quiero ser bueno
PONTE A ENTRENAR

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§1 Problemas

Problem 1.1 (3556283025270446335)**.** Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively. Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

Problem 1.2 (220345421587712)**.** Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point of the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

Problem 1.3 (645068477920006)**.** There are several gentlemen in the meeting of the Diogenes Club, some of which are friends with each other (friendship is mutual). Let's name a participant unsociable if he has exactly one friend among those present at the meeting. By the club rules, the only friend of any unsociable member can leave the meeting (gentlemen leave the meeting one at a time). The purpose of the meeting is to achieve a situation in which that there are no friends left among the participants. Prove that if the goal is achievable, then the number of participants remaining at the meeting does not depend on who left and in what order.

Problem 1.4 (318208660266829737)**.** Let ABC be an acute-angled triangle with $AB \neq AC$, and let I and O be its incenter and circumcenter, respectively. Let the incircle touch BC, CA and AB at D, E and F, respectively. Assume that the line through I parallel to EF, the line through D parallel toAO, and the altitude from A are concurrent. Prove that the concurrency point is the orthocenter of the triangle ABC.

Problem 1.5 (6377764165704184464)**.** Celeste has an unlimited amount of each type of n types of candy, numerated type 1, type 2, ... type n. Initially she takes $m > 0$ candy pieces and places them in a row on a table. Then, she chooses one of the following operations (if available) and executes it:

1. She eats a candy of type k, and in its position in the row she places one candy type $k-1$ followed by one candy type $k+1$ (we consider type $n+1$ to be type 1, and type 0 to be type n).
2. She chooses two consecutive candies which are the same type, and eats them.

Find all positive integers n for which Celeste can leave the table empty for any value of m and any configuration of candies on the table.

Problem 1.6 (7801987958529111131)**.** Allen and Alan play a game. A nonconstant polynomial $P(x, y)$ with real coefficients and a positive integer d greater than the degree of P are known to both Allen and Alan. Alan thinks of a polynomial $Q(x, y)$ with real coefficients and degree at most d and keeps it secret. Allen can make queries of the form (s, t), where s and t are real numbers such that $P(s, t) \neq 0$. Alan must respond with the value $Q(s, t)$. Allen's goal is to determine whether P divides Q. Find (in terms of P and d) the smallest positive integer, g, such that Allen can always achieve this goal making no more than g queries.

Problem 1.7 (8209367948889736949)**.** For every ordered pair of integers (i, j), not necessarily positive, we wish to select a point $P_{i,j}$ in the Cartesian plane whose coordinates lie inside the unit square defined by

$$i < x < i + 1, \qquad j < y < j + 1.$$

Find all real numbers $c > 0$ for which it's possible to choose these points such that for all integers i and j, the (possibly concave or degenerate) quadrilateral $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ has perimeter strictly less than c.

Problem 1.8 (648819281604044)**.** Let n be a positive integer and let $S \subseteq \{0, 1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their majority is defined as the binary string $y \in \{0, 1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k + 1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

Problem 1.9 (4364014706118582858)**.** Let ABC be an equilateral triangle. From the vertex A we draw a ray towards the interior of the triangle such that the ray reaches one of the sides of the triangle. When the ray reaches a side, it then bounces off following the law of reflection, that is, if it arrives with a directed angle α, it leaves with a directed angle $180^\circ - \alpha$. After n bounces, the ray returns to A without ever landing on any of the other two vertices. Find all possible values of n.

Problem 1.10 (732021656607287)**.** Let $m > 1$ be an integer. A sequence $a_1, a_2, a_3, \dots$ is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \geq 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.

Problem 1.11 (351896324490208)**.** Let n be a positive integer. For a permutation $a_1, a_2, \dots, a_n$ of the numbers $1, 2, \dots, n$ we define

$$b_k = \min_{1 \leq i \leq k} a_i + \max_{1 \leq j \leq k} a_j$$

We say that the permutation $a_1, a_2, \dots, a_n$ is guadiana if the sequence $b_1, b_2, \dots, b_n$ does not contain two consecutive equal terms. How many guadiana permutations exist?

Problem 1.12 (5886572081531632011)**.** $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i-th player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

Problem 1.13 (63514716280156)**.** Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$. Show that lines PE and QD meet on ω. Prove that if ω passes through H, then lines PD and QE meet on ω as well.

Problem 1.14 (5897111412933990257)**.** Let ABC be a triangle with circumcircle Γ, and points E and F are chosen from sides CA, AB, respectively. Let the circumcircle of triangle AEF and Γ intersect again at point X. Let the circumcircles of triangle ABE and ACF intersect again at point K. Line AK intersect with Γ again at point M other than A, and N be the reflection point of M with respect to line BC. Let XN intersect with Γ again at point S other that X.

Prove that SM is parallel to BC.

Problem 1.15 (709130660277794345)**.** Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \leq x < \sqrt{a}$. Show that $A = B$.

Problem 1.16 (1547794310266184263)**.** Let k be a positive integer. Lexi has a dictionary $\mathbb{D}$ consisting of some k-letter strings containing only the letters A and B. Lexi would like to write either the letter A or the letter B in each cell of a $k \times k$ grid so that each column contains a string from $\mathbb{D}$ when read from top-to-bottom and each row contains a string from $\mathbb{D}$ when read from left-to-right. What is the smallest integer m such that if $\mathbb{D}$ contains at least m different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathbb{D}$?

Problem 1.17 (1248852037865425410)**.** Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied: Each number in the table is congruent to 1 modulo n. The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2. Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that the sums $R_1 + \ldots R_n$ and $C_1 + \ldots C_n$ are congruent modulo n^4.

Problem 1.18 (4479133443678014025)**.** Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2.$$

Problem 1.19 (676918769934959)**.** Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \dots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \dots, a_i + a_{i+1} + \dots + a_{i+n-1}$$

is divisible by n. Here, we let $a_i = a_{i-n}$ when $i > n$.

Problem 1.20 (771774560036862)**.** Find all pairs of positive integers (a, b) with the following property: there exists an integer N such that for any integers $m \geq N$ and $n \geq N$, every $m \times n$ grid of unit squares may be partitioned into $a \times b$ rectangles and fewer than ab unit squares.

Problem 1.21 (86986480818494)**.** Given a scalene triangle ABC inscribed in the circle (O). Let (I) be its incircle and BI, CI cut AC, AB at E, F respectively. A circle passes through E and touches OB at B cuts (O) again at M. Similarly, a circle passes through F and touches OC at C cuts (O) again at N. ME, NF cut (O) again at P, Q. Let K be the intersection of EF and BC and let PQ cuts BC and EF at G, H, respectively. Show that the median correspond to G of the triangle GHK is perpendicular to IO.

Problem 1.22 (537574018594693)**.** Let ABC be a triangle with O as its circumcenter. A circle Γ tangents OB, OC at B, C, respectively. Let D be a point on Γ other than B with $CB = CD$, E be the second intersection of DO and Γ, and F be the second intersection of EA and Γ. Let X be a point on the line AC so that $XB \perp BD$. Show that one half of $\angle ADF$ is equal to one of $\angle BDX$ and $\angle BXD$.

Problem 1.23 (4320337590540710547)**.** An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions: The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.) No two beams have intersecting interiors. The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam. What is the smallest positive number of beams that can be placed to satisfy these conditions?

Problem 1.24 (6116877365036470315)**.** Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions: (i) $f(n) \neq 0$ for at least one n; (ii) $f(xy) = f(x) + f(y)$ for every positive integers x and y; (iii) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

Problem 1.25 (857047923056144)**.** Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \dots$ by the following rules: (a) On every move of his B passes 1 coin from every box to an adjacent box. (b) On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to an adjacent box. Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Problem 1.26 (899785005954032)**.** The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a move, Vera may flip over one of the coins in the row, subject to the following rules: On the first move, Vera may flip over any of the 2023 coins. On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.) Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

Problem 1.27 (119687225328684)**.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Problem 1.28 (116786407849814)**.** A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Problem 1.29 (620629352845047)**.** As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta : \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$, $\theta(p+1) = \theta(p)+1$, and if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

Problem 1.30 (7021355208717803796)**.** Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Problem 1.31 (3838489129977355762)**.** Two triangles ABC and $A'B'C'$ are on the plane. It is known that each side length of triangle ABC is not less than a, and each side length of triangle $A'B'C'$ is not less than a'. Prove that we can always choose two points in the two triangles respectively such that the distance between them is not less than $\sqrt{\dfrac{a^2 + a'^2}{3}}$.

Problem 1.32 (395315144480173)**.** Let a, b, c be positive reals such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Show that

$$a^a bc + b^b ca + c^c ab \geq 27bc + 27ca + 27ab.$$

Problem 1.33 (829271701496996)**.** Pasha and Vova play the following game, making moves in turn; Pasha moves first. Initially, they have a large piece of plasticine. By a move, Pasha cuts one of the existing pieces into three(of arbitrary sizes), and Vova merges two existing pieces into one. Pasha wins if at some point there appear to be 100 pieces of equal weights. Can Vova prevent Pasha's win?

Problem 1.34 (874415503743541)**.** An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold: (1) no one stands between the two tallest players, (2) no one stands between the third and fourth tallest players, $\vdots$ (N) no one stands between the two shortest players.

Show that this is always possible.

Problem 1.35 (5066939379306191291)**.** Let ABC be an acute triangle with circumcenter O and circumcircle Ω. Choose points D, E from sides AB, AC, respectively, and let ℓ be the line passing through A and perpendicular to DE. Let ℓ intersect the circumcircle of triangle ADE and Ω again at points P, Q, respectively. Let N be the intersection of OQ and BC, S be the intersection of OP and DE, and W be the orthocenter of triangle SAO.

Prove that the points S, N, O, W are concyclic.

Problem 1.36 (457934969594281)**.** A positive integer n is given. A cube $3 \times 3 \times 3$ is built from 26 white and 1 black cubes $1 \times 1 \times 1$ such that the black cube is in the center of $3 \times 3 \times 3$-cube. A cube $3n \times 3n \times 3n$ is formed by n^3 such $3 \times 3 \times 3$-cubes. What is the smallest number of white cubes which should be colored in red in such a way that every white cube will have at least one common vertex with a red one.

Problem 1.37 (8383644831210009641)**.** A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \le M$ for all positive integers n.

Problem 1.38 (1427062131747349943)**.** Let ABC be a triangle with circumcenter O and orthocenter H such that OH is parallel to BC. Let AH intersects again with the circumcircle of ABC at X, and let XB, XC intersect with OH at Y, Z, respectively. If the projections of Y, Z to AB, AC are P, Q, respectively, show that PQ bisects BC.

Problem 1.39 (499788610931519)**.** Andryusha has 100 stones of different weight and he can distinguish the stones by appearance, but does not know their weight. Every evening, Andryusha can put exactly 10 stones on the table and at night the brownie will order them in increasing weight. But, if the drum also lives in the house then surely he will in the morning change the places of some 2 stones.Andryusha knows all about this but does not know if there is a drum in his house. Can he find out?

Problem 1.40 (895654249061658)**.** For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \ge 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Problem 1.41 (6576585943791349484)**.** Regular hexagon is divided to equal rhombuses, with sides, parallels to hexagon sides. On the three sides of the hexagon, among which there are no neighbors, is set directions in order of traversing the hexagon against hour hand. Then, on each side of the rhombus, an arrow directed just as the side of the hexagon parallel to this side. Prove that there is not a closed path going along the arrows.

Problem 1.42 (8493928466779199543)**.** Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f)}_{k}(n)\ldots))$.

Problem 1.43 (88510326676078)**.** Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

Show that $MK = ML$.

Problem 1.44 (447212157564770)**.** Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

Problem 1.45 (723726912323207)**.** Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i.10^i$$

. The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

Problem 1.46 (3744894000085761569)**.** Call a point in the Cartesian plane with integer coordinates a *lattice point*. Given a finite set $\mathcal{S}$ of lattice points we repeatedly perform the following operation: given two distinct lattice points A, B in $\mathcal{S}$ and two distinct lattice points C, D not in $\mathcal{S}$ such that $ACBD$ is a parallelogram with $AB > CD$, we replace A, B by C, D. Show that only finitely many such operations can be performed.

Problem 1.47 (182831966962001)**.** Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

Problem 1.48 (973663451075571)**.** Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

Problem 1.49 (866307541115519)**.** In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

Problem 1.50 (290912955085727393)**.** Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

Problem 1.51 (844684477828422)**.** Let point H be the orthocenter of a scalene triangle ABC. Line AH intersects with the circumcircle Ω of triangle ABC again at point P. Line BH, CH meets with AC, AB at point E and F, respectively. Let PE, PF meet Ω again at point Q, R, respectively. Point Y lies on Ω so that lines AY, QR and EF are concurrent. Prove that PY bisects EF.

Problem 1.52 (883811987981100)**.** Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 1.53 (357249331453104)**.** Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be nice if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Problem 1.54 (4527883777563937913)**.** 10000 children came to a camp; every of them is friend of exactly eleven other children in the camp (friendship is mutual). Every child wears T-shirt of one of seven rainbow's colours; every two friends' colours are different. Leaders demanded that some children (at least one) wear T-shirts of other colours (from those seven colours). Survey pointed that 100 children didn't want to change their colours [translator's comment: it means that any of these 100 children (and only them) can't change his (her) colour such that still every two friends' colours will be different]. Prove that some of other children can change colours of their T-shirts such that as before every two friends' colours will be different.

Problem 1.55 (834743022162424)**.** Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:

$$2023f(f(x)) + 2022x^2 = 2022f(x) + 2023[f(x)]^2 + 1$$

for each integer x.

Problem 1.56 (5514383858686655851)**.** Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

Problem 1.57 (952584318797289)**.** Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

Problem 1.58 (5173505438503336781)**.** Find all polynomials $P(x)$ with integer coefficients such that for all real numbers s and t, if $P(s)$ and $P(t)$ are both integers, then $P(st)$ is also an integer.

Problem 1.59 (303061622555285)**.** A teacher and her 30 students play a game on an infinite cell grid. The teacher starts first, then each of the 30 students makes a move, then the teacher and so on. On one move the person can color one unit segment on the grid. A segment cannot be colored twice. The teacher wins if, after the move of one of the 31 players, there is a 1×2 or 2×1 rectangle , such that each segment from it's border is colored, but the segment between the two adjacent squares isn't colored. Prove that the teacher can win.

Problem 1.60 (3031913484181592371)**.** Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

Problem 1.61 (6193947856984766386)**.** Let $ABCD$ be a cyclic quadrilateral. Assume that the points Q, A, B, P are collinear in this order, in such a way that the line AC is tangent to the circle ADQ, and the line BD is tangent to the circle BCP. Let M and N be the midpoints of segments BC and AD, respectively. Prove that the following three lines are concurrent: line CD, the tangent of circle ANQ at point A, and the tangent to circle BMP at point B.

Problem 1.62 (308110166188097)**.** Let A, B be two fixed points on the unit circle ω, satisfying $\sqrt{2} < AB < 2$. Let P be a point that can move on the unit circle, and it can move to anywhere on the unit circle satisfying $\triangle ABP$ is acute and $AP > AB > BP$. Let H be the orthocenter of $\triangle ABP$ and S be a point on the minor arc AP satisfying $SH = AH$. Let T be a point on the minor arc AB satisfying $TB || AP$. Let $ST \cap BP = Q$.

Show that (recall P varies) the circle with diameter HQ passes through a fixed point.

Problem 1.63 (6953300922471087070)**.** There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 1.64 (816618498838890)**.** Prove that for each real number $r > 2$, there are exactly two or three positive real numbers x satisfying the equation $x^2 = r\lfloor x \rfloor$.

Problem 1.65 (961350373727093)**.** Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

Problem 1.66 (117986541208663)**.** Given a triangle ABC. D is a moving point on the edge BC. Point E and Point F are on the edge AB and AC, respectively, such that $BE = CD$ and $CF = BD$. The circumcircle of $\triangle BDE$ and $\triangle CDF$ intersects at another point P other than D. Prove that there exists a fixed point Q, such that the length of QP is constant.

Problem 1.67 (6098711912608423295)**.** Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Problem 1.68 (822507508246664)**.** We say that a positive integer n is m-expressible if it is possible to get n from some m digits and the six operations $+, -, \times, \div$, exponentiation ^, and concatenation $\oplus$. For example, 5625 is 3-expressible (in two ways): both $5 \oplus (5\hat{}4)$ and $(7 \oplus 5)\hat{}2$ yield 5625.

Does there exist a positive integer N such that all positive integers with N digits are $(N-1)$-expressible?

Problem 1.69 (1690019174311406035)**.** Let S be a nonempty set of positive integers such that, for any (not necessarily distinct) integers a and b in S, the number $ab+1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Problem 1.70 (308215997593136)**.** Misha came to country with n cities, and every 2 cities are connected by the road. Misha want visit some cities, but he doesn't visit one city two time. Every time, when Misha goes from city A to city B, president of country destroy k roads from city B(president can't destroy road, where Misha goes). What maximal number of cities Misha can visit, no matter how president does?

Problem 1.71 (6360153743145135128)**.** Find all functions $f\colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y)+f(x,y-1)}{2}.$$

Problem 1.72 (15195306726194)**.** There are two piles of stones: 1703 stones in one pile and 2022 in the other. Sasha and Olya play the game, making moves in turn, Sasha starts. Let before the player's move the heaps contain a and b stones, with $a \geq b$. Then, on his own move, the player is allowed take from the pile with a stones any number of stones from 1 to b. A player loses if he can't make a move. Who wins?

Remark: For 10.4, the initial numbers are $(444, 999)$

Problem 1.73 (576014113251153)**.** For a finite set C of integer numbers, we define $S(C)$ as the sum of the elements of C. Find two non-empty sets A and B whose intersection is empty, whose union is the set $\{1, 2, \dots, 2021\}$ and such that the product $S(A)S(B)$ is a perfect square.

Problem 1.74 (685485832068823)**.** A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time. (i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it. (ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

Problem 1.75 (3442101705279585713)**.** An integer $n \geq 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ Shiny if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny n-tuple $(x_1, x_2, \ldots, x_n)$.

Problem 1.76 (816006272568007)**.** Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

Problem 1.77 (181878217485192)**.** 1000 children, no two of the same height, lined up. Let us call a pair of different children (a, b) good if between them there is no child whose height is greater than the height of one of a and b, but less than the height of the other. What is the greatest number of good pairs that could be formed? (Here, (a, b) and (b, a) are considered the same pair.)

Problem 1.78 (5613759320855594939)**.** Petya has $10,000$ balls, among them there are no two balls of equal weight. He also has a device, which works as follows: if he puts exactly 10 balls on it, it will report the sum of the weights of some two of them (but he doesn't know which ones). Prove that Petya can use the device a few times so that after a while he will be able to choose one of the balls and accurately tell its weight.

Problem 1.79 (8528437132500966626)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 1.80 (958328158026487)**.** Alice is performing a magic trick. She has a standard deck of 52 cards, which she may order beforehand. She invites a volunteer to pick an integer $0 \leq n \leq 52$, and cuts the deck into a pile with the top n cards and a pile with the remaining $52 - n$. She then gives both piles to the volunteer, who riffles them together and hands the deck back to her face down. (Thus, in the resulting deck, the cards that were in the deck of size n appear in order, as do the cards that were in the deck of size $52 - n$.)

Alice then flips the cards over one-by-one from the top. Before flipping over each card, she may choose to guess the color of the card she is about to flip over. She stops if she guesses incorrectly. What is the maximum number of correct guesses she can guarantee?

Problem 1.81 (1620616963605432410)**.** Given an isosceles triangle $\triangle ABC$, $AB = AC$. A line passes through M, the midpoint of BC, and intersects segment AB and ray CA at D and E, respectively. Let F be a point of ME such that $EF = DM$, and K be a point on MD. Let Γ_1 be the circle passes through B, D, K and Γ_2 be the circle passes through C, E, K. Γ_1 and Γ_2 intersect again at $L \neq K$. Let ω_1 and ω_2 be the circumcircle of $\triangle LDE$ and $\triangle LKM$. Prove that, if ω_1 and ω_2 are symmetric wrt L, then BF is perpendicular to BC.

Problem 1.82 (296367141382799)**.** Given a triangle $\triangle ABC$ with orthocenter H. On its circumcenter, choose an arbitrary point P (other than A, B, C) and let M be the midpoint of HP. Now, we find three points D, E, F on the line BC, CA, AB, respectively, such that $AP \parallel HD, BP \parallel HE, CP \parallel HF$. Show that D, E, F, M are colinear.

Problem 1.83 (15317350224055)**.** Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Determine the minimum value of $(a - b)(b - c)(c - d)(d - a)$ and determine all values of (a, b, c, d) such that the minimum value is achived.

Problem 1.84 (597832355221478)**.** Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcenter of triangle APQ, H the orthocenter of triangle ABC, and S the point of intersection of the lines BQ and CP. Prove that T, H, and S are collinear.

Problem 1.85 (80567267310692)**.** Let n be a positive integer. Given is a subset A of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from A such that $c + 2a > 3b$.

Problem 1.86 (262105369827306)**.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

Problem 1.87 (1440964279096111130)**.** Let a be a positive integer. We say that a positive integer b is a-good if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a-good, but $b+2$ is not a-good. Prove that $b + 1$ is prime.

Problem 1.88 (740814477661493)**.** Determine the greatest positive integer n for which there exists a sequence of distinct positive integers $s_1, s_2, \ldots, s_n$ satisfying

$$s_1^{s_2} = s_2^{s_3} = \cdots = s_{n-1}^{s_n}.$$

Problem 1.89 (6122338123883323140)**.** Determine all nonempty finite sets of positive integers $\{a_1, \ldots, a_n\}$ such that $a_1 \cdots a_n$ divides $(x + a_1) \cdots (x + a_n)$ for every positive integer x.

Problem 1.90 (496656338551810)**.** Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either m by n or n by m is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid (If this never occurs, Freyja has lost the game and Tsvety wins).

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions suffice.

Problem 1.91 (5450879444672277193)**.** Let $n \geq 2$ be a positive integer. Let $\mathcal{R}$ be a connected set of unit squares on a grid. A bar is a rectangle of length or width 1 which is fully contained in $\mathcal{R}$. A bar is special if it is not fully contained within any larger bar. Given that $\mathcal{R}$ contains special bars of sizes $1 \times 2, 1 \times 3, \dots, 1 \times n$, find the smallest possible number of unit squares in $\mathcal{R}$.

Problem 1.92 (2252133047011954512)**.** On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

Problem 1.93 (902621191535073)**.** Given six points A, B, C, D, E, F such that $\triangle BCD \overset{+}{\sim} \triangle ECA \overset{+}{\sim} \triangle BFA$ and let I be the incenter of $\triangle ABC$. Prove that the circumcenter of $\triangle AID, \triangle BIE, \triangle CIF$ are collinear.

Problem 1.94 (623590906176957)**.** The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 1.95 (314213229221479)**.** Given is a natural number $n > 5$. On a circular strip of paper is written a sequence of zeros and ones. For each sequence w of n zeros and ones we count the number of ways to cut out a fragment from the strip on which is written w. It turned out that the largest number M is achieved for the sequence 1100...0 ($n-2$ zeros) and the smallest - for the sequence 00...011 ($n-2$ zeros). Prove that there is another sequence of n zeros and ones that occurs exactly M times.

Problem 1.96 (409146991986056)**.** For each prime p, construct a graph G_p on $\{1, 2, \dots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2+1-n)(n^2+1-m)$. Prove that G_p is disconnected for infinitely many p

Problem 1.97 (8765929309402693604)**.** Define the function $f : (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \geq \frac{1}{2} \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

Problem 1.98 (15595788767204175)**.** Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem 1.99 (3333337471825030029)**.** A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Problem 1.100 (733773583946080)**.** AB and AC are tangents to a circle ω with center O at B, C respectively. Point P is a variable point on minor arc BC. The tangent at P to ω meets AB, AC at D, E respectively. AO meets BP, CP at U, V respectively. The line through P perpendicular to AB intersects DV at M, and the line through P perpendicular to AC intersects EU at N. Prove that as P varies, MN passes through a fixed point.

Problem 1.101 (69707766974981)**.** For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root. Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \leq 1.$$

Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

Problem 1.102 (402654566950359)**.** Let a_1, a_2, $\ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 1.103 (6020628633767269011)**.** Let $ABCDE$ be a regular pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that as P varies, R lies on a fixed line.

Problem 1.104 (2989958142304279488)**.** Given is a set of $2n$ cards numbered $1, 2, \cdots, n$, each number appears twice. The cards are put on a table with the face down. A set of cards is called good if no card appears twice. Baron Munchausen claims that he can specify 80 sets of n cards, of which at least one is sure to be good. What is the maximal n for which the Baron's words could be true?

Problem 1.105 (8559783288978563338)**.** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Problem 1.106 (567108152004136)**.** Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Problem 1.107 (6702571883743406545)**.** Bethan is playing a game on an $n \times n$ grid consisting of n^2 cells. A move consists of placing a counter in an unoccupied cell C where the $2n - 2$ other cells in the same row or column as C contain an even number of counters. After making M moves Bethan realises she cannot make any more moves. Determine the minimum value of M.

Problem 1.108 (1053677942605812231)**.** Determine all Functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(a) - b) + bf(2a)$ is a perfect square for all integers a and b.

Problem 1.109 (233001122289340)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$?

Problem 1.110 (2100441935415071480)**.** Let T be a finite set of squarefree integers.

(a) Show that there exists an integer polynomial $P(x)$ such that the set of squarefree numbers in the range of $P(n)$ across all $n \in \mathbb{Z}$ is exactly T.

(b) Suppose that T is allowed to be infinite. Is it still true that for all choices of T, such an integer polynomial $P(x)$ exists?

Problem 1.111 (1154252954200953594)**.** Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

Problem 1.112 (583277702191991)**.** The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

Problem 1.113 (760426813975831)**.** Let ABC be a triangle with $AB + AC = 3BC$. The B-excircle touches side AC and line BC at E and D, respectively. The C-excircle touches side AB at F. Let lines CF and DE meet at P. Prove that $\angle PBC = 90°$.

Problem 1.114 (3838873685857064127)**.** Let m be a positive integer. Find, in terms of m, all polynomials $P(x)$ with integer coefficients such that for every integer n, there exists an integer k such that $P(k) = n^m$.

Problem 1.115 (7948249970111159954)**.** A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \le t_1 < \ldots < t_k \le 2022$ so that $t_{i+1} - t_i \le 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \ge C.$$

Problem 1.116 (4306507392377162131)**.** Let n be a positive integer such that the number

$$\frac{1^k + 2^k + \cdots + n^k}{n}$$

is an integer for any $k \in \{1, 2, \ldots, 99\}$. Prove that n has no divisors between 2 and 100, inclusive.

Problem 1.117 (869501852347427)**.** Let a, b, and n be positive integers. A lemonade stand owns n cups, all of which are initially empty. The lemonade stand has a filling machine and an emptying machine, which operate according to the following rules: If at any moment, a completely empty cups are available, the filling machine spends the next a minutes filling those a cups simultaneously and doing nothing else. If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else. Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. Find, in terms of a and b, the least possible value of n.

Problem 1.118 (723258861624579)**.** Let $n \ge 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_k} (a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Problem 1.119 (6852995499544467)**.** Find all pairs (a, b) of positive integers such that $a^2 \mid b^3 + 1$ and $b^2 \mid a^3 + 1$.

Problem 1.120 (6174780824971319633)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Problem 1.121 (2350680529866748619)**.** Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

Problem 1.122 (6322745101407512634)**.** Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}, \overline{CA}, \overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Problem 1.123 (4603228855421380865)**.** Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

Problem 1.124 (790369865925596)**.** For a nonnegative integer n define $\operatorname{rad}(n) = 1$ if $n = 0$ or $n = 1$, and $\operatorname{rad}(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $\operatorname{rad}(f(n))$ divides $\operatorname{rad}(f(n^{\operatorname{rad}(n)}))$ for every nonnegative integer n.

Problem 1.125 (6654677204410680146)**.** In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

Problem 1.126 (258585206260584)**.** Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Problem 1.127 (8443582325423683781)**.** Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC including A). Let I be the incenter of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the lines through the intersections of ω_b and ω_c, meet on Ω.

Problem 1.128 (66110871669579)**.** Determine all pairs of positive integers (m, n) for which there exists a bijective function

$$f : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$$

such that the vectors $f(\mathbf{v}) + \mathbf{v}$, as $\mathbf{v}$ runs through all of $\mathbb{Z}_m \times \mathbb{Z}_n$, are pairwise distinct.

(For any integers a and b, the vectors $[a, b], [a+m, b]$ and $[a, b+n]$ are treated as equal.)

Problem 1.129 (3706706337726127226)**.** Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n diffferent triangulations.

(A triangulation is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon)

Problem 1.130 (263704170707884)**.** Consider a triangle ABC with altitudes AD, BE, and CF, and orthocenter H. Let the perpendicular line from H to EF intersects EF, AB and AC at P, T and L, respectively. Point K lies on the side BC such that $BD = KC$. Let ω be a circle that passes through H and P, that is tangent to AH. Prove that circumcircle of triangle ATL and ω are tangent, and KH passes through the tangency point.

Problem 1.131 (291724488494808)**.** Given a positive integer k, find all polynomials P of degree k with integer coefficients such that for all positive integers n where all of $P(n)$,

$P(2024n)$, $P(2024^2n)$ are nonzero, we have

$$\frac{\gcd(P(2024n), P(2024^2n))}{\gcd(P(n), P(2024n))} = 2024^k.$$

Problem 1.132 (725882523060129). Assume three circles mutually outside each other with the property that every line separating two of them have intersection with the interior of the third one. Prove that the sum of pairwise distances between their centers is at most $2\sqrt{2}$ times the sum of their radii. (A line separates two circles, whenever the circles do not have intersection with the line and are on different sides of it.) Note. Weaker results with $2\sqrt{2}$ replaced by some other c may be awarded points depending on the value of $c > 2\sqrt{2}$

Problem 1.133 (7423980435672455501). A permutation of the integers $1, 2, \ldots, m$ is called fresh if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \ldots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \ldots, m$.

Prove that $f_n \geq n \cdot f_{n-1}$ for all $n \geq 3$.

For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.

Problem 1.134 (211625179383762). Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

Problem 1.135 (8330669807899443473). Let ABC be an acute scalene triangle, and let A_1, B_1, C_1 be the feet of the altitudes from A, B, C. Let A_2 be the intersection of the tangents to the circle ABC at B, C and define B_2, C_2 similarly. Let A_2A_1 intersect the circle $A_2B_2C_2$ again at A_3 and define B_3, C_3 similarly. Show that the circles AA_1A_3, BB_1B_3, and CC_1C_3 all have two common points, X_1 and X_2 which both lie on the Euler line of the triangle ABC.

Problem 1.136 (9184583066675086219). An integer a is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers. a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$. b) Decide whether $a = 2$ is friendly.

Problem 1.137 (548248988934632). Let ABC be a triangle with incenter I. Let segment AI intersect the incircle of triangle ABC at point D. Suppose that line BD is perpendicular to line AC. Let P be a point such that $\angle BPA = \angle PAI = 90°$. Point Q lies on segment BD such that the circumcircle of triangle ABQ is tangent to line BI. Point X lies on line PQ such that $\angle IAX = \angle XAC$. Prove that $\angle AXP = 45°$.

Problem 1.138 (5182115879210719670). Convex circumscribed quadrilateral $ABCD$ with its incenter I is given such that its incircle is tangent to $\overline{AD}, \overline{DC}, \overline{CB}$, and $\overline{BA}$ at K, L, M, and N. Lines $\overline{AD}$ and $\overline{BC}$ meet at E and lines $\overline{AB}$ and $\overline{CD}$ meet at F. Let $\overline{KM}$ intersects $\overline{AB}$ and $\overline{CD}$ at X, Y, respectively. Let $\overline{LN}$ intersects $\overline{AD}$ and $\overline{BC}$ at Z, T, respectively. Prove that the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI are tangent if and only if the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI are tangent.

Problem 1.139 (683710365849473). On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Problem 1.140 (852531542088551)**.** Given a triangle ABC for which $\angle BAC \neq 90°$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

Problem 1.141 (2798224660835368817)**.** Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y . Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$.

Problem 1.142 (685138775901874)**.** The cells of a 100×100 table are colored white. In one move, it is allowed to select some 99 cells from the same row or column and recolor each of them with the opposite color. What is the smallest number of moves needed to get a table with a chessboard coloring?

Problem 1.143 (3579058550991835669)**.** There are $n!$ empty baskets in a row, labelled $1, 2, ..., n!$. Caesar first puts a stone in every basket. Caesar then puts 2 stones in every second basket. Caesar continues similarly until he has put n stones into every nth basket. In other words, for each $i = 1, 2, ..., n$, Caesar puts i stones into the baskets labelled $i, 2i, 3i, ..., n!$. Let x_i be the number of stones in basket i after all these steps. Show that $n! \cdot n^2 \leq \sum_{i=1}^{n!} x_i^2 \leq n! \cdot n^2 \cdot \sum_{i=1}^{n} \frac{1}{i}$

Problem 1.144 (988108242834730)**.** Given some monic polynomials $P_1, \ldots, P_n$ with real coefficients, for any real number y, let S_y be the set of real number x such that $y = P_i(x)$ for some $i = 1, 2, ..., n$. If the sets S_{y_1}, S_{y_2} have the same size for any two real numbers y_1, y_2, show that $P_1, \ldots, P_n$ have the same degree.

Problem 1.145 (741259148493039)**.** Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Problem 1.146 (1918325703156767787)**.** Let T_n denotes the least natural such that

$$n \mid 1 + 2 + 3 + \cdots + T_n = \sum_{i=1}^{T_n} i$$

Find all naturals m such that $m \geq T_m$.

Problem 1.147 (458902414604417)**.** A class has 25 students. The teacher wants to stock N candies, hold the Olympics and give away all N candies for success in it (those who solve equally tasks should get equally, those who solve less get less, including, possibly, zero candies). At what smallest N this will be possible, regardless of the number of tasks on Olympiad and the student successes?

Problem 1.148 (6444187106925350071)**.** An infinite sequence of positive integers $a_1, a_2, \ldots$ is called *good* if (1) a_1 is a perfect square, and (2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n$$

is a perfect square. Prove that for any good sequence $a_1, a_2, \ldots$, there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.

Problem 1.149 (857598260795435)**.** Let $ABCD$ be a rhombus with center O. P is a point lying on the side AB. Let I, J, and L be the incenters of triangles PCD, PAD, and PBC, respectively. Let H and K be orthocenters of triangles PLB and PJA, respectively.

Prove that $OI \perp HK$.

Problem 1.150 (692237787009642)**.** Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions: Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns. Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns. They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

Problem 1.151 (3104932449951237120)**.** Let ABC be an acute triangle with circumcircle Γ. Let P and Q be points in the half plane defined by BC containing A, such that BP and CQ are tangents to Γ and $PB = BC = CQ$. Let K and L be points on the external bisector of the angle $\angle CAB$, such that $BK = BA, CL = CA$. Let M be the intersection point of the lines PK and QL. Prove that $MK = ML$.

Problem 1.152 (5407986531182333567)**.** Call admissible a set A of integers that has the following property: If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k. Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Problem 1.153 (8972547734710795566)**.** Let incircle (I) of triangle ABC touch the sides BC, CA, AB at D, E, F respectively. Let (O) be the circumcircle of ABC. Ray EF meets (O) at M. Tangents at M and A of (O) meet at S. Tangents at B and C of (O) meet at T. Line TI meets OA at J. Prove that $\angle ASJ = \angle IST$.

Problem 1.154 (6253841118919498374)**.** The n contestant of EGMO are named $C_1, C_2, \cdots C_n$. After the competition, they queue in front of the restaurant according to the following rules. The Jury chooses the initial order of the contestants in the queue. Every minute, the Jury chooses an integer i with $1 \leq i \leq n$. If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions. If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and process ends. Prove that the process cannot continue indefinitely, regardless of the Jury's choices. Determine for every n the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

Problem 1.155 (965885167255885)**.** A 3×3 grid of unit cells is given. A snake of length k is an animal which occupies an ordered k-tuple of cells in this grid, say $(s_1, \ldots, s_k)$.

These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. After being placed in a finite $n \times n$ grid, if the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has turned around if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead.

Find the largest integer k such that one can place some snake of length k in a 3×3 grid which can turn around.

Problem 1.156 (742686070320805)**.** Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

Problem 1.157 (915478364939250)**.** Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 1.158 (227872694827710)**.** Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \cdots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \cdots a_1}$ is a perfect square.

Problem 1.159 (260347681948452)**.** Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2 b + c = b^2 c + a = c^2 a + b.$$

Problem 1.160 (1862468241301875616)**.** Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Problem 1.161 (264456837378391)**.** Let ABC be a triangle such that the angular bisector of $\angle BAC$, the B-median and the perpendicular bisector of AB intersect at a single point X. Let H be the orthocenter of ABC. Show that $\angle BXH = 90°$.

Problem 1.162 (572967976964328)**.** Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Problem 1.163 (7203789790519658258)**.** Let ABC be a triangle and let P be a point not lying on any of the three lines AB, BC, or CA. Distinct points D, E, and F lie on lines BC, AC, and AB, respectively, such that $\overline{DE} \parallel \overline{CP}$ and $\overline{DF} \parallel \overline{BP}$. Show that there exists a point Q on the circumcircle of $\triangle AEF$ such that $\triangle BAQ$ is similar to $\triangle PAC$.

Problem 1.164 (175119746688413)**.** The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant.

The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Problem 1.165 (2265193939454652363)**.** A circle ω with radius 1 is given. A collection T of triangles is called good, if the following conditions hold: each triangle from T is inscribed in ω; no two triangles from T have a common interior point. Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

Problem 1.166 (4948608980214807448)**.** Let ABC be a scalene triangle with circumcenter O and orthocenter H. Let AYZ be another triangle sharing the vertex A such that its circumcenter is H and its orthocenter is O. Show that if Z is on BC, then A, H, O, Y are concyclic.

Problem 1.167 (23047452603115)**.** Let ABC be a triangle. Let θ be a fixed angle for which

$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$

Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

Problem 1.168 (557323499571799)**.** For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1\le i\le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

Problem 1.169 (944096417683669)**.** For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 1.170 (8799177804774743019)**.** In each square of a garden shaped like a 2022×2022 board, there is initially a tree of height 0. A gardener and a lumberjack alternate turns playing the following game, with the gardener taking the first turn: The gardener chooses a square in the garden. Each tree on that square and all the surrounding squares (of which there are at most eight) then becomes one unit taller. The lumberjack then chooses four different squares on the board. Each tree of positive height on those squares then becomes one unit shorter. We say that a tree is majestic if its height is at least 10^6. Determine the largest K such that the gardener can ensure there are eventually K majestic trees on the board, no matter how the lumberjack plays.

Problem 1.171 (836212333854709)**.** Let $A_1, \ldots, A_{2022}$ be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

In each round, Bob chooses a point in the plane (not necessarily among $A_1, \ldots, A_{2022}$) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points $A_1, \ldots, A_{2022}$ in Q rounds?

Problem 1.172 (4037864050528368034)**.** Find the largest integer $N \in \{1, 2, \ldots, 2019\}$ such that there exists a polynomial $P(x)$ with integer coefficients satisfying the following property: for each positive integer k, $P^k(0)$ is divisible by 2020 if and only if k is divisible by N. Here P^k means P applied k times, so $P^1(0) = P(0), P^2(0) = P(P(0))$, etc.

Problem 1.173 (7914233989480462690)**.** Let ABC be a triangle with incenter I, and A-excenter Γ. Let A_1, B_1, C_1 be the points of tangency of Γ with BC, AC and AB, respectively. Suppose IA_1, IB_1 and IC_1 intersect Γ for the second time at points A_2, B_2, C_2, respectively. M is the midpoint of segment AA_1. If the intersection of A_1B_1 and A_2B_2 is X, and the intersection of A_1C_1 and A_2C_2 is Y, prove that $MX = MY$.

Problem 1.174 (3780160396229984886)**.** Let $\lfloor \bullet \rfloor$ denote the floor function. For nonnegative integers a and b, their bitwise xor, denoted $a \oplus b$, is the unique nonnegative integer such that

$$\left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor - \left\lfloor \frac{a \oplus b}{2^k} \right\rfloor$$

is even for every $k \geq 0$. Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

Problem 1.175 (6064010778487493566)**.** Vulcan and Neptune play a turn-based game on an infinite grid of unit squares. Before the game starts, Neptune chooses a finite number of cells to be flooded. Vulcan is building a levee, which is a subset of unit edges of the grid (called walls) forming a connected, non-self-intersecting path or loop*.

The game then begins with Vulcan moving first. On each of Vulcan's turns, he may add up to three new walls to the levee (maintaining the conditions for the levee). On each of Neptune's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well. Prove that Vulcan can always, in a finite number of turns, build the levee into a closed loop such that all flooded cells are contained in the interior of the loop, regardless of which cells Neptune initially floods.

*More formally, there must exist lattice points $A_0, A_1, \ldots, A_k$, pairwise distinct except possibly $A_0 = A_k$, such that the set of walls is exactly $\{A_0A_1, A_1A_2, \ldots, A_{k-1}A_k\}$. Once a wall is built it cannot be destroyed; in particular, if the levee is a closed loop (i.e. $A_0 = A_k$) then Vulcan cannot add more walls. Since each wall has length 1, the length of the levee is k.

Problem 1.176 (4992489807901310938)**.** Let ABC be a triangle and ℓ_1, ℓ_2 be two parallel lines. Let ℓ_i intersects line BC, CA, AB at X_i, Y_i, Z_i, respectively. Let Δ_i be the triangle formed by the line passed through X_i and perpendicular to BC, the line passed through Y_i and perpendicular to CA, and the line passed through Z_i and perpendicular to AB. Prove that the circumcircles of Δ_1 and Δ_2 are tangent.

Problem 1.177 (8700346175921432509)**.** 2021 points on the plane in the convex position, no three collinear and no four concyclic, are given. Prove that there exist two of them such that every circle passing through these two points contains at least 673 of the other points in its interior. (A finite set of points on the plane are in convex position if the points are the vertices of a convex polygon.)

Problem 1.178 (7384014966956792204)**.** In acute triangle $ABC, \angle A = 45^o$. Points O, H are the circumcenter and the orthocenter of ABC, respectively. D is the foot of altitude from B. Point X is the midpoint of arc AH of the circumcircle of triangle ADH that contains D. Prove that $DX = DO$.

Problem 1.179 (6195404266254375127)**.** In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Problem 1.180 (552612087321706)**.** For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Problem 1.181 (858562234779712)**.** Let $n > 2$ be a positive integer. Given is a horizontal row of n cells where each cell is painted blue or red. We say that a block is a sequence of consecutive boxes of the same color. Arepito the crab is initially standing at the leftmost cell. On each turn, he counts the number m of cells belonging to the largest block containing the square he is on, and does one of the following:

If the square he is on is blue and there are at least m squares to the right of him, Arepito moves m squares to the right;

If the square he is in is red and there are at least m squares to the left of him, Arepito moves m cells to the left;

In any other case, he stays on the same square and does not move any further.

For each n, determine the smallest integer k for which there is an initial coloring of the row with k blue cells, for which Arepito will reach the rightmost cell.

Problem 1.182 (6919176010062551987)**.** Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q\leq n, p,q \text{ primes}} (p+q)$$

Problem 1.183 (5395714337110519657)**.** The vertices of a convex 2550-gon are colored black and white as follows: black, white, two black, two white, three black, three white, ..., 50 black, 50 white. Dania divides the polygon into quadrilaterals with diagonals that have no common points. Prove that there exists a quadrilateral among these, in which two adjacent vertices are black and the other two are white.

Problem 1.184 (423456312616928)**.** Let BC be a fixed segment in the plane, and let A be a variable point in the plane not on the line BC. Distinct points X and Y are chosen on the rays $CA^{\rightarrow}$ and $BA^{\rightarrow}$, respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of ABC at B and C meet line XY at P and Q, respectively, such that the points X, P, Y and Q are pairwise distinct and lie on the same side of BC. Let Ω_1 be the circle through X and P centred on BC. Similarly, let Ω_2 be the circle through Y and Q centred on BC. Prove that Ω_1 and Ω_2 intersect at two fixed points as A varies.

Problem 1.185 (261061984301321)**.** Let $P = \{p_1, p_2, \ldots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all the integers greater than 1 so that their prime decomposition only contains primes of P. The elements of A are colored in such a way that: each element of P has a different color, if $m, n \in A$, then mn is the same color of m or n, for any pair of different colors $\mathcal{R}$ and $\mathcal{S}$, there are no $j, k, m, n \in A$ (not necessarily distinct from one another), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, so that j is a divisor of m and n is a divisor of k, simultaneously. Prove that there exists a prime of P so that all its multiples in A are the same color.

Problem 1.186 (8782897210450267045)**.** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Problem 1.187 (1168447466971762345)**.** Let I, O, ω, Ω be the incenter, circumcenter, the incircle, and the circumcircle, respectively, of a scalene triangle ABC. The incircle ω is tangent to side BC at point D. Let S be the point on the circumcircle Ω such that AS, OI, BC are concurrent. Let H be the orthocenter of triangle BIC. Point T lies on Ω such that $\angle ATI$ is a right angle. Prove that the points D, T, H, S are concyclic.

Problem 1.188 (409149115429190)**.** On the $n \times n$ checker board, several cells were marked in such a way that lower left (L) and upper right(R) cells are not marked and that for any knight-tour from L to R, there is at least one marked cell. For which $n > 3$, is it possible that there always exists three consective cells going through diagonal for which at least two of them are marked?

Problem 1.189 (33618537498844)**.** Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

Problem 1.190 (300334293164389)**.** Kid and Karlsson play a game. Initially they have a square piece of chocolate 2019×2019 grid with 1×1 cells . On every turn Kid divides an arbitrary piece of chololate into three rectanglular pieces by cells, and then Karlsson chooses one of them and eats it. The game finishes when it's impossible to make a legal move. Kid wins if there was made an even number of moves, Karlsson wins if there was made an odd number of moves. Who has the winning strategy?

Problem 1.191 (284766145954043)**.** Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of

S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

Problem 1.192 (4875666253256352039)**.** Supppose that there are roads AB and CD but there are no roads BC and AD between four cities A, B, C, and D. Define restructing to be the changing a pair of roads AB and CD to the pair of roads BC and AD. Initially there were some cities in a country, some of which were connected by roads and for every city there were exactly 100 roads starting in it. The minister drew a new scheme of roads, where for every city there were also exactly 100 roads starting in it. It's known also that in both schemes there were no cities connected by more than one road. Prove that it's possible to obtain the new scheme from the initial after making a finite number of restructings.

Problem 1.193 (326164407850848)**.** Two boys are given a bag of potatoes, each bag containing 150 tubers. They take turns transferring the potatoes, where in each turn they transfer a non-zero tubers from their bag to the other boy's bag. Their moves must satisfy the following condition: In each move, a boy must move more tubers than he had in his bag before any of his previous moves (if there were such moves). So, with his first move, a boy can move any non-zero quantity, and with his fifth move, a boy can move 200 tubers, if before his first, second, third and fourth move, the numbers of tubers in his bag was less than 200. What is the maximal total number of moves the two boys can do?

Problem 1.194 (8866273454792491736)**.** Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x - y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

Problem 1.195 (2477568457295629780)**.** Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

Problem 1.196 (942176258255049)**.** Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

Problem 1.197 (669395675904242)**.** Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 1.198 (814823180113879)**.** Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Problem 1.199 (143039642317874)**.** Find all positive integers k for which there exist a, b, and c positive integers such that

$$|(a-b)^3 + (b-c)^3 + (c-a)^3| = 3 \cdot 2^k.$$

Problem 1.200 (2792820689505589235)**.** The number 2024 is written on a blackboard. Each second, if there exist positive integers a, b, k such that $a^k + b^k$ is written on the blackboard, you may write $a^{k'} + b^{k'}$ on the blackboard for any positive integer k'. Find all positive integers that you can eventually write on the blackboard.

Problem 1.201 (748238852463934)**.** Let ABC be a triangle with incenter I and let AI meet BC at D. Let E be a point on the segment AC, such that $CD = CE$ and let F be on the segment AB such that $BF = BD$. Let $(CEI) \cap (DFI) = P \neq I$ and $(BFI) \cap (DEI) = Q \neq I$. Prove that $PQ \perp BC$.

Problem 1.202 (3427992889083230961)**.** Let $f(x)$ and $g(x)$ be given by $f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$ $g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}$.

Prove that $|f(x) - g(x)| > 2$ for any non-integer real number x satisfying $0 < x < 2018$.

Problem 1.203 (2599680620339408367)**.** Let p and q be relatively prime positive odd integers such that $1 < p < q$. Let A be a set of pairs of integers (a, b), where $0 \leq a \leq p-1, 0 \leq b \leq q-1$, containing exactly one pair from each of the sets

$$\{(a, b), (a+1, b+1)\}, \{(a, q-1), (a+1, 0)\}, \{(p-1, b), (0, b+1)\}$$

whenever $0 \leq a \leq p-2$ and $0 \leq b \leq q-2$. Show that A contains at least $(p-1)(q+1)/8$ pairs whose entries are both even.

Problem 1.204 (887161908366621)**.** Determine all integers $n \geq 3$ for which there exists a conguration of n points in the plane, no three collinear, that can be labelled 1 through n in two different ways, so that the following condition be satisfied: For every triple $(i, j, k), 1 \leq i < j < k \leq n$, the triangle ijk in one labelling has the same orientation as the triangle labelled ijk in the other, except for $(i, j, k) = (1, 2, 3)$.

Problem 1.205 (613109155420064)**.** Let m be a fixed positive integer. The infinite sequence $\{a_n\}_{n \geq 1}$ is defined in the following way: a_1 is a positive integer, and for every integer $n \geq 1$ we have

$$a_{n+1} = \begin{cases} a_n^2 + 2^m & \text{if } a_n < 2^m \\ a_n/2 & \text{if } a_n \geq 2^m \end{cases}$$

For each m, determine all possible values of a_1 such that every term in the sequence is an integer.

Problem 1.206 (4059278924956282558)**.** In a card game, each card is associated with a numerical value from 1 to 100, with each card beating less, with one exception: 1 beats 100. The player knows that 100 cards with different values lie in front of him. The dealer who knows the order of these cards can tell the player which card beats the other for any pair of cards he draws. Prove that the dealer can make one hundred such messages, so that after that the player can accurately determine the value of each card.

Problem 1.207 (8059760967121829853)**.** Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Problem 1.208 (727078403801409)**.** Let ABC be a triangle with incenter I and circumcircle Ω. A point X on Ω which is different from A satisfies $AI = XI$. The incircle touches AC and AB at E, F, respectively. Let M_a, M_b, M_c be the midpoints of sides BC, CA, AB, respectively. Let T be the intersection of the lines M_bF and M_cE. Suppose that AT intersects Ω again at a point S.

Prove that X, M_a, S, T are concyclic.

Problem 1.209 (937132258882447)**.** n coins lies in the circle. If two neighbour coins lies both head up or both tail up, then we can flip both. How many variants of coins are available that can not be obtained from each other by applying such operations?

Problem 1.210 (9137209985622350774)**.** In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

Problem 1.211 (8417327567048605288)**.** Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 1.212 (4278278843148290847)**.** Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Problem 1.213 (6246999615324043054)**.** A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 1.214 (5026826170538858627)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

Problem 1.215 (176031103945886)**.** X is a set of 2020 distinct real numbers. Prove that there exist $a, b \in \mathbb{R}$ and $A \subset X$ such that

$$\sum_{x\in A}(x-a)^2 + \sum_{x\in X\setminus A}(x-b)^2 \le \frac{1009}{1010}\sum_{x\in X} x^2$$

Problem 1.216 (792975361721939)**.** Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold: The sides of each rectangle are parallel to the sides of the unit square. Each point in S is not in the interior of any rectangle. Each point in the interior of the unit square but not in S is in the interior of at least one of the k rectangles (The interior of a polygon does not contain its boundary.)

Problem 1.217 (4618034848035557)**.** Let $f : \mathbb{Z} \to \{1, 2, \ldots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x-y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m+x, n)$ for all integers x.

Problem 1.218 (6302540840099076878)**.** Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90^\circ$.

Problem 1.219 (18644549011438)**.** Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

Problem 1.220 (6666334949338369993)**.** Choose positive integers $b_1, b_2, \ldots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

Problem 1.221 (6978535805224432571)**.** The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \ge 1$. Given an integer $n \ge 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Problem 1.222 (7412933249652771804)**.** Let ABC an acute triangle and D, E and F be the feet of altitudes from A, B and C, respectively. The line EF and the circumcircle of ABC intersect at P, such that F it´s between E and P. Lines BP and DF intersect at Q. Prove that if $ED = EP$, then CQ and DP are parallel.

Problem 1.223 (7583686967751031247)**.** Find all positive integers d for which there exists a degree d polynomial P with real coefficients such that there are at most d different values among $P(0), P(1), P(2), \cdots, P(d^2-d)$.

Problem 1.224 (57940096937913)**.** Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

Problem 1.225 (514210607042538)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + |y|) = x^2 + f(y)$$

Problem 1.226 (4118541811915047639)**.** In a country there are $n > 100$ cities and initially no roads. The government randomly determined the cost of building a two-way road between any two cities, using all amounts from 1 to $\frac{n(n-1)}{2}$ thalers once (all options are equally likely). The mayor of each city chooses the cheapest of the $n-1$ roads emanating from that city and it is built (this may be the mutual desired of the mayors of both cities being connected, or only one of the two). After the construction of these roads, the cities are divided into M connected components (between cities of the same connected component, you can get along the constructed roads, possibly via other cities, but this is not possible for cities of different components). Find the expected value of the random variable M.

Problem 1.227 (1222382895728709073)**.** Given a triangle ABC, a circle Ω is tangent to AB, AC at B, C, respectively. Point D is the midpoint of AC, O is the circumcenter of triangle ABC. A circle Γ passing through A, C intersects the minor arc BC on Ω at P, and intersects AB at Q. It is known that the midpoint R of minor arc PQ satisfies that $CR \perp AB$. Ray PQ intersects line AC at L, M is the midpoint of AL, N is the midpoint of DR, and X is the projection of M onto ON. Prove that the circumcircle of triangle DNX passes through the center of Γ.

Problem 1.228 (3813623497653179264)**.** The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

Problem 1.229 (9130156978935948779)**.** Let n be a positive integer, and let $\mathcal{C}$ be a collection of subsets of $\{1, 2, \ldots, 2^n\}$ satisfying both of the following conditions: Every $(2^n - 1)$-element subset of $\{1, 2, \ldots, 2^n\}$ is a member of $\mathcal{C}$, and Every non-empty member C of $\mathcal{C}$ contains an element c such that $C \setminus \{c\}$ is again a member of $\mathcal{C}$. Determine the smallest size $\mathcal{C}$ may have.

Problem 1.230 (27464517430039)**.** Let ABC be an acute triangle with $AC > AB > BC$. The perpendicular bisectors of AC and AB cut line BC at D and E respectively. Let P and Q be points on lines AC and AB respectively, both different from A, such that $AB = BP$ and $AC = CQ$, and let K be the intersection of lines EP and DQ. Let M be the midpoint of BC. Show that $\angle DKA = \angle EKM$.

Problem 1.231 (5867489266334805897)**.** Let $ABCDE$ be a pentagon inscribed in a circle Ω. A line parallel to the segment BC intersects AB and AC at points S and T, respectively. Let X be the intersection of the line BE and DS, and Y be the intersection of the line CE and DT.

Prove that, if the line AD is tangent to the circle $\odot(DXY)$, then the line AE is tangent to the circle $\odot(EXY)$.

Problem 1.232 (2258867823273260514)**.** Elmo has 2023 cookie jars, all initially empty. Every day, he chooses two distinct jars and places a cookie in each. Every night, Cookie Monster finds a jar with the most cookies and eats all of them. If this process continues indefinitely, what is the maximum possible number of cookies that the Cookie Monster could eat in one night?

Problem 1.233 (712971117639738)**.** Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i + j + k \geq n$.

Problem 1.234 (5790808043328490922)**.** Find all $f(x) \in \mathbb{Z}(x)$ that satisfies the following condition, with the lowest degree. Condition: There exists $g(x), h(x) \in \mathbb{Z}(x)$ such that

$$f(x)^4 + 2f(x) + 2 = (x^4 + 2x^2 + 2)g(x) + 3h(x)$$

Problem 1.235 (671689594281308077)**.** There are n line segments on the plane, no three intersecting at a point, and each pair intersecting once in their respective interiors. Tony and his $2n-1$ friends each stand at a distinct endpoint of a line segment. Tony wishes to send Christmas presents to each of his friends as follows: First, he chooses an endpoint of each segment as a "sink". Then he places the present at the endpoint of the segment he is at. The present moves as follows : • If it is on a line segment, it moves towards the sink. • When it reaches an intersection of two segments, it changes the line segment it travels on and starts moving towards the new sink. If the present reaches an endpoint, the friend on that endpoint can receive their present. Prove that Tony can send presents to exactly n of his $2n-1$ friends.

Problem 1.236 (141708904596471)**.** Let r be a positive integer, and let $a_0, a_1, \cdots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}$$

Prove that the sequence is periodic, i.e. there exists some $p \geq 1$ such that $a_{n+p} = a_n$ for all $n \geq 0$.

Problem 1.237 (221644122066923)**.** A straight road consists of green and red segments in alternating colours, the first and last segment being green. Suppose that the lengths of all segments are more than a centimeter and less than a meter, and that the length of each subsequent segment is larger than the previous one. A grasshopper wants to jump forward along the road along these segments, stepping on each green segment at least once an without stepping on any red segment (or the border between neighboring segments). Prove that the grasshopper can do this in such a way that among the lengths of his jumps no more than 8 different values occur.

Problem 1.238 (7618489197525)**.** Two circles Γ_1 and Γ_2 meet at two distinct points A and B. A line passing through A meets Γ_1 and Γ_2 again at C and D respectively, such that A lies between C and D. The tangent at A to Γ_2 meets Γ_1 again at E. Let F be a point on Γ_2 such that F and A lie on different sides of BD, and $2\angle AFC = \angle ABC$. Prove that the tangent at F to Γ_2, and lines BD and CE are concurrent.

Problem 1.239 (7902258516875436315)**.** Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that: in each row and each column, one third of the entries are I, one third are M and one third are O; and in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O. Note. The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.

Problem 1.240 (259897104343709)**.** There is a queue of n girls on one side of a tennis table, and a queue of n boys on the other side. Both the girls and the boys are numbered from 1 to n in the order they stand. The first game is played by the girl and the boy with the number 1 and then, after each game, the loser goes to the end of their queue, and the winner remains at the table. After a while, it turned out that each girl played exactly one game with each boy. Prove that if n is odd, then a girl and a boy with odd numbers played in the last game.

Problem 1.241 (584014589745861)**.** Let M, N, P be midpoints of BC, AC and AB of triangle $\triangle ABC$ respectively. E and F are two points on the segment $\overline{BC}$ so that $\angle NEC = \frac{1}{2}\angle AMB$ and $\angle PFB = \frac{1}{2}\angle AMC$. Prove that $AE = AF$.

Problem 1.242 (6703839677147050695)**.** In a plane we have n lines, no two of which are parallel or perpendicular, and no three of which are concurrent. A cartesian system of coordinates is chosen for the plane with one of the lines as the x-axis. A point P is located at the origin of the coordinate system and starts moving along the positive x-axis with constant velocity. Whenever P reaches the intersection of two lines, it continues along the line it just reached in the direction that increases its x-coordinate. Show that it is possible to choose the system of coordinates in such a way that P visits points from all n lines.

Problem 1.243 (934985329440054)**.** In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90^\circ$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 1.244 (134403212065462)**.** Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Problem 1.245 (913214378150707)**.** In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

Problem 1.246 (257453182523555)**.** Let a and b be positive integers. The cells of an $(a+b+1) \times (a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

Problem 1.247 (770681078031656)**.** Find all functions $f : R^+ \to R^+$ such that for all positive reals x and y

$$4f(x+yf(x)) = f(x)f(2y)$$

Problem 1.248 (521339998508550)**.** There are 998 cities in a country. Some pairs of cities are connected by two-way flights. According to the law, between any pair cities should be no more than one flight. Another law requires that for any group of cities there will be no more than $5k+10$ flights connecting two cities from this group, where k is the number number of cities in the group. Prove that several new flights can be introduced so that laws still hold and the total number of flights in the country is equal to 5000.

Problem 1.249 (3175174607535531817)**.** Let ABC be a triangle with bisectors BE and CF meet at I. Let D be the projection of I on the BC. Let M and N be the orthocenters of triangles AIF and AIE, respectively. Lines EM and FN meet at P. Let X be the midpoint of BC. Let Y be the point lying on the line AD such that $XY \perp IP$. Prove that line AI bisects the segment XY.

Problem 1.250 (4934938474754667779)**.** Let ABC be a triangle and let H be the orthogonal projection of A on the line BC. Let K be a point on the segment AH such that $AH = 3KH$. Let O be the circumcenter of triangle ABC and let M and N be the midpoints of sides AC and AB respectively. The lines KO and MN meet at a point Z and the perpendicular at Z to OK meets lines AB, AC at X and Y respectively. Show that $\angle XKY = \angle CKB$.

Problem 1.251 (4429559846138102630)**.** An interstellar hotel has 100 rooms with capacities $101, 102, \ldots, 200$ people. These rooms are occupied by n people in total. Now a VIP guest is about to arrive and the owner wants to provide him with a personal room. On that purpose, the owner wants to choose two rooms A and B and move all guests from A to B without exceeding its capacity. Determine the largest n for which the owner can be sure that he can achieve his goal no matter what the initial distribution of the guests is.

Problem 1.252 (682786464566571)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.253 (8463873707700703744)**.** Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 1$, and let $c > 1$ be an integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Show that there are infinitely many positive integers n such that $\gcd(x_n, n + c) = 1$.

Problem 1.254 (3440185808972009200)**.** Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 1.255 (406898817113614)**.** Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Problem 1.256 (1366302870241512636)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 1.257 (1736102587052874498)**.** Some language has only three letters - A, B and C. A sequence of letters is called a word iff it contains exactly 100 letters such that exactly 40 of them are consonants and other 60 letters are all A. What is the maximum numbers of words one can pick such that any two picked words have at least one position where they both have consonants, but different consonants?

Problem 1.258 (16776483958513)**.** Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 1.259 (639126468624733)**.** Let $ABCDEF$ be a hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that segments BD and DF intersect CE at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to CE.

Problem 1.260 (945040565828830)**.** Let P, Q, R, S be non constant polynomials with real coefficients, such that $P(Q(x)) = R(S(x))$ and the degree of P is multiple of the degree of R. Prove that there exists a polynomial T with real coefficients such that

$$P(x) = R(T(x))$$

Problem 1.261 (4451072691230235426)**.** A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.

Problem 1.262 (215375559035207)**.** ABC is an isosceles triangle, with $AB = AC$. D is a moving point such that $AD \parallel BC$, $BD > CD$. Moving point E is on the arc of BC in circumcircle of ABC not containing A, such that $EB < EC$. Ray BC contains point

F with $\angle ADE = \angle DFE$. If ray FD intersects ray BA at X, and intersects ray CA at Y, prove that $\angle XEY$ is a fixed angle.

Problem 1.263 (878429961754697605). Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f((c+1)x + f(y)) = f(x+2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

Problem 1.264 (6029540617185205962). On a social network, no user has more than ten friends (the state "friendship" is symmetrical). The network is connected: if, upon learning interesting news a user starts sending it to its friends, and these friends to their own friends and so on, then at the end, all users hear about the news. Prove that the network administration can divide users into groups so that the following conditions are met: each user is in exactly one group each group is connected in the above sense one of the groups contains from 1 to 100 members and the remaining from 100 to 900.

Problem 1.265 (4000488814786935591). A group of 100 kids has a deck of 101 cards numbered by $0, 1, 2, \ldots, 100$. The first kid takes the deck, shuffles it, and then takes the cards one by one; when he takes a card (not the last one in the deck), he computes the average of the numbers on the cards he took up to that moment, and writes down this average on the blackboard. Thus, he writes down 100 numbers, the first of which is the number on the first taken card. Then he passes the deck to the second kid which shuffles the deck and then performs the same procedure, and so on. This way, each of 100 kids writes down 100 numbers. Prove that there are two equal numbers among the 10000 numbers on the blackboard.

Problem 1.266 (543318535845123). Show that $r = 2$ is the largest real number r which satisfies the following condition:

If a sequence a_1, a_2, ... of positive integers fulfills the inequalities

$$a_n \leq a_{n+2} \leq \sqrt{a_n^2 + r a_{n+1}}$$

for every positive integer n, then there exists a positive integer M such that $a_{n+2} = a_n$ for every $n \geq M$.

Problem 1.267 (8534263250311217423). In acute triangle $\triangle ABC$, $\angle A > \angle B > \angle C$. $\triangle AC_1B$ and $\triangle CB_1A$ are isosceles triangles such that $\triangle AC_1B \overset{+}{\sim} \triangle CB_1A$. Let lines BB_1, CC_1 intersects at T. Prove that if all points mentioned above are distinct, $\angle ATC$ isn't a right angle.

Problem 1.268 (2040194717643782420). $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Problem 1.269 (587866144613888). The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to$ ...

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 1.270 (208479683430579745). There are 18 children in the class. Parents decided to give children from this class a cake. To do this, they first learned from each child the area of the piece he wants to get. After that, they showed a square-shaped cake, the area of which is exactly equal to the sum of 18 named numbers. However, when they saw the cake, the children wanted their pieces to be squares too. The parents cut the cake with lines parallel to the sides of the cake (cuts do not have to start or end on the side of the cake). For what maximum k the parents are guaranteed to cut out k square pieces from the cake, which you can give to k children so that each of them gets what they want?

Problem 1.271 (244533208775214844). A finite set S of points in the coordinate plane is called overdetermined if $|S| \geq 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S| - 2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \geq 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is not overdetermined, but has k overdetermined subsets.

Problem 1.272 (47893544380608). Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \ldots, n\}$ that are red. Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.

Problem 1.273 (6734490609685717062). Let I, G, O be the incenter, centroid and the circumcenter of triangle ABC, respectively. Let X, Y, Z be on the rays BC, CA, AB respectively so that $BX = CY = AZ$. Let F be the centroid of XYZ.

Show that FG is perpendicular to IO.

Problem 1.274 (2153848747665754338). Four points A, B, C and D lie on a circle ω such that $AB = BC = CD$. The tangent line to ω at point C intersects the tangent line to ω at A and the line AD at K and L. The circle ω and the circumcircle of triangle KLA intersect again at M. Prove that $MA = ML$.

Problem 1.275 (233559801569582). Let n be a positive integer. Find the number of permutations a_1, a_2, $\ldots a_n$ of the sequence 1, 2, $\ldots$, n satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

Problem 1.276 (896559847059784). Consider a 2018×2019 board with integers in each unit square. Two unit squares are said to be neighbours if they share a common edge. In each turn, you choose some unit squares. Then for each chosen unit square the average of all its neighbours is calculated. Finally, after these calculations are done, the number in each chosen unit square is replaced by the corresponding average. Is it always possible to make the numbers in all squares become the same after finitely many turns?

Problem 1.277 (194924255136905). Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers $c_1, c_2, c_3, \ldots$, Turbo successively crawls distances $c_1, c_2, c_3, \ldots$ around the circle, each time choosing to crawl either clockwise or counterclockwise. Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers $c_1, c_2, c_3, \ldots$ with $c_i < C$ for all i, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

Problem 1.278 (3951159057888736589)**.** Let $n \geq 5$ be an integer. Consider n squares with side lengths $1, 2, \ldots, n$, respectively. The squares are arranged in the plane with their sides parallel to the x and y axes. Suppose that no two squares touch, except possibly at their vertices. Show that it is possible to arrange these squares in a way such that every square touches exactly two other squares.

Problem 1.279 (5135909621527561588)**.** For all positive integers n, k, let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.) Find all positive integers n such that for every positive integer k, the number $f(n, 2k)$ is odd.

Problem 1.280 (451078820354844)**.** Let $ABCD$ be a quadrilateral inscribed in a circle with center O and E be the intersection of segments AC and BD. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of BCE. The tangent to ω_1 at A and the tangent to ω_2 at C meet at P. The tangent to ω_1 at D and the tangent to ω_2 at B meet at Q. Show that $OP = OQ$.

Problem 1.281 (836909183133087)**.** Given a triangle $\triangle ABC$ with circumcircle Ω. Denote its incenter and A-excenter by I, J, respectively. Let T be the reflection of J w.r.t BC and P is the intersection of BC and AT. If the circumcircle of $\triangle AIP$ intersects BC at $X \neq P$ and there is a point $Y \neq A$ on Ω such that $IA = IY$. Show that $\odot(IXY)$ tangents to the line AI.

Problem 1.282 (1700188229005727470)**.** Let ABC be a scalene triangle with circumcircle Γ. Let M be the midpoint of BC. A variable point P is selected in the line segment AM. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.

Problem 1.283 (822921222405372)**.** Let $n \geq 3$ be a fixed integer. There are $m \geq n+1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n+1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

Problem 1.284 (227919487650283)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Ω. Let M be the midpoint of side BC. Point D is chosen from the minor arc BC on Γ such that $\angle BAD = \angle MAC$. Let E be a point on Γ such that DE is perpendicular to AM, and F be a point on line BC such that DF is perpendicular to BC. Lines HF and AM intersect at point N, and point R is the reflection point of H with respect to N.

Prove that $\angle AER + \angle DFR = 180°$.

Problem 1.285 (2583236079961296677)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$, such that $f(a)f(a+b) - ab$ is a perfect square for all $a, b \in \mathbb{N}$.

Problem 1.286 (967014444176640)**.** Let $m, n \geqslant 2$ be integers, let X be a set with n elements, and let $X_1, X_2, \ldots, X_m$ be pairwise distinct non-empty, not necessary disjoint subset of X. A function $f \colon X \to \{1, 2, \ldots, n+1\}$ is called nice if there exists an index k such that

$$\sum_{x \in X_k} f(x) > \sum_{x \in X_i} f(x) \quad \text{for all } i \neq k.$$

Prove that the number of nice functions is at least n^n.

Problem 1.287 (2918584823978789760). A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Problem 1.288 (4892352754475215646). We say that a set S of integers is rootiful if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 1.289 (8700998965901287095). Let ABC be an acute triangle with circumcircle ω. Let P be a variable point on the arc BC of ω not containing A. Squares $BPDE$ and $PCFG$ are constructed such that A, D, E lie on the same side of line BP and A, F, G lie on the same side of line CP. Let H be the intersection of lines DE and FG. Show that as P varies, H lies on a fixed circle.

Problem 1.290 (448881061747528). A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \cdots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

Problem 1.291 (951015231425815). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x^2 + y^2) = (x + f(y))^2 + f(x - f(y))^2$ for all $x, y \in \mathbb{R}$.

Problem 1.292 (4298196647118074747). Find all integers $n \geq 3$ such that the following property holds: if we list the divisors of $n!$ in increasing order as $1 = d_1 < d_2 < \cdots < d_k = n!$, then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq \cdots \leq d_k - d_{k-1}.$$

Problem 1.293 (42799615327279). We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .

Problem 1.294 (7268978143074030034). Given two circles ω_1 and ω_2 where ω_2 is inside ω_1. Show that there exists a point P such that for any line ℓ not passing through P, if ℓ intersects circle ω_1 at A, B and ℓ intersects circle ω_2 at C, D, where A, C, D, B lie on ℓ in this order, then $\angle APC = \angle BPD$.

Problem 1.295 (3004928220875310213). For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

Problem 1.296 (1121095467606378762). Let $\Gamma, \Gamma_1, \Gamma_2$ be mutually tangent circles. The three circles are also tangent to a line l. Let Γ, Γ_1 be tangent to each other at B_1, Γ, Γ_2 be tangent to each other at B_2, Γ_1, Γ_2 be tangent to each other at C. $\Gamma, \Gamma_1, \Gamma_2$ are tangent to l at A, A_1, A_2 respectively, where A is between A_1, A_2. Let $D_1 = A_1C \cap A_2B_2, D_2 = A_2C \cap A_1B_1$. Prove that D_1D_2 is parallel to l.

Problem 1.297 (3486221094563725571). Given an acute non-isosceles triangle ABC with circumcircle Γ. M is the midpoint of segment BC and N is the midpoint of arc BC of Γ (the one that doesn't contain A). X and Y are points on Γ such that $BX \parallel CY \parallel AM$. Assume there exists point Z on segment BC such that circumcircle of triangle XYZ is tangent to BC. Let ω be the circumcircle of triangle ZMN. Line AM meets ω for the second time at P. Let K be a point on ω such that $KN \parallel AM$, ω_b be a circle that passes through B, X and tangents to BC and ω_c be a circle that passes through C, Y and tangents to BC. Prove that circle with center K and radius KP is tangent to 3 circles ω_b, ω_c and Γ.

Problem 1.298 (677860185151955). The checker moves from the lower left corner of the board 100×100 to the right top corner, moving at each step one cell to the right or one cell up. Let a be the number of paths in which exactly 70 steps the checker take under the diagonal going from the lower left corner to the upper right corner, and b is the number of paths in which such steps are exactly 110. What is more: a or b?

Problem 1.299 (221552874820768). The incircle of a scalene triangle ABC touches the sides BC, CA, and AB at points D, E, and F, respectively. Triangles APE and AQF are constructed outside the triangle so that

$$AP = PE, AQ = QF, \angle APE = \angle ACB, \text{ and } \angle AQF = \angle ABC.$$

Let M be the midpoint of BC. Find $\angle QMP$ in terms of the angles of the triangle ABC.

Problem 1.300 (6190379360381554657). Let $ABCD$ be a parallelogram. Point E lies on segment CD such that

$$2\angle AEB = \angle ADB + \angle ACB,$$

and point F lies on segment BC such that

$$2\angle DFA = \angle DCA + \angle DBA.$$

Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

Problem 1.301 (674938537981329). Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Problem 1.302 (5968448186928885521). Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2.$$

Problem 1.303 (712950951787328). Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Problem 1.304 (2594275832195659804). Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.

[asy]size(10cm); real w = 2*Sin(18); real h = 0.10 * w; real d = 0.33 * h; picture wht; picture blk;

draw(wht, (0,0)–(w,0)–(w+d,h)–(-d,h)–cycle); fill(blk, (0,0)–(w,0)–(w+d,h)–(-d,h)–cycle, black);

// draw(unitcircle, blue+dotted);

// Original polygon add(shift(dir(108))*blk); add(shift(dir(72))*rotate(324)*blk); add(shift(dir(36))*rota add(shift(dir(0))*rotate(252)*blk); add(shift(dir(324))*rotate(216)*wht);

add(shift(dir(288))*rotate(180)*blk); add(shift(dir(252))*rotate(144)*blk); add(shift(dir(216))*rotate(10 add(shift(dir(180))*rotate(72)*blk); add(shift(dir(144))*rotate(36)*wht);

// White shifted real Wk = 1.2; pair W1 = (1.8,0.1); pair W2 = W1 + w*dir(36); pair W3 = W2 + w*dir(108); pair W4 = W3 + w*dir(216); path Wgon = W1–W2–W3–W4–cycle; draw(Wgon); pair WO = (W1+W3)/2; transform Wt = shift(WO)*scale(Wk)*shift(-WO); draw(Wt * Wgon); label("W", WO); /* draw(W1–Wt*W1); draw(W2–Wt*W2); draw(W3–Wt*W3); draw(W4–Wt*W4); */

// Black shifted real Bk = 1.10; pair B1 = (1.5,-0.1); pair B2 = B1 + w*dir(0); pair B3 = B2 + w*dir(324); pair B4 = B3 + w*dir(252); pair B5 = B4 + w*dir(180); pair B6 = B5 + w*dir(144); path Bgon = B1–B2–B3–B4–B5–B6–cycle; pair BO = (B1+B4)/2; transform Bt = shift(BO)*scale(Bk)*shift(-BO); fill(Bt * Bgon, black); fill(Bgon, white); label("B", BO);[/asy]

Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Problem 1.305 (9035270739275883393)**.** Let k and N be integers such that $k > 1$ and $N > 2k + 1$. A number of N persons sit around the Round Table, equally spaced. Each person is either a knight (always telling the truth) or a liar (who always lies). Each person sees the nearest k persons clockwise, and the nearest k persons anticlockwise. Each person says: "I see equally many knights to my left and to my right." Establish, in terms of k and N, whether the persons around the Table are necessarily all knights.

Problem 1.306 (945532205287762)**.** Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Problem 1.307 (15982883825901733390)**.** Let $\mathbb{N}$ denote the set of positive integers. A function $f\colon \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers

$$f(m+1), f(m+2), \ldots, f(m+f(n))$$

is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

Problem 1.308 (2662630172971476475)**.** Consider fractions $\frac{a}{b}$ where a and b are positive integers. (a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}+1$. (b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$ and $b \le \sqrt{n}$.

Problem 1.309 (7456007547971566183)**.** Circles ω_1 and ω_2 have centres O_1 and O_2, respectively. These two circles intersect at points X and Y. AB is common tangent line of these two circles such that A lies on ω_1 and B lies on ω_2. Let tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 for the second time at M and line AK intersects ω_1 for the second time at N. Prove that lines AM, BN and O_1O_2 concur.

Problem 1.310 (762174477377522)**.** Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

Problem 1.311 (813804034055493)**.** In a circle there are 2019 plates, on each lies one cake. Petya and Vasya are playing a game. In one move, Petya points at a cake and calls number from 1 to 16, and Vasya moves the specified cake to the specified number of check clockwise or counterclockwise (Vasya chooses the direction each time). Petya wants at least some k pastries to accumulate on one of the plates and Vasya wants to stop him. What is the largest k Petya can succeed?

Problem 1.312 (315251261850257)**.** Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

Problem 1.313 (528087142744727)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Problem 1.314 (797215984506934)**.** Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Problem 1.315 (960400012939961)**.** For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n, such that all the differences

$$t(n+a) - t(n); t(n+a+1) - t(n+1), \ldots, t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

Problem 1.316 (857386332886077)**.** Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Problem 1.317 (514046395982396)**.** A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Problem 1.318 (915997916422887)**.** Let ABC and $A'B'C'$ be two triangles so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ form a triangle as well. Suppose that for any point X on the circumcircle of ABC, there exists exactly one point X' on the circumcircle of $A'B'C'$ so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ and $\overline{XX'}$ are concyclic. Show that ABC is similar to $A'B'C'$.

Problem 1.319 (629259075127282)**.** Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

Problem 1.320 (678030172296176)**.** Determine the smallest value of M for which for any choice of positive integer n and positive real numbers $x_1 < x_2 < \ldots < x_n \leq 2023$ the inequality

$$\sum_{1 \leq i < j \leq n, x_j - x_i \geq 1} 2^{i-j} \leq M$$

holds.

Problem 1.321 (4415914581303660291)**.** 24 students attend a mathematical circle. For any team consisting of 6 students, the teacher considers it to be either GOOD or OK. For the tournament of mathematical battles, the teacher wants to partition all the students into 4 teams of 6 students each. May it happen that every such partition contains either 3 GOOD teams or exactly one GOOD team and both options are present?

Problem 1.322 (218743543617334)**.** Let $n \geq 3$ be a positive integer. Find the maximum number of diagonals in a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

Problem 1.323 (7503515175847762748)**.** Let n be a positive integer, and let A be a subset of $\{1, \cdots, n\}$. An A-partition of n into k parts is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \cdots, a_k$ belong to A and are not necessarily distinct. The number of different parts in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \cdots, a_k\}$. We say that an A-partition of n into k parts is optimal if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

Problem 1.324 (406431313842688)**.** Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way. (A domino is a tile of size 2×1 or 1×2. Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be adjacent if they are different and share a common side.)

Problem 1.325 (634298954927697)**.** Let $ABCD$ be a trapezoid with $AB \parallel CD$ and inscribed in a circumference Γ. Let P and Q be two points on segment AB (A, P, Q, B appear in that order and are distinct) such that $AP = QB$. Let E and F be the second intersection points of lines CP and CQ with Γ, respectively. Lines AB and EF intersect at G. Prove that line DG is tangent to Γ.

Problem 1.326 (4582918044793570936)**.** Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Problem 1.327 (5441518070935718077)**.** Let ABC be an acute-angled triangle. The line through C perpendicular to AC meets the external angle bisector of $\angle ABC$ at D. Let H be the foot of the perpendicular from D onto BC. The point K is chosen on AB so that $KH \parallel AC$. Let M be the midpoint of AK. Prove that $MC = MB + BH$.

Problem 1.328 (9230571119761900018)**.** In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Problem 1.329 (591652153716935)**.** Let M be the midpoint of BC of triangle ABC. The circle with diameter BC, ω, meets AB, AC at D, E respectively. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PAC, \angle PCA = \angle PAB$, and $2PM \cdot DE = BC^2$. Point X lies outside ω such that $XM \parallel AP$, and $\frac{XB}{XC} = \frac{AB}{AC}$. Prove that $\angle BXC + \angle BAC = 90^\circ$.

Problem 1.330 (4835329555526569551)**.** Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7 \\
&\vdots & &\vdots
\end{aligned}$$

$$a_{2n-1} = \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, \qquad a_{2n} = a_{2n-1} + a_1$$

Problem 1.331 (161342796381450)**.** For each integer $n \geq 1$, compute the smallest possible value of
$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$
over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

Problem 1.332 (1965233157265405983)**.** Given a triangle $\triangle ABC$. Denote its incircle and circumcircle by ω, Ω, respectively. Assume that ω tangents the sides AB, AC at F, E, respectively. Then, let the intersections of line EF and Ω to be P, Q. Let M to be the mid-point of BC. Take a point R on the circumcircle of $\triangle MPQ$, say Γ, such that $MR \perp EF$. Prove that the line AR, ω and Γ intersect at one point.

Problem 1.333 (521969466382456)**.** Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

Problem 1.334 (329519083206921)**.** Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that
$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Problem 1.335 (942225649898797)**.** Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$
Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Problem 1.336 (8024569764169071557)**.** 12 schoolchildren are engaged in a circle of patriotic songs, each of them knows a few songs (maybe none). We will say that a group of schoolchildren can sing a song if at least one member of the group knows it. Supervisor the circle noticed that any group of 10 circle members can sing exactly 20 songs, and any group of 8 circle members - exactly 16 songs. Prove that the group of all 12 circle members can sing exactly 24 songs.

Problem 1.337 (5562895031008938211)**.** A lattice point in the Cartesian plane is a point whose coordinates are both integers. A lattice polygon is a polygon all of whose vertices are lattice points.

Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Ω such that the vertices of Γ all lie on the boundary of Ω, and exactly one vertex of Ω is not a vertex of Γ.

Problem 1.338 (785479468600231)**.** Let n and k be positive integers and G be a complete graph on n vertices. Each edge of G is colored one of k colors such that every triangle consists of either three edges of the same color or three edges of three different colors. Furthermore, there exist two different-colored edges. Prove that $n \leq (k-1)^2$.

Problem 1.339 (716406996122549)**.** Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xf(x-y)) + yf(x) = x + y + f(x^2),$$
for all real numbers x and y.

Problem 1.340 (607556370102952)**.** Let Ω be the circumcircle of an acute triangle ABC. Points D, E, F are the midpoints of the inferior arcs BC, CA, AB, respectively, on Ω. Let G be the antipode of D in Ω. Let X be the intersection of lines GE and AB, while Y the intersection of lines FG and CA. Let the circumcenters of triangles BEX and CFY be points S and T, respectively. Prove that D, S, T are collinear.

Problem 1.341 (3192129869376364982)**.** Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

Problem 1.342 (6566978587694479725)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that the following conditions are true for every pair of positive integers (x, y): (i): x and $f(x)$ have the same number of positive divisors. (ii): If $x \nmid y$ and $y \nmid x$, then:

$$\gcd(f(x), f(y)) > f(\gcd(x, y))$$

Problem 1.343 (8005762280394288133)**.** A school has 450 students. Each student has at least 100 friends among the others and among any 200 students, there are always two that are friends. Prove that 302 students can be sent on a kayak trip such that each of the 151 two seater kayaks contain people who are friends.

Problem 1.344 (8048961544243923335)**.** Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

Problem 1.345 (958427699872884)**.** Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 1.346 (6558910862034852540)**.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2$$

Problem 1.347 (5757441138678056478)**.** Does there exist an infinite sequence of integers a_0, a_1, a_2, $\ldots$ such that $a_0 \neq 0$ and, for any integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

has n distinct real roots?

Problem 1.348 (6980917169184912998)**.** Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Problem 1.349 (365155864249414)**.** Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Problem 1.350 (3626448942281457521)**.** Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Problem 1.351 (2052086795458)**.** In triangle ABC ($\angle A \neq 90°$), let O, H be the circumcenter and the foot of the altitude from A respectively. Suppose M, N are the midpoints of BC, AH respectively. Let D be the intersection of AO and BC and let H' be the reflection of H about M. Suppose that the circumcircle of $OH'D$ intersects the circumcircle of BOC at E. Prove that NO and AE are concurrent on the circumcircle of BOC.

Problem 1.352 (5664985199661230516)**.** In every row of a grid $100 \times n$ is written a permutation of the numbers $1, 2 \ldots, 100$. In one move you can choose a row and swap two non-adjacent numbers with difference 1. Find the largest possible n, such that at any moment, no matter the operations made, no two rows may have the same permutations.

Problem 1.353 (143601603071770)**.** Let ABC be an acute angled triangle and let P, Q be points on AB, AC respectively, such that PQ is parallel to BC. Points X, Y are given on line segments BQ, CP respectively, such that $\angle AXP = \angle XCB$ and $\angle AYQ = \angle YBC$. Prove that $AX = AY$.

Problem 1.354 (905557261061260)**.** Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Problem 1.355 (3569981165307602347)**.** Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all positive real numbers x and y,

$$f(xy + 1) = f(x) f\left(\frac{1}{x} + f\left(\frac{1}{y}\right)\right).$$

Problem 1.356 (6513083395063379 42)**.** Given a convex pentagon $ABCDE$. Let A_1 be the intersection of BD with CE and define B_1, C_1, D_1, E_1 similarly, A_2 be the second intersection of $\odot(ABD_1), \odot(AEC_1)$ and define B_2, C_2, D_2, E_2 similarly. Prove that $AA_2, BB_2, CC_2, DD_2, EE_2$ are concurrent.

Problem 1.357 (966139221944695)**.** Stierlitz wants to send an encryption to the Center, which is a code containing 100 characters, each a "dot" or a "dash". The instruction he received from the Center the day before about conspiracy reads:

i) when transmitting encryption over the radio, exactly 49 characters should be replaced with their opposites;

ii) the location of the "wrong" characters is decided by the transmitting side and the Center is not informed of it.

Prove that Stierlitz can send 10 encryptions, each time choosing some 49 characters to flip, such that when the Center receives these 10 ciphers, it may unambiguously restore the original code.

Problem 1.358 (592243963244567)**.** Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1}+y$ and $x+y^{p-1}$ are both powers of p.

Problem 1.359 (181463134716189)**.** In kindergarten, nurse took $n > 1$ identical cardboard rectangles and distributed them to n children; every child got one rectangle. Every child cut his (her) rectangle into several identical squares (squares of different children could be different). Finally, the total number of squares was prime. Prove that initial rectangles was squares.

Problem 1.360 (9200700111246490890)**.** Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Problem 1.361 (340033255492200)**.** Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

Problem 1.362 (1613309914397651478)**.** Let $ABCD$ be a convex quadrilateral with $\angle B < \angle A < 90^o$. Let I be the midpoint of AB and S the intersection of AD and BC. Let R be a variable point inside the triangle SAB such that $\angle ASR = \angle BSR$. On the straight lines AR, BR , take the points E, F, respectively so that BE, AF are parallel to RS. Suppose that EF intersects the circumcircle of triangle SAB at points H, K. On the segment AB, take points M, N such that $\angle AHM = \angle BHI$, $\angle BKN = \angle AKI$.

a) Prove that the center J of the circumcircle of triangle SMN lies on a fixed line.

b) On BE, AF , take the points P, Q respectively so that CP is parallel to SE and DQ is parallel to SF. The lines SE, SF intersect the circle (SAB), respectively, at U, V. Let G be the intersection of AU and BV. Prove that the median of vertex G of the triangle GPQ always passes through a fixed point .

Problem 1.363 (57065759079551)**.** Let p be an odd prime number. Suppose P and Q are polynomials with integer coefficients such that $P(0) = Q(0) = 1$, there is no nonconstant polynomial dividing both P and Q, and

$$1 + \cfrac{x}{1 + \cfrac{2x}{\ddots \, 1 + \cfrac{}{1 + (p-1)x}}} = \frac{P(x)}{Q(x)}.$$

Show that all coefficients of P except for the constant coefficient are divisible by p, and all coefficients of Q are *not* divisible by p.

Problem 1.364 (888579900722065)**.** Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Problem 1.365 (7017112574129036660)**.** Let ABC be a triangle with $AB < AC$, and let I_a be its A-excenter. Let D be the projection of I_a to BC. Let X be the intersection of AI_a and BC, and let Y, Z be the points on AC, AB, respectively, such that X, Y, Z are on a line perpendicular to AI_a. Let the circumcircle of AYZ intersect AI_a again at U. Suppose that the tangent of the circumcircle of ABC at A intersects BC at T, and the segment TU intersects the circumcircle of ABC at V. Show that $\angle BAV = \angle DAC$.

Problem 1.366 (4056351287962212080)**.** Let a, b, c, d be positive integers such that $ad \neq bc$ and $gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 1.367 (8265057113266691052)**.** Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$

Problem 1.368 (653910026918142375)**.** Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let the internal angle bisectors at A and B meet at X, the internal angle bisectors at B and C meet at Y, the internal angle bisectors at C and D meet at Z, and the internal angle bisectors at D and A meet at W. Further, let AC and BD meet at P. Suppose that the points X, Y, Z, W, O, and P are distinct. Prove that O, X, Y, Z, W lie on the same circle if and only if P, X, Y, Z, and W lie on the same circle.

Problem 1.369 (7003931234708262274)**.** There are $n \geq 3$ positive real numbers $a_1, a_2, \ldots, a_n$. For each $1 \leq i \leq n$ we let $b_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ (here we define a_0 to be a_n and a_{n+1} to be a_1). Assume that for all i and j in the range 1 to n, we have $a_i \leq a_j$ if and only if $b_i \leq b_j$. Prove that $a_1 = a_2 = \cdots = a_n$.

Problem 1.370 (3214097809181137769)**.** A set X of positive integers is said to be iberic if X is a subset of $\{2, 3, \ldots, 2018\}$, and whenever m, n are both in X, $\gcd(m, n)$ is also in X. An iberic set is said to be olympic if it is not properly contained in any other iberic set. Find all olympic iberic sets that contain the number 33.

Problem 1.371 (120014342762916)**.** Let I be the incenter of $\triangle ABC$ and BX, CY are its two angle bisectors. M is the midpoint of arc $\overset{\frown}{BAC}$. It is known that $MXIY$ are concyclic. Prove that the area of quadrilateral $MBIC$ is equal to that of pentagon $BXIYC$.

Problem 1.372 (8892145789808454835)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is essentially increasing if $f(s) \leq f(t)$ holds whenever $s \leq t$ are real numbers such that $f(s) \neq 0$ and $f(t) \neq 0$.

Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \ldots, x_{2022}$, there exist k essentially increasing functions $f_1, \ldots, f_k$ such that

$$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \ldots 2022.$$

Problem 1.373 (613633329435671)**.** Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n - a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bob's win?

Problem 1.374 (776638240838060)**.** We call a positive integer n peculiar if, for any positive divisor d of n the integer $d(d+1)$ divides $n(n+1)$. Prove that for any four different peculiar positive integers A, B, C and D the following holds:

$$\gcd(A, B, C, D) = 1.$$

Problem 1.375 (497699112554737)**.** Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Problem 1.376 (447976536517137)**.** A collection of n squares on the plane is called tri-connected if the following criteria are satisfied:

(i) All the squares are congruent. (ii) If two squares have a point P in common, then P is a vertex of each of the squares. (iii) Each square touches exactly three other squares.

How many positive integers n are there with $2018 \leq n \leq 3018$, such that there exists a collection of n squares that is tri-connected?

Problem 1.377 (122001240071629)**.** Vasya has 100 cards of 3 colors, and there are not more than 50 cards of same color. Prove that he can create 10×10 square, such that every cards of same color have not common side.

Problem 1.378 (6412565047152896593)**.** We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL. Show that line KL passes through the orthocenter of triangle ABC.

Problem 1.379 (57940527352528)**.** Determine all polynomials P with real coefficients satisfying the following condition: whenever x and y are real numbers such that $P(x)$ and $P(y)$ are both rational, so is $P(x+y)$.

Problem 1.380 (5261846980754565299)**.** Let A, B, C be the midpoints of the three sides $B'C', C'A', A'B'$ of the triangle $A'B'C'$ respectively. Let P be a point inside ΔABC, and AP, BP, CP intersect with BC, CA, AB at P_a, P_b, P_c, respectively. Lines P_aP_b, P_aP_c intersect with $B'C'$ at R_b, R_c respectively, lines P_bP_c, P_bP_a intersect with $C'A'$ at S_c, S_a respectively. and lines P_cP_a, P_cP_b intersect with $A'B'$ at T_a, T_b, respectively. Given that S_c, S_a, T_a, T_b are all on a circle centered at O.

Show that $OR_b = OR_c$.

Problem 1.381 (16134758174084)**.** Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z)-1$ have absolute value 1.

Problem 1.382 (741862231001118)**.** Find all positive integers n such that the following statement holds: Suppose real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfy $|a_k|+|b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, each of which is either -1 or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leq 1.$$

Problem 1.383 (848370325196914)**.** Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$

Problem 1.384 (655207782865052)**.** $n \geq 2$ is a given positive integer. $i \leq a_i \leq n$ satisfies for all $1 \leq i \leq n$, and S_i is defined as $a_1 + a_2 + ... + a_i (S_0 = 0)$. Show that there exists such $1 \leq k \leq n$ that satisfies $a_k^2 + S_{n-k} < 2S_n - \frac{n(n+1)}{2}$.

Problem 1.385 (7208752288636072458)**.** Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i+1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.

Problem 1.386 (3159161448000677570)**.** Let $a > 1$ be a positive integer and $d > 1$ be a positive integer coprime to a. Let $x_1 = 1$, and for $k \geq 1$, define

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \text{ does not divide } x_k \\ x_k / a & \text{if } a \text{ divides } x_k \end{cases}$$

Find, in terms of a and d, the greatest positive integer n for which there exists an index k such that x_k is divisible by a^n.

Problem 1.387 (6250022811186392279)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Problem 1.388 (989812634983805)**.** Let $n > 2$ be a positive integer. Masha writes down n natural numbers along a circle. Next, Taya performs the following operation: Between any two adjacent numbers a and b, she writes a divisor of the number $a + b$ greater than 1, then Taya erases the original numbers and obtains a new set of n numbers along the circle. Can Taya always perform these operations in such a way that after some number of operations, all the numbers are equal?

Problem 1.389 (205642765475865)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all surjective functions $g : \mathbb{Z} \to \mathbb{Z}$, $f + g$ is also surjective. (A function g is surjective over $\mathbb{Z}$ if for all integers y, there exists an integer x such that $g(x) = y$.)

Problem 1.390 (633974672407561)**.** Let $(a_n)_{n \geq 1}$ be a sequence of positive real numbers with the property that

$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$

for all positive integers n. Show that $a_{2022} \leq 1$.

Problem 1.391 (307861271235140)**.** Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

Problem 1.392 (3600625270766782129)**.** A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that no three points in P lie on a line and no two points in P lie on a line through the origin. A triangle with vertices in P is fat if O is strictly inside the triangle. Find the maximum number of fat triangles.

Problem 1.393 (5101270312905584526)**.** The exam has 25 topics, each of which has 8 questions. On a test, there are 4 questions of different topics. Is it possible to make 50 tests so that each question was asked exactly once, and for any two topics there is a test where are questions of both topics?

Problem 1.394 (869040684570675)**.** Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Problem 1.395 (37921131297270)**.** You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leq r \leq 2n - 2$ you can choose a subset of the blocks whose total weight is at least r but at most $r + 2$.

Problem 1.396 (3707562559770315754)**.** A sequence $P_1, \ldots, P_n$ of points in the plane (not necessarily diferent) is carioca if there exists a permutation $a_1, \ldots, a_n$ of the numbers $1, \ldots, n$ for which the segments

$$P_{a_1}P_{a_2}, P_{a_2}P_{a_3}, \ldots, P_{a_n}P_{a_1}$$

are all of the same length.

Determine the greatest number k such that for any sequence of k points in the plane, $2023 - k$ points can be added so that the sequence of 2023 points is carioca.

Problem 1.397 (453277275848272)**.** Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

Problem 1.398 (8752098831819609857)**.** For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Problem 1.399 (6568001756330762063)**.** Define the mexth of k sets as the kth smallest positive integer that none of them contain, if it exists. Does there exist a family $\mathcal{F}$ of sets of positive integers such that for any nonempty finite subset $\mathcal{G}$ of $\mathcal{F}$, the mexth of $\mathcal{G}$ exists, and for any positive integer n, there is exactly one nonempty finite subset $\mathcal{G}$ of $\mathcal{F}$ such that n is the mexth of $\mathcal{G}$.

Problem 1.400 (453148723429253)**.** We are given an acute triangle ABC with $AB \neq AC$. Let D be a point of BC such that DA is tangent to the circumcircle of ABC. Let E and F be the circumcenters of triangles ABD and ACD, respectively, and let M be the midpoints EF. Prove that the line tangent to the circumcircle of AMD through D is also tangent to the circumcircle of ABC.

Problem 1.401 (704326412238502)**.** Let ABC be a triangle with incenter I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A across the line EF lies on the line BC.

Problem 1.402 (5707875418806483255)**.** For each positive integer n, let $s(n)$ be the sum of the squares of the digits of n. For example, $s(15) = 1^2 + 5^2 = 26$. Determine all integers $n \geq 1$ such that $s(n) = n$.

Problem 1.403 (1856371892766039579)**.** Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \ldots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Problem 1.404 (3923745101517032298)**.** Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.

Problem 1.405 (2139114147569608698)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 1.406 (239934686230450)**.** Let triangle$ABC(AB < AC)$ with incenter I circumscribed in $\odot O$. Let M, N be midpoint of arc $\widehat{BAC}$ and $\widehat{BC}$, respectively. D lies on $\odot O$ so that $AD//BC$, and E is tangency point of A-excircle of $\triangle ABC$. Point F is in $\triangle ABC$ so that $FI//BC$ and $\angle BAF = \angle EAC$. Extend NF to meet $\odot O$ at G, and extend AG to meet line IF at L. Let line AF and DI meet at K. Proof that $ML \perp NK$.

Problem 1.407 (3906812380515301028)**.** Given a triangle $\triangle ABC$. Denote its incenter and orthocenter by I, H, respectively. If there is a point K with

$$AH + AK = BH + BK = CH + CK$$

Show that H, I, K are collinear.

Problem 1.408 (580405361636802)**.** Let the real numbers a, b, c, d satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$36 \leq 4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \leq 48.$$

Problem 1.409 (579228243242060)**.** Let $ABCD$ be a parallelogram. A line through C crosses the side AB at an interior point X, and the line AD at Y. The tangents of the circle AXY at X and Y, respectively, cross at T. Prove that the circumcircles of triangles ABD and TXY intersect at two points, one lying on the line AT and the other one lying on the line CT.

Problem 1.410 (35724831608408)**.** We will say that a set of real numbers $A = (a_1, ..., a_{17})$ is stronger than the set of real numbers $B = (b_1, ..., b_{17})$, and write $A > B$ if among all inequalities $a_i > b_j$ the number of true inequalities is at least 3 times greater than the number of false. Prove that there is no chain of sets $A_1, A_2, ..., A_N$ such that $A_1 > A_2 > \cdots A_N > A_1$.

Remark: For 11.4, the constant 3 is changed to 2 and $N = 3$ and 17 is changed to m and n in the definition (the number of elements don't have to be equal).

Problem 1.411 (412405546768537)**.** Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a non-negative integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every positive integer m (i.e. $m > 0$).

Problem 1.412 (819328919046836)**.** Which positive integers n make the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \frac{n^2(n-1)}{4}$$

true?

Problem 1.413 (313143209359080)**.** The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions. (An example of one possible layout of the park is shown to the left below, in which there are six junctions and nine trails.) IMAGE A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Problem 1.414 (849916170311036)**.** Find all ordered triplets (p, q, r) of positive integers such that p and q are two (not necessarily distinct) primes, r is even, and

$$p^3 + q^2 = 4r^2 + 45r + 103.$$

Problem 1.415 (723162974888793)**.** Call a number n good if it can be expressed as $2^x + y^2$ for where x and y are nonnegative integers. (a) Prove that there exist infinitely many sets of 4 consecutive good numbers. (b) Find all sets of 5 consecutive good numbers.

Problem 1.416 (7427384519403100799)**.** Let n be a positive integer. Initially, a bishop is placed in each square of the top row of a $2^n \times 2^n$ chessboard; those bishops are numbered from 1 to 2^n from left to right. A jump is a simultaneous move made by all bishops such that each bishop moves diagonally, in a straight line, some number of squares, and at the end of the jump, the bishops all stand in different squares of the same row.

Find the total number of permutations σ of the numbers $1, 2, \ldots, 2^n$ with the following property: There exists a sequence of jumps such that all bishops end up on the bottom row arranged in the order $\sigma(1), \sigma(2), \ldots, \sigma(2^n)$, from left to right.

Problem 1.417 (637496989440645)**.** Is there exist a sequence $a_0, a_1, a_2, \cdots$ consisting of non-zero integers that satisfies the following condition?

Condition: For all integers n (≥ 2020), equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

has a real root with its absolute value larger than 2.001.

Problem 1.418 (3245291910836201005)**.** Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

Problem 1.419 (5458049157791318449)**.** An infinite sequence of positive integers $a_1, a_2, \ldots$ is called *good* if (1) a_1 is a perfect square, and (2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n$$

is a perfect square. Prove that for any good sequence $a_1, a_2, \ldots$, there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.

Problem 1.420 (2566019241385820279)**.** Consider an integer $n \geq 2$ and write the numbers $1, 2, \ldots, n$ down on a board. A move consists in erasing any two numbers a and b, then writing down the numbers $a+b$ and $|a-b|$ on the board, and then removing repetitions (e.g., if the board contained the numbers $2, 5, 7, 8$, then one could choose the numbers $a = 5$ and $b = 7$, obtaining the board with numbers $2, 8, 12$). For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

Problem 1.421 (596902679696332)**.** Find all positive integers $n \geqslant 2$ for which there exist n real numbers $a_1 < \cdots < a_n$ and a real number $r > 0$ such that the $\frac{1}{2}n(n-1)$ differences $a_j - a_i$ for $1 \leqslant i < j \leqslant n$ are equal, in some order, to the numbers $r^1, r^2, \ldots, r^{\frac{1}{2}n(n-1)}$.

Problem 1.422 (4829488265746237263)**.** Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a given increasing function that takes positive values. For any pair (m, n) of positive integers, we call it disobedient if $f(mn) \neq f(m)f(n)$. For any positive integer m, we call it ultra-disobedient if for any nonnegative integer N, there are always infinitely many positive integers n satisfying that $(m, n), (m, n+1), \ldots, (m, n+N)$ are all disobedient pairs.

Show that if there exists some disobedient pair, then there exists some ultra-disobedient positive integer.

Problem 1.423 (699399831701585)**.** Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

Problem 1.424 (162858780891462)**.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any positive integers a and b, the following two conditions hold: (1) $f(ab) = f(a)f(b)$, and (2) at least two of the numbers $f(a)$, $f(b)$, and $f(a+b)$ are equal.

Problem 1.425 (192311188438770)**.** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)\,(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Problem 1.426 (5392114638976928066)**.** Show that there exists a set $\mathcal{C}$ of 2020 distinct, positive integers that satisfies simultaneously the following properties: • When one computes the greatest common divisor of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct. • When one computes the least common multiple of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct.

Problem 1.427 (939535945446129)**.** In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \le \max\{\angle BAC, \angle ABC\}$.

Problem 1.428 (240654526717277)**.** Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Problem 1.429 (352746613208735)**.** Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and $u_{k+1} = u_k + a_k u_{k-1}$, $v_{k+1} = v_k + a_{n-k} v_{k-1}$ for $k = 1, \ldots, n-1$.

Prove that $u_n = v_n$.

Problem 1.430 (518384374486289)**.** Let O be the center of the equilateral triangle ABC. Pick two points P_1 and P_2 other than B, O, C on the circle $\odot(BOC)$ so that on this circle B, P_1, P_2, O, C are placed in this order. Extensions of BP_1 and CP_1 intersects respectively with side CA and AB at points R and S. Line AP_1 and RS intersects at point Q_1. Analogously point Q_2 is defined. Let $\odot(OP_1Q_1)$ and $\odot(OP_2Q_2)$ meet again at point U other than O.

Prove that $2\,\angle Q_2UQ_1 + \angle Q_2OQ_1 = 360^\circ$.

Remark. $\odot(XYZ)$ denotes the circumcircle of triangle XYZ.

Problem 1.431 (231259391294064)**.** Every two of the n cities of Ruritania are connected by a direct flight of one from two airlines. Promonopoly Committee wants at least k flights performed by one company. To do this, he can at least every day to choose any three cities and change the ownership of the three flights connecting these cities each other (that is, to take each of these flights from a company that performs it, and pass the other). What is the largest k committee knowingly will be able to achieve its goal in no time, no matter how the flights are distributed hour?

Problem 1.432 (4514183051583887150)**.** Let $n \ge 2$ be an integer. A sequence $\alpha = (a_1, a_2, ..., a_n)$ of n integers is called Lima if $\gcd\{a_i - a_j \text{ such that } a_i > a_j \text{ and } 1 \le i, j \le n\} = 1$, that is, if the greatest common divisor of all the differences $a_i - a_j$ with $a_i > a_j$ is 1. One operation consists of choosing two elements a_k and a_ℓ from a sequence, with $k \ne \ell$, and replacing a_ℓ by $a'_\ell = 2a_k - a_\ell$. Show that, given a collection of $2^n - 1$ Lima sequences, each one formed by n integers, there are two of them, say β and γ, such that it is possible to transform β into γ through a finite number of operations.

Notes. The sequences $(1, 2, 2, 7)$ and $(2, 7, 2, 1)$ have the same elements but are different. If all the elements of a sequence are equal, then that sequence is not Lima.

Problem 1.433 (5949258338135822858)**.** In 10×10 square we choose n cells. In every chosen cell we draw one arrow from the angle to opposite angle. It is known, that for any two arrows, or the end of one of them coincides with the beginning of the other, or the distance between their ends is at least 2. What is the maximum possible value of n?

Problem 1.434 (643666520789113)**.** Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Problem 1.435 (799773800583372)**.** A square grid 100×100 is tiled in two ways - only with dominoes and only with squares 2×2. What is the least number of dominoes that are entirely inside some square 2×2?

Problem 1.436 (1696528644272897376)**.** Prove that for all sufficiently large positive integers d, at least 99% of the polynomials of the form

$$\sum_{i \leqslant d} \sum_{j \leqslant d} \pm x^i y^j$$

are irreducible over the integers.

Problem 1.437 (467342110469005)**.** Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

Problem 1.438 (702587891849077)**.** Given an integer $n \geqslant 2$. Suppose there is a point P inside a convex cyclic $2n$-gon $A_1 \ldots A_{2n}$ satisfying

$$\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_{2n}A_1,$$

prove that

$$\prod_{i=1}^{n} |A_{2i-1}A_{2i}| = \prod_{i=1}^{n} |A_{2i}A_{2i+1}|,$$

where $A_{2n+1} = A_1$.

Problem 1.439 (697545974967766)**.** In triangle ABC points M and N are the midpoints of sides AC and AB, respectively and D is the projection of A into BC. Point O is the circumcenter of ABC and circumcircles of BOC, DMN intersect at points R, T. Lines DT, DR intersect line MN at E and F, respectively. Lines CT, BR intersect at K. A point P lies on KD such that PK is the angle bisector of $\angle BPC$. Prove that the circumcircles of ART and PEF are tangent.

Problem 1.440 (8959954456910482516)**.** Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

Problem 1.441 (1527496195334546428)**.** On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k, count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially, there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A?

Problem 1.442 (4679791554410865501)**.** For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

Problem 1.443 (4375421764909014892)**.** Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

Problem 1.444 (3102273497351946473)**.** Let $f : \mathbb{N} \to \mathbb{N}$. Show that $f(m)+n \mid f(n)+m$ for all positive integers $m \leq n$ if and only if $f(m) + n \mid f(n) + m$ for all positive integers $m \geq n$.

Problem 1.445 (102296866595865)**.** Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove that there exists a point E, lying outside quadrilateral $ABCD$, such that ray EH bisects both angles $\angle BES$, $\angle TED$, and $\angle BEN = \angle MED$.

Problem 1.446 (5299971832672937326)**.** Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $\triangle ANK, \triangle BKL, \triangle CLM, \triangle DMN$.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

Problem 1.447 (7220404010846068686)**.** Let ABC be a acute, non-isosceles triangle. D, E, F are the midpoints of sides AB, BC, AC, resp. Denote by (O), (O') the circumcircle and Euler circle of ABC. An arbitrary point P lies inside triangle DEF and DP, EP, FP intersect (O') at D', E', F', resp. Point A' is the point such that D' is the midpoint of AA'. Points B', C' are defined similarly. a. Prove that if $PO = PO'$ then $O \in (A'B'C')$; b. Point A' is mirrored by OD, its image is X. Y, Z are created in the same manner. H is the orthocenter of ABC and XH, YH, ZH intersect BC, AC, AB at M, N, L resp. Prove that M, N, L are collinear.

Problem 1.448 (428632191392819)**.** Initially, 10 ones are written on a blackboard. Grisha and Gleb are playing game, by taking turns; Grisha goes first. On one move Grisha squares some 5 numbers on the board. On his move, Gleb picks a few (perhaps none) numbers on the board and increases each of them by 1. If in $10,000$ moves on the board a number divisible by 2023 appears, Gleb wins, otherwise Grisha wins. Which of the players has a winning strategy?

Problem 1.449 (8690567757444826166)**.** A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Problem 1.450 (8569243655022492300)**.** Given a $\triangle ABC$ and a point P. Let O, D, E, F be the circumcenter of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB$, respectively and let T be the intersection of BC with EF. Prove that the reflection of O in EF lies on the perpendicular from D to PT.

Problem 1.451 (6612845742708555351)**.** Cyclic quadrilateral $ABCD$ has circumcircle (O). Points M and N are the midpoints of BC and CD, and E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Let EN meet FM at P. Denote S as the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

Problem 1.452 (402139377468684)**.** For a positive integer k, let $s(k)$ denote the number of 1s in the binary representation of k. Prove that for any positive integer n,

$$\sum_{i=1}^{n}(-1)^{s(3i)} > 0.$$

Problem 1.453 (6497483389877629432)**.** Initially, a word of 250 letters with 125 letters A and 125 letters B is written on a blackboard. In each operation, we may choose a contiguous string of any length with equal number of letters A and equal number of letters B, reverse those letters and then swap each B with A and each A with B (Example: $ABABBA$ after the operation becomes $BAABAB$). Decide if it possible to choose initial word, so that after some operations, it will become the same as the first word, but in reverse order.

Problem 1.454 (719467452801051)**.** Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

Problem 1.455 (770421031902562)**.** A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s, t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

Problem 1.456 (2212576839999739806)**.** One hundred sages play the following game. They are waiting in some fixed order in front of a room. The sages enter the room one after another. When a sage enters the room, the following happens - the guard in the room chooses two arbitrary distinct numbers from the set $1, 2, 3$, and announces them to the sage in the room. Then the sage chooses one of those numbers, tells it to the guard, and leaves the room, and the next enters, and so on. During the game, before a sage chooses a number, he can ask the guard what were the chosen numbers of the previous two sages. During the game, the sages cannot talk to each other. At the end, when everyone has finished, the game is considered as a failure if the sum of the 100 chosen numbers is exactly 200; else it is successful. Prove that the sages can create a strategy, by which they can win the game.

Problem 1.457 (8617608868051245066)**.** The columns and the row of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x+y$ is $0, 1$ or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color. Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d+2$ from its original position, and each square contains a token with the same color as the square.

Problem 1.458 (3470579368412517052)**.** A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either: the rabbit cannot move; or the hunter can determine the cell in which the rabbit started. Decide whether there exists a winning strategy for the hunter.

Problem 1.459 (8152181601565653036)**.** Let D be a point on segment PQ. Let ω be a fixed circle passing through D, and let A be a variable point on ω. Let X be the intersection of the tangent to the circumcircle of $\triangle ADP$ at P and the tangent to the circumcircle of $\triangle ADQ$ at Q. Show that as A varies, X lies on a fixed line.

Problem 1.460 (6287115858827066074)**.** Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?

The expression $\lfloor x \rfloor$ denotes the integer part (or floor) of the real number x. Thus $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \pi \rfloor = \lfloor 22/7 \rfloor = 3$, $\lfloor 42 \rfloor = 42$, and $\lfloor 0 \rfloor = 0$.

Problem 1.461 (255228327462897)**.** In the game of Ring Mafia, there are 2019 counters arranged in a circle. 673 of these counters are mafia, and the remaining 1346 counters are town. Two players, Tony and Madeline, take turns with Tony going first. Tony does not know which counters are mafia but Madeline does.

On Tony's turn, he selects any subset of the counters (possibly the empty set) and removes all counters in that set. On Madeline's turn, she selects a town counter which is adjacent to a mafia counter and removes it. Whenever counters are removed, the remaining counters are brought closer together without changing their order so that they still form a circle. The game ends when either all mafia counters have been removed, or all town counters have been removed.

Is there a strategy for Tony that guarantees, no matter where the mafia counters are placed and what Madeline does, that at least one town counter remains at the end of the game?

Problem 1.462 (165465510156789)**.** Let Ω be the circumcircle of an isosceles trapezoid $ABCD$, in which AD is parallel to BC. Let X be the reflection point of D with respect to BC. Point Q is on the arc BC of Ω that does not contain A. Let P be the intersection of DQ and BC. A point E satisfies that EQ is parallel to PX, and EQ bisects $\angle BEC$. Prove that EQ also bisects $\angle AEP$.

Problem 1.463 (888114441475156)**.** Consider infinite sequences $a_1, a_2, \dots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Problem 1.464 (664494485253935)**.** Determine all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ satisfies the equation $f(a^2 + ab + f(b^2)) = af(b) + b^2 + f(a^2)\, \forall a, b \in \mathbb{R}$

Problem 1.465 (6819074419096549446)**.** Ann and Beto play with a two pan balance scale. They have 2023 dumbbells labeled with their weights, which are the numbers $1, 2, \dots, 2023$, with none of them repeating themselves. Each player, in turn, chooses a dumbbell that was not yet placed on the balance scale and places it on the pan with the least weight at the moment. If the scale is balanced, the player places it on any pan. Ana starts the game, and they continue in this way alternately until all the dumbbells are placed. Ana wins if at the end the scale is balanced, otherwise Beto win. Determine which of the players has a winning strategy and describe the strategy.

Problem 1.466 (504512181993018)**.** For a positive integer n denote by $P_0(n)$ the product of all non-zero digits of n. Let N_0 be the set of all positive integers n such that $P_0(n)|n$. Find the largest possible value of ℓ such that N_0 contains infinitely many strings of ℓ consecutive integers.

Problem 1.467 (6276002868513182227)**.** Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 1.468 (5873161915777778529)**.** In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

Problem 1.469 (1302548092028853470)**.** Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions: The frog visits only points in $\{1, 2, \dots, 2^n - 1\}$, each at most once. The length of each hop is in $\{2^0, 2^1, 2^2, \dots\}$. (The hops may be either direction, left or right.) Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

Problem 1.470 (472882074231586)**.** Let $G = (V, E)$ be a finite simple graph on n vertices. An edge e of G is called a bottleneck if one can partition V into two disjoint sets A and B such that at most 100 edges of G have one endpoint in A and one endpoint in B; and the edge e is one such edge (meaning the edge e also has one endpoint in A and one endpoint in B). Prove that at most $100n$ edges of G are bottlenecks.

Problem 1.471 (7550072974614174968)**.** Let $n \geqslant 3$ be an integer, and let $x_1, x_2, \dots, x_n$ be real numbers in the interval $[0, 1]$. Let $s = x_1 + x_2 + \ldots + x_n$, and assume that $s \geqslant 3$. Prove that there exist integers i and j with $1 \leqslant i < j \leqslant n$ such that

$$2^{j-i} x_i x_j > 2^{s-3}.$$

Problem 1.472 (8402748184217471405)**.** In ΔABC, $AD \perp BC$ at D. E, F lie on line AB, such that $BD = BE = BF$. Let I, J be the incenter and A-excenter. Prove that there exist two points P, Q on the circumcircle of ΔABC , such that $PB = QC$, and $\Delta PEI \sim \Delta QFJ$.

Problem 1.473 (4576482737766940742)**.** Consider the isosceles right triangle ABC with $\angle BAC = 90°$. Let ℓ be the line passing through B and the midpoint of side AC. Let Γ be the circumference with diameter AB. The line ℓ and the circumference Γ meet at point P, different from B. Show that the circumference passing through A, C and P is tangent to line BC at C.

Problem 1.474 (3866807698726339637)**.** Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbors - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Problem 1.475 (3982719612496247400)**.** Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

Problem 1.476 (6206024898840097202)**.** Let ABC be a triangle with a right angle at C. Let I be the incentre of triangle ABC, and let D be the foot of the altitude from C to AB. The incircle ω of triangle ABC is tangent to sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in lines C_1A_1 and C_1B_1, respectively. Let K and L be the reflections of D in lines C_1A_1 and C_1B_1, respectively.

Prove that the circumcircles of triangles A_1EI, B_1FI, and C_1KL have a common point.

Problem 1.477 (254414643421075)**.** Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

Problem 1.478 (569685816807741)**.** Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

Problem 1.479 (371185267312965)**.** Acute-angled triangle ABC with circumcircle ω is given. Let D be the midpoint of AC, E be the foot of altitude from A to BC, and F be the intersection point of AB and DE. Point H lies on the arc BC of ω (the one that does not contain A) such that $\angle BHE = \angle ABC$. Prove that $\angle BHF = 90°$.

Problem 1.480 (6002187361907355959)**.** Consider the triangle ABC with $\angle BCA > 90°$. The circumcircle Γ of ABC has radius R. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R. The perpendicular bisector of PB intersects Γ at the points D and E.

Prove P is the incentre of triangle CDE.

Problem 1.481 (781756252908608)**.** Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i, j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

Problem 1.482 (1251781469282726042)**.** An acute triangle ABC is given and H and O be its orthocenter and circumcenter respectively. Let K be the midpoint of AH and ℓ be a line through O. Let P and Q be the projections of B and C on ℓ. Prove that

$$KP + KQ \geq BC$$

Problem 1.483 (4674406086325821196)**.** Circles ω_1 and ω_2 intersect each other at points A and B. Point C lies on the tangent line from A to ω_1 such that $\angle ABC = 90°$. Arbitrary line ℓ passes through C and cuts ω_2 at points P and Q. Lines AP and AQ cut ω_1 for the second time at points X and Z respectively. Let Y be the foot of altitude from A to ℓ. Prove that points X, Y and Z are collinear.

Problem 1.484 (602995508900984)**.** Two ants are moving along the edges of a convex polyhedron. The route of every ant ends in its starting point, so that one ant does not pass through the same point twice along its way. On every face F of the polyhedron are written the number of edges of F belonging to the route of the first ant and the number of edges of F belonging to the route of the second ant. Is there a polyhedron and a pair of routes described as above, such that only one face contains a pair of distinct numbers?

Problem 1.485 (844043529345657 44)**.** Determine the largest integer $n \geq 3$ for which the edges of the complete graph on n vertices can be assigned pairwise distinct non-negative integers such that the edges of every triangle have numbers which form an arithmetic progression.

Problem 1.486 (208441124738479)**.** Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m + n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

Problem 1.487 (1634257707699822785)**.** Let a, b, c be fixed positive integers. There are $a + b + c$ ducks sitting in a circle, one behind the other. Each duck picks either rock, paper, or scissors, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors. A move consists of an operation of one of the following three forms: If a duck picking rock sits behind a duck picking scissors, they switch places. If a duck picking paper sits behind a duck picking rock, they switch places. If a duck picking scissors sits behind a duck picking paper, they switch places. Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

Problem 1.488 (6269154814902278202)**.** Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 1.489 (8489819892706651399)**.** For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G.

Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

Problem 1.490 (684771433215596)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 1.491 (745968391440822)**.** Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Problem 1.492 (6209707374283278028)**.** Let ABC be a triangle and D be a point inside triangle ABC. Γ is the circumcircle of triangle ABC, and DB, DC meet Γ again at E, F , respectively. Γ_1, Γ_2 are the circumcircles of triangle ADE and ADF respectively. Assume X is on Γ_2 such that BX is tangent to Γ_2. Let BX meets Γ again at Z. Prove that the line CZ is tangent to Γ_1 .

Problem 1.493 (1580707630770476037)**.** Two triangles intersect to form seven finite disjoint regions, six of which are triangles with area 1. The last region is a hexagon with area A. Compute the minimum possible value of A.

Problem 1.494 (861953008482666)**.** Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leq 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leq 2|y|.$$

Determine all possible values of $P(0)$.

Problem 1.495 (827629029640194)**.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ so that the equality

$$f(x + yf(x + y)) + xf(x) = f(xf(x + y + 1)) + y^2$$

is true for any real numbers x, y.

Problem 1.496 (891406366009347)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that the equation

$$f(xf(x) + y) = f(y) + x^2$$

holds for all rational numbers x and y.

Here, $\mathbb{Q}$ denotes the set of rational numbers.

Problem 1.497 (8757490679465390171)**.** Color every vertex of 2008-gon with two colors, such that adjacent vertices have different color. If sum of angles of vertices of first color is same as sum of angles of vertices of second color, than we call 2008-gon as interesting. Convex 2009-gon one vertex is marked. It is known, that if remove any unmarked vertex, then we get interesting 2008-gon. Prove, that if we remove marked vertex, then we get interesting 2008-gon too.

Problem 1.498 (653200526211133)**.** Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1.

Define $p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ and $q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$. Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

Problem 1.499 (2749225075653830789)**.** In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \dots, Q_{24}$ whose corners are vertices of the 100-gon, so that the quadrilaterals $Q_1, \dots, Q_{24}$ are pairwise disjoint, and every quadrilateral Q_i has three corners of one color and one corner of the other color.

Problem 1.500 (3075694960611200431)**.** Adithya and Bill are playing a game on a connected graph with $n > 2$ vertices, two of which are labeled A and B, so that A and B are distinct and non-adjacent and known to both players. Adithya starts on vertex A and Bill starts on B. Each turn, both players move simultaneously: Bill moves to an adjacent vertex, while Adithya may either move to an adjacent vertex or stay at his current vertex. Adithya loses if he is on the same vertex as Bill, and wins if he reaches B alone. Adithya cannot see where Bill is, but Bill can see where Adithya is. Given that Adithya has a winning strategy, what is the maximum possible number of edges the graph may have? (Your answer may be in terms of n.)

Problem 1.501 (9103148252094553273)**.** The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a path from X to Y is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called diverse if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Problem 1.502 (825542457780626)**.** Yuri is looking at the great Mayan table. The table has 200 columns and 2^{200} rows. Yuri knows that each cell of the table depicts the sun or the moon, and any two rows are different (i.e. differ in at least one column). Each cell of the table is covered with a sheet. The wind has blown aways exactly two sheets from each row. Could it happen that now Yuri can find out for at least 10000 rows what is depicted in each of them (in each of the columns)?

Problem 1.503 (3048608408918882691)**.** Is it possible to arrange everything in all cells of an infinite checkered plane all natural numbers (once) so that for each n in each square $n \times n$ the sum of the numbers is a multiple of n?

Problem 1.504 (8639636622304457736)**.** Let $\triangle ABC$ be a triangle, and let S and T be the midpoints of the sides BC and CA, respectively. Suppose M is the midpoint of the segment ST and the circle ω through A, M and T meets the line AB again at N. The tangents of ω at M and N meet at P. Prove that P lies on BC if and only if the triangle ABC is isosceles with apex at A.

Problem 1.505 (8129091008921005997)**.** Let a, b, c, x, y, z be real numbers such that

$$a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = (a+b)^2 + (x+y)^2 = (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

Problem 1.506 (302438226120877)**.** Given triangle ABC. Let $BPCQ$ be a parallelogram (P is not on BC). Let U be the intersection of CA and BP, V be the intersection of AB and CP, X be the intersection of CA and the circumcircle of triangle ABQ distinct from A, and Y be the intersection of AB and the circumcircle of triangle ACQ distinct from A. Prove that $\overline{BU} = \overline{CV}$ if and only if the lines AQ, BX, and CY are concurrent.

Problem 1.507 (616860610609120)**.** A few (at least 5) integers are put on a circle, such that each of them is divisible by the sum of its neighbors. If the sum of all numbers is positive, what is its minimal value?

Problem 1.508 (212792614834869)**.** Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The distance between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

Problem 1.509 (5841938333292270043)**.** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Problem 1.510 (12311699525330)**.** Suppose $a_1 < a_2 < \cdots < a_{2024}$ is an arithmetic sequence of positive integers, and $b_1 < b_2 < \cdots < b_{2024}$ is a geometric sequence of positive integers. Find the maximum possible number of integers that could appear in both sequences, over all possible choices of the two sequences.

Problem 1.511 (5990443173263547430)**.** Given a fixed circle (O) and two fixed points B, C on that circle, let A be a moving point on (O) such that $\triangle ABC$ is acute and scalene. Let I be the midpoint of BC and let AD, BE, CF be the three heights of $\triangle ABC$. In two rays $\overrightarrow{FA}, \overrightarrow{EA}$, we pick respectively M, N such that $FM = CE, EN = BF$. Let L be the intersection of MN and EF, and let $G \neq L$ be the second intersection of (LEN) and (LFM).

a) Show that the circle (MNG) always goes through a fixed point.

b) Let AD intersects (O) at $K \neq A$. In the tangent line through D of (DKI), we pick P, Q such that $GP \parallel AB, GQ \parallel AC$. Let T be the center of (GPQ). Show that GT always goes through a fixed point.

Problem 1.512 (711016608896725)**.** Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(S)$ denote the number of ways to draw 8 lines with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

Problem 1.513 (6728439333021242021)**.** Let $S = \{13, 133, \cdots\}$ be the set of the positive integers of the form $133\cdots3$. Consider a horizontal row of 2022 cells. Ana and Borja play a game: they alternatively write a digit on the leftmost empty cell, starting with Ana. When the row is filled, the digits are read from left to right to obtain a 2022-digit number N. Borja wins if N is divisible by a number in S, otherwise Ana wins. Find which player has a winning strategy and describe it.

Problem 1.514 (297728211754501)**.** The board used for playing a game consists of the left and right parts. In each part there are several fields and there're several segments connecting two fields from different parts (all the fields are connected.) Initially, there is a violet counter on a field in the left part, and a purple counter on a field in the right part. Lyosha and Pasha alternatively play their turn, starting from Pasha, by moving their chip (Lyosha-violet, and Pasha-purple) over a segment to other field that has no chip. It's prohibited to repeat a position twice, i.e. can't move to position that already been occupied by some earlier turns in the game. A player losses if he can't make a move. Is there a board and an initial positions of counters that Pasha has a winning strategy?

Problem 1.515 (978369715927760)**.** Point D is selected inside acute $\triangle ABC$ so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD so that $AE = EC$. Let M be the midpoint of BC.

Show that line AB is tangent to the circumcircle of triangle BEM.

Problem 1.516 (6566259136811987209)**.** Let Ω be the A-excircle of triangle ABC, and suppose that Ω is tangent to lines BC, CA, and AB at points D, E, and F, respectively. Let M be the midpoint of segment EF. Two more points P and Q are on Ω such that EP and FQ are both parallel to DM. Let BP meet CQ at point X. Prove that the line AM is the angle bisector of $\angle XAD$.

Problem 1.517 (7997372712267182584)**.** Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Problem 1.518 (105422576188851)**.** A short-sighted rook is a rook that beats all squares in the same column and in the same row for which he can not go more than 60-steps. What is the maximal amount of short-sighted rooks that don't beat each other that can be put on a 100×100 chessboard.

Problem 1.519 (5129113369150286745)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

Problem 1.520 (890162155331408920)**.** Let $a_0 < a_1 < a_2 < \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 1.521 (5871948911817167044)**.** Determine all polynomials $P(x)$ with degree $n \geq 1$ and integer coefficients so that for every real number x the following condition is satisfied

$$P(x) = (x - P(0))(x - P(1))(x - P(2)) \cdots (x - P(n-1))$$

Problem 1.522 (2134021625648303394)**.** The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily distinct) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$. Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

Problem 1.523 (7335226310540156292)**.** Let ABC be a right triangle with $\angle B = 90°$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the seconf intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

Problem 1.524 (684265043263216)**.** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Problem 1.525 (4389998719836463980)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.526 (1837105952530316058)**.** Let $k \geq 2$ be an integer. Find the smallest integer $n \geq k + 1$ with the property that there exists a set of n distinct real numbers such that each of its elements can be written as a sum of k other distinct elements of the set.

Problem 1.527 (7351162576557167474)**.** Consider an acute-angled triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of the triangle CEF and Γ meet at X and C, with $X \neq C$. The line BX and the tangent to Γ through A meet at Y. Let P be the point on segment AB so that $YP = YA$, with $P \neq A$, and let Q be the point where AB and the parallel to BC through Y meet each other. Show that F is the midpoint of PQ.

Problem 1.528 (7710975676761169567)**.** Two different integers u and v are written on a board. We perform a sequence of steps. At each step we do one of the following two operations:

(i) If a and b are different integers on the board, then we can write $a + b$ on the board, if it is not already there. (ii) If a, b and c are three different integers on the board, and if an integer x satisfies $ax^2 + bx + c = 0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u, v) from which any integer can eventually be written on the board after a finite sequence of steps.

Problem 1.529 (3458270318471332488)**.** Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

Problem 1.530 (403529216204023)**.** A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a + 1), P(a + 2), \ldots, P(a + b)\}$$

is fragrant?

Problem 1.531 (120105730464462)**.** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 1.532 (437956241529021)**.** In a country, there are N cities and $N(N-1)$ one-way roads: one road from X to Y for each ordered pair of cities $X \neq Y$. Every road has a maintenance cost. For each $k = 1, \dots, N$ let's consider all the ways to select k cities and $N-k$ roads so that from each city it is possible to get to some selected city, using only selected roads.

We call such a system of cities and roads with the lowest total maintenance cost k-optimal. Prove that cities can be numbered from 1 to N so that for each $k = 1, \dots, N$ there is a k-optimal system of roads with the selected cities numbered $1, \dots, k$.

Problem 1.533 (1159469125385582912)**.** Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n | P(m) - P(k)$ implies $n | m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n. (a) Find a pair (a, b) which is 51-good, but not very good. (b) Show that all 2010-good pairs are very good.

Problem 1.534 (443006607452241)**.** Let $x_1, x_2, \dots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Problem 1.535 (7431104394604748426)**.** Given a positive integer N, determine all positive integers n, satisfying the following condition: for any list $d_1, d_2, \dots, d_k$ of (not necessarily distinct) divisors of n such that $\frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_k} > N$, some of the fractions $\frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_k}$ add up to exactly N.

Problem 1.536 (8851048763094130212)**.** Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines AC, DT, and BK are concurrent.

Problem 1.537 (8053761138620448460)**.** Let ABC be a scalene triangle, and points O and H be its circumcenter and orthocenter, respectively. Point P lies inside triangle AHO and satisfies $\angle AHP = \angle POA$. Let M be the midpoint of segment $\overline{OP}$. Suppose that BM and CM intersect with the circumcircle of triangle ABC again at X and Y, respectively.

Prove that line XY passes through the circumcenter of triangle APO.

Problem 1.538 (6851509563331617580)**.** There are several discs whose radii are no more that 1, and whose centers all lie on a segment with length l. Prove that the union of all the discs has a perimeter not exceeding $4l + 8$.

Problem 1.539 (9156814072173030162)**.** Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

Problem 1.540 (876239022447910)**.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

Problem 1.541 (689941395946854)**.** In convex quadrilateral $ABCD$, let diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let the circumcircles of ADE and BCE intersect $\overline{AB}$ again at $P \neq A$ and $Q \neq B$, respectively. Let the circumcircle of ACP intersect $\overline{AD}$ again at $R \neq A$, and let the circumcircle of BDQ intersect $\overline{BC}$ again at $S \neq B$. Prove that A, B, R, and S are concyclic.

Problem 1.542 (7205358409203299180)**.** Ana plays a game on a 100×100 chessboard. Initially, there is a white pawn on each square of the bottom row and a black pawn on each square of the top row, and no other pawns anywhere else. Each white pawn moves toward the top row and each black pawn moves toward the bottom row in one of the following ways: it moves to the square directly in front of it if there is no other pawn on it; it captures a pawn on one of the diagonally adjacent squares in the row immediately in front of it if there is a pawn of the opposite color on it. (We say a pawn P captures a pawn Q of the opposite color if we remove Q from the board and move P to the square that Q was previously on.)

Ana can move any pawn (not necessarily alternating between black and white) according to those rules. What is the smallest number of pawns that can remain on the board after no more moves can be made?

Problem 1.543 (6837149463099766937)**.** Let $n \geq 3$ be an odd integer. In a $2n \times 2n$ board, we colour $2(n-1)^2$ cells. What is the largest number of three-square corners that can surely be cut out of the uncoloured figure?

Problem 1.544 (119129720704350)**.** Let H be the orthocenter of a given triangle ABC. Let BH and AC meet at a point E, and CH and AB meet at F. Suppose that X is a point on the line BC. Also suppose that the circumcircle of triangle BEX and the line AB intersect again at Y, and the circumcircle of triangle CFX and the line AC intersect again at Z. Show that the circumcircle of triangle AYZ is tangent to the line AH.

Problem 1.545 (201785415121070)**.** Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

(The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 1.546 (8807076875709895728)**.** Call a positive integer emphatic if it can be written in the form $a^2 + b!$, where a and b are positive integers. Prove that there are infinitely many positive integers n such that n, $n+1$, and $n+2$ are all emphatic.

Problem 1.547 (8609709793627283757)**.** Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Problem 1.548 (545015136325290)**.** Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 1.549 (267446035349026955). Fix integers $n \geq k \geq 2$. We call a collection of integral valued coins $n - diverse$ if no value occurs in it more than n times. Given such a collection, a number S is $n - reachable$ if that collection contains n coins whose sum of values equals S. Find the least positive integer D such that for any n-diverse collection of D coins there are at least k numbers that are n-reachable.

Problem 1.550 (571373387028298). Let ABC be a triangle with $\angle BAC > 90°$, and let O be its circumcenter and ω be its circumcircle. The tangent line of ω at A intersects the tangent line of ω at B and C respectively at point P and Q. Let D, E be the feet of the altitudes from P, Q onto BC, respectively. F, G are two points on $\overline{PQ}$ different from A, so that A, F, B, E and A, G, C, D are both concyclic. Let M be the midpoint of $\overline{DE}$. Prove that DF, OM, EG are concurrent.

Problem 1.551 (158732792334122). Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

Problem 1.552 (255403454348745096). Given is an equilateral triangle ABC with circumcenter O. Let D be a point on to minor arc BC of its circumcircle such that $DB > DC$. The perpendicular bisector of OD meets the circumcircle at E, F, with E lying on the minor arc BC. The lines BE and CF meet at P. Prove that $PD \perp BC$.

Problem 1.553 (853206838493072). Let $\triangle ABC$ be an acute-angled triangle with its incenter I. Suppose that N is the midpoint of the arc BAC of the circumcircle of triangle $\triangle ABC$, and P is a point such that $ABPC$ is a parallelogram.Let Q be the reflection of A over N and R the projection of A on $\overline{QI}$. Show that the line $\overline{AI}$ is tangent to the circumcircle of triangle $\triangle PQR$

Problem 1.554 (473848321984972370З). On a circle there're 1000 marked points, each colored in one of k colors. It's known that among any 5 pairwise intersecting segments, endpoints of which are 10 distinct marked points, there're at least 3 segments, each of which has its endpoints colored in different colors. Determine the smallest possible value of k for which it's possible.

Problem 1.555 (168250003841029). Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Problem 1.556 (274933009357884). Let $n \geq 3$ be an integer. We say that an arrangement of the numbers 1, 2, $\ldots$, n^2 in a $n \times n$ table is row-valid if the numbers in each row can be permuted to form an arithmetic progression, and column-valid if the numbers in each column can be permuted to form an arithmetic progression. For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

Problem 1.557 (7553717274310387624). Let ABC be a triangle with incentre I and circumcircle ω. The incircle of the triangle ABC touches the sides BC, CA and AB at D, E and F, respectively. The circumcircle of triangle ADI crosses ω again at P, and the lines PE and PF cross ω again at $X$and Y, respectively. Prove that the lines AI, BX and CY are concurrent.

Problem 1.558 (5326267355571829268)**.** A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$ $x_1 + x_2 + \ldots + x_n \geq 0$.

Problem 1.559 (1473691226426629581)**.** A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Problem 1.560 (5835156231907738776)**.** Given triangle ABC with A-excenter I_A, the foot of the perpendicular from I_A to BC is D. Let the midpoint of segment I_AD be M, T lies on arc BC(not containing A) satisfying $\angle BAT = \angle DAC$, I_AT intersects the circumcircle of ABC at $S \neq T$. If SM and BC intersect at X, the perpendicular bisector of AD intersects AC, AB at Y, Z respectively, prove that AX, BY, CZ are concurrent.

Problem 1.561 (6783316811528119504)**.** Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Problem 1.562 (46260042068525)**.** Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we have any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

Problem 1.563 (466409818083772)**.** Find all integers n satisfying $n \geq 2$ and $\dfrac{\sigma(n)}{p(n) - 1} = n$, in which $\sigma(n)$ denotes the sum of all positive divisors of n, and $p(n)$ denotes the largest prime divisor of n.

Problem 1.564 (166169225490521)**.** The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

Problem 1.565 (120381541018683)**.** Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x \in F} 1/x \neq r$ for all finite subsets F of S.

Problem 1.566 (7796424663887996427)**.** Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Problem 1.567 (2358076615453535648)**.** Let m be a positive integer, and consider a $m \times m$ checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

Problem 1.568 (9085006630991620229)**.** Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and $60°$ rotational symmetry (that is, there is a point O such that a $60°$ rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2+3n+1$ citizens at $3n^2+3n+1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

Problem 1.569 (4381532748791402633)**.** Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Problem 1.570 (4190798556185983491)**.** Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

Problem 1.571 (5073004669687570949)**.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

Problem 1.572 (810041368501810)**.** Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively. a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A+B, B+C, C+A$ are disjoint? b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A+B, B+C, C+A$ are disjoint?

Here $X+Y$ denotes the set $\{x+y : x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and for $X, Y \subseteq \mathbb{Q}$.

Problem 1.573 (1978345856029698287)**.** Let $S_1, S_2, \ldots, S_{100}$ be finite sets of integers whose intersection is not empty. For each non-empty $T \subseteq \{S_1, S_2, \ldots, S_{100}\}$, the size of the intersection of the sets in T is a multiple of the number of sets in T. What is the least possible number of elements that are in at least 50 sets?

Problem 1.574 (4785409545704689551)**.** Let ABC be a triangle, and let ω_1, ω_2 be centered at O_1, O_2 and tangent to line BC at B, C respectively. Let line AB intersect ω_1 again at X and let line AC intersect ω_2 again at Y. If Q is the other intersection of the circumcircles of triangles ABC and AXY, then prove that lines AQ, BC, and O_1O_2 either concur or are all parallel.

Problem 1.575 (973095234047520)**.** Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Problem 1.576 (9153191064326230951)**.** Let scalene triangle ABC have altitudes AD, BE, CF and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines PE at $X \neq P$ and PF at $Y \neq P$. Prove that $XY \parallel BC$.

Problem 1.577 (4752965628566204727)**.** Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

Problem 1.578 (397912644922719)**.** Find all real numbers $x_1, \ldots, x_{2016}$ that satisfy the following equation for each $1 \leq i \leq 2016$. (Here $x_{2017} = x_1$.)

$$x_i^2 + x_i - 1 = x_{i+1}$$

Problem 1.579 (919147551255493)**.** Let $m, n \geq 2$ be distinct positive integers. In an infinite grid of unit squares, each square is filled with exactly one real number so that In each $m \times m$ square, the sum of the numbers in the m^2 cells is equal. In each $n \times n$ square, the sum of the numbers in the n^2 cells is equal. There exist two cells in the grid that do not contain the same number. Let S be the set of numbers that appear in at least one square on the grid. Find, in terms of m and n, the least possible value of $|S|$.

Problem 1.580 (620564216459483)**.** Let ABC be an acute scalene triangle such that $AB < AC$. The midpoints of sides AB and AC are M and N, respectively. Let P and Q be points on the line MN such that $\angle CBP = \angle ACB$ and $\angle QCB = \angle CBA$. The circumscribed circle of triangle ABP intersects line AC at D ($D \neq A$) and the circumscribed circle of triangle AQC intersects line AB at E ($E \neq A$). Show that lines BC, DP, and EQ are concurrent.

Problem 1.581 (4742951979457606021)**.** There are 2021 points on a circle. Kostya marks a point, then marks the adjacent point to the right, then he marks the point two to its right, then three to the next point's right, and so on. Which move will be the first time a point is marked twice?

Problem 1.582 (8059299736482475200)**.** Let $N \geq 2$ be an integer, and let $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots b_N)$ be sequences of non-negative integers. For each integer $i \notin \{1, \ldots, N\}$, let $a_i = a_k$ and $b_i = b_k$, where $k \in \{1, \ldots, N\}$ is the integer such that $i - k$ is divisible by n. We say $\mathbf{a}$ is $\mathbf{b}$-harmonic if each a_i equals the following arithmetic mean:

$$a_i = \frac{1}{2b_i + 1} \sum_{s=-b_i}^{b_i} a_{i+s}.$$

Suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is a constant sequence, and that both $\mathbf{a}$ is $\mathbf{b}$-harmonic and $\mathbf{b}$ is $\mathbf{a}$-harmonic.

Prove that at least $N + 1$ of the numbers $a_1, \ldots, a_N, b_1, \ldots, b_N$ are zero.

Problem 1.583 (1637184643761804371)**.** Initially, on the lower left and right corner of a 2018×2018 board, there're two horses, red and blue, respectively. A and B alternatively play their turn, A start first. Each turn consist of moving their horse (A-red, and B-blue) by, simultaneously, 20 cells respect to one coordinate, and 17 cells respect to the other; while preserving the rule that the horse can't occupied the cell that ever occupied by any horses in the game. The player who can't make the move loss, who has the winning strategy?

Problem 1.584 (748681263295975)**.** We are given an acute triangle ABC. The angle bisector of $\angle BAC$ cuts BC at P. Points D and E lie on segments AB and AC, respectively, so that $BC \parallel DE$. Points K and L lie on segments PD and PE, respectively, so that points A, D, E, K, L are concyclic. Prove that points B, C, K, L are also concyclic.

Problem 1.585 (908587245178389)**.** Let I be the incenter of triangle ABC, and ℓ be the perpendicular bisector of AI. Suppose that P is on the circumcircle of triangle ABC, and line AP and ℓ intersect at point Q. Point R is on ℓ such that $\angle IPR = 90°$.Suppose that line IQ and the midsegment of ABC that is parallel to BC intersect at M. Show that $\angle AMR = 90°$

(Note: In a triangle, a line connecting two midpoints is called a midsegment.)

Problem 1.586 (315159980103862)**.** Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

Problem 1.587 (969197144236847)**.** Each girl among 100 girls has 100 balls; there are in total 10000 balls in 100 colors, from each color there are 100 balls. On a move, two girls can exchange a ball (the first gives the second one of her balls, and vice versa). The operations can be made in such a way, that in the end, each girl has 100 balls, colored in the 100 distinct colors. Prove that there is a sequence of operations, in which each ball is exchanged no more than 1 time, and at the end, each girl has 100 balls, colored in the 100 colors.

Problem 1.588 (914387802726278)**.** Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied: $x_1 \le x_2 \le \ldots \le x_{2020}$; $x_{2020} \le x_1 + 1$; there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ is a permutation of $(1, 2, 2)$, and they are both permutations of $(2, 2, 1)$. Note that any list is a permutation of itself.

Problem 1.589 (8317584744128058138)**.** One side of a square sheet of paper is colored red, the other - in blue. On both sides, the sheet is divided into n^2 identical square cells. In each of these $2n^2$ cells is written a number from 1 to k. Find the smallest k,for which the following properties hold simultaneously: (i) on the red side, any two numbers in different rows are distinct; (ii) on the blue side, any two numbers in different columns are different; (iii) for each of the n^2 squares of the partition, the number on the blue side is not equal to the number on the red side.

Problem 1.590 (549237375256018)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions : 1) $f(x+y) - f(x) - f(y) \in \{0, 1\}$ for all $x, y \in \mathbb{R}$ 2) $\lfloor f(x) \rfloor = \lfloor x \rfloor$ for all real x.

Problem 1.591 (5379858391330892049)**.** Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Problem 1.592 (8725820796958956406)**.** Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remain at H, inradius of triangle ABC remains unchanged.

Problem 1.593 (2211812924503059239)**.** We are given n coins of different weights and n balances, $n > 2$. On each turn one can choose one balance, put one coin on the right pan and one on the left pan, and then delete these coins out of the balance. It's known that one balance is wrong (but it's not known ehich exactly), and it shows an arbitrary result on every turn. What is the smallest number of turns required to find the heaviest coin?

Problem 1.594 (9083308405590075982)**.** Let a, b, c, x, y, z be positive reals such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that

$$a^x + b^y + c^z \geq \frac{4abcxyz}{(x+y+z-3)^2}.$$

Problem 1.595 (3173124324482060330)**.** Consider a n-sided regular polygon, $n \geq 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r-3) \geq n$, then there exist at least two congruent triangles whose vertices belong to V.

Problem 1.596 (139398523212430)**.** Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Problem 1.597 (6598717146370603O8)**.** Let ABC be a triangle with circumcenter O and circumcircle ω. Let D be the foot of the altitude from A to $\overline{BC}$. Let P and Q be points on the circumcircles of triangles AOB and AOC, respectively, such that A, P, and Q are collinear. Prove that if the circumcircle of triangle OPQ is tangent to ω at T, then $\angle BTD = \angle CAP$.

Problem 1.598 (210358073900610)**.** Let triangle ABC have altitudes BE and CF which meet at H. The reflection of A over BC is A'. Let (ABC) meet $(AA'E)$ at P and $(AA'F)$ at Q. Let BC meet PQ at R. Prove that $EF \parallel HR$.

Problem 1.599 (8895719454292056765)**.** Given a non-right triangle ABC with $BC > AC > AB$. Two points $P_1 \neq P_2$ on the plane satisfy that, for $i = 1, 2$, if AP_i, BP_i and CP_i intersect the circumcircle of the triangle ABC at D_i, E_i, and F_i, respectively, then $D_iE_i \perp D_iF_i$ and $D_iE_i = D_iF_i \neq 0$. Let the line P_1P_2 intersects the circumcircle of ABC at Q_1 and Q_2. The Simson lines of Q_1, Q_2 with respect to ABC intersect at W.

Prove that W lies on the nine-point circle of ABC.

Problem 1.600 (25177681716771)**.** Determine all prime numbers p and all positive integers x and y satisfying
$$x^3 + y^3 = p(xy + p).$$

Problem 1.601 (3386683349955795885)**.** For every positive integer $M \geq 2$, find the smallest real number C_M such that for any integers $a_1, a_2, \ldots, a_{2023}$, there always exist some integer $1 \leq k < M$ such that
$$\left\{\frac{ka_1}{M}\right\} + \left\{\frac{ka_2}{M}\right\} + \cdots + \left\{\frac{ka_{2023}}{M}\right\} \leq C_M.$$
Here, $\{x\}$ is the unique number in the interval $[0, 1)$ such that $x - \{x\}$ is an integer.

Problem 1.602 (6183425212304704085)**.** A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

Problem 1.603 (1989615889874190156)**.** Given a 32×32 table, we put a mouse (facing up) at the bottom left cell and a piece of cheese at several other cells. The mouse then starts moving. It moves forward except that when it reaches a piece of cheese, it eats a part of it, turns right, and continues moving forward. We say that a subset of cells containing cheese is good if, during this process, the mouse tastes each piece of cheese exactly once and then falls off the table. Show that:

(a) No good subset consists of 888 cells. (b) There exists a good subset consisting of at least 666 cells.

Problem 1.604 (3201334969593516l3)**.** Let n be a positive integer and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be integers satisfying the following condition: the numbers $x_1, \ldots, x_n$ are pairwise distinct and for every positive integer m there exists a polynomial P_m with integer coefficients such that $P_m(x_i) - y_i$, $i = 1, \ldots, n$, are all divisible by m. Prove that there exists a polynomial P with integer coefficients such that $P(x_i) = y_i$ for all $i = 1, \ldots, n$.

Problem 1.605 (5891289107244537458)**.** Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

Problem 1.606 (563612490071424)**.** Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Problem 1.607 (2672133756769464425)**.** Is there a scalene triangle ABC similar to triangle IHO, where I, H, and O are the incenter, orthocenter, and circumcenter, respectively, of triangle ABC?

Problem 1.608 (575901379082524)**.** Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Problem 1.609 (4439711278400170990)**.** N oligarchs built a country with N cities with each one of them owning one city. In addition, each oligarch built some roads such that the maximal amount of roads an oligarch can build between two cities is 1 (note that there can be more than 1 road going through two cities, but they would belong to different oligarchs). A total of d roads were built. Some oligarchs wanted to create a corporation by combining their cities and roads so that from any city of the corporation you can go to any city of the corporation using only corporation roads (roads can go to other cities outside corporation) but it turned out that no group of less than N oligarchs can create a corporation. What is the maximal amount that d can have?

Problem 1.610 (7494618588207758150)**.** An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \cdots + 2018$?

Problem 1.611 (551619066390682)**.** Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Problem 1.612 (1560607598563435210)**.** Let ABC be an acute triangle with $\angle ACB > 2\angle ABC$. Let I be the incenter of ABC, K is the reflection of I in line BC. Let line BA and KC intersect at D. The line through B parallel to CI intersects the minor arc BC on the circumcircle of ABC at $E(E \neq B)$. The line through A parallel to BC intersects the line BE at F. Prove that if $BF = CE$, then $FK = AD$.

Problem 1.613 (236318831875052)**.** Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Problem 1.614 (574687232505662)**.** Find all pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

Problem 1.615 (549441013338848)**.** What is the minimal number of operations needed to repaint a entirely white grid 100×100 to be entirely black, if on one move we can choose 99 cells from any row or column and change their color?

Problem 1.616 (307733682720311)**.** Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define

O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Problem 1.617 (6254579538196178032)**.** Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$

Problem 1.618 (457324036151847)**.** Let O and H be the circumcenter and the orthocenter, respectively, of an acute triangle ABC. Points D and E are chosen from sides AB and AC, respectively, such that A, D, O, E are concyclic. Let P be a point on the circumcircle of triangle ABC. The line passing P and parallel to OD intersects AB at point X, while the line passing P and parallel to OE intersects AC at Y. Suppose that the perpendicular bisector of $\overline{HP}$ does not coincide with XY, but intersect XY at Q, and that points A, Q lies on the different sides of DE. Prove that $\angle EQD = \angle BAC$.

Problem 1.619 (140536805208587401)**.** Let P be a polynomial of degree greater than or equal to 4 with integer coefficients. An integer x is called P-representable if there exists integer numbers a and b such that $x = P(a) - P(b)$. Prove that, if for all $N \geq 0$, more than half of the integers of the set $\{0, 1, \ldots, N\}$ are P-representable, then all the even integers are P-representable or all the odd integers are P-representable.

Problem 1.620 (37302962546151)**.** Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Problem 1.621 (175452544956824)**.** In the city built are 2019 metro stations. Some pairs of stations are connected. tunnels, and from any station through the tunnels you can reach any other. The mayor ordered to organize several metro lines: each line should include several different stations connected in series by tunnels (several lines can pass through the same tunnel), and in each station must lie at least on one line. To save money no more than k lines should be made. It turned out that the order of the mayor is not feasible. What is the largest k it could to happen?

Problem 1.622 (1891712635906763103)**.** Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle\, BMC$ so that $\angle KBC = 90°$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.

Problem 1.623 (7500559455615129254)**.** For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N} + 1}{2}} \leqslant b_N(x-1)^2 + x.$$

Problem 1.624 (8354552322611949357)**.** We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Problem 1.625 (299125558562230)**.** Ana and Bety play a game alternating turns. Initially, Ana chooses an odd possitive integer and composite n such that $2^j < n < 2^{j+1}$ with $2 < j$. In her first turn Bety chooses an odd composite integer n_1 such that

$$n_1 \leq \frac{1^n + 2^n + \cdots + (n-1)^n}{2(n-1)^{n-1}}.$$

Then, on her other turn, Ana chooses a prime number p_1 that divides n_1. If the prime that Ana chooses is 3, 5 or 7, the Ana wins; otherwise Bety chooses an odd composite positive integer n_2 such that

$$n_2 \leq \frac{1^{p_1} + 2^{p_1} + \cdots + (p_1-1)^{p_1}}{2(p_1-1)^{p_1-1}}.$$

After that, on her turn, Ana chooses a prime p_2 that divides n_2,, if p_2 is 3, 5, or 7, Ana wins, otherwise the process repeats. Also, Ana wins if at any time Bety cannot choose an odd composite positive integer in the corresponding range. Bety wins if she manages to play at least $j-1$ turns. Find which of the two players has a winning strategy.

Problem 1.626 (18109155851115304730)**.** Given a scalene triangle $\triangle ABC$. B', C' are points lie on the rays $\overrightarrow{AB}, \overrightarrow{AC}$ such that $\overline{AB'} = \overline{AC}, \overline{AC'} = \overline{AB}$. Now, for an arbitrary point P in the plane. Let Q be the reflection point of P w.r.t $\overline{BC}$. The intersections of $\odot(BB'P)$ and $\odot(CC'P)$ is P' and the intersections of $\odot(BB'Q)$ and $\odot(CC'Q)$ is Q'. Suppose that O, O' are circumcenters of $\triangle ABC, \triangle AB'C'$ Show that

1. O', P', Q' are colinear
2. $\overline{O'P'} \cdot \overline{O'Q'} = \overline{OA}^2$

Problem 1.627 (503121367540901)**.** Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Problem 1.628 (119253293150446)**.** In the plane, 2022 points are chosen such that no three points lie on the same line. Each of the points is coloured red or blue such that each triangle formed by three distinct red points contains at least one blue point. What is the largest possible number of red points?

Problem 1.629 (437645166165639)**.** Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2.$$

Problem 1.630 (9000483733039705317)**.** Fix an integer $n \geq 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in $\mathcal{S}$, the triangle ABC is nice for AB if $[ABC] \leq [ABX]$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles). A triangle is beautiful if its vertices are all in $\mathcal{S}$ and is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

Problem 1.631 (6438524243840428787)**.** Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 1.632 (284109588966873)**.** Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

Problem 1.633 (9162230842142232349)**.** Let ABC be a triangle. Distinct points D, E, F lie on sides BC, AC, and AB, respectively, such that quadrilaterals $ABDE$ and $ACDF$ are cyclic. Line AD meets the circumcircle of $\triangle ABC$ again at P. Let Q denote the reflection of P across BC. Show that Q lies on the circumcircle of $\triangle AEF$.

Problem 1.634 (856916153770874)**.** Find all pairs (n, d) of positive integers such that $d \mid n^2$ and $(n-d)^2 < 2d$.

Problem 1.635 (4678973565823282552)**.** Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

Problem 1.636 (2270109693486247508)**.** Let $\mathbb{Z}$ denote the set of all integers. Find all polynomials $P(x)$ with integer coefficients that satisfy the following property:

For any infinite sequence $a_1, a_2, \dots$ of integers in which each integer in $\mathbb{Z}$ appears exactly once, there exist indices $i < j$ and an integer k such that $a_i + a_{i+1} + \cdots + a_j = P(k)$.

Problem 1.637 (757902621276461)**.** Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Problem 1.638 (6340105142765788083)**.** In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

Problem 1.639 (599825051147866097)**.** Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Problem 1.640 (806540218855542)**.** Let ABC be an acute triangle with $AB < AC$. Denote by P and Q points on the segment BC such that $\angle BAP = \angle CAQ < \frac{\angle BAC}{2}$. B_1 is a point on segment AC. BB_1 intersects AP and AQ at P_1 and Q_1, respectively. The angle bisectors of $\angle BAC$ and $\angle CBB_1$ intersect at M. If $PQ_1 \perp AC$ and $QP_1 \perp AB$, prove that AQ_1MPB is cyclic.

Problem 1.641 (1073572769363152471)**.** Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

Note that $\angle A = \angle FAB$. The other interior angles of the hexagon are similarly described.

Problem 1.642 (796349431725149)**.** An acute, non-isosceles triangle ABC is inscribed in a circle with centre O. A line go through O and midpoint I of BC intersects AB, AC

at E, F respectively. Let D, G be reflections to A over O and circumcentre of (AEF), respectively. Let K be the reflection of O over circumcentre of (OBC). a) Prove that D, G, K are collinear. b) Let M, N are points on KB, KC that $IM \perp AC$, $IN \perp AB$. The midperpendiculars of IK intersects MN at H. Assume that IH intersects AB, AC at P, Q respectively. Prove that the circumcircle of $\triangle APQ$ intersects (O) the second time at a point on AI.

Problem 1.643 (574223786384294)**.** Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

Problem 1.644 (8592236630142322398)**.** Let n be a positive integer and consider an $n \times n$ square grid. For $1 \leq k \leq n$, a python of length k is a snake that occupies k consecutive cells in a single row, and no other cells. Similarly, an anaconda of length k is a snake that occupies k consecutive cells in a single column, and no other cells.

The grid contains at least one python or anaconda, and it satisfies the following properties: No cell is occupied by multiple snakes. If a cell in the grid is immediately to the left or immediately to the right of a python, then that cell must be occupied by an anaconda. If a cell in the grid is immediately to above or immediately below an anaconda, then that cell must be occupied by a python.

Prove that the sum of the squares of the lengths of the snakes is at least n^2.

Problem 1.645 (920619320023657807)**.** Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

Problem 1.646 (7016087217872166929)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(yf(x)) + f(x - 1) = f(x)f(y)$ and $|f(x)| < 2022$ for all $0 < x < 1$.

Problem 1.647 (7146141883280672441)**.** Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations: Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell. Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell. At the end of the game, one cell contains 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

Problem 1.648 (512148051997527)**.** Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is sum-free if, whenever x and y are (not necessarily distinct) members of A, their sum $x + y$ does not belong to A. Determine all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a) : a \in A\}$ is also sum-free.

Note: a function $f : \mathbb{N} \to \mathbb{N}$ is surjective if, for every positive integer n, there exists a positive integer m such that $f(m) = n$.

Problem 1.649 (2201137214247796233)**.** A neighborhood consists of 10×10 squares. On New Year's Eve it snowed for the first time and since then exactly 10 cm of snow fell on each square every night (and snow fell only at night). Every morning, the janitor selects one row or column and shovels all the snow from there onto one of the adjacent rows or columns (from each cell to the adjacent side). For example, he can select the seventh column and from each of its cells shovel all the snow into the cell of the left of it. You cannot shovel snow outside the neighborhood. On the evening of the 100th day of the year, an inspector will come to the city and find the cell with the snowdrift of maximal height. The goal of the janitor is to ensure that this height is minimal. What height of snowdrift will the inspector find?

Problem 1.650 (2210005554575274405)**.** On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered marked if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive). Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference. Alina finds that the $N+1$ yellow labels take each value $0, 1, ..., N$ exactly once. Show that at least one blue label is a multiple of 3. (A chord is a line segment joining two different points on a circle.)

Problem 1.651 (236181624113090)**.** Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Problem 1.652 (3353450172272500341)**.** Let $ABCD$ be a cyclic quadrilateral. Let DA and BC intersect at E and let AB and CD intersect at F. Assume that A, E, F all lie on the same side of BD. Let P be on segment DA such that $\angle CPD = \angle CBP$, and let Q be on segment CD such that $\angle DQA = \angle QBA$. Let AC and PQ meet at X. Prove that, if $EX = EP$, then EF is perpendicular to AC.

Problem 1.653 (645930596871591)**.** Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is stable if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \leq x$ and $y' \leq y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

Problem 1.654 (8255863576892581507)**.** Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

Problem 1.655 (7088779505939683183)**.** Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

Problem 1.656 (896600029778859256)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

Problem 1.657 (8963205841174892420)**.** Let $ABCD$ be a convex quadrilateral with pairwise distinct side lengths such that $AC \perp BD$. Let O_1, O_2 be the circumcenters of $\Delta ABD, \Delta CBD$, respectively. Show that AO_2, CO_1, the Euler line of ΔABC and the Euler line of ΔADC are concurrent.

(Remark: The Euler line of a triangle is the line on which its circumcenter, centroid, and orthocenter lie.)

Problem 1.658 (423911944927735)**.** In acute $\triangle ABC$, O is the circumcenter, I is the incenter. The incircle touches BC, CA, AB at D, E, F. And the points K, M, N are the midpoints of BC, CA, AB respectively.

a) Prove that the lines passing through D, E, F in parallel with IK, IM, IN respectively are concurrent.

b) Points T, P, Q are the middle points of the major arc BC, CA, AB on $\odot ABC$. Prove that the lines passing through D, E, F in parallel with IT, IP, IQ respectively are concurrent.

Problem 1.659 (5113543632741494138)**.** Find all positive integers (a, b, c) such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

Problem 1.660 (456772085666528)**.** Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Problem 1.661 (2003233604438068678)**.** Given a triangle ABC and a point O on a plane. Let Γ be the circumcircle of ABC. Suppose that CO intersects with AB at D, and BO and CA intersect at E. Moreover, suppose that AO intersects with Γ at A, F. Let I be the other intersection of Γ and the circumcircle of ADE, and Y be the other intersection of BE and the circumcircle of CEI, and Z be the other intersection of CD and the circumcircle of BDI. Let T be the intersection of the two tangents of Γ at B, C, respectively. Lastly, suppose that TF intersects with Γ again at U, and the reflection of U w.r.t. BC is G.

Show that F, I, G, O, Y, Z are concyclic.

Problem 1.662 (249336393279214231)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers $x \neq 1$,

$$f(x - f(x)) + f(x) = \frac{x^2 - x + 1}{x - 1}.$$

Find all possible values of $f(2023)$.

Problem 1.663 (579345048538257)**.** Given a triangle ABC with incenter I. The incircle of triangle ABC is tangent to BC at D. Let P and Q be points on the side BC such that $\angle PAB = \angle BCA$ and $\angle QAC = \angle ABC$, respectively. Let K and L be the incenter of triangles ABP and ACQ, respectively. Prove that AD is the Euler line of triangle IKL.

Problem 1.664 (817429246000759)**.** Find all integers $n \geq 2$ for which there exists a sequence of $2n$ pairwise distinct points $(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$ in the plane satisfying the following four conditions: no three of the $2n$ points are collinear; $P_iP_{i+1} \geq 1$ for all

$i = 1, 2, \ldots, n$, where $P_{n+1} = P_1$; $Q_iQ_{i+1} \geq 1$ for all $i = 1, 2, \ldots, n$, where $Q_{n+1} = Q_1$; and $P_iQ_j \leq 1$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Problem 1.665 (8811824418974048155)**.** $ABCDE$ is a cyclic pentagon, with circumcentre O. $AB = AE = CD$. I midpoint of BC. J midpoint of DE. F is the orthocentre of $\triangle ABE$, and G the centroid of $\triangle AIJ.CE$ intersects BD at H, OG intersects FH at M. Show that $AM \perp CD$.

Problem 1.666 (3859961452154270883)**.** A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

Problem 1.667 (6135851041251773220)**.** 200 natural numbers are written in a row. For any two adjacent numbers of the row, the right one is either 9 times greater than the left one, 2 times smaller than the left one. Can the sum of all these 200 numbers be equal to 24^{2022}?

Problem 1.668 (512052756136271)**.** Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals of AC and BD. Let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

Problem 1.669 (531504969275602705)**.** Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Problem 1.670 (803002459788170506)**.** Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$. Suppose that the three line segments B_1C_2, C_1A_2, A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

Problem 1.671 (297274918587198)**.** Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

Problem 1.672 (6955756846906975678)**.** If there are several heaps of stones on the table, it is said that there are *many* stones on the table, if we can find 50 piles and number them with the numbers from 1 to 50 so that the first pile contains at least one stone, the second - at least two stones,..., the 50-th has at least 50 stones. Let the table be initially contain 100 piles of 100 stones each. Find the largest $n \leq 10000$ such that after removing any n stones, there will still be *many* stones left on the table.

Problem 1.673 (604188725177670)**.** Let $n \geq 2$ be a positive integer. There are n real coefficient polynomials $P_1(x), P_2(x), \cdots, P_n(x)$ which is not all the same, and their leading coefficients are positive. Prove that

$$\deg(P_1^n + P_2^n + \cdots + P_n^n - nP_1P_2\cdots P_n) \geq (n-2)\max_{1\leq i\leq n}(\deg P_i)$$

and find when the equality holds.

Problem 1.674 (6975633259976638169)**.** On the round necklace there are $n > 3$ beads, each painted in red or blue. If a bead has adjacent beads painted the same color, it can be repainted (from red to blue or from blue to red). For what n for any initial coloring of beads it is possible to make a necklace in which all beads are painted equally?

Problem 1.675 (5341232263014748696)**.** Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that $m = 1$ and $l = 2k$; or $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Problem 1.676 (6583158985288167725)**.** Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$; (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$; (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 1.677 (2667130530962382147)**.** We say that a function $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is great if for any nonnegative integers m and n,

$$f(m+1, n+1)f(m, n) - f(m+1, n)f(m, n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

Problem 1.678 (744695293387960)**.** Let $m, n \geq 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid.* He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m-1$ points that divide segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m, n) can Carl necessarily accomplish his task, regardless of which n points he is given?

*Here, the centroid of n points with coordinates $(x_1, y_1), \dots, (x_n, y_n)$ is the point with coordinates $\left(\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n}\right)$.

Problem 1.679 (8840567523125912282)**.** Let $ABCD$ be a trapezoid with $AB \parallel CD$. Its diagonals intersect at a point P. The line passing through P parallel to AB intersects AD and BC at Q and R, respectively. Exterior angle bisectors of angles DBA, DCA intersect at X. Let S be the foot of X onto BC. Prove that if quadrilaterals $ABPQ$, $CDQP$ are circumcribed, then $PR = PS$.

Problem 1.680 (8156079118189111754)**.** Let n be positive integer and fix $2n$ distinct points on a circle. Determine the number of ways to connect the points with n arrows (oriented line segments) such that all of the following conditions hold: each of the $2n$ points is a startpoint or endpoint of an arrow; no two arrows intersect; and there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that A, B, C and D appear in clockwise order around the circle (not necessarily consecutively).

Problem 1.681 (56332281758558)**.** Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Problem 1.682 (5180896359975323937)**.** For every pair (m, n) of positive integers, a positive real number $a_{m,n}$ is given. Assume that

$$a_{m+1,n+1} = \frac{a_{m,n+1}a_{m+1,n} + 1}{a_{m,n}}$$

for all positive integers m and n. Suppose further that $a_{m,n}$ is an integer whenever $\min(m, n) \leq 2$. Prove that $a_{m,n}$ is an integer for all positive integers m and n.

Problem 1.683 (697045850918084)**.** In the country there're N cities and some pairs of cities are connected by two-way airlines (each pair with no more than one). Every airline belongs to one of k companies. It turns out that it's possible to get to any city from any other, but it fails when we delete all airlines belonging to any one of the companies. What is the maximum possible number of airlines in the country ?

Problem 1.684 (7904897494032012729)**.** Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions: $m > 1$ and $\gcd(m, n) = 1$; the numbers $P(0)$, $P^2(0)$, $\ldots$, $P^{m-1}(0)$ are not divisible by n; and $P^m(0)$ is divisible by n. Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

Problem 1.685 (361772755079059)**.** Let $\mathbb{R}_{>0}$ be the set of all positive real numbers. Find all strictly monotone (increasing or decreasing) functions $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that there exists a two-variable polynomial $P(x, y)$ with real coefficients satisfying

$$f(xy) = P(f(x), f(y))$$

for all $x, y \in \mathbb{R}_{>0}$.

Problem 1.686 (132497611943266)**.** Suppose that a, b, c, d are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 1.687 (7144061033013)**.** Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Problem 1.688 (204202362084074)**.** For each integer $n \geq 2$, find all integer solutions of the following system of equations:

$$\begin{gathered} x_1 = (x_2 + x_3 + x_4 + \ldots + x_n)^{2018} \\ x_2 = (x_1 + x_3 + x_4 + \ldots + x_n)^{2018} \\ \vdots \\ x_n = (x_1 + x_2 + x_3 + \ldots + x_{n-1})^{2018} \end{gathered}$$

Problem 1.689 (5347245479409093202)**.** Let G be a graph with 400 vertices. For any edge AB we call a cuttlefish the set of all edges from A and B (including AB). Each edge of the graph is assigned a value of 1 or -1. It is known that the sum of edges at any cuttlefish is greater than or equal to 1. Prove that the sum of the numbers at all edges is at least -10^4.

Problem 1.690 (3435532350205377704)**.** Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.

Problem 1.691 (7636650160414045108)**.** Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leq m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is n, and bad otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

Problem 1.692 (8126547357118301633)**.** An infinite sequence $a_1, a_2, a_3, \dots$ of real numbers satisfies

$$a_{2n-1} + a_{2n} > a_{2n+1} + a_{2n+2} \qquad \text{and} \qquad a_{2n} + a_{2n+1} < a_{2n+2} + a_{2n+3}$$

for every positive integer n. Prove that there exists a real number C such that $a_n a_{n+1} < C$ for every positive integer n.

Problem 1.693 (199006625390154)**.** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Problem 1.694 (680158311639624)**.** Find all positive integers $k < 202$ for which there exist a positive integers n such that

$$\left\{\frac{n}{202}\right\} + \left\{\frac{2n}{202}\right\} + \cdots + \left\{\frac{kn}{202}\right\} = \frac{k}{2}$$

Problem 1.695 (106666027438734)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(x + f(y)) + xy = f(x)f(y) + f(x) + y.$$

Problem 1.696 (1612300762204186997)**.** For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m) - 1}{m - 1} = \frac{\sigma(n) - 1}{n - 1} = \frac{\sigma(mn) - 1}{mn - 1}.$$

Problem 1.697 (613302970238472)**.** Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Problem 1.698 (3032245772349874005)**.** Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonals AC meets the segments DP at Q. The line through P parallel to CD mmets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent .

Problem 1.699 (811235233671414145)**.** Let m and n be positive integers. A circular necklace contains mn beads, each either red or blue. It turned out that no matter how the necklace was cut into m blocks of n consecutive beads, each block had a distinct number of red beads. Determine, with proof, all possible values of the ordered pair (m, n).

Problem 1.700 (344307741773187)**.** Let m and n be integers greater than 2, and let A and B be non-constant polynomials with complex coefficients, at least one of which has a degree greater than 1. Prove that if the degree of the polynomial $A^m - B^n$ is less than $\min(m, n)$, then $A^m = B^n$.

Problem 1.701 (8948164820835424145)**.** Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

Problem 1.702 (20663652231924)**.** Consider pairs of functions (f, g) from the set of nonnegative integers to itself such that $f(0) + f(1) + f(2) + \cdots + f(42) \leq 2022$; for any integers $a \geq b \geq 0$, we have $g(a + b) \leq f(a) + f(b)$. Determine the maximum possible value of $g(0) + g(1) + g(2) + \cdots + g(84)$ over all such pairs of functions.

Problem 1.703 (522601446762373)**.** Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

Problem 1.704 (7689980261025088265)**.** Let $ABCD$ be a parallelogram. Let $W, X, Y,$ and Z be points on sides $AB, BC, CD,$ and DA, respectively, such that the incenters of triangles $AWZ, BXW, CYX,$ and DZY form a parallelogram. Prove that $WXYZ$ is a parallelogram.

Problem 1.705 (317862961000833)**.** Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 1.706 (718838419070287)**.** Consider an odd prime p and a positive integer $N < 50p$. Let $a_1, a_2, \ldots, a_N$ be a list of positive integers less than p such that any specific value occurs at most $\frac{51}{100}N$ times and $a_1 + a_2 + \cdots + a_N$ is not divisible by p. Prove that there exists a permutation $b_1, b_2, \ldots, b_N$ of the a_i such that, for all $k = 1, 2, \ldots, N$, the sum $b_1 + b_2 + \cdots + b_k$ is not divisible by p.

Problem 1.707 (3104172479883832933)**.** Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy)$$

for all real numbers x and y.

Problem 1.708 (7284124089748055531)**.** Let n be a positive integer. The following 35 multiplication are performed:

$$1 \cdot n, 2 \cdot n, \dots, 35 \cdot n.$$

Show that in at least one of these results the digit 7 appears at least once.

Problem 1.709 (1790114062253914451)**.** Given a triangle $\triangle ABC$ and a point O. X is a point on the ray $\overrightarrow{AC}$. Let X' be a point on the ray $\overrightarrow{BA}$ so that $\overline{AX} = \overline{AX_1}$ and A lies in the segment $\overline{BX_1}$. Then, on the ray $\overrightarrow{BC}$, choose X_2 with $\overline{X_1X_2} \parallel \overline{OC}$.

Prove that when X moves on the ray $\overrightarrow{AC}$, the locus of circumcenter of $\triangle BX_1X_2$ is a part of a line.

Problem 1.710 (8916142707013964275)**.** Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

Problem 1.711 (3753289685429929419)**.** Determine all polynomials f with integer coefficients such that $f(p)$ is a divisor of $2^p - 2$ for every odd prime p.

Problem 1.712 (1557927271810341706)**.** Let $A_1A_2 \dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

Problem 1.713 (467943798848835)**.** Elmo and Elmo's clone are playing a game. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, who wins? (Your answer may be in terms of n.)

Problem 1.714 (792293045512112)**.** Let ABC be an acute triangle inscribed in a circle ω with center O. Points E, F lie on its side AC, AB, respectively, such that O lies on EF and $BCEF$ is cyclic. Let R, S be the intersections of EF with the shorter arcs AB, AC of ω, respectively. Suppose K, L are the reflection of R about C and the reflection of S about B, respectively. Suppose that points P and Q lie on the lines BS and RC, respectively, such that PK and QL are perpendicular to BC. Prove that the circle with center P and radius PK is tangent to the circumcircle of RCE if and only if the circle with center Q and radius QL is tangent to the circumcircle of BFS.

Problem 1.715 (213513857758059)**.** Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Problem 1.716 (2886276736199315342)**.** Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0, 0)$ and ending at a point on the line $x + y = n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths. *

Problem 1.717 (5363953658134647103)**.** Let ABC be a triangle with incenter I. The line through I, perpendicular to AI, intersects the circumcircle of ABC at points P and Q. It turns out there exists a point T on the side BC such that $AB + BT = AC + CT$ and $AT^2 = AB \cdot AC$. Determine all possible values of the ratio IP/IQ.

Problem 1.718 (3232480961068145020)**.** Find all pair of constants (a, b) such that there exists real-coefficient polynomial $p(x)$ and $q(x)$ that satisfies the condition below.

Condition: $\forall x \in \mathbb{R},\ \ p(x^2)q(x+1) - p(x+1)q(x^2) = x^2 + ax + b$

Problem 1.719 (552933284268039)**.** Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

Problem 1.720 (528504335909385)**.** Given a triangle $\triangle ABC$ whose incenter is I and A-excenter is J. A' is point so that AA' is a diameter of $\odot(\triangle ABC)$. Define H_1, H_2 to be the orthocenters of $\triangle BIA'$ and $\triangle CJA'$. Show that $H_1H_2 \parallel BC$

Problem 1.721 (287986230573307)**.** Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Problem 1.722 (5952830561616844902)**.** We are given a positive integer $s \geq 2$. For each positive integer k, we define its twist k' as follows: write k as $as + b$, where a, b are non-negative integers and $b < s$, then $k' = bs + a$. For the positive integer n, consider the infinite sequence $d_1, d_2, \ldots$ where $d_1 = n$ and d_{i+1} is the twist of d_i for each positive integer i. Prove that this sequence contains 1 if and only if the remainder when n is divided by $s^2 - 1$ is either 1 or s.

Problem 1.723 (183354438240037)**.** Let I, O, H, and Ω be the incenter, circumcenter, orthocenter, and the circumcircle of the triangle ABC, respectively. Assume that line AI intersects with Ω again at point $M \neq A$, line IH and BC meets at point D, and line MD intersects with Ω again at point $E \neq M$. Prove that line OI is tangent to the circumcircle of triangle IHE.

Problem 1.724 (8971817929368411167)**.** 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex); (ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Problem 1.725 (282712203118607)**.** Let ABC be an acute-angled triangle with $AC > AB$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC, and AB at W, X, and Y, respectively. The circumcircles of triangles AXY and ABC intersect again at $Z \neq A$. Prove that if $W \neq D$ and $OW = OD$, then DZ is tangent to the circle AXY.

Problem 1.726 (9055967412808709037)**.** Baron Munchhausen has a collection of stones, such that they are of 1000 distinct whole weights, 2^{1000} stones of every weight. Baron states that if one takes exactly one stone of every weight, then the weight of all these

1000 stones chosen will be less than 2^{1010}, and there is no other way to obtain this weight by picking another set of stones of the collection. Can this statement happen to be true?

Problem 1.727 (835565816078264)**.** Let $a_1, a_2, \dots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

Problem 1.728 (1986632843459004559)**.** Consider an acute-angled triangle $\triangle ABC$ $(AC > AB)$ with its orthocenter H and circumcircle Γ.Points M,P are midpoints of BC and AH respectively.The line $\overline{AM}$ meets Γ again at X and point N lies on the line $\overline{BC}$ so that $\overline{NX}$ is tangent to Γ. Points J and K lie on the circle with diameter MP such that $\angle AJP = \angle HNM$ (B and J lie one the same side of $\overline{AH}$) and circle ω_1, passing through K, H, and J, and circle ω_2 passing through K, M, and N, are externally tangent to each other. Prove that the common external tangents of ω_1 and ω_2 meet on the line $\overline{NH}$.

Problem 1.729 (5713525138564417722)**.** A cyclic quadrilateral $ABCD$ has circumcircle Γ, and $AB + BC = AD + DC$. Let E be the midpoint of arc BCD, and $F(\neq C)$ be the antipode of A wrt Γ. Let I, J, K be the incenter of $\triangle ABC$, the A-excenter of $\triangle ABC$, the incenter of $\triangle BCD$, respectively. Suppose that a point P satisfies $\triangle BIC \overset{+}{\sim} \triangle KPJ$. Prove that EK and PF intersect on Γ.

Problem 1.730 (1266870846109464791)**.** Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Problem 1.731 (651490142085731)**.** Let I be the incenter of triangle ABC, and let ω be its incircle. Let E and F be the points of tangency of ω with CA and AB, respectively. Let X and Y be the intersections of the circumcircle of BIC and ω. Take a point T on BC such that $\angle AIT$ is a right angle. Let G be the intersection of EF and BC, and let Z be the intersection of XY and AT. Prove that AZ, ZG, and AI form an isosceles triangle.

Problem 1.732 (9026100911884959358)**.** Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Problem 1.733 (1336030836839904136)**.** Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if area (BCD) + area (ADE) = area (ABD) + area (ABP).

Problem 1.734 (5062971667185317512)**.** Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and

CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

Problem 1.735 (709204825099641)**.** Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Problem 1.736 (894895504790373)**.** Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers $a_1, a_2, \cdots, a_{20}$ such that, for each $k = 1, 2, \cdots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k : k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Problem 1.737 (173979142158596)**.** Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Problem 1.738 (441177656992348)**.** Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints: each cell contains a distinct divisor; the sums of all rows are equal; and the sums of all columns are equal.

Problem 1.739 (3812208515075577730)**.** Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder that b^n leaves upon division by a^n. Assume there exists a positive integer N such that $r_n < \frac{2^n}{n}$ for all integers $n \geq N$. Prove that a divides b.

Problem 1.740 (1872712387771032593)**.** Let H be the orthocenter of triangle ABC, and AD, BE, CF be the three altitudes of triangle ABC. Let G be the orthogonal projection of D onto EF, and DD' be the diameter of the circumcircle of triangle DEF. Line AG and the circumcircle of triangle ABC intersect again at point X. Let Y be the intersection of GD' and BC, while Z be the intersection of AD' and GH. Prove that X, Y, and Z are collinear.

Problem 1.741 (275429739915708)**.** Consider a 100×100 square unit lattice $\mathbf{L}$ (hence $\mathbf{L}$ has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in $\mathbf{L}$ and every point in $\mathbf{L}$ is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

Problem 1.742 (596300332016249)**.** Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

Problem 1.743 (232495612059721)**.** A domino is a 1×2 or 2×1 tile. Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. The value of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called balanced if there exists some $k \geq 1$ such that each row and each column has a value of k. Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

Problem 1.744 (7229423492681245326)**.** Find the smallest constant $C > 1$ such that the following statement holds: for every integer $n \geq 2$ and sequence of non-integer positive real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1,$$

it's possible to choose positive integers b_i such that (i) for each $i = 1, 2, \ldots, n$, either $b_i = \lfloor a_i \rfloor$ or $b_i = \lfloor a_i \rfloor + 1$, and (ii) we have

$$1 < \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \leq C.$$

(Here $\lfloor \bullet \rfloor$ denotes the floor function, as usual.)

Problem 1.745 (758429597657132)**.** Let n be a positive integer. Find the number of sequences $a_0, a_1, a_2, \ldots, a_{2n}$ of integers in the range $[0, n]$ such that for all integers $0 \leq k \leq n$ and all nonnegative integers m, there exists an integer $k \leq i \leq 2k$ such that $\lfloor k/2^m \rfloor = a_i$.

Problem 1.746 (372825050751557)**.** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n}, \text{ for } n \geq 1$$

Show that, if there exists at least one term among every million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ that is an integer, then there exists some k such that $b_k > 2021^{2021}$.

Problem 1.747 (487703623613277)**.** Let ABC be a triangle with $AC > AB$, and denote its circumcircle by Ω and incentre by I. Let its incircle meet sides BC, CA, AB at D, E, F respectively. Let X and Y be two points on minor arcs $\widehat{DF}$ and $\widehat{DE}$ of the incircle, respectively, such that $\angle BXD = \angle DYC$. Let line XY meet line BC at K. Let T be the point on Ω such that KT is tangent to Ω and T is on the same side of line BC as A. Prove that lines TD and AI meet on Ω.

Problem 1.748 (6051857606097163028)**.** Let $a_1, a_2, \ldots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor$$

holds for all integers $n > N$.

Problem 1.749 (8705251856251359603)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(1) \neq f(-1)$ and

$$f(m+n)^2 \mid f(m) - f(n)$$

for all integers m, n.

Problem 1.750 (493735785757154)**.** Given is a graph G of $n + 1$ vertices, which is constructed as follows: initially there is only one vertex v, and one a move we can add a vertex and connect it to exactly one among the previous vertices. The vertices have non-negative real weights such that v has weight 0 and each other vertex has a weight not exceeding the avarage weight of its neighbors, increased by 1. Prove that no weight can exceed n^2.

Problem 1.751 (600298381529685). Find all pairs of positive integers (a, b) satisfying the following conditions: a divides $b^4 + 1$, b divides $a^4 + 1$, $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor$.

Problem 1.752 (680055064158556). Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

Problem 1.753 (3975075785518808190). Determine all positive integers k for which there exist a positive integer m and a set S of positive integers such that any integer $n > m$ can be written as a sum of distinct elements of S in exactly k ways.

Problem 1.754 (646424364467534). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) - 9f(y)) = (x + 3y)^2 f(x - 3y)$$

for all $x, y \in \mathbb{R}$.

Problem 1.755 (736279317663030). The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

(i) $1 \le a_j \le 2015$ for all $j \ge 1$, (ii) $k + a_k \ne \ell + a_\ell$ for all $1 \le k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

for all integers m and n such that $n > m \ge N$.

Problem 1.756 (736821043753990). Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

Problem 1.757 (4777015574921577837). Let $n \ge 2$ be an integer, and let $a_1, a_2, \cdots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \cdots, b_n$ satisfying the following three conditions:

(A) $a_i \le b_i$ for $i = 1, 2, \cdots, n$;

(B) the remainders of $b_1, b_2, \cdots, b_n$ on division by n are pairwise different; and

(C) $b_1 + b_2 + \cdots b_n \le n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1+a_2+\cdots a_n}{n} \right\rfloor\right)$

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x, that is, the largest integer that does not exceed x.)

Problem 1.758 (8612979541975584705). Let G be a connected graph and let X, Y be two disjoint subsets of its vertices, such that there are no edges between them. Given that G/X has m connected components and G/Y has n connected components, what is the minimal number of connected components of the graph $G/(X \cup Y)$?

Problem 1.759 (2694660444585153591). Find all binary operations $\diamondsuit : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$, the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and if $a \ge 1$ then $a \diamondsuit a \ge 1$.

Problem 1.760 (2143833415170817930)**.** Let B and C be two fixed points in the plane. For each point A of the plane, outside of the line BC, let G be the barycenter of the triangle ABC. Determine the locus of points A such that $\angle BAC + \angle BGC = 180^\circ$.

Note: The locus is the set of all points of the plane that satisfies the property.

Problem 1.761 (14852916670686)**.** Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \le n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn. (ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn. (iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Problem 1.762 (3417358984411200361)**.** Let ABC be a triangle with circumcircle Ω, circumcenter O and orthocenter H. Let S lie on Ω and P lie on BC such that $\angle ASP = 90^\circ$, line SH intersects the circumcircle of $\triangle APS$ at $X \neq S$. Suppose OP intersects CA, AB at Q, R, respectively, QY, RZ are the altitude of $\triangle AQR$. Prove that X, Y, Z are collinear.

Problem 1.763 (402288800658108)**.** Let $n \ge 3$ be a positive integer, and let S be a set of n distinct points in the plane. Call an unordered pair of distinct points A, B tasty if there exists a circle passing through A and B not passing through or containing any other point in S. Find the maximum number of tasty pairs over all possible sets S of n points.

Problem 1.764 (6489054720541585180)**.** Let ABC be a triangle such that $\angle BAC = 90^\circ$ and $AB = AC$. Let M be the midpoint of BC. A point $D \neq A$ is chosen on the semicircle with diameter BC that contains A. The circumcircle of triangle DAM cuts lines DB and DC at E and F respectively. Show that $BE = CF$.

Problem 1.765 (2302470517258475835)**.** Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Problem 1.766 (660403976209529)**.** A number is called Norwegian if it has three distinct positive divisors whose sum is equal to 2022. Determine the smallest Norwegian number. (Note: The total number of positive divisors of a Norwegian number is allowed to be larger than 3.)

Problem 1.767 (2117883853443241027)**.** On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices of which are colored in the same color. Could Petya prevent him?

Problem 1.768 (247248446755838)**.** Let $a_1, a_2, \ldots, a_{2019}$ be positive integers and P a polynomial with integer coefficients such that, for every positive integer n,

$$P(n) \text{ divides } a_1^n + a_2^n + \cdots + a_{2019}^n.$$

Prove that P is a constant polynomial.

Problem 1.769 (7094618843236371 20). Among 16 coins there are 8 heavy coins with weight of 11 g, and 8 light coins with weight of 10 g, but it's unknown what weight of any coin is. One of the coins is anniversary. How to know, is anniversary coin heavy or light, via three weighings on scales with two cups and without any weight?

Problem 1.770 (5726273084626389998). Fix an integer $n \geq 2$. A fairy chess piece leopard may move one cell up, or one cell to the right, or one cell diagonally down-left. A leopard is placed onto some cell of a $3n \times 3n$ chequer board. The leopard makes several moves, never visiting a cell twice, and comes back to the starting cell. Determine the largest possible number of moves the leopard could have made.

Problem 1.771 (155530102293601). A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

i) No line passes through any point of the configuration.

ii) No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 1.772 (8540244741312291150). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + f(x)) + f(y) = xf(y) + f(x + y)$$

for all real numbers x and y.

Problem 1.773 (2649132917657979429). Let $\mathcal{S}$ be a set of 10 points in a plane that lie within a disk of radius 1 billion. Define a *move* as picking a point $P \in \mathcal{S}$ and reflecting it across $\mathcal{S}$'s centroid. Does there always exist a sequence of at most 1500 moves after which all points of $\mathcal{S}$ are contained in a disk of radius 10?

Problem 1.774 (700438823364 55). Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is not marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Problem 1.775 (6405240413257919216). Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 1.776 (2650659158441459375). Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Problem 1.777 (5204026586393077531)**.** Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

Problem 1.778 (1293772592063302344)**.** In non-isosceles acute $\triangle ABC$, AP, BQ, CR is the height of the triangle. A_1 is the midpoint of BC, AA_1 intersects QR at K, QR intersects a straight line that crosses A and is parallel to BC at point D, the line connecting the midpoint of AH and K intersects DA_1 at A_2. Similarly define B_2, C_2. $\triangle A_2B_2C_2$ is known to be non-degenerate, and its circumscribed circle is ω. Prove that: there are circles $\odot A'$, $\odot B'$, $\odot C'$ tangent to and INSIDE ω satisfying: (1) $\odot A'$ is tangent to AB and AC, $\odot B'$ is tangent to BC and BA, and $\odot C'$ is tangent to CA and CB. (2) A', B', C' are different and collinear.

Problem 1.779 (6025085618534905645)**.** Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_BO_D}$.

Problem 1.780 (689874125173032)**.** Let ω_1, ω_2 be two non-intersecting circles, with circumcenters O_1, O_2 respectively, and radii r_1, r_2 respectively where $r_1 < r_2$. Let AB, XY be the two internal common tangents of ω_1, ω_2, where A, X lie on ω_1, B, Y lie on ω_2. The circle with diameter AB meets ω_1, ω_2 at P and Q respectively. If

$$\angle AO_1P + \angle BO_2Q = 180^\circ,$$

find the value of $\frac{PX}{QY}$ (in terms of r_1, r_2).

Problem 1.781 (526922799283626)**.** For each $1 \le i \le 9$ and $T \in \mathbb{N}$, define $d_i(T)$ to be the total number of times the digit i appears when all the multiples of 1829 between 1 and T inclusive are written out in base 10.

Show that there are infinitely many $T \in \mathbb{N}$ such that there are precisely two distinct values among $d_1(T)$, $d_2(T)$, $\dots$, $d_9(T)$

Problem 1.782 (536523538808893)**.** Find all polynomials $P(x)$ with real coefficients such that for all nonzero real numbers x,

$$P(x) + P\left(\frac{1}{x}\right) = \frac{P\left(x + \frac{1}{x}\right) + P\left(x - \frac{1}{x}\right)}{2}.$$

Problem 1.783 (4018921933875333744)**.** Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

Problem 1.784 (653318686726030)**.** Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard: In the first line, Gugu writes down every number of the form $a - b$, where a and b are two (not necessarily distinct) numbers on his napkin. In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the first line. In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the first line. Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

Problem 1.785 (156471770451237). Let ABC be an acute, scalene triangle. Let H be the orthocenter and O be the circumcenter of triangle ABC, and let P be a point interior to the segment HO. The circle with center P and radius PA intersects the lines AB and AC again at R and S, respectively. Denote by Q the symmetric point of P with respect to the perpendicular bisector of BC. Prove that points P, Q, R and S lie on the same circle.

Problem 1.786 (495587557940069). Let the excircle of a triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively. Denote by γ the circumcircle of triangle $A_1B_1C_1$ and assume that γ passes through vertex A. Show that $\overline{AA_1}$ is a diameter of γ. Show that the incenter of $\triangle ABC$ lies on line B_1C_1.

Problem 1.787 (4218160072471349910). Given is a natural number $n > 4$. There are n points marked on the plane, no three of which lie on the same line. Vasily draws one by one all the segments connecting pairs of marked points. At each step, drawing the next segment S, Vasily marks it with the smallest natural number, which hasn't appeared on a drawn segment that has a common end with S. Find the maximal value of k, for which Vasily can act in such a way that he can mark some segment with the number k?

Problem 1.788 (931951248564234). Let $n > 3$ be a positive integer. Suppose that n children are arranged in a circle, and n coins are distributed between them (some children may have no coins). At every step, a child with at least 2 coins may give 1 coin to each of their immediate neighbors on the right and left. Determine all initial distributions of the coins from which it is possible that, after a finite number of steps, each child has exactly one coin.

Problem 1.789 (141955509989127). Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \le i, j \le n$, such that: for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

Problem 1.790 (8823022869500312410). Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, 4, \cdots \right\}.$$

Prove that every integer $x \ge 2$ can be written as the product of one or more elements of A, which are not necessarily different. For every integer $x \ge 2$ let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different. Prove that there exist infinitely many pairs (x, y) of integers with $x \ge 2$, $y \ge 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \ne x_2$ or $y_1 \ne y_2$).

Problem 1.791 (4308913658510445082). Let $ABCD$ be a convex quadrilateral, the incenters of $\triangle ABC$ and $\triangle ADC$ are I, J, respectively. It is known that AC, BD, IJ

concurrent at a point P. The line perpendicular to BD through P intersects with the outer angle bisector of $\angle BAD$ and the outer angle bisector $\angle BCD$ at E, F, respectively. Show that $PE = PF$.

Problem 1.792 (436681276656848)**.** For the quadrilateral $ABCD$, let AC and BD intersect at E, AB and CD intersect at F, and AD and BC intersect at G. Additionally, let W, X, Y, and Z be the points of symmetry to E with respect to AB, BC, CD, and DA respectively. Prove that one of the intersection points of $\odot(FWY)$ and $\odot(GXZ)$ lies on the line FG.

Problem 1.793 (6848161986234395515)**.** Call a rational number short if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is $m-$tastic if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \le k < t$. Let $S(m)$ be the set of $m-$tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

Problem 1.794 (988671418474826)**.** Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

Problem 1.795 (3037670535896233971)**.** Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Problem 1.796 (522990139281725)**.** For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \ldots)$ is a p-sequence, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$. (a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n? (b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

Problem 1.797 (4330093832251809273)**.** Find all pairs (a, b) of positive integers such that a^3 is multiple of b^2 and $b - 1$ is multiple of $a - 1$.

Problem 1.798 (5901329049595563801)**.** Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f \colon \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

Problem 1.799 (106106949450397)**.** Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

Problem 1.800 (556895401643484982). Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 1.801 (614247648874042). Misha has a 100x100 chessboard and a bag with 199 rooks. In one move he can either put one rook from the bag on the lower left cell of the grid, or remove two rooks which are on the same cell, put one of them on the adjacent square which is above it or right to it, and put the other in the bag. Misha wants to place exactly 100 rooks on the board, which don't beat each other. Will he be able to achieve such arrangement?

Problem 1.802 (728988632553727). Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

Problem 1.803 (380257662603408). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Problem 1.804 (2988718857225198152). For a polynomial P and a positive integer n, define P_n as the number of positive integer pairs (a, b) such that $a < b \leq n$ and $|P(a)| - |P(b)|$ is divisible by n. Determine all polynomial P with integer coefficients such that $P_n \leq 2021$ for all positive integers n.

Problem 1.805 (748293992911976). A infinite sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfy $a_n \geq n^2$. Suppose that for each $i, j \geq 0$ there exist k, l with $(i, j) \neq (k, l)$, $l - k = j - i$, and $a_l - a_k = a_j - a_i$. Prove that $a_n \geq (n + 2016)^2$ for some n.

Problem 1.806 (1743818063911276331). Let $n \geq 3$ be a fixed integer. The number 1 is written n times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers a and b, replacing them with the numbers 1 and $a + b$, then adding one stone to the first bucket and $\gcd(a, b)$ stones to the second bucket. After some finite number of moves, there are s stones in the first bucket and t stones in the second bucket, where s and t are positive integers. Find all possible values of the ratio $\frac{t}{s}$.

Problem 1.807 (6306108494297192985). Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Problem 1.808 (541615131309445). Let Γ be the circumcircle of triangle ABC. The line parallel to AC passing through B meets Γ at D ($D \neq B$), and the line parallel to AB passing through C intersects Γ to E ($E \neq C$). Lines AB and CD meet at P, and lines AC and BE meet at Q. Let M be the midpoint of DE. Line AM meets Γ at Y

($Y \neq A$) and line PQ at J. Line PQ intersects the circumcircle of triangle BCJ at Z ($Z \neq J$). If lines BQ and CP meet each other at X, show that X lies on the line YZ.

Problem 1.809 (2008341270346760748)**.** Find the least positive integer n for which there exists a set $\{s_1, s_2, \dots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\cdots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

Problem 1.810 (1689010605544198840)**.** In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain.

Problem 1.811 (846826818545123)**.** Let n be a fixed positive integer. Ben is playing a computer game. The computer picks a tree T such that no vertex of T has degree 2 and such that T has exactly n leaves, labeled $v_1, \dots, v_n$. The computer then puts an integer weight on each edge of T, and shows Ben neither the tree T nor the weights. Ben can ask queries by specifying two integers $1 \leq i < j \leq n$, and the computer will return the sum of the weights on the path from v_i to v_j. At any point, Ben can guess whether the tree's weights are all zero. He wins the game if he is correct, and loses if he is incorrect.

(a) Show that if Ben asks all $\binom{n}{2}$ possible queries, then he can guarantee victory. (b) Does Ben have a strategy to guarantee victory in less than $\binom{n}{2}$ queries?

Problem 1.812 (8778540732652162753)**.** Let ABC be a triangle. Suppose that D, E, and F are points on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively such that triangles AEF, BFD, and CDE have equal inradii. Prove that the sum of the inradii of $\triangle AEF$ and $\triangle DEF$ is equal to the inradius of $\triangle ABC$.

Problem 1.813 (4824592143913840)**.** On a table with 25 columns and 300 rows, Kostya painted all its cells in three colors. Then, Lesha, looking at the table, for each row names one of the three colors and marks in that row all cells of that color (if there are no cells of that color in that row, he does nothing). After that, all columns that have at least a marked square will be deleted. Kostya wants to be left as few as possible columns in the table, and Lesha wants there to be as many as possible columns in the table. What is the largest number of columns Lesha can guarantee to leave?

Problem 1.814 (12700532379080534448)**.** For a sequence $a_1 < a_2 < \cdots < a_n$ of integers, a pair (a_i, a_j) with $1 \leq i < j \leq n$ is called interesting if there exists a pair (a_k, a_l) of integers with $1 \leq k < l \leq n$ such that

$$\frac{a_l - a_k}{a_j - a_i} = 2.$$

For each $n \geq 3$, find the largest possible number of interesting pairs in a sequence of length n.

Problem 1.815 (8670333331361701457)**.** Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil\frac{n}{1}\right\rceil+\left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{n}{3}\right\rceil+\cdots+\left\lceil\frac{n}{n}\right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problem 1.816 (586194373652638)**.** Let $m, n, a_1, a_2, \ldots, a_n$ be positive integers and r be a real number. Prove that the equation

$$\lfloor a_1 x \rfloor + \lfloor a_2 x \rfloor + \cdots + \lfloor a_n x \rfloor = sx + r$$

has exactly ms solutions in x, where $s = a_1 + a_2 + \cdots + a_n + \frac{1}{m}$.

Problem 1.817 (2974998787723554962)**.** There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled $A_1, A_2, \ldots, A_{2022}$ in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

Problem 1.818 (2556841339462610604)**.** Suppose that (a_1, b_1), (a_2, b_2), $\ldots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Problem 1.819 (172839066140251)**.** Let n be a positive integer. Let S be a set of ordered pairs (x, y) such that $1 \le x \le n$ and $0 \le y \le n$ in each pair, and there are no pairs (a, b) and (c, d) of different elements in S such that $a^2 + b^2$ divides both $ac + bd$ and $ad - bc$. In terms of n, determine the size of the largest possible set S.

Problem 1.820 (748616641641895)**.** Let ABC be a triangle. Let ABC_1, BCA_1, CAB_1 be three equilateral triangles that do not overlap with ABC. Let P be the intersection of the circumcircles of triangle ABC_1 and CAB_1. Let Q be the point on the circumcircle of triangle CAB_1 so that PQ is parallel to BA_1. Let R be the point on the circumcircle of triangle ABC_1 so that PR is parallel to CA_1.

Show that the line connecting the centroid of triangle ABC and the centroid of triangle PQR is parallel to BC.

Problem 1.821 (6751071460392744865)**.** Don Miguel places a token in one of the $(n+1)^2$ vertices determined by an $n \times n$ board. A move consists of moving the token from the vertex on which it is placed to an adjacent vertex which is at most $\sqrt{2}$ away, as long as it stays on the board. A path is a sequence of moves such that the token was in each one of the $(n+1)^2$ vertices exactly once. What is the maximum number of diagonal moves (those of length $\sqrt{2}$) that a path can have in total?

Problem 1.822 (7243491713649826569)**.** In the triangle ABC let B' and C' be the midpoints of the sides AC and AB respectively and H the foot of the altitude passing through the vertex A. Prove that the circumcircles of the triangles $AB'C'$,$BC'H$, and $B'CH$ have a common point I and that the line HI passes through the midpoint of the segment $B'C'$.

Problem 1.823 (173010886819234)**.** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

Problem 1.824 (521941955566221852)**.** Let $\mathbb{Q}_{>1}$ be the set of rational numbers greater than 1. Let $f : \mathbb{Q}_{>1} \to \mathbb{Z}$ be a function that satisfies

$$f(q) = \begin{cases} q - 3 & \text{if } q \text{ is an integer,} \\ \lceil q \rceil - 3 + f\left(\frac{1}{\lceil q \rceil - q}\right) & \text{otherwise.} \end{cases}$$

Show that for any $a, b \in \mathbb{Q}_{>1}$ with $\frac{1}{a} + \frac{1}{b} = 1$, we have $f(a) + f(b) = -2$.

Problem 1.825 (8982900673855870942)**.** Let there be an equilateral triangle ABC and a point P in its plane such that $AP < BP < CP$. Suppose that the lengths of segments AP, BP and CP uniquely determine the side of ABC. Prove that P lies on the circumcircle of triangle ABC.

Problem 1.826 (162618813015033)**.** In $\triangle ABC$, tangents of the circumcircle $\odot O$ at B, C and at A, B intersects at X, Y respectively. AX cuts BC at D and CY cuts AB at F. Ray DF cuts arc AB of the circumcircle at P. Q, R are on segments AB, AC such that P, Q, R are collinear and $QR \parallel BO$. If $PQ^2 = PR \cdot QR$, find $\angle ACB$.

Problem 1.827 (409530198849693)**.** In a cyclic convex hexagon $ABCDEF$, AB and DC intersect at G, AF and DE intersect at H. Let M, N be the circumcenters of BCG and EFH, respectively. Prove that the BE, CF and MN are concurrent.