogousely, from the lemma 4.2.2.2 we know, that $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n$. Let us assume (for a contradiction), that $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order $k > n$. But then from the lemma 4.2.2.1 we have that $x$ is a condensation point of the set $\sigma^b(D^2)$ of order at least $n+2$ and that is a contradiction. $\square$

**Lemma 4.2.2.4.** *For all $n \in \mathbb{N}$ all condensation points of the set $\sigma^b(D^2)$ of order $n$ are in $\sigma^b(D^2)$.*

**Proof.**
Inductive
• $n = 0$: Clear, as they are isolated points of $\sigma^b(D^2)$.
• inductive step: Let $x$ be a condensation point of the set$\sigma^b(D^2)$ of order $n > 0$. From the lemma 4.2.2.3 point $x + \frac{1}{2}$ is a condensation point of the set $\sigma^b(D^2)$ of order $n-1$. From the inductive assumption $x + \frac{1}{2} \in \sigma^b(D^2)$. Then $x \in \sigma^b(D^2)$. $\square$

**Lemma 4.2.2.5.** *If $A, B \subseteqq \mathbb{R}$ have no infinite ascending sequences, then set $A+B := \{a+b \mid a \in A, b \in B\}$ also have no infinite ascending sequences.*

**Proof.**
Let $A$, $B$ have no infinite ascending sequences. Let $c_n \in A+B$ are elements of some sequence. With a sequence $c_n$ there are two associated sequences $a_n$, $b_n$, such that, for all $n$, we have $a_n \in A$, $b_n \in B$ and $a_n + b_n = c_n$. Assume (for contradicition), that $c_n$ is an infinite ascending sequence. Then $\forall_n \; a_{n+1} > a_n \;\vee\; b_{n+1} > b_n$. From the assumtion $a_n$ has no infinite ascending sequence, so $a_n$ has a weakly decreasing subsequence $a_{n_k}$. But then subsequence $b_{n_k}$ must be strictly increasing, what gives us a contradiction. ↯$\square$

**Lemma 4.2.2.6.** *In $\sigma^b(D^2)$ there are no infinite ascending sequences.*

**Proof.**
Let us denote by $A_n$ the set of all possible Euler orbicharacteristics realised by orbifolds of type $*b_1, \ldots, b_n$. Then $A_0 = \{1\}$ and $A_{n+1} = A_n + \{-\frac{n-1}{2n} \mid n \geqslant 2\}$. From that, from the lemma 4.2.2.5, for all $n$, we have that $A_n$ do not have infinite ascending sequence. $\sigma^b(D^2) = \bigcup\limits_{n=0}^{\infty} A_n$. Let us also observe, that for all $n$, we have $A_n \subseteqq [1 - \frac{n}{4}, 1 - \frac{n}{2}]$. From that we have $\sigma^b(D^2)$ do not have infinite ascending sequences. $\square$

**Theorem 4.2.2.7.** *The biggest condensation point of the set$\sigma^b(D^2)$ of order $n$ is $1 - \frac{n}{2}$.*

**Proof.**
Inductive
• $n = 0$: $1 \in \sigma^b(D^2)$ and 1 is the biggest element of $\sigma^b(D^2)$.
• an inductive step: From the inductive assumption we know that $1 - \frac{n}{2}$ is the biggest condensation point of the set$\sigma^b(D^2)$ of order $n$. From the lemma 4.2.2.3 we have then that $1 - \frac{n+1}{2}$ is a condensation point of the set$\sigma^b(D^2)$ of order $n + 1$. Let us assume (for a contradiction), that there exist a bigger condensation point of order $n + 1$ equal to $y > 1 - \frac{n+1}{2}$. But then, from lemma 4.2.2.3, point $y + \frac{1}{2}$ would be a condensation point of order $n$, what gives a contradiction, because $y + \frac{1}{2} > 1 - \frac{n}{2}$. □

**Theorem 4.2.2.8.** *The order type of the set of possible Euler orbicharacteristics of two dimentional orbifolds is $\omega^\omega$.*

**Proof.**
From the lemma 4.2.2.6 we know, that $\sigma^b(D^2)$ is well ordered. From this and from the theorem 4.2.2.7 we know, that for the point $1 - \frac{n}{2}$ there exist a neighborhood $U = (1 - \frac{n}{2} - \varepsilon, 1 - \frac{n}{2} + \varepsilon)$ such that $U \cap \sigma^b(D^2)$ is homeomorphic to $\omega^n$. From this, and again from theorem 4.2.2.7 we have that $\sigma^b(D^2) \cap [1, 1 - \frac{n}{2})$ is homeomorphic with $\omega^n$. From this $\sigma^b(D^2)$ is homeomorphic with $\omega^\omega$. From this $\sigma^I(S^2)$ is homeomorphic with $\omega^\omega$.
$\sigma^I(S^2) = 2\sigma^b(D^2)$, so for all $n \in \mathbb{N}$ set $\sigma^I(S^2) \cap [2, n)$ has a lower order type then $\sigma^b(D^2) \cap [2, n)$. From this, we have that $\sigma^I(S^2) \cup \sigma^b(D^2) \cong \omega^\omega$. □

**Theorem 4.2.2.9.** *The first (biggest) negative condensation point of the setof all possible Euler orbicharacteristicof two dimentional orbifolds is $\frac{1}{12}$. It is the condensation point of order* 1.

**Proof.**
We will show, that $-\frac{1}{12}$ is the biggest negative condensation point of the set $\sigma^b(D^2)$. From this we will obtain the thesis, as the set of all possible Euler orbicharacteristics of two dimentional orbifolds is equal to $\sigma^I(S^2) \cup \sigma^b(D^2)$ and $\sigma^I(S^2) = 2\sigma^b(D^2)$, so the biggest negative point of the set $\sigma^I(S^2)$ is smaller than the biggest negative condensation point of the set $\sigma^b(D^2)$.
• $-\frac{1}{12} = \chi^{orb}((2, 3)) - \frac{1}{2}$, from this we have that $-\frac{1}{12}$ a condensation point of the set $\sigma^b(D^2)$ of order at least 1.
• Let us assume (for the contradiction), that there exist bigger, negative condensation point of the set $\sigma^b(D^2)$ of order at least 1. Let us denote it by $x$.
However, then, from th lemma 4.2.2.3 point $x + \frac{1}{2}$ is the condensation point of the set $\sigma^b(D^2)$. What is more, since $x \in (0, -\frac{1}{12})$, then $x + \frac{1}{2} \in (\frac{1}{2}, \frac{5}{12}$. From the lemma 4.2.2.4 we have that $x$ is in $\sigma^b(D^2)$. But orbifolds of the type $*b_1$ can have Euler orbicharacteristiconly greater or equal $\frac{1}{2}$. Orbifolds of the type $*b_1b_2$ can only have Euler orbicharacteristic$\frac{1}{2}$, $\frac{5}{12}$ and some smaller. Orbifolds of the type $*b_1b_2b_3\ldots$ can have Euler orbicharacteristiconly lower than $\frac{1}{4}$. This analysis of the cases leads us to the conclusion, that $(\frac{1}{2}, \frac{5}{12}) \cap \sigma^b(D^2) = \emptyset$ and to the contradiction.
• Above analysis of the cases leads us also to the conclusion, that $\frac{5}{12}$ is an isolated point of the set $\sigma^b(D^2)$, from this $-\frac{1}{12}$ is a condensation point of order 1 of the set $\sigma^b(D^2)$. □

# Chapter 5

# Decidability

## 5.1 Algorithm

Here we will show the proof that the problem of decideng whether a given rational number is in an Euler orbicharacteristic's spectrum or not is decidable by showing algorithm for doing this.
We start with $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

# Chapter 6

# Counting occurrences

abcd

## 6.1 Deformations on orbifolds?

# Chapter 7

# Power series and generating functions

# Chapter 8

# Conclusions

# Bibliography


[1] John Conway and Daniel Huson. The orbifold notation for two-dimensional groups. *Structural Chemistry*, 13, 08 2002.

[2] Chaim Goodman-Strauss John H. Conway, Heidi Burgiel. *The Symmetries of Things*. A K Peters, 1 edition, 2008.

[3] William P Thurston. *The geometry and topology of three-manifolds*. s.n, 1979.