quieroserbueno

PONTE A ENTRENAR

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§1 Problemas

Problem 1.1 (46260042068525)**.** Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we have any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

Problem 1.2 (60640107784874935 66)**.** Vulcan and Neptune play a turn-based game on an infinite grid of unit squares. Before the game starts, Neptune chooses a finite number of cells to be flooded. Vulcan is building a levee, which is a subset of unit edges of the grid (called walls) forming a connected, non-self-intersecting path or loop*.

The game then begins with Vulcan moving first. On each of Vulcan's turns, he may add up to three new walls to the levee (maintaining the conditions for the levee). On each of Neptune's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well. Prove that Vulcan can always, in a finite number of turns, build the levee into a closed loop such that all flooded cells are contained in the interior of the loop, regardless of which cells Neptune initially floods. *More formally, there must exist lattice points $A_0, A_1, \ldots, A_k$, pairwise distinct except possibly $A_0 = A_k$, such that the set of walls is exactly $\{A_0A_1, A_1A_2, \ldots, A_{k-1}A_k\}$. Once a wall is built it cannot be destroyed; in particular, if the levee is a closed loop (i.e. $A_0 = A_k$) then Vulcan cannot add more walls. Since each wall has length 1, the length of the levee is k.

Problem 1.3 (10113478786976456 66)**.** On a circle we write $2n$ real numbers with a positive sum. For each number, there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both these sets.

Problem 1.4 (876239022447910)**.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

Problem 1.5 (14736912264266295 81)**.** A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Problem 1.6 (4298196647118074747)**.** Find all integers $n \geq 3$ such that the following property holds: if we list the divisors of $n!$ in increasing order as $1 = d_1 < d_2 < \cdots < d_k = n!$, then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq \cdots \leq d_k - d_{k-1}.$$

Problem 1.7 (684265043263216)**.** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 1.8 (6322745101407512634)**.** Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}, \overline{CA}, \overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Problem 1.9 (259897104343709)**.** There is a queue of n girls on one side of a tennis table, and a queue of n boys on the other side. Both the girls and the boys are numbered from 1 to n in the order they stand. The first game is played by the girl and the boy with the number 1 and then, after each game, the loser goes to the end of their queue, and the winner remains at the table. After a while, it turned out that each girl played exactly one game with each boy. Prove that if n is odd, then a girl and a boy with odd numbers played in the last game.

Problem 1.10 (1837105952530316058)**.** Let $k \geq 2$ be an integer. Find the smallest integer $n \geq k+1$ with the property that there exists a set of n distinct real numbers such that each of its elements can be written as a sum of k other distinct elements of the set.

Problem 1.11 (8317584744128058138)**.** One side of a square sheet of paper is colored red, the other - in blue. On both sides, the sheet is divided into n^2 identical square cells. In each of these $2n^2$ cells is written a number from 1 to k. Find the smallest k,for which the following properties hold simultaneously: (i) on the red side, any two numbers in different rows are distinct; (ii) on the blue side, any two numbers in different columns are different; (iii) for each of the n^2 squares of the partition, the number on the blue side is not equal to the number on the red side.

Problem 1.12 (2212576839999739806)**.** One hundred sages play the following game. They are waiting in some fixed order in front of a room. The sages enter the room one after another. When a sage enters the room, the following happens - the guard in the room chooses two arbitrary distinct numbers from the set $1, 2, 3$, and announces them to the sage in the room. Then the sage chooses one of those numbers, tells it to the guard, and leaves the room, and the next enters, and so on. During the game, before a sage chooses a number, he can ask the guard what were the chosen numbers of the previous two sages. During the game, the sages cannot talk to each other. At the end, when everyone has finished, the game is considered as a failure if the sum of the 100 chosen numbers is exactly 200; else it is successful. Prove that the sages can create a strategy, by which they can win the game.

Problem 1.13 (561375932085594939)**.** Petya has $10,000$ balls, among them there are no two balls of equal weight. He also has a device, which works as follows: if he puts exactly 10 balls on it, it will report the sum of the weights of some two of them (but he doesn't know which ones). Prove that Petya can use the device a few times so that after a while he will be able to choose one of the balls and accurately tell its weight.

Problem 1.14 (210358073900610)**.** Let triangle ABC have altitudes BE and CF which meet at H. The reflection of A over BC is A'. Let (ABC) meet $(AA'E)$ at P and $(AA'F)$ at Q. Let BC meet PQ at R. Prove that $EF \parallel HR$.

Problem 1.15 (302701281403387)**.** Each point of a three-dimensional space is colored with one of two colors such that whenever an isosceles triangle ABC with $AB = AC$ has vertices of the same color c it follows that the midpoint of BC also is colored with c. Prove that there exists a perpendicular square prism with all vertices of equal color.

Problem 1.16 (8866273454792491736)**.** Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x - y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

Problem 1.17 (6654677204410680146)**.** In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

Problem 1.18 (1856371892766039579)**.** Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \ldots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Problem 1.19 (8534263250311217423)**.** In acute triangle $\triangle ABC$, $\angle A > \angle B > \angle C$. $\triangle AC_1B$ and $\triangle CB_1A$ are isosceles triangles such that $\triangle AC_1B \overset{+}{\sim} \triangle CB_1A$. Let lines BB_1, CC_1 intersects at T. Prove that if all points mentioned above are distinct, $\angle ATC$ isn't a right angle.

Problem 1.20 (5101270312905584526)**.** The exam has 25 topics, each of which has 8 questions. On a test, there are 4 questions of different topics. Is it possible to make 50 tests so that each question was asked exactly once, and for any two topics there is a test where are questions of both topics?

Problem 1.21 (600298381529685)**.** Find all pairs of positive integers (a, b) satisfying the following conditions: a divides $b^4 + 1$, b divides $a^4 + 1$, $\lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$.

Problem 1.22 (945532205287762)**.** Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Problem 1.23 (2265193939454652363)**.** A circle ω with radius 1 is given. A collection T of triangles is called good, if the following conditions hold: each triangle from T is inscribed in ω; no two triangles from T have a common interior point. Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

Problem 1.24 (526922799283626)**.** For each $1 \leq i \leq 9$ and $T \in \mathbb{N}$, define $d_i(T)$ to be the total number of times the digit i appears when all the multiples of 1829 between 1 and T inclusive are written out in base 10.

Show that there are infinitely many $T \in \mathbb{N}$ such that there are precisely two distinct values among $d_1(T)$, $d_2(T)$, $\ldots$, $d_9(T)$

Problem 1.25 (8053761138620448460)**.** Let ABC be a scalene triangle, and points O and H be its circumcenter and orthocenter, respectively. Point P lies inside triangle AHO and satisfies $\angle AHP = \angle POA$. Let M be the midpoint of segment $\overline{OP}$. Suppose that BM and CM intersect with the circumcircle of triangle ABC again at X and Y, respectively.

Prove that line XY passes through the circumcenter of triangle APO.

Problem 1.26 (6135851041251773220)**.** 200 natural numbers are written in a row. For any two adjacent numbers of the row, the right one is either 9 times greater than the left one, 2 times smaller than the left one. Can the sum of all these 200 numbers be equal to 24^{2022}?

Problem 1.27 (8489819892706651399)**.** For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G.

Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

Problem 1.28 (213513857758059)**.** Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Problem 1.29 (409146991986056)**.** For each prime p, construct a graph G_p on $\{1, 2, \ldots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2 + 1 - n)(n^2 + 1 - m)$. Prove that G_p is disconnected for infinitely many p

Problem 1.30 (15595788767204175)**.** Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem 1.31 (6666334949338369993)**.** Choose positive integers $b_1, b_2, \ldots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \geq r$ for all positive integers n.

What are the possible values of r across all possible choices of the sequence (b_n)?

Problem 1.32 (1978345856029698287)**.** Let $S_1, S_2, \ldots, S_{100}$ be finite sets of integers whose intersection is not empty. For each non-empty $T \subseteq \{S_1, S_2, \ldots, S_{100}\}$, the size of the intersection of the sets in T is a multiple of the number of sets in T. What is the least possible number of elements that are in at least 50 sets?

Problem 1.33 (496656338551810)**.** Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either m by n or n by m is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid (If this never occurs, Freyja has lost the game and Tsvety wins).

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions suffice.

Problem 1.34 (9691971442236847)**.** Each girl among 100 girls has 100 balls; there are in total 10000 balls in 100 colors, from each color there are 100 balls. On a move, two girls can exchange a ball (the first gives the second one of her balls, and vice versa). The operations can be made in such a way, that in the end, each girl has 100 balls, colored in the 100 distinct colors. Prove that there is a sequence of operations, in which each ball is exchanged no more than 1 time, and at the end, each girl has 100 balls, colored in the 100 colors.

Problem 1.35 (1754525449568 24)**.** In the city built are 2019 metro stations. Some pairs of stations are connected. tunnels, and from any station through the tunnels you can reach any other. The mayor ordered to organize several metro lines: each line should include several different stations connected in series by tunnels (several lines can pass through the same tunnel), and in each station must lie at least on one line. To save money no more than k lines should be made. It turned out that the order of the mayor is not feasible. What is the largest k it could to happen?

Problem 1.36 (5441518070935718077)**.** Let ABC be an acute-angled triangle. The line through C perpendicular to AC meets the external angle bisector of $\angle ABC$ at D. Let H be the foot of the perpendicular from D onto BC. The point K is chosen on AB so that $KH \parallel AC$. Let M be the midpoint of AK. Prove that $MC = MB + BH$.

Problem 1.37 (7094618843236371 20)**.** Among 16 coins there are 8 heavy coins with weight of 11 g, and 8 light coins with weight of 10 g, but it's unknown what weight of any coin is. One of the coins is anniversary. How to know, is anniversary coin heavy or light, via three weighings on scales with two cups and without any weight?

Problem 1.38 (503121367540901)**.** Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Problem 1.39 (8059760967121829853)**.** Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Problem 1.40 (883811987981100)**.** Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 1.41 (8255863576892581507)**.** Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

Problem 1.42 (448881061747528)**.** A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \cdots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

Problem 1.43 (6576585943791349484)**.** Regular hexagon is divided to equal rhombuses, with sides, parallels to hexagon sides. On the three sides of the hexagon, among which there are no neighbors, is set directions in order of traversing the hexagon against hour hand. Then, on each side of the rhombus, an arrow directed just as the side of the hexagon parallel to this side. Prove that there is not a closed path going along the arrows.

Problem 1.44 (931951248564234)**.** Let $n > 3$ be a positive integer. Suppose that n children are arranged in a circle, and n coins are distributed between them (some children may have no coins). At every step, a child with at least 2 coins may give 1 coin to each of their immediate neighbors on the right and left. Determine all initial distributions of the coins from which it is possible that, after a finite number of steps, each child has exactly one coin.

Problem 1.45 (7220404010846068686)**.** Let ABC be a acute, non-isosceles triangle. $D,\ E,\ F$ are the midpoints of sides $AB,\ BC,\ AC$, resp. Denote by $(O),\ (O')$ the circumcircle and Euler circle of ABC. An arbitrary point P lies inside triangle DEF and $DP,\ EP,\ FP$ intersect (O') at $D',\ E',\ F'$, resp. Point A' is the point such that D' is the midpoint of AA'. Points $B',\ C'$ are defined similarly. a. Prove that if $PO = PO'$ then $O \in (A'B'C')$; b. Point A' is mirrored by OD, its image is X. $Y,\ Z$ are created in the same manner. H is the orthocenter of ABC and $XH,\ YH,\ ZH$ intersect BC, AC, AB at $M,\ N,\ L$ resp. Prove that $M,\ N,\ L$ are collinear.

Problem 1.46 (3192129869376364982)**.** Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

Problem 1.47 (792975361721939)**.** Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold: The sides of each rectangle are parallel to the sides of the unit square. Each point in S is not in the interior of any rectangle. Each point in the interior of the unit square but not in S is in the interior of at least one of the k rectangles (The interior of a polygon does not contain its boundary.)

Problem 1.48 (296367141382799)**.** Given a triangle $\triangle ABC$ with orthocenter H. On its circumcenter, choose an arbitrary point P (other than A, B, C) and let M be the midpoint of HP. Now, we find three points D, E, F on the line BC, CA, AB, respectively, such that $AP \parallel HD, BP \parallel HE, CP \parallel HF$. Show that D, E, F, M are colinear.

Problem 1.49 (813804034055493)**.** In a circle there are 2019 plates, on each lies one cake. Petya and Vasya are playing a game. In one move, Petya points at a cake and calls number from 1 to 16, and Vasya moves the specified cake to the specified number of check clockwise or counterclockwise (Vasya chooses the direction each time). Petya wants at least some k pastries to accumulate on one of the plates and Vasya wants to stop him. What is the largest k Petya can succeed?

Problem 1.50 (47893544380608)**.** Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \ldots, n\}$ that are red. Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

Problem 1.51 (4510788820354844)**.** Let $ABCD$ be a quadrilateral inscribed in a circle with center O and E be the intersection of segments AC and BD. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of BCE. The tangent to ω_1 at A and the tangent to ω_2 at C meet at P. The tangent to ω_1 at D and the tangent to ω_2 at B meet at Q. Show that $OP = OQ$.

Problem 1.52 (2989958142304279488)**.** Given is a set of $2n$ cards numbered $1, 2, \cdots, n$, each number appears twice. The cards are put on a table with the face down. A set of cards is called good if no card appears twice. Baron Munchausen claims that he can specify 80 sets of n cards, of which at least one is sure to be good. What is the maximal n for which the Baron's words could be true?

Problem 1.53 (1222382895728709073)**.** Given a triangle ABC, a circle Ω is tangent to AB, AC at B, C, respectively. Point D is the midpoint of AC, O is the circumcenter of triangle ABC. A circle Γ passing through A, C intersects the minor arc BC on Ω at P, and intersects AB at Q. It is known that the midpoint R of minor arc PQ satisfies that $CR \perp AB$. Ray PQ intersects line AC at L, M is the midpoint of AL, N is the midpoint of DR, and X is the projection of M onto ON. Prove that the circumcircle of triangle DNX passes through the center of Γ.

Problem 1.54 (5878661446138888)**.** The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 1.55 (712971117639738). Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i+j+k \geq n$.

Problem 1.56 (105422576188851). A short-sighted rook is a rook that beats all squares in the same column and in the same row for which he can not go more than 60-steps. What is the maximal amount of short-sighted rooks that don't beat each other that can be put on a 100×100 chessboard.

Problem 1.57 (723258861624579). Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Problem 1.58 (4375421764909014892). Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

Problem 1.59 (245448917471703). In a $2 \times n$ array of positive real numbers, the sum of the two real numbers in each of the n columns is 1. Prove that it's possible to select one number from each column such that the sum of the selected numbers in each row is at most $\frac{n+1}{4}$.

Problem 1.60 (545015136325290). Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 1.61 (7268978143074030034). Given two circles ω_1 and ω_2 where ω_2 is inside ω_1. Show that there exists a point P such that for any line ℓ not passing through P, if ℓ intersects circle ω_1 at A, B and ℓ intersects circle ω_2 at C, D, where A, C, D, B lie on ℓ in this order, then $\angle APC = \angle BPD$.

Problem 1.62 (937132258882447). n coins lies in the circle. If two neighbour coins lies both head up or both tail up, then we can flip both. How many variants of coins are available that can not be obtained from each other by applying such operations?

Problem 1.63 (161342796381450). For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

Problem 1.64 (8608387455131778331). Calvin and Hobbes play a game. First, Hobbes picks a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, 2020\}$, known to both players. Then, Calvin and

Hobbes take turns choosing a number from $\{1, 2, \ldots, 2020\}$ which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of $\mathcal{F}$, otherwise Hobbes wins. What is the largest possible size of a family $\mathcal{F}$ that Hobbes could pick while still having a winning strategy?

Problem 1.65 (760426813975831)**.** Let ABC be a triangle with $AB + AC = 3BC$. The B-excircle touches side AC and line BC at E and D, respectively. The C-excircle touches side AB at F. Let lines CF and DE meet at P. Prove that $\angle PBC = 90°$.

Problem 1.66 (5990443173263547430)**.** Given a fixed circle (O) and two fixed points B, C on that circle, let A be a moving point on (O) such that $\triangle ABC$ is acute and scalene. Let I be the midpoint of BC and let AD, BE, CF be the three heights of $\triangle ABC$. In two rays $\overrightarrow{FA}, \overrightarrow{EA}$, we pick respectively M, N such that $FM = CE, EN = BF$. Let L be the intersection of MN and EF, and let $G \neq L$ be the second intersection of (LEN) and (LFM).

a) Show that the circle (MNG) always goes through a fixed point.

b) Let AD intersects (O) at $K \neq A$. In the tangent line through D of (DKI), we pick P, Q such that $GP \parallel AB, GQ \parallel AC$. Let T be the center of (GPQ). Show that GT always goes through a fixed point.

Problem 1.67 (569685816807741)**.** Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

Problem 1.68 (571373387028298)**.** Let ABC be a triangle with $\angle BAC > 90°$, and let O be its circumcenter and ω be its circumcircle. The tangent line of ω at A intersects the tangent line of ω at B and C respectively at point P and Q. Let D, E be the feet of the altitudes from P, Q onto BC, respectively. F, G are two points on $\overline{PQ}$ different from A, so that A, F, B, E and A, G, C, D are both concyclic. Let M be the midpoint of $\overline{DE}$. Prove that DF, OM, EG are concurrent.

Problem 1.69 (4389998719836463980)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.70 (2134021625648303394)**.** The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily distinct) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$. Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

Problem 1.71 (648819281604044)**.** Let n be a positive integer and let $S \subseteq \{0, 1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \ldots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their majority is defined as the binary string $y \in \{0, 1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \ldots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k + 1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

Problem 1.72 (4059278924956282558)**.** In a card game, each card is associated with a numerical value from 1 to 100, with each card beating less, with one exception: 1 beats 100. The player knows that 100 cards with different values lie in front of him. The dealer who knows the order of these cards can tell the player which card beats the other for any pair of cards he draws. Prove that the dealer can make one hundred such messages, so that after that the player can accurately determine the value of each card.

Problem 1.73 (819328919046836)**.** Which positive integers n make the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\lfloor\frac{ij}{n+1}\right\rfloor=\frac{n^2(n-1)}{4}$$

true?

Problem 1.74 (4948608980214807448)**.** Let ABC be a scalene triangle with circumcenter O and orthocenter H. Let AYZ be another triangle sharing the vertex A such that its circumcenter is H and its orthocenter is O. Show that if Z is on BC, then A, H, O, Y are concyclic.

Problem 1.75 (102296866595865)**.** Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove that there exists a point E, lying outside quadrilateral $ABCD$, such that ray EH bisects both angles $\angle BES$, $\angle TED$, and $\angle BEN = \angle MED$.

Problem 1.76 (3458270318471332488)**.** Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

Problem 1.77 (1891712635906763103)**.** Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle\ BMC$ so that $\angle KBC = 90°$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.

Problem 1.78 (9153191064326230951)**.** Let scalene triangle ABC have altitudes AD, BE, CF and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines PE at $X \neq P$ and PF at $Y \neq P$. Prove that $XY \parallel BC$.

Problem 1.79 (9103148252094553273)**.** The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a path from X to Y is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called diverse if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Problem 1.80 (2974998787723554962)**.** There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled $A_1, A_2, \ldots, A_{2022}$ in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

Problem 1.81 (829271701496996)**.** Pasha and Vova play the following game, making moves in turn; Pasha moves first. Initially, they have a large piece of plasticine. By a move, Pasha cuts one of the existing pieces into three(of arbitrary sizes), and Vova merges two existing pieces into one. Pasha wins if at some point there appear to be 100 pieces of equal weights. Can Vova prevent Pasha's win?

Problem 1.82 (2672133756769464425)**.** Is there a scalene triangle ABC similar to triangle IHO, where I, H, and O are the incenter, orthocenter, and circumcenter, respectively, of triangle ABC?

Problem 1.83 (257453182523555)**.** Let a and b be positive integers. The cells of an $(a+b+1)\times(a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

Problem 1.84 (6340105142765788083)**.** In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

Problem 1.85 (3866807698726339637)**.** Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbors - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Problem 1.86 (2139114147569608698)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 1.87 (942176258255049)**.** Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

Problem 1.88 (8811824418974048155)**.** $ABCDE$ is a cyclic pentagon, with circumcentre O. $AB = AE = CD$. I midpoint of BC. J midpoint of DE. F is the orthocentre of $\triangle ABE$, and G the centroid of $\triangle AIJ.CE$ intersects BD at H, OG intersects FH at M. Show that $AM \perp CD$.

Problem 1.89 (35724831608408)**.** We will say that a set of real numbers $A = (a_1, ..., a_{17})$ is stronger than the set of real numbers $B = (b_1, ..., b_{17})$, and write $A > B$ if among all inequalities $a_i > b_j$ the number of true inequalities is at least 3 times greater than the number of false. Prove that there is no chain of sets $A_1, A_2, ..., A_N$ such that $A_1 > A_2 > \cdots A_N > A_1$.

Remark: For 11.4, the constant 3 is changed to 2 and $N = 3$ and 17 is changed to m and n in the definition (the number of elements don't have to be equal).

Problem 1.90 (8690567757444826166)**.** A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Problem 1.91 (899785005954032)**.** The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a move, Vera may flip over one of the coins in the row, subject to the following rules: On the first move, Vera may flip over any of the 2023 coins. On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.) Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

Problem 1.92 (822921222405372)**.** Let $n \geq 3$ be a fixed integer. There are $m \geq n+1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n+1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

Problem 1.93 (3182086602668829737)**.** Let ABC be an acute-angled triangle with $AB \neq AC$, and let I and O be its incenter and circumcenter, respectively. Let the incircle touch BC, CA and AB at D, E and F, respectively. Assume that the line through I parallel to EF, the line through D parallel toAO, and the altitude from A are concurrent. Prove that the concurrency point is the orthocenter of the triangle ABC.

Problem 1.94 (2886276736199315342)**.** Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths. *

Problem 1.95 (896600029778859256)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

Problem 1.96 (1121095467606378762)**.** Let $\Gamma, \Gamma_1, \Gamma_2$ be mutually tangent circles. The three circles are also tangent to a line l. Let Γ, Γ_1 be tangent to each other at B_1, Γ, Γ_2 be tangent to each other at B_2, Γ_1, Γ_2 be tangent to each other at C. $\Gamma, \Gamma_1, \Gamma_2$ are tangent to l at A, A_1, A_2 respectively, where A is between A_1, A_2. Let $D_1 = A_1C \cap A_2B_2, D_2 = A_2C \cap A_1B_1$. Prove that D_1D_2 is parallel to l.

Problem 1.97 (428632191392819). Initially, 10 ones are written on a blackboard. Grisha and Gleb are playing game, by taking turns; Grisha goes first. On one move Grisha squares some 5 numbers on the board. On his move, Gleb picks a few (perhaps none) numbers on the board and increases each of them by 1. If in 10,000 moves on the board a number divisible by 2023 appears, Gleb wins, otherwise Grisha wins. Which of the players has a winning strategy?

Problem 1.98 (8609709793627283757). Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Problem 1.99 (836909183133087). Given a triangle $\triangle ABC$ with circumcircle Ω. Denote its incenter and A-excenter by I, J, respectively. Let T be the reflection of J w.r.t BC and P is the intersection of BC and AT. If the circumcircle of $\triangle AIP$ intersects BC at $X \neq P$ and there is a point $Y \neq A$ on Ω such that $IA = IY$. Show that $\odot(IXY)$ tangents to the line AI.

Problem 1.100 (233559801569582). Let n be a positive integer. Find the number of permutations $a_1,\ a_2,\ \ldots a_n$ of the sequence $1,\ 2,\ \ldots\ ,\ n$ satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

Problem 1.101 (6975633259976638169). On the round necklace there are $n > 3$ beads, each painted in red or blue. If a bead has adjacent beads painted the same color, it can be repainted (from red to blue or from blue to red). For what n for any initial coloring of beads it is possible to make a necklace in which all beads are painted equally?

Problem 1.102 (472882074231586). Let $G = (V, E)$ be a finite simple graph on n vertices. An edge e of G is called a bottleneck if one can partition V into two disjoint sets A and B such that at most 100 edges of G have one endpoint in A and one endpoint in B; and the edge e is one such edge (meaning the edge e also has one endpoint in A and one endpoint in B). Prove that at most $100n$ edges of G are bottlenecks.

Problem 1.103 (302438226120877). Given triangle ABC. Let $BPCQ$ be a parallelogram (P is not on BC). Let U be the intersection of CA and BP, V be the intersection of AB and CP, X be the intersection of CA and the circumcircle of triangle ABQ distinct from A, and Y be the intersection of AB and the circumcircle of triangle ACQ distinct from A. Prove that $\overline{BU} = \overline{CV}$ if and only if the lines AQ, BX, and CY are concurrent.

Problem 1.104 (8963205841174892420). Let $ABCD$ be a convex quadrilateral with pairwise distinct side lengths such that $AC \perp BD$. Let O_1, O_2 be the circumcenters of $\Delta ABD, \Delta CBD$, respectively. Show that AO_2, CO_1, the Euler line of ΔABC and the Euler line of ΔADC are concurrent.

(Remark: The Euler line of a triangle is the line on which its circumcenter, centroid, and orthocenter lie.)

Problem 1.105 (18644549011438). Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n) \text{ times}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

Problem 1.106 (579228243242060)**.** Let $ABCD$ be a parallelogram. A line through C crosses the side AB at an interior point X, and the line AD at Y. The tangents of the circle AXY at X and Y, respectively, cross at T. Prove that the circumcircles of triangles ABD and TXY intersect at two points, one lying on the line AT and the other one lying on the line CT.

Problem 1.107 (781756252908608)**.** Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i, j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

Problem 1.108 (4678973565823282552)**.** Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

Problem 1.109 (5897111412933990257)**.** Let ABC be a triangle with circumcircle Γ, and points E and F are chosen from sides CA, AB, respectively. Let the circumcircle of triangle AEF and Γ intersect again at point X. Let the circumcircles of triangle ABE and ACF intersect again at point K. Line AK intersect with Γ again at point M other than A, and N be the reflection point of M with respect to line BC. Let XN intersect with Γ again at point S other that X.

Prove that SM is parallel to BC.

Problem 1.110 (1637184643761804371)**.** Initially, on the lower left and right corner of a 2018×2018 board, there're two horses, red and blue, respectively. A and B alternatively play their turn, A start first. Each turn consist of moving their horse (A-red, and B-blue) by, simultaneously, 20 cells respect to one coordinate, and 17 cells respect to the other; while preserving the rule that the horse can't occupied the cell that ever occupied by any horses in the game. The player who can't make the move loss, who has the winning strategy?

Problem 1.111 (7553717274310387624)**.** Let ABC be a triangle with incentre I and circumcircle ω. The incircle of the triangle ABC touches the sides BC, CA and AB at D, E and F, respectively. The circumcircle of triangle ADI crosses ω again at P, and the lines PE and PF cross ω again at $X$and Y, respectively. Prove that the lines AI, BX and CY are concurrent.

Problem 1.112 (4934938474754667779)**.** Let ABC be a triangle and let H be the orthogonal projection of A on the line BC. Let K be a point on the segment AH such that $AH = 3KH$. Let O be the circumcenter of triangle ABC and let M and N be the midpoints of sides AC and AB respectively. The lines KO and MN meet at a point Z and the perpendicular at Z to OK meets lines AB, AC at X and Y respectively. Show that $\angle XKY = \angle CKB$.

Problem 1.113 (1810915585111530473)**.** Given a scalene triangle $\triangle ABC$. B', C' are points lie on the rays $\overrightarrow{AB}, \overrightarrow{AC}$ such that $\overline{AB'} = \overline{AC}, \overline{AC'} = \overline{AB}$. Now, for an arbitrary point P in the plane. Let Q be the reflection point of P w.r.t $\overline{BC}$. The intersections of $\odot(BB'P)$ and $\odot(CC'P)$ is P' and the intersections of $\odot(BB'Q)$ and $\odot(CC'Q)$ is Q'. Suppose that O, O' are circumcenters of $\triangle ABC, \triangle AB'C'$ Show that

1. O', P', Q' are colinear
2. $\overline{O'P'} \cdot \overline{O'Q'} = \overline{OA}^2$

Problem 1.114 (852531542088551)**.** Given a triangle ABC for which $\angle BAC \neq 90°$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

Problem 1.115 (4742951979457606021)**.** There are 2021 points on a circle. Kostya marks a point, then marks the adjacent point to the right, then he marks the point two to its right, then three to the next point's right, and so on. Which move will be the first time a point is marked twice?

Problem 1.116 (287986230573307)**.** Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Problem 1.117 (548248988934632)**.** Let ABC be a triangle with incenter I. Let segment AI intersect the incircle of triangle ABC at point D. Suppose that line BD is perpendicular to line AC. Let P be a point such that $\angle BPA = \angle PAI = 90°$. Point Q lies on segment BD such that the circumcircle of triangle ABQ is tangent to line BI. Point X lies on line PQ such that $\angle IAX = \angle XAC$. Prove that $\angle AXP = 45°$.

Problem 1.118 (8916142707013964275)**.** Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

Problem 1.119 (719467452801051)**.** Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

Problem 1.120 (1598288382590173390)**.** Let $\mathbb{N}$ denote the set of positive integers. A function $f\colon \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers

$$f(m+1), f(m+2), \ldots, f(m+f(n))$$

is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

Problem 1.121 (297274918587198)**.** Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

Problem 1.122 (274933009357884)**.** Let $n \geq 3$ be an integer. We say that an arrangement of the numbers 1, 2, ..., n^2 in a $n \times n$ table is row-valid if the numbers in each row can be permuted to form an arithmetic progression, and column-valid if the numbers in each column can be permuted to form an arithmetic progression. For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

Problem 1.123 (4451072691230235426)**.** A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.

Problem 1.124 (5395714337110519657)**.** The vertices of a convex 2550-gon are colored black and white as follows: black, white, two black, two white, three black, three white, ..., 50 black, 50 white. Dania divides the polygon into quadrilaterals with diagonals that have no common points. Prove that there exists a quadrilateral among these, in which two adjacent vertices are black and the other two are white.

Problem 1.125 (645068477920006)**.** There are several gentlemen in the meeting of the Diogenes Club, some of which are friends with each other (friendship is mutual). Let's name a participant unsociable if he has exactly one friend among those present at the meeting. By the club rules, the only friend of any unsociable member can leave the meeting (gentlemen leave the meeting one at a time). The purpose of the meeting is to achieve a situation in which that there are no friends left among the participants. Prove that if the goal is achievable, then the number of participants remaining at the meeting does not depend on who left and in what order.

Problem 1.126 (5867489266334805897)**.** Let $ABCDE$ be a pentagon inscribed in a circle Ω. A line parallel to the segment BC intersects AB and AC at points S and T, respectively. Let X be the intersection of the line BE and DS, and Y be the intersection of the line CE and DT.

Prove that, if the line AD is tangent to the circle $\odot(DXY)$, then the line AE is tangent to the circle $\odot(EXY)$.

Problem 1.127 (461803484803557)**.** Let $f : \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

Problem 1.128 (855628849330783)**.** Let m and n be positive integers. The cells of a $2m \times 2n$ grid are colored black and white. Suppose that for any cell c, a rook placed on c attacks more cells of the opposite color (not including c itself). Prove that every row and column contains an equal number of black and white cells.

Problem 1.129 (300334293164389)**.** Kid and Karlsson play a game. Initially they have a square piece of chocolate 2019×2019 grid with 1×1 cells . On every turn Kid divides an arbitrary piece of chololate into three rectanglular pieces by cells, and then Karlsson chooses one of them and eats it. The game finishes when it's impossible to make a legal move. Kid wins if there was made an even number of moves, Karlsson wins if there was made an odd number of moves. Who has the winning strategy?

Problem 1.130 (633974672407561)**.** Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers with the property that
$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$
for all positive integers n. Show that $a_{2022} \leq 1$.

Problem 1.131 (8005762280394288133)**.** A school has 450 students. Each student has at least 100 friends among the others and among any 200 students, there are always two that are friends. Prove that 302 students can be sent on a kayak trip such that each of the 151 two seater kayaks contain people who are friends.

Problem 1.132 (1612300762204186997)**.** For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying
$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

Problem 1.133 (493735785757154)**.** Given is a graph G of $n+1$ vertices, which is constructed as follows: initially there is only one vertex v, and one a move we can add a vertex and connect it to exactly one among the previous vertices. The vertices have non-negative real weights such that v has weight 0 and each other vertex has a weight not exceeding the avarage weight of its neighbors, increased by 1. Prove that no weight can exceed n^2.

Problem 1.134 (7500559455615129254)**.** For every positive integer N, determine the smallest real number b_N such that, for all real x,
$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

Problem 1.135 (857598260795435)**.** Let $ABCD$ be a rhombus with center O. P is a point lying on the side AB. Let I, J, and L be the incenters of triangles PCD, PAD, and PBC, respectively. Let H and K be orthocenters of triangles PLB and PJA, respectively.

Prove that $OI \perp HK$.

Problem 1.136 (162618813015033)**.** In $\triangle ABC$, tangents of the circumcircle $\odot O$ at B, C and at A, B intersects at X, Y respectively. AX cuts BC at D and CY cuts AB at F. Ray DF cuts arc AB of the circumcircle at P. Q, R are on segments AB, AC such that P, Q, R are collinear and $QR \parallel BO$. If $PQ^2 = PR \cdot QR$, find $\angle ACB$.

Problem 1.137 (942225649898797)**.** Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$
Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Problem 1.138 (6190379360381554657)**.** Let $ABCD$ be a parallelogram. Point E lies on segment CD such that
$$2\angle AEB = \angle ADB + \angle ACB,$$
and point F lies on segment BC such that
$$2\angle DFA = \angle DCA + \angle DBA.$$
Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

Problem 1.139 (528504335909385)**.** Given a triangle $\triangle ABC$ whose incenter is I and A-excenter is J. A' is point so that AA' is a diameter of $\odot(\triangle ABC)$. Define H_1, H_2 to be the orthocenters of $\triangle BIA'$ and $\triangle CJA'$. Show that $H_1H_2 \parallel BC$

Problem 1.140 (4603228855421380865)**.** Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

Problem 1.141 (623590906176957)**.** The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 1.142 (23047452603115)**.** Let ABC be a triangle. Let θ be a fixed angle for which

$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$

Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

Problem 1.143 (9187468721920084868)**.** Each lattice point of $\mathbb{Z}^2$ is colored with one of three colors, with every color used at least once. Show that one can find a right triangle with pairwise distinct colored vertices.

Problem 1.144 (797215984506934)**.** Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Problem 1.145 (625002281186392279)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Problem 1.146 (614247648874042)**.** Misha has a 100x100 chessboard and a bag with 199 rooks. In one move he can either put one rook from the bag on the lower left cell of the grid, or remove two rooks which are on the same cell, put one of them on the adjacent square which is above it or right to it, and put the other in the bag. Misha wants to place exactly 100 rooks on the board, which don't beat each other. Will he be able to achieve such arrangement?

Problem 1.147 (711016608896725)**.** Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(S)$ denote the number of ways to draw 8 lines with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

Problem 1.148 (799773800583372)**.** A square grid 100×100 is tiled in two ways - only with dominoes and only with squares 2×2. What is the least number of dominoes that are entirely inside some square 2×2?

Problem 1.149 (5835156231907738776)**.** Given triangle ABC with A-excenter I_A, the foot of the perpendicular from I_A to BC is D. Let the midpoint of segment I_AD be M, T lies on arc BC(not containing A) satisfying $\angle BAT = \angle DAC$, I_AT intersects the circumcircle of ABC at $S \neq T$. If SM and BC intersect at X, the perpendicular bisector of AD intersects AC, AB at Y, Z respectively, prove that AX, BY, CZ are concurrent.

Problem 1.150 (825542457780626)**.** Yuri is looking at the great Mayan table. The table has 200 columns and 2^{200} rows. Yuri knows that each cell of the table depicts the sun or the moon, and any two rows are different (i.e. differ in at least one column). Each cell of the table is covered with a sheet. The wind has blown aways exactly two sheets from each row. Could it happen that now Yuri can find out for at least 10000 rows what is depicted in each of them (in each of the columns)?

Problem 1.151 (5664985199661230516)**.** In every row of a grid $100 \times n$ is written a permutation of the numbers $1, 2 \ldots, 100$. In one move you can choose a row and swap two non-adjacent numbers with difference 1. Find the largest possible n, such that at any moment, no matter the operations made, no two rows may have the same permutations.

Problem 1.152 (120381541018683)**.** Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x \in F} 1/x \neq r$ for all finite subsets F of S.

Problem 1.153 (119129720704350)**.** Let H be the orthocenter of a given triangle ABC. Let BH and AC meet at a point E, and CH and AB meet at F. Suppose that X is a point on the line BC. Also suppose that the circumcircle of triangle BEX and the line AB intersect again at Y, and the circumcircle of triangle CFX and the line AC intersect again at Z. Show that the circumcircle of triangle AYZ is tangent to the line AH.

Problem 1.154 (227919487650283)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Ω. Let M be the midpoint of side BC. Point D is chosen from the minor arc BC on Γ such that $\angle BAD = \angle MAC$. Let E be a point on Γ such that DE is perpendicular to AM, and F be a point on line BC such that DF is perpendicular to BC. Lines HF and AM intersect at point N, and point R is the reflection point of H with respect to N.

Prove that $\angle AER + \angle DFR = 180°$.

Problem 1.155 (4738483219849723703)**.** On a circle there're 1000 marked points, each colored in one of k colors. It's known that among any 5 pairwise intersecting segments, endpoints of which are 10 distinct marked points, there're at least 3 segments, each of which has its endpoints colored in different colors. Determine the smallest possible value of k for which it's possible.

Problem 1.156 (8569243655022492300)**.** Given a $\triangle ABC$ and a point P. Let O, D, E, F

be the circumcenter of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB$, respectively and let T be the intersection of BC with EF. Prove that the reflection of O in EF lies on the perpendicular from D to PT.

Problem 1.157 (5968448186928885521)**.** Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n}\left(\frac{1}{k^2}+\frac{1}{k^3}\right) \geq m \cdot \left(\sum_{k=m}^{n}\frac{1}{k^2}\right)^2.$$

Problem 1.158 (6029540617185205962)**.** On a social network, no user has more than ten friends (the state "friendship" is symmetrical). The network is connected: if, upon learning interesting news a user starts sending it to its friends, and these friends to their own friends and so on, then at the end, all users hear about the news. Prove that the network administration can divide users into groups so that the following conditions are met: each user is in exactly one group each group is connected in the above sense one of the groups contains from 1 to 100 members and the remaining from 100 to 900.

Problem 1.159 (326164407850848)**.** Two boys are given a bag of potatoes, each bag containing 150 tubers. They take turns transferring the potatoes, where in each turn they transfer a non-zero tubers from their bag to the other boy's bag. Their moves must satisfy the following condition: In each move, a boy must move more tubers than he had in his bag before any of his previous moves (if there were such moves). So, with his first move, a boy can move any non-zero quantity, and with his fifth move, a boy can move 200 tubers, if before his first, second, third and fourth move, the numbers of tubers in his bag was less than 200. What is the maximal total number of moves the two boys can do?

Problem 1.160 (748616641641895)**.** Let ABC be a triangle. Let ABC_1, BCA_1, CAB_1 be three equilateral triangles that do not overlap with ABC. Let P be the intersection of the circumcircles of triangle ABC_1 and CAB_1. Let Q be the point on the circumcircle of triangle CAB_1 so that PQ is parallel to BA_1. Let R be the point on the circumcircle of triangle ABC_1 so that PR is parallel to CA_1.

Show that the line connecting the centroid of triangle ABC and the centroid of triangle PQR is parallel to BC.

Problem 1.161 (2084796834305797 45)**.** There are 18 children in the class. Parents decided to give children from this class a cake. To do this, they first learned from each child the area of the piece he wants to get. After that, they showed a square-shaped cake, the area of which is exactly equal to the sum of 18 named numbers. However, when they saw the cake, the children wanted their pieces to be squares too. The parents cut the cake with lines parallel to the sides of the cake (cuts do not have to start or end on the side of the cake). For what maximum k the parents are guaranteed to cut out k square pieces from the cake, which you can give to k children so that each of them gets what they want?

Problem 1.162 (844684477828422)**.** Let point H be the orthocenter of a scalene triangle ABC. Line AH intersects with the circumcircle Ω of triangle ABC again at point P. Line BH, CH meets with AC, AB at point E and F, respectively. Let PE, PF meet Ω again at point Q, R, respectively. Point Y lies on Ω so that lines AY, QR and EF are concurrent. Prove that PY bisects EF.

Problem 1.163 (2667130530962382147)**.** We say that a function $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is great if for any nonnegative integers m and n,

$$f(m+1, n+1)f(m, n) - f(m+1, n)f(m, n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

Problem 1.164 (3470579368412517052)**.** A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either: the rabbit cannot move; or the hunter can determine the cell in which the rabbit started. Decide whether there exists a winning strategy for the hunter.

Problem 1.165 (4439711278400170990)**.** N oligarchs built a country with N cities with each one of them owning one city. In addition, each oligarch built some roads such that the maximal amount of roads an oligarch can build between two cities is 1 (note that there can be more than 1 road going through two cities, but they would belong to different oligarchs). A total of d roads were built. Some oligarchs wanted to create a corporation by combining their cities and roads so that from any city of the corporation you can go to any city of the corporation using only corporation roads (roads can go to other cities outside corporation) but it turned out that no group of less than N oligarchs can create a corporation. What is the maximal amount that d can have?

Problem 1.166 (967014444176640)**.** Let $m, n \geqslant 2$ be integers, let X be a set with n elements, and let $X_1, X_2, \dots, X_m$ be pairwise distinct non-empty, not necessary disjoint subset of X. A function $f \colon X \to \{1, 2, \dots, n+1\}$ is called nice if there exists an index k such that

$$\sum_{x \in X_k} f(x) > \sum_{x \in X_i} f(x) \quad \text{for all } i \neq k.$$

Prove that the number of nice functions is at least n^n.

Problem 1.167 (596902679696332)**.** Find all positive integers $n \geqslant 2$ for which there exist n real numbers $a_1 < \cdots < a_n$ and a real number $r > 0$ such that the $\frac{1}{2}n(n-1)$ differences $a_j - a_i$ for $1 \leqslant i < j \leqslant n$ are equal, in some order, to the numbers $r^1, r^2, \dots, r^{\frac{1}{2}n(n-1)}$.

Problem 1.168 (16342577076998227785)**.** Let a, b, c be fixed positive integers. There are $a + b + c$ ducks sitting in a circle, one behind the other. Each duck picks either rock, paper, or scissors, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors. A move consists of an operation of one of the following three forms: If a duck picking rock sits behind a duck picking scissors, they switch places. If a duck picking paper sits behind a duck picking rock, they switch places. If a duck picking scissors sits behind a duck picking paper, they switch places. Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

Problem 1.169 (499788610931519)**.** Andryusha has 100 stones of different weight and he can distinguish the stones by appearance, but does not know their weight. Every evening, Andryusha can put exactly 10 stones on the table and at night the brownie will order them in increasing weight. But, if the drum also lives in the house then surely he will in the morning change the places of some 2 stones.Andryusha knows all about this but does not know if there is a drum in his house. Can he find out?

Problem 1.170 (5363953658134647103)**.** Let ABC be a triangle with incenter I. The line through I, perpendicular to AI, intersects the circumcircle of ABC at points P and Q. It turns out there exists a point T on the side BC such that $AB + BT = AC + CT$ and $AT^2 = AB \cdot AC$. Determine all possible values of the ratio IP/IQ.

Problem 1.171 (8528437132500966626)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 1.172 (7997372712267182584)**.** Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Problem 1.173 (329519083206921)**.** Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Problem 1.174 (12311699525330)**.** Suppose $a_1 < a_2 < \cdots < a_{2024}$ is an arithmetic sequence of positive integers, and $b_1 < b_2 < \cdots < b_{2024}$ is a geometric sequence of positive integers. Find the maximum possible number of integers that could appear in both sequences, over all possible choices of the two sequences.

Problem 1.175 (965885167255885)**.** A 3×3 grid of unit cells is given. A snake of length k is an animal which occupies an ordered k-tuple of cells in this grid, say $(s_1, \ldots, s_k)$. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. After being placed in a finite $n \times n$ grid, if the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has turned around if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead.

Find the largest integer k such that one can place some snake of length k in a 3×3 grid which can turn around.

Problem 1.176 (5998250511147866097)**.** Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Problem 1.177 (3813623497653179264)**.** The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

Problem 1.178 (6246999615324043054)**.** A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 1.179 (63514716280156)**.** Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that

$AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$. Show that lines PE and QD meet on ω. Prove that if ω passes through H, then lines PD and QE meet on ω as well.

Problem 1.180 (2211812924503059239)**.** We are given n coins of different weights and n balances, $n > 2$. On each turn one can choose one balance, put one coin on the right pan and one on the left pan, and then delete these coins out of the balance. It's known that one balance is wrong (but it's not known ehich exactly), and it shows an arbitrary result on every turn. What is the smallest number of turns required to find the heaviest coin?

Problem 1.181 (669395675904242)**.** Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 1.182 (796313598765903)**.** There are three boxes of stones. Each hour, Sisyphus moves a stone from one box to another. For each transfer of a stone, he receives from Zeus a number of coins equal to the number of stones in the box from which the stone is drawn minus the number of stones in the recipient box, with the stone Sisyphus just carried not counted. If this number is negative, Sisyphus pays the corresponding amount (and can pay later if he is broke).

After 8760 hours, all the stones lie in their initial boxes. What is the greatest possible earning of Sisyphus at that moment, in terms of the initial quantities in the three boxes?

Problem 1.183 (697045850918084)**.** In the country there're N cities and some pairs of cities are connected by two-way airlines (each pair with no more than one). Every airline belongs to one of k companies. It turns out that it's possible to get to any city from any other, but it fails when we delete all airlines belonging to any one of the companies. What is the maximum possible number of airlines in the country ?

Problem 1.184 (817429246000759)**.** Find all integers $n \geq 2$ for which there exists a sequence of $2n$ pairwise distinct points $(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$ in the plane satisfying the following four conditions: no three of the $2n$ points are collinear; $P_iP_{i+1} \geq 1$ for all $i = 1, 2, \ldots, n$, where $P_{n+1} = P_1$; $Q_iQ_{i+1} \geq 1$ for all $i = 1, 2, \ldots, n$, where $Q_{n+1} = Q_1$; and $P_iQ_j \leq 1$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Problem 1.185 (240654526717277)**.** Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Problem 1.186 (915997916422887)**.** Let ABC and $A'B'C'$ be two triangles so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ form a triangle as well. Suppose that for any point X on the circumcircle of ABC, there exists exactly one point X' on the circumcircle of $A'B'C'$ so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ and $\overline{XX'}$ are concyclic. Show that ABC is similar to $A'B'C'$.

Problem 1.187 (6978535805224432571)**.** The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Problem 1.188 (6837149463099766937)**.** Let $n \geq 3$ be an odd integer. In a $2n \times 2n$ board, we colour $2(n-1)^2$ cells. What is the largest number of three-square corners that can surely be cut out of the uncoloured figure?

Problem 1.189 (37921131297270)**.** You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leq r \leq 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

Problem 1.190 (7017112574129036660)**.** Let ABC be a triangle with $AB < AC$, and let I_a be its A-excenter. Let D be the projection of I_a to BC. Let X be the intersection of AI_a and BC, and let Y, Z be the points on AC, AB, respectively, such that X, Y, Z are on a line perpendicular to AI_a. Let the circumcircle of AYZ intersect AI_a again at U. Suppose that the tangent of the circumcircle of ABC at A intersects BC at T, and the segment TU intersects the circumcircle of ABC at V. Show that $\angle BAV = \angle DAC$.

Problem 1.191 (6783316811528119504)**.** Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Problem 1.192 (4320337590540710547)**.** An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions: The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.) No two beams have intersecting interiors. The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam. What is the smallest positive number of beams that can be placed to satisfy these conditions?

Problem 1.193 (303061622555285)**.** A teacher and her 30 students play a game on an infinite cell grid. The teacher starts first, then each of the 30 students makes a move, then the teacher and so on. On one move the person can color one unit segment on the grid. A segment cannot be colored twice. The teacher wins if, after the move of one of the 31 players, there is a 1×2 or 2×1 rectangle , such that each segment from it's border is colored, but the segment between the two adjacent squares isn't colored. Prove that the teacher can win.

Problem 1.194 (221644122066923)**.** A straight road consists of green and red segments in alternating colours, the first and last segment being green. Suppose that the lengths of all segments are more than a centimeter and less than a meter, and that the length of each subsequent segment is larger than the previous one. A grasshopper wants to jump forward along the road along these segments, stepping on each green segment at least once an without stepping on any red segment (or the border between neighboring segments). Prove that the grasshopper can do this in such a way that among the lengths of his jumps no more than 8 different values occur.

Problem 1.195 (537574018594693)**.** Let ABC be a triangle with O as its circumcenter. A circle Γ tangents OB, OC at B, C, respectively. Let D be a point on Γ other than

B with $CB = CD$, E be the second intersection of DO and Γ, and F be the second intersection of EA and Γ. Let X be a point on the line AC so that $XB \perp BD$. Show that one half of $\angle ADF$ is equal to one of $\angle BDX$ and $\angle BXD$.

Problem 1.196 (4992489807901310938)**.** Let ABC be a triangle and ℓ_1, ℓ_2 be two parallel lines. Let ℓ_i intersects line BC, CA, AB at X_i, Y_i, Z_i, respectively. Let Δ_i be the triangle formed by the line passed through X_i and perpendicular to BC, the line passed through Y_i and perpendicular to CA, and the line passed through Z_i and perpendicular to AB. Prove that the circumcircles of Δ_1 and Δ_2 are tangent.

Problem 1.197 (215375559035207)**.** ABC is an isosceles triangle, with $AB = AC$. D is a moving point such that $AD \parallel BC$, $BD > CD$. Moving point E is on the arc of BC in circumcircle of ABC not containing A, such that $EB < EC$. Ray BC contains point F with $\angle ADE = \angle DFE$. If ray FD intersects ray BA at X, and intersects ray CA at Y, prove that $\angle XEY$ is a fixed angle.

Problem 1.198 (117986541208663)**.** Given a triangle ABC. D is a moving point on the edge BC. Point E and Point F are on the edge AB and AC, respectively, such that $BE = CD$ and $CF = BD$. The circumcircle of $\triangle BDE$ and $\triangle CDF$ intersects at another point P other than D. Prove that there exists a fixed point Q, such that the length of QP is constant.

Problem 1.199 (4000488814786935591)**.** A group of 100 kids has a deck of 101 cards numbered by $0, 1, 2, \ldots, 100$. The first kid takes the deck, shuffles it, and then takes the cards one by one; when he takes a card (not the last one in the deck), he computes the average of the numbers on the cards he took up to that moment, and writes down this average on the blackboard. Thus, he writes down 100 numbers, the first of which is the number on the first taken card. Then he passes the deck to the second kid which shuffles the deck and then performs the same procedure, and so on. This way, each of 100 kids writes down 100 numbers. Prove that there are two equal numbers among the 10000 numbers on the blackboard.

Problem 1.200 (308215997593136)**.** Misha came to country with n cities, and every 2 cities are connected by the road. Misha want visit some cities, but he doesn‘t visit one city two time. Every time, when Misha goes from city A to city B, president of country destroy k roads from city B(president can‘t destroy road, where Misha goes). What maximal number of cities Misha can visit, no matter how president does?

Problem 1.201 (645930596871591)**.** Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is stable if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \le x$ and $y' \le y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

Problem 1.202 (315159980103862)**.** Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

Problem 1.203 (689874125173032)**.** Let ω_1, ω_2 be two non-intersecting circles, with circumcenters O_1, O_2 respectively, and radii r_1, r_2 respectively where $r_1 < r_2$. Let AB, XY be the two internal common tangents of ω_1, ω_2, where A, X lie on ω_1, B, Y lie on ω_2. The circle with diameter AB meets ω_1, ω_2 at P and Q respectively. If

$$\angle AO_1P + \angle BO_2Q = 180^\circ,$$

find the value of $\frac{PX}{QY}$ (in terms of r_1, r_2).

Problem 1.204 (275429739915708)**.** Consider a 100×100 square unit lattice $\mathbf{L}$ (hence $\mathbf{L}$ has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in $\mathbf{L}$ and every point in $\mathbf{L}$ is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

Problem 1.205 (8851048763094130212)**.** Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines AC, DT, and BK are concurrent.

Problem 1.206 (5299971832672937326)**.** Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $\triangle ANK, \triangle BKL, \triangle CLM, \triangle DMN$.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

Problem 1.207 (1620616963605432410)**.** Given an isosceles triangle $\triangle ABC$, $AB = AC$. A line passes through M, the midpoint of BC, and intersects segment AB and ray CA at D and E, respectively. Let F be a point of ME such that $EF = DM$, and K be a point on MD. Let Γ_1 be the circle passes through B, D, K and Γ_2 be the circle passes through C, E, K. Γ_1 and Γ_2 intersect again at $L \neq K$. Let ω_1 and ω_2 be the circumcircle of $\triangle LDE$ and $\triangle LKM$. Prove that, if ω_1 and ω_2 are symmetric wrt L, then BF is perpendicular to BC.

Problem 1.208 (2302470517258475835)**.** Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Problem 1.209 (1560607598563435 21)**.** Let ABC be an acute triangle with $\angle ACB > 2\angle ABC$. Let I be the incenter of ABC, K is the reflection of I in line BC. Let line BA and KC intersect at D. The line through B parallel to CI intersects the minor arc BC on the circumcircle of ABC at $E(E \neq B)$. The line through A parallel to BC intersects the line BE at F. Prove that if $BF = CE$, then $FK = AD$.

Problem 1.210 (4679791554410865501)**.** For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

Problem 1.211 (290912955085727393)**.** Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

Problem 1.212 (3923745101517032298). Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.

Problem 1.213 (402654566950359). Let a_1, a_2, $\ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 1.214 (4118541811915047639). In a country there are $n > 100$ cities and initially no roads. The government randomly determined the cost of building a two-way road between any two cities, using all amounts from 1 to $\frac{n(n-1)}{2}$ thalers once (all options are equally likely). The mayor of each city chooses the cheapest of the $n - 1$ roads emanating from that city and it is built (this may be the mutual desired of the mayors of both cities being connected, or only one of the two). After the construction of these roads, the cities are divided into M connected components (between cities of the same connected component, you can get along the constructed roads, possibly via other cities, but this is not possible for cities of different components). Find the expected value of the random variable M.

Problem 1.215 (9162230842142232349). Let ABC be a triangle. Distinct points D, E, F lie on sides BC, AC, and AB, respectively, such that quadrilaterals $ABDE$ and $ACDF$ are cyclic. Line AD meets the circumcircle of $\triangle ABC$ again at P. Let Q denote the reflection of P across BC. Show that Q lies on the circumcircle of $\triangle AEF$.

Problem 1.216 (3417358984411200361). Let ABC be a triangle with circumcircle Ω, circumcenter O and orthocenter H. Let S lie on Ω and P lie on BC such that $\angle ASP = 90°$, line SH intersects the circumcircle of $\triangle APS$ at $X \neq S$. Suppose OP intersects CA, AB at Q, R, respectively, QY, RZ are the altitude of $\triangle AQR$. Prove that X, Y, Z are collinear.

Problem 1.217 (2201137214247796233). A neighborhood consists of 10×10 squares. On New Year's Eve it snowed for the first time and since then exactly 10 cm of snow fell on each square every night (and snow fell only at night). Every morning, the janitor selects one row or column and shovels all the snow from there onto one of the adjacent rows or columns (from each cell to the adjacent side). For example, he can select the seventh column and from each of its cells shovel all the snow into the cell of the left of it. You cannot shovel snow outside the neighborhood. On the evening of the 100th day of the year, an inspector will come to the city and find the cell with the snowdrift of maximal height. The goal of the janitor is to ensure that this height is minimal. What height of snowdrift will the inspector find?

Problem 1.218 (811235233671414145). Let m and n be positive integers. A circular necklace contains mn beads, each either red or blue. It turned out that no matter how the necklace was cut into m blocks of n consecutive beads, each block had a distinct number of red beads. Determine, with proof, all possible values of the ordered pair (m, n).

Problem 1.219 (6734490609685717062)**.** Let I, G, O be the incenter, centroid and the circumcenter of triangle ABC, respectively. Let X, Y, Z be on the rays BC, CA, AB respectively so that $BX = CY = AZ$. Let F be the centroid of XYZ.

Show that FG is perpendicular to IO.

Problem 1.220 (620629352845047)**.** As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta : \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$, $\theta(p+1) = \theta(p) + 1$, and if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

Problem 1.221 (5180896359975323937)**.** For every pair (m, n) of positive integers, a positive real number $a_{m,n}$ is given. Assume that

$$a_{m+1,n+1} = \frac{a_{m,n+1}a_{m+1,n} + 1}{a_{m,n}}$$

for all positive integers m and n. Suppose further that $a_{m,n}$ is an integer whenever $\min(m, n) \leq 2$. Prove that $a_{m,n}$ is an integer for all positive integers m and n.

Problem 1.222 (2749225075653830789)**.** In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and every quadrilateral Q_i has three corners of one color and one corner of the other color.

Problem 1.223 (2644568373783917)**.** Let ABC be a triangle such that the angular bisector of $\angle BAC$, the B-median and the perpendicular bisector of AB intersect at a single point X. Let H be the orthocenter of ABC. Show that $\angle BXH = 90^\circ$.

Problem 1.224 (8948164820835424145)**.** Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

Problem 1.225 (16134758174084)**.** Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

Problem 1.226 (2556841339462610604)**.** Suppose that (a_1, b_1), (a_2, b_2), $\ldots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Problem 1.227 (3838489129977355762)**.** Two triangles ABC and $A'B'C'$ are on the plane. It is known that each side length of triangle ABC is not less than a, and each side length of triangle $A'B'C'$ is not less than a'. Prove that we can always choose two points in the two triangles respectively such that the distance between them is not less than $\sqrt{\dfrac{a^2 + a'^2}{3}}$.

Problem 1.228 (5873161915777778529)**.** In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

Problem 1.229 (966139221944695)**.** Stierlitz wants to send an encryption to the Center, which is a code containing 100 characters, each a "dot" or a "dash". The instruction he received from the Center the day before about conspiracy reads:

i) when transmitting encryption over the radio, exactly 49 characters should be replaced with their opposites;

ii) the location of the "wrong" characters is decided by the transmitting side and the Center is not informed of it.

Prove that Stierlitz can send 10 encryptions, each time choosing some 49 characters to flip, such that when the Center receives these 10 ciphers, it may unambiguously restore the original code.

Problem 1.230 (8024569764169071557)**.** 12 schoolchildren are engaged in a circle of patriotic songs, each of them knows a few songs (maybe none). We will say that a group of schoolchildren can sing a song if at least one member of the group knows it. Supervisor the circle noticed that any group of 10 circle members can sing exactly 20 songs, and any group of 8 circle members - exactly 16 songs. Prove that the group of all 12 circle members can sing exactly 24 songs.

Problem 1.231 (770421031902562)**.** A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

Problem 1.232 (57065759079551)**.** Let p be an odd prime number. Suppose P and Q are polynomials with integer coefficients such that $P(0) = Q(0) = 1$, there is no nonconstant polynomial dividing both P and Q, and

$$1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{\ddots}{1 + (p-1)x}}} = \frac{P(x)}{Q(x)}.$$

Show that all coefficients of P except for the constant coefficient are divisible by p, and all coefficients of Q are *not* divisible by p.

Problem 1.233 (8700998965901287095)**.** Let ABC be an acute triangle with circumcircle ω. Let P be a variable point on the arc BC of ω not containing A. Squares $BPDE$ and $PCFG$ are constructed such that A, D, E lie on the same side of line BP and A, F, G lie on the same side of line CP. Let H be the intersection of lines DE and FG. Show that as P varies, H lies on a fixed circle.

Problem 1.234 (141955509989127)**.** Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \le i, j \le n$, such that: for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

Problem 1.235 (132497611943266)**.** Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 1.236 (6497483389877629432)**.** Initially, a word of 250 letters with 125 letters A and 125 letters B is written on a blackboard. In each operation, we may choose a contiguous string of any length with equal number of letters A and equal number of letters B, reverse those letters and then swap each B with A and each A with B (Example: $ABABBA$ after the operation becomes $BAABAB$). Decide if it possible to choose initial word, so that after some operations, it will become the same as the first word, but in reverse order.

Problem 1.237 (258585206260584)**.** Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Problem 1.238 (4218160072471349910)**.** Given is a natural number $n > 4$. There are n points marked on the plane, no three of which lie on the same line. Vasily draws one by one all the segments connecting pairs of marked points. At each step, drawing the next segment S, Vasily marks it with the smallest natural number, which hasn't appeared on a drawn segment that has a common end with S. Find the maximal value of k, for which Vasily can act in such a way that he can mark some segment with the number k?

Problem 1.239 (231259391294064)**.** Every two of the n cities of Ruritania are connected by a direct flight of one from two airlines. Promonopoly Committee wants at least k flights performed by one company. To do this, he can at least every day to choose any three cities and change the ownership of the three flights connecting these cities each other (that is, to take each of these flights from a company that performs it, and pass the other). What is the largest k committee knowingly will be able to achieve its goal in no time, no matter how the flights are distributed hour?

Problem 1.240 (6360153743145135128)**.** Find all functions $f\colon \mathbb{Z}^2 \to [0, 1]$ such that for any integers x and y,

$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

Problem 1.241 (4429559846138102630)**.** An interstellar hotel has 100 rooms with capacities $101, 102, \ldots, 200$ people. These rooms are occupied by n people in total. Now a VIP guest is about to arrive and the owner wants to provide him with a personal room. On that purpose, the owner wants to choose two rooms A and B and move all guests from A to B without exceeding its capacity. Determine the largest n for which the owner can be sure that he can achieve his goal no matter what the initial distribution of the guests is.

Problem 1.242 (457934969594281)**.** A positive integer n is given. A cube $3 \times 3 \times 3$ is built from 26 white and 1 black cubes $1 \times 1 \times 1$ such that the black cube is in the center of $3 \times 3 \times 3$-cube. A cube $3n \times 3n \times 3n$ is formed by n^3 such $3 \times 3 \times 3$-cubes. What is the smallest number of white cubes which should be colored in red in such a way that every white cube will have at least one common vertex with a red one.

Problem 1.243 (4527883777563937913)**.** 10000 children came to a camp; every of them is friend of exactly eleven other children in the camp (friendship is mutual). Every child wears T-shirt of one of seven rainbow's colours; every two friends' colours are different. Leaders demanded that some children (at least one) wear T-shirts of other colours (from those seven colours). Survey pointed that 100 children didn't want to change their colours [translator's comment: it means that any of these 100 children (and only them) can't change his (her) colour such that still every two friends' colours will be different]. Prove that some of other children can change colours of their T-shirts such that as before every two friends' colours will be different.

Problem 1.244 (952584318797289)**.** Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

Problem 1.245 (684771433215596)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 1.246 (6612845742708555351)**.** Cyclic quadrilateral $ABCD$ has circumcircle (O). Points M and N are the midpoints of BC and CD, and E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Let EN meet FM at P. Denote S as the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

Problem 1.247 (423911944927735)**.** In acute $\triangle ABC$, O is the circumcenter, I is the incenter. The incircle touches BC, CA, AB at D, E, F. And the points K, M, N are the midpoints of BC, CA, AB respectively.

a) Prove that the lines passing through D, E, F in parallel with IK, IM, IN respectively are concurrent.

b) Points T, P, Q are the middle points of the major arc BC, CA, AB on $\odot ABC$. Prove that the lines passing through D, E, F in parallel with IT, IP, IQ respectively are concurrent.

Problem 1.248 (6513083395063379420)**.** Given a convex pentagon $ABCDE$. Let A_1 be the intersection of BD with CE and define B_1, C_1, D_1, E_1 similarly, A_2 be the second intersection of $\odot(ABD_1), \odot(AEC_1)$ and define B_2, C_2, D_2, E_2 similarly. Prove that $AA_2, BB_2, CC_2, DD_2, EE_2$ are concurrent.

Problem 1.249 (437645166165639)**.** Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2.$$

Problem 1.250 (8048961544243923335)**.** Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

Problem 1.251 (456772085666528)**.** Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Problem 1.252 (9137209985622350774)**.** In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Problem 1.253 (4306507392377162131)**.** Let n be a positive integer such that the number

$$\frac{1^k + 2^k + \cdots + n^k}{n}$$

is an integer for any $k \in \{1, 2, \ldots, 99\}$. Prove that n has no divisors between 2 and 100, inclusive.

Problem 1.254 (2117883853443241027)**.** On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Petya prevent him?

Problem 1.255 (3245291910836201005)**.** Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

Problem 1.256 (7088779505939683183)**.** Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

Problem 1.257 (651490142085731)**.** Let I be the incenter of triangle ABC, and let ω be its incircle. Let E and F be the points of tangency of ω with CA and AB, respectively. Let X and Y be the intersections of the circumcircle of BIC and ω. Take a point T on BC such that $\angle AIT$ is a right angle. Let G be the intersection of EF and BC, and let Z be the intersection of XY and AT. Prove that AZ, ZG, and AI form an isosceles triangle.

Problem 1.258 (1736102587052874498)**.** Some language has only three letters - A, B and C. A sequence of letters is called a word iff it contains exactly 100 letters such that exactly 40 of them are consonants and other 60 letters are all A. What is the maximum numbers of words one can pick such that any two picked words have at least one position where they both have consonants, but different consonants?

Problem 1.259 (6116877365036470315)**.** Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions: (i) $f(n) \neq 0$ for at least one n; (ii) $f(xy) = f(x) + f(y)$ for every positive integers x and y; (iii) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

Problem 1.260 (8972547734710795566)**.** Let incircle (I) of triangle ABC touch the sides BC, CA, AB at D, E, F respectively. Let (O) be the circumcircle of ABC. Ray EF meets (O) at M. Tangents at M and A of (O) meet at S. Tangents at B and C of (O) meet at T. Line TI meets OA at J. Prove that $\angle ASJ = \angle IST$.

Problem 1.261 (8209367948889736949)**.** For every ordered pair of integers (i, j), not necessarily positive, we wish to select a point $P_{i,j}$ in the Cartesian plane whose coordinates lie inside the unit square defined by

$$i < x < i + 1, \qquad j < y < j + 1.$$

Find all real numbers $c > 0$ for which it's possible to choose these points such that for all integers i and j, the (possibly concave or degenerate) quadrilateral $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ has perimeter strictly less than c.

Problem 1.262 (284109588966873)**.** Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

Problem 1.263 (6558910862034852540)**.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2$$

Problem 1.264 (607556370102952)**.** Let Ω be the circumcircle of an acute triangle ABC. Points D, E, F are the midpoints of the inferior arcs BC, CA, AB, respectively, on Ω. Let G be the antipode of D in Ω. Let X be the intersection of lines GE and AB, while Y the intersection of lines FG and CA. Let the circumcenters of triangles BEX and CFY be points S and T, respectively. Prove that D, S, T are collinear.

Problem 1.265 (4278278843148290847)**.** Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \dots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Problem 1.266 (8126547357118301633)**.** An infinite sequence a_1, a_2, a_3, $\dots$ of real numbers satisfies

$$a_{2n-1} + a_{2n} > a_{2n+1} + a_{2n+2} \qquad \text{and} \qquad a_{2n} + a_{2n+1} < a_{2n+2} + a_{2n+3}$$

for every positive integer n. Prove that there exists a real number C such that $a_n a_{n+1} < C$ for every positive integer n.

Problem 1.267 (522990139281725)**.** For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \dots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \dots)$ is a p-sequence, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$. (a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n? (b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

Problem 1.268 (549441013338848)**.** What is the minimal number of operations needed to repaint a entirely white grid 100×100 to be entirely black, if on one move we can choose 99 cells from any row or column and change their color?

Problem 1.269 (457324036151847)**.** Let O and H be the circumcenter and the orthocenter, respectively, of an acute triangle ABC. Points D and E are chosen from sides AB and AC, respectively, such that A, D, O, E are concyclic. Let P be a point on the circumcircle of triangle ABC. The line passing P and parallel to OD intersects AB at point X, while the line passing P and parallel to OE intersects AC at Y. Suppose that the perpendicular bisector of $\overline{HP}$ does not coincide with XY, but intersect XY at Q, and that points A, Q lies on the different sides of DE. Prove that $\angle EQD = \angle BAC$.

Problem 1.270 (181463134716189)**.** In kindergarten, nurse took $n > 1$ identical cardboard rectangles and distributed them to n children; every child got one rectangle. Every child cut his (her) rectangle into several identical squares (squares of different children could be different). Finally, the total number of squares was prime. Prove that initial rectangles was squares.

Problem 1.271 (796349431725149)**.** An acute, non-isosceles triangle ABC is inscribed in a circle with centre O. A line go through O and midpoint I of BC intersects AB, AC at E, F respectively. Let D, G be reflections to A over O and circumcentre of (AEF), respectively. Let K be the reflection of O over circumcentre of (OBC). $a)$ Prove that D, G, K are collinear. $b)$ Let M, N are points on KB, KC that $IM \perp AC$, $IN \perp AB$. The midperpendiculars of IK intersects MN at H. Assume that IH intersects AB, AC at P, Q respectively. Prove that the circumcircle of $\triangle APQ$ intersects (O) the second time at a point on AI.

Problem 1.272 (2003233604438068678)**.** Given a triangle ABC and a point O on a plane. Let Γ be the circumcircle of ABC. Suppose that CO intersects with AB at D, and BO and CA intersect at E. Moreover, suppose that AO intersects with Γ at A, F. Let I be the other intersection of Γ and the circumcircle of ADE, and Y be the other intersection of BE and the circumcircle of CEI, and Z be the other intersection of CD and the circumcircle of BDI. Let T be the intersection of the two tangents of Γ at B, C, respectively. Lastly, suppose that TF intersects with Γ again at U, and the reflection of U w.r.t. BC is G.

Show that F, I, G, O, Y, Z are concyclic.

Problem 1.273 (211625179383762)**.** Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

Problem 1.274 (4892352754475215646)**.** We say that a set S of integers is rootiful if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 1.275 (1168447466971762345)**.** Let I, O, ω, Ω be the incenter, circumcenter, the incircle, and the circumcircle, respectively, of a scalene triangle ABC. The incircle ω is tangent to side BC at point D. Let S be the point on the circumcircle Ω such that AS, OI, BC are concurrent. Let H be the orthocenter of triangle BIC. Point T lies on Ω such that $\angle ATI$ is a right angle. Prove that the points D, T, H, S are concyclic.

Problem 1.276 (443006607452241)**.** Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n} \prod_{j\neq i} \frac{1-x_ix_j}{x_i-x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Problem 1.277 (6025085618534905645)**.** Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_BO_D}$.

Problem 1.278 (57940096937913)**.** Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

Problem 1.279 (314213229221479)**.** Given is a natural number $n > 5$. On a circular strip of paper is written a sequence of zeros and ones. For each sequence w of n zeros and ones we count the number of ways to cut out a fragment from the strip on which is written w. It turned out that the largest number M is achieved for the sequence 1100...0 ($n-2$ zeros) and the smallest - for the sequence 00...011 ($n-2$ zeros). Prove that there is another sequence of n zeros and ones that occurs exactly M times.

Problem 1.280 (677860185151955)**.** The checker moves from the lower left corner of the board 100×100 to the right top corner, moving at each step one cell to the right or one cell up. Let a be the number of paths in which exactly 70 steps the checker take under the diagonal going from the lower left corner to the upper right corner, and b is the number of paths in which such steps are exactly 110. What is more: a or b?

Problem 1.281 (8757490679465390171)**.** Color every vertex of 2008-gon with two colors, such that adjacent vertices have different color. If sum of angles of vertices of first color is same as sum of angles of vertices of second color, than we call 2008-gon as interesting. Convex 2009-gon one vertex is marked. It is known, that if remove any unmarked vertex, then we get interesting 2008-gon. Prove, that if we remove marked vertex, then we get interesting 2008-gon too.

Problem 1.282 (8402748184217471405)**.** In ΔABC, $AD \perp BC$ at D. E, F lie on line AB, such that $BD = BE = BF$. Let I, J be the incenter and A-excenter. Prove that there exist two points P, Q on the circumcircle of ΔABC , such that $PB = QC$, and $\Delta PEI \sim \Delta QFJ$.

Problem 1.283 (695330092247108707)**.** There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 1.284 (728988632553727)**.** Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

Problem 1.285 (329951351081287)**.** You're given an $n \times n$ matrix of real numbers. In an operation, you may negate the entries of any row or column. Prove that in a finite number of operations, you can ensure every row and every column of the matrix has nonnegative sum.

Problem 1.286 (241697479443718)**.** A convex polyhedron is floating in the Aegean sea. Can 90% of its volume be below the water level while more than half of its surface area is above the water level?

Problem 1.287 (5347245479409093202)**.** Let G be a graph with 400 vertices. For any edge AB we call a cuttlefish the set of all edges from A and B (including AB). Each edge of the graph is assigned a value of 1 or -1. It is known that the sum of edges at any cuttlefish is greater than or equal to 1. Prove that the sum of the numbers at all edges is at least -10^4.

Problem 1.288 (653200526211133)**.** Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define $p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ and $q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$. Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

Problem 1.289 (6183425212304704085)**.** A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

Problem 1.290 (836212333854709)**.** Let A_1, …, A_{2022} be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

In each round, Bob chooses a point in the plane (not necessarily among $A_1, \ldots, A_{2022}$) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points $A_1, \ldots, A_{2022}$ in Q rounds?

Problem 1.291 (4415914581303660291)**.** 24 students attend a mathematical circle. For any team consisting of 6 students, the teacher considers it to be either GOOD or OK. For the tournament of mathematical battles, the teacher wants to partition all the students into 4 teams of 6 students each. May it happen that every such partition contains either 3 GOOD teams or exactly one GOOD team and both options are present?

Problem 1.292 (1440964279096111130)**.** Let a be a positive integer. We say that a positive integer b is a-good if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a-good, but $b+2$ is not a-good. Prove that $b+1$ is prime.

Problem 1.293 (402139377468684)**.** For a positive integer k, let $s(k)$ denote the number of 1s in the binary representation of k. Prove that for any positive integer n,

$$\sum_{i=1}^{n} (-1)^{s(3i)} > 0.$$

Problem 1.294 (961350373727093)**.** Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

Problem 1.295 (8799177804774743019)**.** In each square of a garden shaped like a 2022×2022 board, there is initially a tree of height 0. A gardener and a lumberjack alternate turns playing the following game, with the gardener taking the first turn: The gardener chooses a square in the garden. Each tree on that square and all the surrounding squares (of which there are at most eight) then becomes one unit taller. The lumberjack then chooses four different squares on the board. Each tree of positive height on those squares then becomes one unit shorter. We say that a tree is majestic if its height is at least 10^6. Determine the largest K such that the gardener can ensure there are eventually K majestic trees on the board, no matter how the lumberjack plays.

Problem 1.296 (7203789790519658258)**.** Let ABC be a triangle and let P be a point not lying on any of the three lines AB, BC, or CA. Distinct points D, E, and F lie on lines BC, AC, and AB, respectively, such that $\overline{DE} \parallel \overline{CP}$ and $\overline{DF} \parallel \overline{BP}$. Show that there exists a point Q on the circumcircle of $\triangle AEF$ such that $\triangle BAQ$ is similar to $\triangle PAC$.

Problem 1.297 (308110166188097)**.** Let A, B be two fixed points on the unit circle ω, satisfying $\sqrt{2} < AB < 2$. Let P be a point that can move on the unit circle, and it can move to anywhere on the unit circle satisfying $\triangle ABP$ is acute and $AP > AB > BP$. Let H be the orthocenter of $\triangle ABP$ and S be a point on the minor arc AP satisfying $SH = AH$. Let T be a point on the minor arc AB satisfying $TB||AP$. Let $ST \cap BP = Q$.

Show that (recall P varies) the circle with diameter HQ passes through a fixed point.

Problem 1.298 (239934686230450)**.** Let triangle$ABC(AB < AC)$ with incenter I circumscribed in $\odot O$. Let M, N be midpoint of arc $\widehat{BAC}$ and $\widehat{BC}$, respectively. D lies on $\odot O$ so that $AD//BC$, and E is tangency point of A-excircle of $\triangle ABC$. Point F is in $\triangle ABC$ so that $FI//BC$ and $\angle BAF = \angle EAC$. Extend NF to meet $\odot O$ at G, and extend AG to meet line IF at L. Let line AF and DI meet at K. Proof that $ML \perp NK$.

Problem 1.299 (3780160396229984886)**.** Let $\lfloor \bullet \rfloor$ denote the floor function. For nonnegative integers a and b, their bitwise xor, denoted $a \oplus b$, is the unique nonnegative integer such that

$$\left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor - \left\lfloor \frac{a \oplus b}{2^k} \right\rfloor$$

is even for every $k \geq 0$. Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

Problem 1.300 (15195306726194)**.** There are two piles of stones: 1703 stones in one pile and 2022 in the other. Sasha and Olya play the game, making moves in turn, Sasha starts. Let before the player's move the heaps contain a and b stones, with $a \geq b$. Then, on his own move, the player is allowed take from the pile with a stones any number of stones from 1 to b. A player loses if he can't make a move. Who wins?

Remark: For 10.4, the initial numbers are $(444, 999)$

Problem 1.301 (5066939379306191291)**.** Let ABC be an acute triangle with circumcenter O and circumcircle Ω. Choose points D, E from sides AB, AC, respectively, and let ℓ be the line passing through A and perpendicular to DE. Let ℓ intersect the circumcircle of triangle ADE and Ω again at points P, Q, respectively. Let N be the intersection of

OQ and BC, S be the intersection of OP and DE, and W be the orthocenter of triangle SAO.

Prove that the points S, N, O, W are concyclic.

Problem 1.302 (6302540840099076878)**.** Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

Problem 1.303 (165465510156789)**.** Let Ω be the circumcircle of an isosceles trapezoid $ABCD$, in which AD is parallel to BC. Let X be the reflection point of D with respect to BC. Point Q is on the arc BC of Ω that does not contain A. Let P be the intersection of DQ and BC. A point E satisfies that EQ is parallel to PX, and EQ bisects $\angle BEC$. Prove that EQ also bisects $\angle AEP$.

Problem 1.304 (17901140622539144451)**.** Given a triangle $\triangle ABC$ and a point O. X is a point on the ray $\overrightarrow{AC}$. Let X' be a point on the ray $\overrightarrow{BA}$ so that $\overline{AX} = \overline{AX_1}$ and A lies in the segment $\overline{BX_1}$. Then, on the ray $\overrightarrow{BC}$, choose X_2 with $\overline{X_1X_2} \parallel \overline{OC}$.

Prove that when X moves on the ray $\overrightarrow{AC}$, the locus of circumcenter of $\triangle BX_1X_2$ is a part of a line.

Problem 1.305 (6566259136811987209)**.** Let Ω be the A-excircle of triangle ABC, and suppose that Ω is tangent to lines BC, CA, and AB at points D, E, and F, respectively. Let M be the midpoint of segment EF. Two more points P and Q are on Ω such that EP and FQ are both parallel to DM. Let BP meet CQ at point X. Prove that the line AM is the angle bisector of $\angle XAD$.

Problem 1.306 (3435532350205377704)**.** Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.

Problem 1.307 (1302548092028853470)**.** Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions: The frog visits only points in $\{1, 2, \dots, 2^n - 1\}$, each at most once. The length of each hop is in $\{2^0, 2^1, 2^2, \dots\}$. (The hops may be either direction, left or right.) Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

Problem 1.308 (803002459788170506)**.** Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$. Suppose that the three line segments B_1C_2, C_1A_2, A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

Problem 1.309 (4308913658510445082)**.** Let $ABCD$ be a convex quadrilateral, the incenters of $\triangle ABC$ and $\triangle ADC$ are I, J, respectively. It is known that AC, BD, IJ concurrent at a point P. The line perpendicular to BD through P intersects with the outer angle bisector of $\angle BAD$ and the outer angle bisector $\angle BCD$ at E, F, respectively. Show that $PE = PF$.

Problem 1.310 (521339998508550)**.** There are 998 cities in a country. Some pairs of cities are connected by two-way flights. According to the law, between any pair cities should be no more than one flight. Another law requires that for any group of cities there will be no more than $5k + 10$ flights connecting two cities from this group, where k is the number number of cities in the group. Prove that several new flights can be introduced so that laws still hold and the total number of flights in the country is equal to 5000.

Problem 1.311 (8612979541975584705)**.** Let G be a connected graph and let X, Y be two disjoint subsets of its vertices, such that there are no edges between them. Given that G/X has m connected components and G/Y has n connected components, what is the minimal number of connected components of the graph $G/(X \cup Y)$?

Problem 1.312 (9055967412808709037)**.** Baron Munchhausen has a collection of stones, such that they are of 1000 distinct whole weights, 2^{1000} stones of every weight. Baron states that if one takes exactly one stone of every weight, then the weight of all these 1000 stones chosen will be less than 2^{1010}, and there is no other way to obtain this weight by picking another set of stones of the collection. Can this statement happen to be true?

Problem 1.313 (1293772592063302344)**.** In non-isosceles acute $\triangle ABC$, AP, BQ, CR is the height of the triangle. A_1 is the midpoint of BC, AA_1 intersects QR at K, QR intersects a straight line that crosses A and is parallel to BC at point D, the line connecting the midpoint of AH and K intersects DA_1 at A_2. Similarly define B_2, C_2. $\triangle A_2B_2C_2$ is known to be non-degenerate, and its circumscribed circle is ω. Prove that: there are circles $\odot A'$, $\odot B'$, $\odot C'$ tangent to and INSIDE ω satisfying: (1) $\odot A'$ is tangent to AB and AC, $\odot B'$ is tangent to BC and BA, and $\odot C'$ is tangent to CA and CB. (2) A', B', C' are different and collinear.

Problem 1.314 (7948249970111159954)**.** A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 < \ldots < t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left|\sum_{i=1}^{k} a_{t_i}\right| \geq C.$$

Problem 1.315 (13360308368399041365)**.** Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if area (BCD) + area (ADE) = area (ABD) + area (ABP).

Problem 1.316 (15807076307704760375)**.** Two triangles intersect to form seven finite disjoint regions, six of which are triangles with area 1. The last region is a hexagon with area A. Compute the minimum possible value of A.

Problem 1.317 (482459214391384)**.** On a table with 25 columns and 300 rows, Kostya painted all its cells in three colors. Then, Lesha, looking at the table, for each row names one of the three colors and marks in that row all cells of that color (if there are no cells of that color in that row, he does nothing). After that, all columns that have at least a marked square will be deleted. Kostya wants to be left as few as possible columns in the table, and Lesha wants there to be as many as possible columns in the table. What is the largest number of columns Lesha can guarantee to leave?

Problem 1.318 (591652153716935)**.** Let M be the midpoint of BC of triangle ABC. The circle with diameter BC, ω, meets AB, AC at D, E respectively. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PAC, \angle PCA = \angle PAB$, and $2PM \cdot DE = BC^2$. Point X

lies outside ω such that $XM \parallel AP$, and $\frac{XB}{XC} = \frac{AB}{AC}$. Prove that $\angle BXC + \angle BAC = 90°$.

Problem 1.319 (627600286851318227)**.** Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 1.320 (908587245178389)**.** Let I be the incenter of triangle ABC, and ℓ be the perpendicular bisector of AI. Suppose that P is on the circumcircle of triangle ABC, and line AP and ℓ intersect at point Q. Point R is on ℓ such that $\angle IPR = 90°$.Suppose that line IQ and the midsegment of ABC that is parallel to BC intersect at M. Show that $\angle AMR = 90°$

(Note: In a triangle, a line connecting two midpoints is called a midsegment.)

Problem 1.321 (297728211754501)**.** The board used for playing a game consists of the left and right parts. In each part there are several fields and there're several segments connecting two fields from different parts (all the fields are connected.) Initially, there is a violet counter on a field in the left part, and a purple counter on a field in the right part. Lyosha and Pasha alternatively play their turn, starting from Pasha, by moving their chip (Lyosha-violet, and Pasha-purple) over a segment to other field that has no chip. It's prohibited to repeat a position twice, i.e. can't move to position that already been occupied by some earlier turns in the game. A player losses if he can't make a move. Is there a board and an initial positions of counters that Pasha has a winning strategy?

Problem 1.322 (1613309914397651478)**.** Let $ABCD$ be a convex quadrilateral with $\angle B < \angle A < 90^o$. Let I be the midpoint of AB and S the intersection of AD and BC. Let R be a variable point inside the triangle SAB such that $\angle ASR = \angle BSR$. On the straight lines AR, BR , take the points E, F, respectively so that BE, AF are parallel to RS. Suppose that EF intersects the circumcircle of triangle SAB at points H, K. On the segment AB, take points M, N such that $\angle AHM = \angle BHI$, $\angle BKN = \angle AKI$.

a) Prove that the center J of the circumcircle of triangle SMN lies on a fixed line.

b) On BE, AF , take the points P, Q respectively so that CP is parallel to SE and DQ is parallel to SF. The lines SE, SF intersect the circle (SAB), respectively, at U, V. Let G be the intersection of AU and BV. Prove that the median of vertex G of the triangle GPQ always passes through a fixed point .

Problem 1.323 (1965233157265405983)**.** Given a triangle $\triangle ABC$. Denote its incircle and circumcircle by ω, Ω, respectively. Assume that ω tangents the sides AB, AC at F, E, respectively. Then, let the intersections of line EF and Ω to be P, Q. Let M to be the mid-point of BC. Take a point R on the circumcircle of $\triangle MPQ$, say Γ, such that $MR \perp EF$. Prove that the line AR, ω and Γ intersect at one point.

Problem 1.324 (4875666253256352039)**.** Supppose that there are roads AB and CD but there are no roads BC and AD between four cities A, B, C, and D. Define restructing to be the changing a pair of roads AB and CD to the pair of roads BC and AD. Initially there were some cities in a country, some of which were connected by roads and for every city there were exactly 100 roads starting in it. The minister drew a new scheme of roads, where for every city there were also exactly 100 roads starting in it. It's known also that in both schemes there were no cities connected by more than one road. Prove that it's possible to obtain the new scheme from the initial after making a finite number of restructings.

Problem 1.325 (260804315613681)**.** Suppose a $a' \times b' \times c'$ rectangular prism fits inside an $a \times b \times c$ rectangular prism. Is it possible that $a' + b' + c' > a + b + c$?

Problem 1.326 (69707766974981)**.** For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root. Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \leq 1.$$

Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

Problem 1.327 (208441124738479)**.** Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

Problem 1.328 (3048608408918882691)**.** Is it possible to arrange everything in all cells of an infinite checkered plane all natural numbers (once) so that for each n in each square $n \times n$ the sum of the numbers is a multiple of n?

Problem 1.329 (495587557940069)**.** Let the excircle of a triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively. Denote by γ the circumcircle of triangle $A_1B_1C_1$ and assume that γ passes through vertex A. Show that $\overline{AA_1}$ is a diameter of γ. Show that the incenter of $\triangle ABC$ lies on line B_1C_1.

Problem 1.330 (436681276656848)**.** For the quadrilateral $ABCD$, let AC and BD intersect at E, AB and CD intersect at F, and AD and BC intersect at G. Additionally, let W, X, Y, and Z be the points of symmetry to E with respect to AB, BC, CD, and DA respectively. Prove that one of the intersection points of $\odot(FWY)$ and $\odot(GXZ)$ lies on the line FG.

Problem 1.331 (9154783649392500)**.** Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 1.332 (8892145789808454835)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is essentially increasing if $f(s) \leq f(t)$ holds whenever $s \leq t$ are real numbers such that $f(s) \neq 0$ and $f(t) \neq 0$.

Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \dots, x_{2022}$, there exist k essentially increasing functions $f_1, \dots, f_k$ such that

$$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \dots 2022.$$

Problem 1.333 (409530198849693)**.** In a cyclic convex hexagon $ABCDEF$, AB and DC intersect at G, AF and DE intersect at H. Let M, N be the circumcenters of BCG and EFH, respectively. Prove that the BE, CF and MN are concurrent.

Problem 1.334 (1248852037865425410)**.** Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied: Each number in the table is congruent to 1 modulo n. The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2. Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that the sums $R_1 + \dots R_n$ and $C_1 + \dots C_n$ are congruent modulo n^4.

Problem 1.335 (7243491713649826569)**.** In the triangle ABC let B' and C' be the midpoints of the sides AC and AB respectively and H the foot of the altitude passing through the vertex A. Prove that the circumcircles of the triangles $AB'C'$,$BC'H$, and $B'CH$ have a common point I and that the line HI passes through the midpoint of the segment $B'C'$.

Problem 1.336 (8330669807899443473)**.** Let ABC be an acute scalene triangle, and let A_1, B_1, C_1 be the feet of the altitudes from A, B, C. Let A_2 be the intersection of the tangents to the circle ABC at B, C and define B_2, C_2 similarly. Let A_2A_1 intersect the circle $A_2B_2C_2$ again at A_3 and define B_3, C_3 similarly. Show that the circles AA_1A_3, BB_1B_3, and CC_1C_3 all have two common points, X_1 and X_2 which both lie on the Euler line of the triangle ABC.

Problem 1.337 (9026100911884959358)**.** Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Problem 1.338 (6209707374283278028)**.** Let ABC be a triangle and D be a point inside triangle ABC. Γ is the circumcircle of triangle ABC, and DB, DC meet Γ again at E, F , respectively. Γ_1, Γ_2 are the circumcircles of triangle ADE and ADF respectively. Assume X is on Γ_2 such that BX is tangent to Γ_2. Let BX meets Γ again at Z. Prove that the line CZ is tangent to Γ_1 .

Problem 1.339 (3626448942281457521)**.** Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Problem 1.340 (3906812380515301028)**.** Given a triangle $\triangle ABC$. Denote its incenter and orthocenter by I, H, respectively. If there is a point K with

$$AH + AK = BH + BK = CH + CK$$

Show that H, I, K are collinear.

Problem 1.341 (3159161448000677570)**.** Let $a > 1$ be a positive integer and $d > 1$ be a positive integer coprime to a. Let $x_1 = 1$, and for $k \geq 1$, define

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \text{ does not divide } x_k \\ x_k/a & \text{if } a \text{ divides } x_k \end{cases}$$

Find, in terms of a and d, the greatest positive integer n for which there exists an index k such that x_k is divisible by a^n.

Problem 1.342 (3031913484181592371)**.** Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

Problem 1.343 (437956241529021)**.** In a country, there are N cities and $N(N-1)$ one-way roads: one road from X to Y for each ordered pair of cities $X \neq Y$. Every road has a maintenance cost. For each $k = 1, \ldots, N$ let's consider all the ways to select k cities and $N - k$ roads so that from each city it is possible to get to some selected city, using only selected roads.

We call such a system of cities and roads with the lowest total maintenance cost k-optimal. Prove that cities can be numbered from 1 to N so that for each $k = 1, \ldots, N$ there is a k-optimal system of roads with the selected cities numbered $1, \ldots, k$.

Problem 1.344 (2594275832195659804)**.** Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.

[asy]size(10cm); real w = 2*Sin(18); real h = 0.10 * w; real d = 0.33 * h; picture wht; picture blk;

draw(wht, (0,0)–(w,0)–(w+d,h)–(-d,h)–cycle); fill(blk, (0,0)–(w,0)–(w+d,h)–(-d,h)–cycle, black);

// draw(unitcircle, blue+dotted);

// Original polygon add(shift(dir(108))*blk); add(shift(dir(72))*rotate(324)*blk); add(shift(dir(36))*rotate add(shift(dir(0))*rotate(252)*blk); add(shift(dir(324))*rotate(216)*wht);

add(shift(dir(288))*rotate(180)*blk); add(shift(dir(252))*rotate(144)*blk); add(shift(dir(216))*rotate(108) add(shift(dir(180))*rotate(72)*blk); add(shift(dir(144))*rotate(36)*wht);

// White shifted real Wk = 1.2; pair W1 = (1.8,0.1); pair W2 = W1 + w*dir(36); pair W3 = W2 + w*dir(108); pair W4 = W3 + w*dir(216); path Wgon = W1–W2–W3–W4–cycle; draw(Wgon); pair WO = (W1+W3)/2; transform Wt = shift(WO)*scale(Wk)*shift(-WO); draw(Wt * Wgon); label("W", WO); /* draw(W1–Wt*W1); draw(W2–Wt*W2); draw(W3–Wt*W3); draw(W4–Wt*W4); */

// Black shifted real Bk = 1.10; pair B1 = (1.5,-0.1); pair B2 = B1 + w*dir(0); pair B3 = B2 + w*dir(324); pair B4 = B3 + w*dir(252); pair B5 = B4 + w*dir(180); pair B6 = B5 + w*dir(144); path Bgon = B1–B2–B3–B4–B5–B6–cycle; pair BO = (B1+B4)/2; transform Bt = shift(BO)*scale(Bk)*shift(-BO); fill(Bt * Bgon, black); fill(Bgon, white); label("B", BO);[/asy]

Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Problem 1.345 (8417327567048605288)**.** Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 1.346 (8782897210450267045)**.** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Problem 1.347 (86986480818494)**.** Given a scalene triangle ABC inscribed in the circle (O). Let (I) be its incircle and BI, CI cut AC, AB at E, F respectively. A circle passes through E and touches OB at B cuts (O) again at M. Similarly, a circle passes through F and touches OC at C cuts (O) again at N. ME, NF cut (O) again at P, Q. Let K be the intersection of EF and BC and let PQ cuts BC and EF at G, H, respectively. Show that the median correspond to G of the triangle GHK is perpendicular to IO.

Problem 1.348 (8895719454292056765)**.** Given a non-right triangle ABC with $BC > AC > AB$. Two points $P_1 \neq P_2$ on the plane satisfy that, for $i = 1, 2$, if AP_i, BP_i and CP_i intersect the circumcircle of the triangle ABC at D_i, E_i, and F_i, respectively, then $D_iE_i \perp D_iF_i$ and $D_iE_i = D_iF_i \neq 0$. Let the line P_1P_2 intersects the circumcircle of ABC at Q_1 and Q_2. The Simson lines of Q_1, Q_2 with respect to ABC intersect at W.

Prove that W lies on the nine-point circle of ABC.

Problem 1.349 (80567267310692)**.** Let n be a positive integer. Given is a subset A of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from A such that $c + 2a > 3b$.

Problem 1.350 (8152181601565653036)**.** Let D be a point on segment PQ. Let ω be a fixed circle passing through D, and let A be a variable point on ω. Let X be the intersection of the tangent to the circumcircle of $\triangle ADP$ at P and the tangent to the circumcircle of $\triangle ADQ$ at Q. Show that as A varies, X lies on a fixed line.

Problem 1.351 (2918584823978789760)**.** A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Problem 1.352 (8639636622304457736)**.** Let $\triangle ABC$ be a triangle, and let S and T be the midpoints of the sides BC and CA, respectively. Suppose M is the midpoint of the segment ST and the circle ω through A, M and T meets the line AB again at N. The tangents of ω at M and N meet at P. Prove that P lies on BC if and only if the triangle ABC is isosceles with apex at A.

Problem 1.353 (6827864645665571)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.354 (6955756846906975678)**.** If there are several heaps of stones on the table, it is said that there are *many* stones on the table, if we can find 50 piles and number them with the numbers from 1 to 50 so that the first pile contains at least one stone, the second - at least two stones,..., the 50-th has at least 50 stones. Let the table be initially contain 100 piles of 100 stones each. Find the largest $n \leq 10000$ such that after removing any n stones, there will still be *many* stones left on the table.

Problem 1.355 (221552874820768)**.** The incircle of a scalene triangle ABC touches the sides BC, CA, and AB at points D, E, and F, respectively. Triangles APE and AQF are constructed outside the triangle so that

$$AP = PE, AQ = QF, \angle APE = \angle ACB, \text{ and } \angle AQF = \angle ABC.$$

Let M be the midpoint of BC. Find $\angle QMP$ in terms of the angles of the triangle ABC.

Problem 1.356 (70043882336455)**.** Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is not marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Problem 1.357 (20663652231924)**.** Consider pairs of functions (f, g) from the set of nonnegative integers to itself such that $f(0) + f(1) + f(2) + \cdots + f(42) \leq 2022$; for any integers $a \geq b \geq 0$, we have $g(a + b) \leq f(a) + f(b)$. Determine the maximum possible value of $g(0) + g(1) + g(2) + \cdots + g(84)$ over all such pairs of functions.

Problem 1.358 (521969466382456)**.** Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

Problem 1.359 (7550072974614174968)**.** Let $n \geqslant 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[0, 1]$. Let $s = x_1 + x_2 + \ldots + x_n$, and assume that $s \geqslant 3$. Prove that there exist integers i and j with $1 \leqslant i < j \leqslant n$ such that

$$2^{j-i} x_i x_j > 2^{s-3}.$$

Problem 1.360 (4018921933875333744)**.** Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

Problem 1.361 (7229423492681245326)**.** Find the smallest constant $C > 1$ such that the following statement holds: for every integer $n \geq 2$ and sequence of non-integer positive real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1,$$

it's possible to choose positive integers b_i such that (i) for each $i = 1, 2, \ldots, n$, either $b_i = \lfloor a_i \rfloor$ or $b_i = \lfloor a_i \rfloor + 1$, and (ii) we have

$$1 < \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \leq C.$$

(Here $\lfloor \bullet \rfloor$ denotes the floor function, as usual.)

Problem 1.362 (3859961452154270883)**.** A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

Problem 1.363 (409149115429190)**.** On the $n \times n$ checker board, several cells were marked in such a way that lower left (L) and upper right(R) cells are not marked and that for any knight-tour from L to R, there is at least one marked cell. For which $n > 3$, is it possible that there always exists three consective cells going through diagonal for which at least two of them are marked?

Problem 1.364 (639126468624733)**.** Let $ABCDEF$ be a hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that segments BD and DF intersect CE at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to CE.

Problem 1.365 (16776483958513)**.** Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 1.366 (702587891849077)**.** Given an integer $n \geqslant 2$. Suppose there is a point P inside a convex cyclic $2n$-gon $A_1 \ldots A_{2n}$ satisfying

$$\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_{2n}A_1,$$

prove that

$$\prod_{i=1}^{n} |A_{2i-1}A_{2i}| = \prod_{i=1}^{n} |A_{2i}A_{2i+1}|,$$

where $A_{2n+1} = A_1$.

Problem 1.367 (978369715927760)**.** Point D is selected inside acute $\triangle ABC$ so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD so that $AE = EC$. Let M be the midpoint of BC.

Show that line AB is tangent to the circumcircle of triangle BEM.

Problem 1.368 (2252133047011954512)**.** On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

Problem 1.369 (571352513856417722)**.** A cyclic quadrilateral $ABCD$ has circumcircle Γ, and $AB + BC = AD + DC$. Let E be the midpoint of arc BCD, and $F (\neq C)$ be the antipode of A wrt Γ. Let I, J, K be the incenter of $\triangle ABC$, the A-excenter of $\triangle ABC$, the incenter of $\triangle BCD$, respectively. Suppose that a point P satisfies $\triangle BIC \overset{+}{\sim} \triangle KPJ$. Prove that EK and PF intersect on Γ.

Problem 1.370 (692237787009642)**.** Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions: Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns. Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns. They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

Problem 1.371 (727078403801409)**.** Let ABC be a triangle with incenter I and circumcircle Ω. A point X on Ω which is different from A satisfies $AI = XI$. The incircle touches AC and AB at E, F, respectively. Let M_a, M_b, M_c be the midpoints of sides BC, CA, AB, respectively. Let T be the intersection of the lines M_bF and M_cE. Suppose that AT intersects Ω again at a point S.

Prove that X, M_a, S, T are concyclic.

Problem 1.372 (14270621317473499943)**.** Let ABC be a triangle with circumcenter O and orthocenter H such that OH is parallel to BC. Let AH intersects again with the circumcircle of ABC at X, and let XB, XC intersect with OH at Y, Z, respectively. If the projections of Y, Z to AB, AC are P, Q, respectively, show that PQ bisects BC.

Problem 1.373 (4037864050528368034)**.** Find the largest integer $N \in \{1, 2, \ldots, 2019\}$ such that there exists a polynomial $P(x)$ with integer coefficients satisfying the following property: for each positive integer k, $P^k(0)$ is divisible by 2020 if and only if k is divisible by N. Here P^k means P applied k times, so $P^1(0) = P(0), P^2(0) = P(P(0))$, etc.

Problem 1.374 (4835329555526569551)**.** Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7 \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1
\end{aligned}$$

Problem 1.375 (6919176010062551987)**.** Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q\leq n, p,q \text{ primes}} (p+q)$$

Problem 1.376 (902621191535073)**.** Given six points A, B, C, D, E, F such that $\triangle BCD \overset{+}{\sim} \triangle ECA \overset{+}{\sim} \triangle BFA$ and let I be the incenter of $\triangle ABC$. Prove that the circumcenter of $\triangle AID, \triangle BIE, \triangle CIF$ are collinear.

Problem 1.377 (6193947856984766386)**.** Let $ABCD$ be a cyclic quadrilateral. Assume that the points Q, A, B, P are collinear in this order, in such a way that the line AC is tangent to the circle ADQ, and the line BD is tangent to the circle BCP. Let M and N be the midpoints of segments BC and AD, respectively. Prove that the following three lines are concurrent: line CD, the tangent of circle ANQ at point A, and the tangent to circle BMP at point B.

Problem 1.378 (660403976209529)**.** A number is called Norwegian if it has three distinct positive divisors whose sum is equal to 2022. Determine the smallest Norwegian number.

(Note: The total number of positive divisors of a Norwegian number is allowed to be larger than 3.)

Problem 1.379 (7494618588207758150)**.** An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1+2+3+\cdots+2018$?

Problem 1.380 (616860610609120)**.** A few (at least 5) integers are put on a circle, such that each of them is divisible by the sum of its neighbors. If the sum of all numbers is positive, what is its minimal value?

Problem 1.381 (282712203118607)**.** Let ABC be an acute-angled triangle with $AC > AB$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC, and AB at W, X, and Y, respectively. The circumcircles of triangles AXY and ABC intersect again at $Z \neq A$. Prove that if $W \neq D$ and $OW = OD$, then DZ is tangent to the circle AXY.

Problem 1.382 (2694660444585153591)**.** Find all binary operations $\diamondsuit : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$, the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and if $a \geq 1$ then $a \diamondsuit a \geq 1$.

Problem 1.383 (5514383858686655851)**.** Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

Problem 1.384 (7904897494032012729)**.** Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions: $m > 1$ and $\gcd(m, n) = 1$; the numbers $P(0)$, $P^2(0)$, …, $P^{m-1}(0)$ are not divisible by n; and $P^m(0)$ is divisible by n. Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

Problem 1.385 (518384374486289)**.** Let O be the center of the equilateral triangle ABC. Pick two points P_1 and P_2 other than B, O, C on the circle $\odot(BOC)$ so that on this circle B, P_1, P_2, O, C are placed in this order. Extensions of BP_1 and CP_1 intersects respectively with side CA and AB at points R and S. Line AP_1 and RS intersects at point Q_1. Analogously point Q_2 is defined. Let $\odot(OP_1Q_1)$ and $\odot(OP_2Q_2)$ meet again at point U other than O.

Prove that $2\angle Q_2UQ_1 + \angle Q_2OQ_1 = 360°$.

Remark. $\odot(XYZ)$ denotes the circumcircle of triangle XYZ.

Problem 1.386 (193788212098506)**.** In a convex n-gon $\mathcal{P}$, we draw several diagonals. A drawn diagonal is *good* if it intersects exactly one other drawn diagonal in the interior of $\mathcal{P}$. Over all choices of diagonals to draw, find the maximum possible number of good diagonals.

Problem 1.387 (6020628633767269011)**.** Let $ABCDE$ be a regular pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that as P varies, R lies on a fixed line.

Problem 1.388 (574223786384294)**.** Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \dots, n$.

Problem 1.389 (3353450172272500341)**.** Let $ABCD$ be a cyclic quadrilateral. Let DA and BC intersect at E and let AB and CD intersect at F. Assume that A, E, F all lie on the same side of BD. Let P be on segment DA such that $\angle CPD = \angle CBP$, and let Q be on segment CD such that $\angle DQA = \angle QBA$. Let AC and PQ meet at X. Prove that, if $EX = EP$, then EF is perpendicular to AC.

Problem 1.390 (1872712387771032593)**.** Let H be the orthocenter of triangle ABC, and AD, BE, CF be the three altitudes of triangle ABC. Let G be the orthogonal projection of D onto EF, and DD' be the diameter of the circumcircle of triangle DEF. Line AG and the circumcircle of triangle ABC intersect again at point X. Let Y be the intersection of GD' and BC, while Z be the intersection of AD' and GH. Prove that X, Y, and Z are collinear.

Problem 1.391 (458902414604417)**.** A class has 25 students. The teacher wants to stock N candies, hold the Olympics and give away all N candies for success in it (those who solve equally tasks should get equally, those who solve less get less, including, possibly, zero candies). At what smallest N this will be possible, regardless of the number of tasks on Olympiad and the student successes?

Problem 1.392 (6306108494297192985)**.** Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Problem 1.393 (8670333331361701457)**.** Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problem 1.394 (736821043753990)**.** Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

Problem 1.395 (989812634983805)**.** Let $n > 2$ be a positive integer. Masha writes down n natural numbers along a circle. Next, Taya performs the following operation: Between any two adjacent numbers a and b, she writes a divisor of the number $a + b$ greater than 1, then Taya erases the original numbers and obtains a new set of n numbers along the circle. Can Taya always perform these operations in such a way that after some number of operations, all the numbers are equal?

Problem 1.396 (313143209359080)**.** The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions. (An example of one possible layout of the park is shown to the left below, in which there are six junctions and nine trails.) https://services.artofproblemsolving.com/download.php?id=YXR0YWNobWVudHMvZS9mLzc1YmNjN2Yx
A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Problem 1.397 (59492583381358822858)**.** In 10×10 square we choose n cells. In every chosen cell we draw one arrow from the angle to opposite angle. It is known, that for any two arrows, or the end of one of them coincides with the beginning of the other, or the distance between their ends is at least 2. What is the maximum possible value of n?

Problem 1.398 (122001240071629)**.** Vasya has 100 cards of 3 colors, and there are not more than 50 cards of same color. Prove that he can create 10×10 square, such that every cards of same color have not common side.

Problem 1.399 (5261846980754565299)**.** Let A, B, C be the midpoints of the three sides $B'C', C'A', A'B'$ of the triangle $A'B'C'$ respectively. Let P be a point inside ΔABC, and AP, BP, CP intersect with BC, CA, AB at P_a, P_b, P_c, respectively. Lines P_aP_b, P_aP_c intersect with $B'C'$ at R_b, R_c respectively, lines P_bP_c, P_bP_a intersect with $C'A'$ at S_c, S_a respectively. and lines P_cP_a, P_cP_b intersect with $A'B'$ at T_a, T_b, respectively. Given that S_c, S_a, T_a, T_b are all on a circle centered at O.

Show that $OR_b = OR_c$.

Problem 1.400 (913214378150707)**.** In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

Problem 1.401 (244533208775214844)**.** A finite set S of points in the coordinate plane is called overdetermined if $|S| \geq 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S|-2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \geq 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is not overdetermined, but has k overdetermined subsets.

Problem 1.402 (528087142744727)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Problem 1.403 (4479133443678014025)**.** Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2 .$$

Problem 1.404 (685138775901874)**.** The cells of a 100×100 table are colored white. In one move, it is allowed to select some 99 cells from the same row or column and recolor each of them with the opposite color. What is the smallest number of moves needed to get a table with a chessboard coloring?

Problem 1.405 (1527496195334546428)**.** On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k, count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially, there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A?

Problem 1.406 (183354438240037)**.** Let I, O, H, and Ω be the incenter, circumcenter, orthocenter, and the circumcircle of the triangle ABC, respectively. Assume that line AI intersects with Ω again at point $M \neq A$, line IH and BC meets at point D, and line MD intersects with Ω again at point $E \neq M$. Prove that line OI is tangent to the circumcircle of triangle IHE.

Problem 1.407 (6136318250466883786)**.** Each of the 41 dashed unit segments below is independently colored either black or white, each randomly with probability $\frac{1}{2}$. The black segments thus form the walls of a maze (but the white segments are passable).

What is the probability that there will exist a path from side A to side B that does not cross any of the black lines?

Problem 1.408 (5901329049595563801)**.** Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f \colon \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

Problem 1.409 (181878217485192)**.** 1000 children, no two of the same height, lined up. Let us call a pair of different children (a, b) good if between them there is no child whose height is greater than the height of one of a and b, but less than the height of the other. What is the greatest number of good pairs that could be formed? (Here, (a, b) and (b, a) are considered the same pair.)