9) $\int \frac{2\ln^2 x^2}{x} dx$

$u = \ln x^2$, $du = \frac{1}{x^2} \cdot 2x dx = \frac{2}{x} dx$

$\int \frac{2\ln^2 x^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\ln^3 x^2}{3} + C$

10) $\int \frac{4\arctan^3 x}{x^2+1} dx$

$u = \arctan x$, $du = \frac{1}{x^2+1} dx$

$\int \frac{4\arctan^3 x}{x^2+1} dx = \int 4u^3 du = 4 \cdot \frac{u^4}{4} + C = u^4 + C = \arctan^4 x + C$

Unidad 3. Tratamiento analítico de las integrales indefinidas.

3.1 Técnicas de Integración

3.1.1 Integración por partes

No podemos separar multiplicaciones al integrar.

$D(uv) = uv' + vu'$

$d(uv) = u\,dv + v\,du$ $\qquad u\,dv = d(uv) - v\,du$

$\int u\,dv = uv - \int v\,du$

Ejemplo

$\int x \cos x \, dx$

Tomamos $u = x$, $dv = \cos x \, dx \Rightarrow du = dx$, $v = \operatorname{sen} x$

$\Rightarrow \int x \cos x \, dx = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x - (-\cos x) + C = x \operatorname{sen} x + \cos x + C$

Pero a veces se queda más difícil, para la elección de u suele jalar

I nversas trigo

L og

A lg

T rigo

E xp.

También se puede usar para integrar cosas que sabemos derivar y no integrar

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$$

$u = \ln x \quad v = x$

$du = \frac{1}{x} dx \quad dv = dx$

A veces te quedan integrales cíclicas

$$\int e^x \operatorname{sen} x \, dx = e^x(-\cos x) - \int (-\cos x \, dx) e^x = -e^x \cos x + \int e^x \cos x \, dx$$

$u_1 = e^x \quad v_1 = -\cos x$

$du_1 = e^x dx \quad dv_1 = \operatorname{sen} x \, dx$

$u_2 = e^x \quad v_2 = \operatorname{sen} x$

$du_2 = e^x dx \quad dv_2 = \cos x \, dx$

$$\int e^x \operatorname{sen} x \, dx = -e^x \cos x + e^x \operatorname{sen} x - \int e^x \operatorname{sen} x \, dx$$

$$\Rightarrow \int e^x \operatorname{sen} x \, dx = \frac{e^x \operatorname{sen} x - e^x \cos x}{2} + C$$

Existe algo llamado Integración por partes tabular, para que sea más compacto

$$\int u \, dv = \int f(x) g(x) \, dx$$

$u = f(x) \quad dv = g(x) \, dx$

$f(x)$ y sus derivadas	signo	$g(x)$ y su integral
→	+	↘
→	−	↘

Ejemplo $\int x^3 \cos x \, dx \quad f(x) = x^3 \quad g(x) = \cos x$

x^3 →	+	$\cos x$
$3x^2$ →	−	$\operatorname{sen} x$
$6x$ →	+	$-\cos x$
6 →	−	$-\operatorname{sen} x$
0	+	$\cos x$

3.1.3 Sustitución Trigonométrica

Caso 1. $\sqrt{x^2+a^2}$

$\theta = arcth\ x$

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\sqrt{x^2+a^2} = a\sec\theta$ En realidad es

$a|\sec\theta|$

dibujarlo siempre

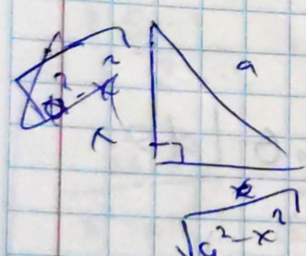
$$\int \frac{1}{\sqrt{x^2+9}} = \int \frac{1}{\cancel{3}\sec\theta}\cancel{3}\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + C$$

$$= \ln|\sqrt{x^2+9}+x| + C$$

Caso 2 $\sqrt{a^2-x^2}$



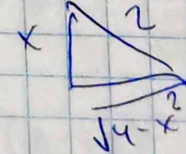
$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

$$\int \frac{x^2}{\sqrt{4-x^2}}dx$$



$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{x^2}{\sqrt{4-x^2}}dx = \int 4\sin^2\theta\, d\theta$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$\theta = \text{Arcsen}\left|\frac{x}{2}\right|$

$$= 2\,\text{Arcsen}\left|\frac{x}{2}\right| - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$= 2\,\text{Arcsen}\left|\frac{x}{2}\right| - \frac{1}{2}x\sqrt{4-x^2} + C$$

Caso 3. $\sqrt{x^2-a^2}$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

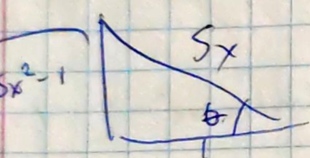
$\sqrt{x^2-a^2} = a\tan\theta$

$$\int \frac{dx}{\sqrt{25x^2-1}}$$

$u = 5x$

$du = 5\,dx$

$$= \frac{1}{5}\int \frac{5\,dx}{\sqrt{25x^2-1}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-1}}$$



$x = \frac{1}{5}\sec\theta$

$dx = \frac{1}{5}\sec\theta\tan\theta\, d\theta$

$\sqrt{25x^2-1} = \tan\theta$

$$\int \frac{dx}{\sqrt{25x^2-1}} = \int \frac{\frac{1}{5}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{\tan\theta}} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta|\, d\theta$$

$$= \frac{1}{5}\ln\left|5x + \sqrt{25x^2-1}\right| + C$$

Tema 8.1.4 Fracciones Parciales.

El método de fracciones parciales de

$$\frac{f(x)}{g(x)} \quad \text{con } \deg(f) < \deg(g)$$

(Si es $\geq$ tomar el residuo)

Factorizar $g(x)$, asignar la suma según la forma de los factores de $g(x)$.

- Sea $(x-r)$ un factor lineal de $g(x)$, sea $(x-r)^n$ la mayor potencia de $x-r$ que divide a $g(x)$ asignar la suma

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_n}{(x-r)^n}$$

- Sea (x^2+px+q) un factor cuadrático irreducible de $g(x)$ y sea $(x^2+px+q)^m$ la mayor potencia de x^2+px+q que divide a $g(x)$, entonces asignar la suma

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_mx+C_m}{(x^2+px+q)^m}$$

Ejemplo:

$$\int \frac{5x-1}{x^2-x-2}\,dx$$

$$\frac{5x-1}{x^2-x-2} = \frac{5x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \rightarrow 5x-1 = A(x+1) + B(x-2)$$

$$5x-1 = Ax+A+Bx-2B = (A+B)x + (A-2B)$$

$$A+B=5 \qquad A-2B=-1$$

$$A+B-A+2B = 5-(-1) = 6 \Rightarrow 3B=6 \Rightarrow \boxed{B=2,\ A=3}$$

$$= \int \frac{3}{x-2}\,dx + \int \frac{3}{x+1}\,dx = 3\ln|x-2| + 2\ln|x+1| + C$$

$$\int \frac{6x+7}{x^2+4x+4}\,dx$$

$$\frac{6x+7}{x^2+4x+4} = \frac{6x+7}{(x+2)^2} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2}$$

$$6x+7 = A_1(x+2) + A_2 = A_1 x + (2A_1 + A_2)$$

$$A_1 = 6 \qquad A_2 = -5$$

$$\int \frac{6x+7}{x^2+4x+4}\,dx = \int \frac{6}{x+2}\,dx + \int \frac{-5}{(x+2)^2}\,dx$$

$$u = x+2$$

$$6\int \frac{du}{u} - 5\int \frac{1}{u^2}\,du = 6\ln|u| - 5\frac{u^{-1}}{-1} + C = 6\ln|u| + \frac{5}{u} + C$$

$$= 6\ln|x+2| + \frac{5}{x+2} + C$$

$$\int \frac{x^4-2x^3+2x^2-4x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \int \left(+ \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\right) dx = \int dx + \int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,d$$

$$\begin{array}{r|l} & x^3 \; -x^2 \; x \; -1 \\ \hline x & x^4 - x^3 \; x^2 - x \\ -1 & -x^3 \; x^2 - x \; 1 \end{array}$$

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1)$$

$$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1x + C_1}{x^2+1}$$

$$-2x+4 = A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+C)(x-1)^2$$

$$= A_1(x^3-x^2+x-1) + A_2(x^2+1) + B(x^3-2x^2+x) + C(x^2-2x+1)$$

$$= x^3(A_1+B) + x^2(-A_1+A_2-2B+C) + x(A_1+B-2C) + (-A_1+A_2+C)$$

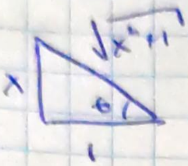
$A_1+B=0,\ -A_1+A_2-2B+C=0,\ A_1+B-2C=-2,\ -A_1+A_2+C=4$

$-2C=-2 \Rightarrow C=1 \qquad -A_1+A_2=3 \Rightarrow 3-2B+1=0 \Rightarrow -2B=-4 \Rightarrow B=2$

$A_1 = -2, \quad A_2 = 1$

$$\int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,dx = -2\int \frac{1}{x-1}\,dx + \int \frac{dx}{(x-1)^2} + \int \frac{2x+1}{x^2+1}\,dx$$

$\frac{2x+1}{x^2+1}$



$\sqrt{x^2+1} = \sec\theta$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{2x+1}{x^2+1}\,dx = \int \frac{2\tan\theta+1}{\cancel{\sec^2\theta}} \cdot \cancel{\sec^2\theta}\, d\theta = \int (2\tan\theta+1)\,d\theta$$

$$\int \frac{x^2+1}{x^3-6x^2+11x-6}\,dx$$

	x^2	$-4x$	3
x	x^3	$-4x^2$	$3x$
-2	$-2x^2$	$8x$	-6

$(x-2)(x-1)(x-3)$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Si $x=1$	Si $x=2$	Si $x=3$
$2 = A(-1)(-2)$	$5 = B(1)(-1)$	$10 = C(2)(1)$
$\Rightarrow A = 1$	$\Rightarrow B = -5$	$\Rightarrow C = 5$

$$\int \frac{dx}{x-1} - 5\int \frac{dx}{x-2} + 5\int \frac{dx}{x-3} = \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$$

$$\int \frac{2x^2+4}{x^3+3x^2+3x+1}dx = \int \frac{2x^2+4}{(x+1)^3}dx$$

$$\frac{2x^2+4}{(x+1)^3} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3}$$

$$2x^2+4 = A_1(x+1)^2 + A_2(x+1) + A_3$$

Si $x=-1$	Derivamos c/v	Derivamos c/v x^2
$\boxed{6 = A_3}$	$4x = 2A_1(x+1) + A_2$	$4 = 2A_1$
	Si $x=-1$	$A_1 = 2$
	$\boxed{-4 = A_2}$	

$$2\int \frac{dx}{x+1} - 4\int \frac{dx}{(x+1)^2} + 6\int \frac{dx}{(x+1)^3} = 2\ln|x+1| + 4(x+1)^{-1} - 3(x+1)^{-2} + C$$

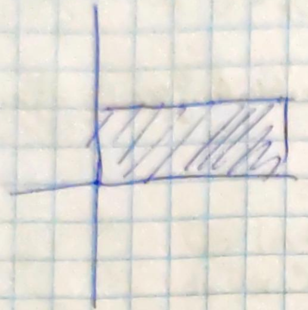
4. Aproximación y cálculo del área bajo la curva por métodos elementales (método de los rectángulos y métodos de los trapecios)

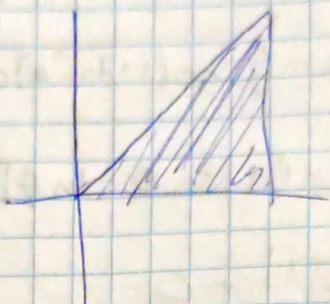
4.1 Área bajo la curva

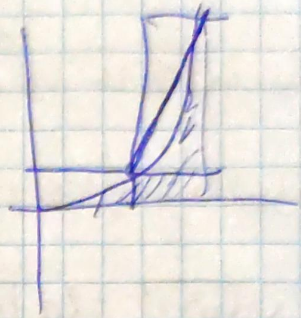
4.1.1 Descripción del cambio en forma gráfica

El área bajo la curva es como la distancia

$v = 2\ m/s$ $\qquad$ $v = t$ $\qquad$ y $v = t^2$







puedes aproximar con ~~cuadrados~~ rectángulos y trapecios

Trabajo en clase
~~Tarea~~ a de Tarea x d

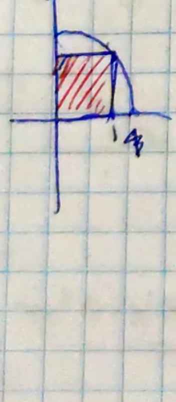
5. Tratamiento Analítico de las Integrales definidas y uso ~~deductivo~~ intuitivo de los procesos infinitos y las situaciones límite

5.1 Integral Definida

5.1.1 Sumas de Riemann

$$S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Ejemplo $f(x) = 2 - x^2$ $\quad 0 \le x \le 1$



$S_1 = f(c_1)\Delta x_1$

$c_1 = 1$

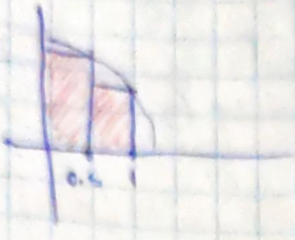
$\Delta x_1 = 1$

$f(c_1) = f(1) = 2 - 1^2 = 1$

$S_1 = (1)(1) = \underline{1}$

$n = 2$

$S_2 = f(c_1)\Delta x_1 + f(c_2)\Delta x_2$



$c_1 = 0,5$

$c_2 = 1$

$f(c_1) = f(0,5) = 2-(0.5)^2 = 1.75$

$f(c_2) = f(1) = 1$

$\Delta x_1 = \Delta x_2 = 0.5$

$S_2 = 1.75(0.5) + 1(0.5) = 1.375$

5.1.2 Interpretación Geométrica

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

El área bajo la curva

$f(x) = 2 - x^2$

$\Delta x_k = \frac{b-a}{n} \Rightarrow \Delta x_k = \frac{1}{n}$

$c_k = a + k\Delta x_k = k\left(\frac{1}{n}\right) = \frac{k}{n}$

$f(c_k) = f\left(\frac{k}{n}\right) = 2 - \left(\frac{k}{n}\right)^2 = 2 - \frac{k^2}{n^2}$

Entonces

$$\int_0^1 (2-x^2)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 - \frac{k^2}{n^2}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2}{n} - \frac{k^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2\right) = \lim_{n\to\infty} \left(\frac{2}{n}\cdot n - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

(L'Hopital)

$$= \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2}$$

y $\lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \lim_{n\to\infty} \frac{4n+3}{12n}$

$= \lim_{n\to\infty} \frac{4}{12} = 1/3$

$$= 2 - 1/3 = \boxed{5/3}$$

5.1.3 Teorema Fundamental del Cálculo

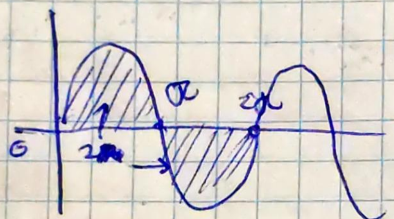
$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$$

$$\int_0^1 (2-x^2)\,dx = 2x-\frac{x^3}{3}\Big|_0^1 = \left[2(1)-\frac{(1)^3}{3}\right]-\left[2(0)-\frac{(0)^3}{3}\right] = 2-\tfrac{1}{3} = \tfrac{5}{3}$$

5.1.4 Interpretación Analítica

$$\int_0^{\pi} \operatorname{sen} x\,dx = -\cos x\Big|_0^{\pi} = [-\cos\pi]-[-\cos 0] = 1+1 = 2$$

$$\int_0^{2\pi} \operatorname{sen} x\,dx = -\cos x\Big|_0^{2\pi} = [-\cos 2\pi]-[-\cos 0] = -1+1 = 0$$



Entonces la integral indefinida no es necesariamente el área, puede ser suma o resta de esas, ~~exacto~~ si es cuando todo está por arriba

5.1.5 Propiedades

- $\int_a^b f(x)\,dx = F(b)-F(a)$
- $\int_b^a f(x)\,dx = F(a)-F(b) = -\int_a^b f(x)\,dx$
- $\int_a^a f(x)\,dx = F(a)-F(a) = 0$

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$g = x^3+1 \quad g' = 3x^2$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{g(-1)}^{g(1)} \sqrt{u}\,du = \int_{0}^{2} \sqrt{u}\,du = \frac{2}{3}\,u^{3/2}\Big|_0^2 = \frac{4\sqrt{2}}{3}$$

S2 Aplicación Inmediata

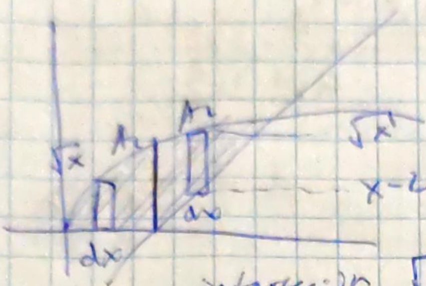
$$\begin{cases} y = \sqrt{x} \\ y = x-2 \\ y = 0 \end{cases}$$

$A = A_1 + A_2$

← intersección $x-2$ y 0

$$A_1 = \int_0^2 \sqrt{x}\,dx = \int_0^2 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^2$$

$$= \frac{4\sqrt{2}}{3}$$



$\sqrt{x} - x + 2$

← intersección $\sqrt{x}$ y $x-2$

$$A_2 = \int_2^4 (\sqrt{x} - x + 2)\,dx = \int_2^4 (x^{1/2} - x + 2)\,dx$$

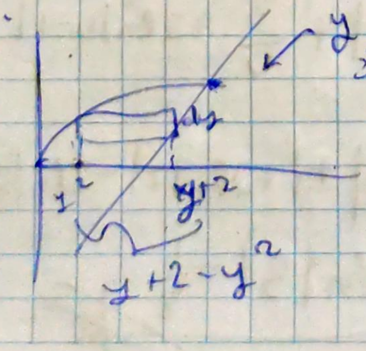
$$= \frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\Big|_2^4 = \frac{16}{3} - 8 + 8 - \left(\frac{4\sqrt{2}}{3} - 2 + 4\right)$$

$$= \frac{16}{3} - 2 - \frac{4\sqrt{2}}{3} = \frac{10}{3} - \frac{4\sqrt{2}}{3}$$

$$A = \frac{4\sqrt{3}}{3} + \frac{10}{3} - \frac{4\sqrt{2}}{3} = \frac{10}{3}$$

- También lo podemos ver como rectángulos

$y = \sqrt{x} \Rightarrow x = y^2$ $\qquad$ $y = x - 2 \Rightarrow x = y + 2$



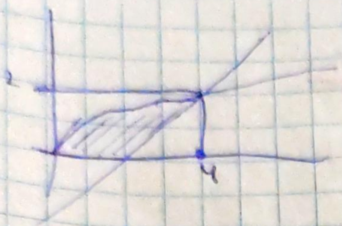
$y^2 = y + 2$

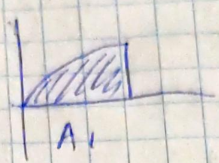
$\Rightarrow y = 2$

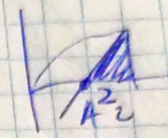
$$A = \int_0^2 (y + 2 - y^2)\,dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= 2 + 4 - \frac{8}{3} = \frac{10}{3}$$

Otra forma xd







$$A = A_1 - A_2$$

$$A_1 = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}$$

$$\cancel{A_2 = \int_2^4} \qquad A_2 = \frac{2 \cdot 2}{2} = 2 \qquad A = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3}$$