Bitacora

Oro IMO 2025

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§1 Problems

§1.1 October

Problem 1.1 (No primitive roots mod 2^n)**.** Show that there are no primitive roots modulo 2^n for $n \geq 3$. That is, show there is no integer g such that g, g^2, g^3, ...covers every odd residue modulo 2^n.

Problem 1.2 (Japan 1996/2)**.** Let m and n be odd positive integers with $\gcd(m, n) = 1$. Evaluate

$$\gcd(5^m + 7^m, 5^n + 7^n).$$

Problem 1.3 (OMM 2020/6)**.** Sea $n \geq 2$ un número entero. Sean $x_1, x_2, \ldots, x_n$ números reales distintos de 0 que satisfacen la ecuación

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_n + \frac{1}{x_1}\right) = \left(x_1^2 + \frac{1}{x_2^2}\right)\left(x_2^2 + \frac{1}{x_3^2}\right)\cdots\left(x_n^2 + \frac{1}{x_1^2}\right)$$

Problem 1.4 (OMM 2007/6)**.** Sea ABC un triángulo tal que $AB > AC > BC$. Sea D un punto sobre el lado AB de tal manera que $CD = BC$, y sea M el punto medio del lado AC. Muestra que $BD = AC$ si y sólo si $\angle BAC = 2\angle ABM$.

Problem 1.5 (IMO 1968/1)**.** Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

§2 Solutions

§2.1 October

§2.1.1 No primitive roots mod 2^n

No primitive roots mod 2^n

Show that there are no primitive roots modulo 2^n for $n \geq 3$. That is, show there is no integer g such that g, g^2, g^3, …covers every odd residue modulo 2^n.

Solution. We have that $g^{2^{n-1}} \equiv 1 \pmod{2^n}$ because $\varphi(2^n) = 2^{n-1}$ then

$$g^{2^{n-2}} \equiv -1 \pmod{2^n}$$

because we have 2^{n-1} different odd residues, and if $g^{2^{n-2}}$ were 1, we would have a cicle of size 2^{n-2} and that's a contradiction.
Then for $n \geq 3$ we have 2^{n-2} is even and $g^{2^{n-2}}$ is a square so -1 is a quadratic residue mod 2^n, so it's a quadratic residue mod 8, but that's false.
Then g doesn't exist. $\square$

§2.1.2 Japan 1996/2

Japan 1996/2

Let m and n be odd positive integers with $\gcd(m, n) = 1$. Evaluate

$$\gcd(5^m + 7^m, 5^n + 7^n).$$

Solution. WLOG $m > n$ (If $m = n = 1$ then the value is 12)
Let $d = \gcd(5^m + 7^m, 5^n + 7^n)$. then $\left(\frac{5}{7}\right)^m \equiv \left(\frac{5}{7}\right)^n \equiv -1 \pmod{d}$.
By Bezout we have x, y integers such that $mx + ny = 1$ and we have

$$\left(\frac{5}{7}\right) \equiv \left(\frac{5}{7}\right)^{mx} \cdot \left(\frac{5}{7}\right)^{ny} \equiv (-1)^{x+y} \pmod{d}$$

If $x + y$ is even we have that x, y have the same parity and mx, ny also have the same parity then $mx + ny$ is even but $mx + ny$ is 1 so this is impossible.
Then

$$\left(\frac{5}{7}\right) \equiv -1 \pmod{d}$$

And $5 \equiv -7 \pmod{d}$ then $d \mid 12$. And we're going to prove that $d = 12$.
First, $4 \mid d$ because

$$5^m + 7^m \equiv 1^m + (-1)^m \equiv 1 - 1 \equiv 0 \pmod{4}$$

and

$$5^m + 7^m \equiv (-1)^m + 1^m \equiv -1 + 1 \equiv 0 \pmod{3}$$

then $12 \mid 5^m + 7^m$ and it's analogously for n, then $12 \mid d \mid 12$ and $d = 12$. $\square$