ME CORTAN EN ENERO

PONTE A ENTRENAR

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§1 Problemas

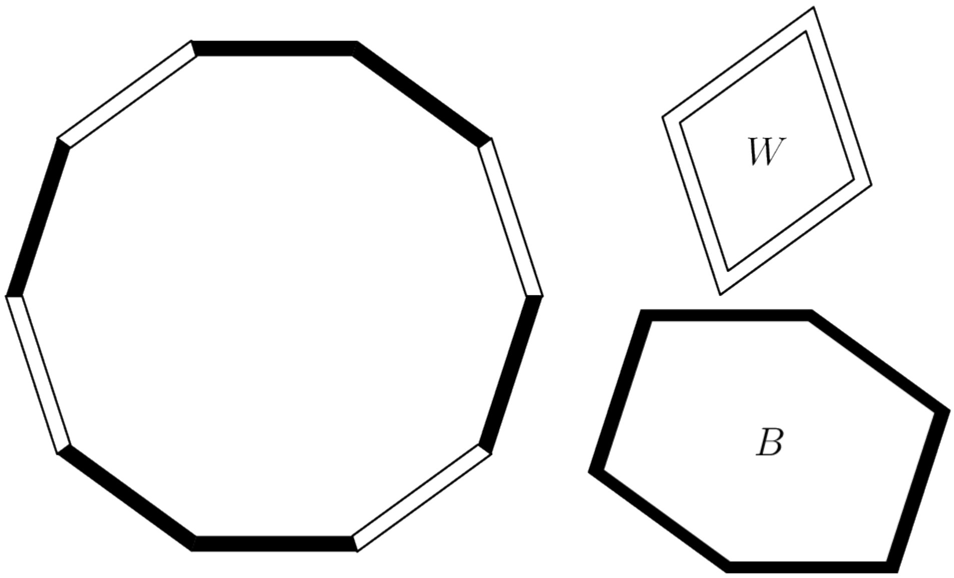
Problem 1.1 (Cono Sur 2016/6)**.** We say that three different integers are friendly if one of them divides the product of the other two. Let n be a positive integer.

a) Show that, between n^2 and $n^2 + n$, exclusive, does not exist any triplet of friendly numbers.

b) Determine if for each n exists a triplet of friendly numbers between n^2 and $n^2 + n + 3\sqrt{n}$, exclusive.

Problem 1.2 (USAMO 2022/2)**.** Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Problem 1.3 (AC ABC387F)**.** https://atcoder.jp/contests/abc387/tasks/abc387_f

Problem 1.4 (Rioplatense 2024/3.5). Let $S = \{2, 3, 4, \dots\}$ be the set of positive integers greater than 1. Find all functions $f : S \to S$ that satisfy

$$\gcd(a, f(b)) \cdot \operatorname{lcm}(f(a), b) = f(ab)$$

for all pairs of integers $a, b \in S$.

Clarification: $\gcd(a, b)$ is the greatest common divisor of a and b, and $\operatorname{lcm}(a, b)$ is the least common multiple of a and b.

Problem 1.5 (India 2024/5). Let points A_1, A_2 and A_3 lie on the circle Γ in a counter-clockwise order, and let P be a point in the same plane. For $i \in \{1, 2, 3\}$, let τ_i denote the counter-clockwise rotation of the plane centred at A_i, where the angle of rotation is equial to the angle at vertex A_i in $\triangle A_1A_2A_3$. Further, define P_i to be the point $\tau_{i+2}(\tau_i(\tau_{i+1}(P)))$, where the indices are taken modulo 3 (i.e., $\tau_4 = \tau_1$ and $\tau_5 = \tau_2$).

Prove that the radius of the circumcircle of $\triangle P_1P_2P_3$ is at most the radius of Γ.

Problem 1.6 (OMM 2018/6). Encuentra todos los enteros $n \geq 3$ tales que existe un polígono convexto de n lados $A_1A_2 \dots A_n$ que tenga las siguientes características:

- Todos los ángulos internos de $A_1A_2 \dots A_n$ son iguales.
- No todos los lados de $A_1A_2 \dots A_n$ son iguales.
- Existe un triángulo T y un punto O en el interior de $A_1A_2 \dots A_n$ tal que los n triángulos $OA_1A_2, OA_2A_3, \dots OA_nA_1$ son todos semejantes a T.

Problem 1.7 (OTIS Russian Combo Russia 2014/9.1). Each lattice point of $\mathbb{Z}^2$ is colored with one of three colors, with every color used at least once. Show that one can find a right triangle with pairwise distinct colored vertices.

Problem 1.8 (Vietnam National Olympiad for High School Student 2022/12). Given Fibonacci sequence (F_n), and a positive integer m, denote $k(m)$ by the smallest positive integer satisfying $F_{n+k(m)} \equiv F_n (\bmod m)$, for all natural numbers n. a) Prove that: For all $m_1, m_2 \in \mathbb{Z}^+$, we have:

$$k([m_1, m_2]) = [k(m_1), k(m_2)].$$

(Here $[a, b]$ is the least common multiple of a, b.)

b) Determine $k(2), k(4), k(5), k(10)$.

Problem 1.9 (OTIS Orders Romania TST 1996). Find all primes p, q such that $\alpha^{3pq} - \alpha \equiv 0 \pmod{3pq}$ for all integers α.

Problem 1.10 (Canada 2023/3). An acute triangle is a triangle that has all angles less than 90° (90° is a Right Angle). Let ABC be an acute triangle with altitudes AD, BE, and CF meeting at H. The circle passing through points D, E, and F meets AD, BE, and CF again at X, Y, and Z respectively. Prove the following inequality:

$$\frac{AH}{DX} + \frac{BH}{EY} + \frac{CH}{FZ} \geq 3.$$

Problem 1.11 (Leer/Aprender Cauchy). Leer/Aprender Cauchy

Problem 1.12 (Bulgaria JBMO TST 2023/2). Let x, y, and z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{x+3}{y+z} + \frac{y+3}{z+x} + \frac{z+3}{x+y} + 3 \geq 27 \cdot \frac{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2}{(x+y+z)^3}.$$

Problem 1.13 (The A-Schwaat Line)**.** Let ABC be a triangle with altitude $\overline{AD}$. Let M and N denote the midpoints of $\overline{AD}$ and $\overline{BC}$. Show that line MN passes through the symmedian point of $\triangle ABC$ (this line is called the A-Schwatt line).

Problem 1.14 (IMO SL 2022/C4)**.** Let $n > 3$ be a positive integer. Suppose that n children are arranged in a circle, and n coins are distributed between them (some children may have no coins). At every step, a child with at least 2 coins may give 1 coin to each of their immediate neighbors on the right and left. Determine all initial distributions of the coins from which it is possible that, after a finite number of steps, each child has exactly one coin.

Problem 1.15 (Argentina Ibero TST 2016/2)**.** Let $ABCD$ be a trapezoid with bases $BC \parallel AD$ and non-parallel sides AB and CD. On the diagonals AC and BD, let P and Q be the points such that AC bisects $\angle BPD$ and BD bisects $\angle AQC$. Prove that $\angle BPD = \angle AQC$.

Problem 1.16 (Iran TST 2015/3.2)**.** Assume that a_1, a_2, a_3 are three given positive integers consider the following sequence: $a_{n+1} = \operatorname{lcm}[a_n, a_{n-1}] - \operatorname{lcm}[a_{n-1}, a_{n-2}]$ for $n \geq 3$ Prove that there exist a positive integer k such that $k \leq a_3 + 4$ and $a_k \leq 0$. ($[a, b]$ means the least positive integer such that$a \mid [a, b], b \mid [a, b]$ also because $\operatorname{lcm}[a, b]$ takes only nonzero integers this sequence is defined until we find a zero number in the sequence)

Problem 1.17 (DIME 2022/11)**.** A positive integer n is called *un-two* if there does not exist an ordered triple of integers (a, b, c) such that exactly two of

$$\frac{7a+b}{n}, \frac{7b+c}{n}, \frac{7c+a}{n}$$

are integers. Find the sum of all un-two positive integers.

Problem 1.18 (Turkey EGMO TST 2024/6)**.** Let ω_1 and ω_2 be two different circles that intersect at two different points, X and Y. Let lines l_1 and l_2 be common tangent lines of these circles such that l_1 is tangent ω_1 at A and ω_2 at C and l_2 is tangent ω_1 at B and ω_2 at D. Let Z be the reflection of Y respect to l_1 and let BC and ω_1 meet at K for the second time. Let AD and ω_2 meet at L for the second time. Prove that the line tangent to ω_1 and passes through K and the line tangent to ω_2 and passes through L meet on the line XZ.

Problem 1.19 (Sudafrica 2012/6)**.** Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that $f(km) + f(kn) - f(k)f(mn) \geq 1$ for all $k, m, n \in \mathbb{N}$.

Problem 1.20 (JBMO 2024/4)**.** Three friends Archie, Billie, and Charlie play a game. At the beginning of the game, each of them has a pile of 2024 pebbles. Archie makes the first move, Billie makes the second, Charlie makes the third and they continue to make moves in the same order. In each move, the player making the move must choose a positive integer n greater than any previously chosen number by any player, take $2n$ pebbles from his pile and distribute them equally to the other two players. If a player cannot make a move, the game ends and that player loses the game. Determine all the players who have a strategy such that, regardless of how the other two players play, they will not lose the game.

Problem 1.21 (Iran TST 2023/1.5)**.** Suppose that $n \geq 2$ and $a_1, a_2, ..., a_n$ are natural numbers that $(a_1, a_2, ..., a_n) = 1$. Find all strictly increasing function $f : \mathbb{Z} \to \mathbb{R}$ that:

$$\forall x_1, x_2, ..., x_n \in \mathbb{Z} : f(\sum_{i=1}^{n} x_i a_i) = \sum_{i=1}^{n} f(x_i a_i)$$

Problem 1.22 (EGMO 2024/1)**.** Two different integers u and v are written on a board. We perform a sequence of steps. At each step we do one of the following two operations:

(i) If a and b are different integers on the board, then we can write $a+b$ on the board, if it is not already there. (ii) If a, b and c are three different integers on the board, and if an integer x satisfies $ax^2+bx+c=0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u, v) from which any integer can eventually be written on the board after a finite sequence of steps.

Problem 1.23 (USA TST 2025/2)**.** Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of real numbers for which $a_1 > b_1$ and

$$a_{n+1} = a_n^2 - 2b_n$$
$$b_{n+1} = b_n^2 - 2a_n$$

for all positive integers n. Prove that $a_1, a_2, \ldots$ is eventually increasing (that is, there exists a positive integer N for which $a_k < a_{k+1}$ for all $k > N$).

Problem 1.24 (OTIS Orders Don Zagier)**.** Let S denote the integers $n \geq 2$ with the property that for any positive integer a we have

$$a^{n+1} \equiv a \pmod{n}.$$

Show that S is finite and determine its elements.

Problem 1.25 (PAGMO 2021/6)**.** Let ABC be a triangle with incenter I, and A-excenter Γ. Let A_1, B_1, C_1 be the points of tangency of Γ with BC, AC and AB, respectively. Suppose IA_1, IB_1 and IC_1 intersect Γ for the second time at points A_2, B_2, C_2, respectively. M is the midpoint of segment AA_1. If the intersection of A_1B_1 and A_2B_2 is X, and the intersection of A_1C_1 and A_2C_2 is Y, prove that $MX = MY$.

Problem 1.26 (EGMO 2024/4)**.** For a sequence $a_1 < a_2 < \cdots < a_n$ of integers, a pair (a_i, a_j) with $1 \leq i < j \leq n$ is called interesting if there exists a pair (a_k, a_l) of integers with $1 \leq k < l \leq n$ such that

$$\frac{a_l - a_k}{a_j - a_i} = 2.$$

For each $n \geq 3$, find the largest possible number of interesting pairs in a sequence of length n.

Problem 1.27 (USACO 2016 JAN BRZ 2)**.** https://usaco.org/index.php?page=viewproblem2&cpid=592

Problem 1.28 (JBMOSL 2021/C4)**.** Alice and Bob play a game together as a team on a 100×100 board with all unit squares initially white. Alice sets up the game by coloring exactly k of the unit squares red at the beginning. After that, a legal move for Bob is to choose a row or column with at least 10 red squares and color all of the remaining squares in it red. What is the smallest k such that Alice can set up a game in such a way that Bob can color the entire board red after finitely many moves?

Problem 1.29 (OTIS Orders HMMT November 2014)**.** Determine all positive integers $1 \leq m \leq 50$ for which there exists an integer n for which m divides $n^{n+1} + 1$.

Problem 1.30 (CF 2052J). https://codeforces.com/problemset/problem/2052/J

Problem 1.31 (AC ARC80A). https://atcoder.jp/contests/abc069/tasks/arc080_a?lang=en

Problem 1.32 (USA TST 2015/1). (Walkthrough OTIS) Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Problem 1.33 (IOI 2023 LONGEST TRIP). https://oj.uz/problem/view/IOI23_longesttrip

Problem 1.34 (AC ABC388E). https://atcoder.jp/contests/abc388/tasks/abc388_e

Problem 1.35 (OTIS Funcionales Indian Postal Set 2016). Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f\left(xf(y) - yf(x)\right) = f(xy) - xy$$

for all real numbers x and y.

Problem 1.36 (AC ABC385E). https://atcoder.jp/contests/abc385/tasks/abc385_e

Problem 1.37 (AC ABC387E). https://atcoder.jp/contests/abc387/tasks/abc387_e