spamgeoysl

PONTE A ENTRENAR

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§1 Problemas

Problem 1.1 (1810915585111530473). Given a scalene triangle $\triangle ABC$. B', C' are points lie on the rays $\overrightarrow{AB}, \overrightarrow{AC}$ such that $\overline{AB'} = \overline{AC}, \overline{AC'} = \overline{AB}$. Now, for an arbitrary point P in the plane. Let Q be the reflection point of P w.r.t $\overline{BC}$. The intersections of $\odot(BB'P)$ and $\odot(CC'P)$ is P' and the intersections of $\odot(BB'Q)$ and $\odot(CC'Q)$ is Q'. Suppose that O, O' are circumcenters of $\triangle ABC, \triangle AB'C'$ Show that

1. O', P', Q' are colinear
2. $\overline{O'P'} \cdot \overline{O'Q'} = \overline{OA}^2$

Problem 1.2 (781756252908608). Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i,j) \,|\, 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j)\in A} a_i a_j < 0.$$

Problem 1.3 (7550072974614174968). Let $n \geqslant 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[0,1]$. Let $s = x_1 + x_2 + \ldots + x_n$, and assume that $s \geqslant 3$. Prove that there exist integers i and j with $1 \leqslant i < j \leqslant n$ such that

$$2^{j-i} x_i x_j > 2^{s-3}.$$

Problem 1.4 (8963205841174892420). Let $ABCD$ be a convex quadrilateral with pairwise distinct side lengths such that $AC \perp BD$. Let O_1, O_2 be the circumcenters of $\Delta ABD, \Delta CBD$, respectively. Show that AO_2, CO_1, the Euler line of ΔABC and the Euler line of ΔADC are concurrent.

(Remark: The Euler line of a triangle is the line on which its circumcenter, centroid, and orthocenter lie.)

Problem 1.5 (1248852037865425410). Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied: Each number in the table is congruent to 1 modulo n. The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2. Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that the sums $R_1 + \ldots R_n$ and $C_1 + \ldots C_n$ are congruent modulo n^4.

Problem 1.6 (4992489807901310938)**.** Let ABC be a triangle and ℓ_1, ℓ_2 be two parallel lines. Let ℓ_i intersects line BC, CA, AB at X_i, Y_i, Z_i, respectively. Let Δ_i be the triangle formed by the line passed through X_i and perpendicular to BC, the line passed through Y_i and perpendicular to CA, and the line passed through Z_i and perpendicular to AB. Prove that the circumcircles of Δ_1 and Δ_2 are tangent.

Problem 1.7 (6246999615324043054)**.** A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 1.8 (6116877365036470315)**.** Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions: (i) $f(n) \neq 0$ for at least one n; (ii) $f(xy) = f(x) + f(y)$ for every positive integers x and y; (iii) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

Problem 1.9 (3470579368412517052)**.** A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either: the rabbit cannot move; or the hunter can determine the cell in which the rabbit started. Decide whether there exists a winning strategy for the hunter.

Problem 1.10 (883811987981100)**.** Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 1.11 (6513083395063379942)**.** Given a convex pentagon $ABCDE$. Let A_1 be the intersection of BD with CE and define B_1, C_1, D_1, E_1 similarly, A_2 be the second intersection of $\odot(ABD_1), \odot(AEC_1)$ and define B_2, C_2, D_2, E_2 similarly. Prove that $AA_2, BB_2, CC_2, DD_2, EE_2$ are concurrent.

Problem 1.12 (8059760967121829853)**.** Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Problem 1.13 (456772085666528)**.** Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Problem 1.14 (587866144613888)**.** The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 1.15 (528087142744727)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Problem 1.16 (16206169636605432410)**.** Given an isosceles triangle $\triangle ABC$, $AB = AC$. A line passes through M, the midpoint of BC, and intersects segment AB and ray CA at D and E, respectively. Let F be a point of ME such that $EF = DM$, and K be a point on MD. Let Γ_1 be the circle passes through B, D, K and Γ_2 be the circle passes through C, E, K. Γ_1 and Γ_2 intersect again at $L \neq K$. Let ω_1 and ω_2 be the circumcircle of $\triangle LDE$ and $\triangle LKM$. Prove that, if ω_1 and ω_2 are symmetric wrt L, then BF is perpendicular to BC.

Problem 1.17 (7088779505939683183)**.** Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

Problem 1.18 (689874125173032)**.** Let ω_1, ω_2 be two non-intersecting circles, with circumcenters O_1, O_2 respectively, and radii r_1, r_2 respectively where $r_1 < r_2$. Let AB, XY be the two internal common tangents of ω_1, ω_2, where A, X lie on ω_1, B, Y lie on ω_2. The circle with diameter AB meets ω_1, ω_2 at P and Q respectively. If

$$\angle AO_1P + \angle BO_2Q = 180^\circ,$$

find the value of $\frac{PX}{QY}$ (in terms of r_1, r_2).

Problem 1.19 (526922799283626)**.** For each $1 \leq i \leq 9$ and $T \in \mathbb{N}$, define $d_i(T)$ to be the total number of times the digit i appears when all the multiples of 1829 between 1 and T inclusive are written out in base 10.

Show that there are infinitely many $T \in \mathbb{N}$ such that there are precisely two distinct values among $d_1(T)$, $d_2(T)$, $\dots$, $d_9(T)$

Problem 1.20 (86986480818494)**.** Given a scalene triangle ABC inscribed in the circle (O). Let (I) be its incircle and BI, CI cut AC, AB at E, F respectively. A circle passes through E and touches OB at B cuts (O) again at M. Similarly, a circle passes through F and touches OC at C cuts (O) again at N. ME, NF cut (O) again at P, Q. Let K be the intersection of EF and BC and let PQ cuts BC and EF at G, H, respectively. Show that the median correspond to G of the triangle GHK is perpendicular to IO.

Problem 1.21 (7243491713649826569)**.** In the triangle ABC let B' and C' be the midpoints of the sides AC and AB respectively and H the foot of the altitude passing through the vertex A. Prove that the circumcircles of the triangles $AB'C'$,$BC'H$, and $B'CH$ have a common point I and that the line HI passes through the midpoint of the segment $B'C'$.

Problem 1.22 (4678973565823282552)**.** Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

Problem 1.23 (902621191535073)**.** Given six points A, B, C, D, E, F such that $\triangle BCD \overset{+}{\sim} \triangle ECA \overset{+}{\sim} \triangle BFA$ and let I be the incenter of $\triangle ABC$. Prove that the circumcenter of $\triangle AID, \triangle BIE, \triangle CIF$ are collinear.

Problem 1.24 (748616641641895)**.** Let ABC be a triangle. Let ABC_1, BCA_1, CAB_1 be three equilateral triangles that do not overlap with ABC. Let P be the intersection of the circumcircles of triangle ABC_1 and CAB_1. Let Q be the point on the circumcircle of triangle CAB_1 so that PQ is parallel to BA_1. Let R be the point on the circumcircle of triangle ABC_1 so that PR is parallel to CA_1.

Show that the line connecting the centroid of triangle ABC and the centroid of triangle PQR is parallel to BC.

Problem 1.25 (7025878918490077)**.** Given an integer $n \geqslant 2$. Suppose there is a point P inside a convex cyclic $2n$-gon $A_1 \ldots A_{2n}$ satisfying

$$\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_{2n}A_1,$$

prove that

$$\prod_{i=1}^{n} |A_{2i-1}A_{2i}| = \prod_{i=1}^{n} |A_{2i}A_{2i+1}|,$$

where $A_{2n+1} = A_1$.

Problem 1.26 (6953300922471087077)**.** There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 1.27 (264456837378391)**.** Let ABC be a triangle such that the angular bisector of $\angle BAC$, the B-median and the perpendicular bisector of AB intersect at a single point X. Let H be the orthocenter of ABC. Show that $\angle BXH = 90°$.

Problem 1.28 (915997916422887)**.** Let ABC and $A'B'C'$ be two triangles so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ form a triangle as well. Suppose that for any point X on the circumcircle of ABC, there exists exactly one point X' on the circumcircle of $A'B'C'$ so that the midpoints of $\overline{AA'}, \overline{BB'}, \overline{CC'}$ and $\overline{XX'}$ are concyclic. Show that ABC is similar to $A'B'C'$.

Problem 1.29 (2672133756769464425)**.** Is there a scalene triangle ABC similar to triangle IHO, where I, H, and O are the incenter, orthocenter, and circumcenter, respectively, of triangle ABC?

Problem 1.30 (836909183133087)**.** Given a triangle $\triangle ABC$ with circumcircle Ω. Denote its incenter and A-excenter by I, J, respectively. Let T be the reflection of J w.r.t BC and P is the intersection of BC and AT. If the circumcircle of $\triangle AIP$ intersects BC at $X \neq P$ and there is a point $Y \neq A$ on Ω such that $IA = IY$. Show that $\odot(IXY)$ tangents to the line AI.

Problem 1.31 (156060759856343521)**.** Let ABC be an acute triangle with $\angle ACB > 2\angle ABC$. Let I be the incenter of ABC, K is the reflection of I in line BC. Let line BA and KC intersect at D. The line through B parallel to CI intersects the minor arc BC on the circumcircle of ABC at $E(E \neq B)$. The line through A parallel to BC intersects the line BE at F. Prove that if $BF = CE$, then $FK = AD$.

Problem 1.32 (4375421764909014892)**.** Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and
$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

Problem 1.33 (297274918587198)**.** Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \dots, d_k)$ such that for $i = 1, 2, \dots, k$, the number $d_1 + d_2 + \dots + d_i$ is a perfect square.

Problem 1.34 (8916142707013964275)**.** Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

Problem 1.35 (6612845742708555351)**.** Cyclic quadrilateral $ABCD$ has circumcircle (O). Points M and N are the midpoints of BC and CD, and E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Let EN meet FM at P. Denote S as the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

Problem 1.36 (6734490609685717062)**.** Let I, G, O be the incenter, centroid and the circumcenter of triangle ABC, respectively. Let X, Y, Z be on the rays BC, CA, AB respectively so that $BX = CY = AZ$. Let F be the centroid of XYZ.

Show that FG is perpendicular to IO.

Problem 1.37 (165465510156789)**.** Let Ω be the circumcircle of an isosceles trapezoid $ABCD$, in which AD is parallel to BC. Let X be the reflection point of D with respect to BC. Point Q is on the arc BC of Ω that does not contain A. Let P be the intersection of DQ and BC. A point E satisfies that EQ is parallel to PX, and EQ bisects $\angle BEC$. Prove that EQ also bisects $\angle AEP$.

Problem 1.38 (952584318797289)**.** Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

Problem 1.39 (5742237863842294)**.** Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

Problem 1.40 (579228243242060)**.** Let $ABCD$ be a parallelogram. A line through C crosses the side AB at an interior point X, and the line AD at Y. The tangents of the circle AXY at X and Y, respectively, cross at T. Prove that the circumcircles of triangles ABD and TXY intersect at two points, one lying on the line AT and the other one lying on the line CT.

Problem 1.41 (162618813015033)**.** In $\triangle ABC$, tangents of the circumcircle $\odot O$ at B, C and at A, B intersects at X, Y respectively. AX cuts BC at D and CY cuts AB at F. Ray DF cuts arc AB of the circumcircle at P. Q, R are on segments AB, AC such that P, Q, R are collinear and $QR \parallel BO$. If $PQ^2 = PR \cdot QR$, find $\angle ACB$.

Problem 1.42 (57940096937913)**.** Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

Problem 1.43 (4451072691230235426)**.** A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.

Problem 1.44 (961350373727093)**.** Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

Problem 1.45 (5441518070935718077)**.** Let ABC be an acute-angled triangle. The line through C perpendicular to AC meets the external angle bisector of $\angle ABC$ at D. Let H be the foot of the perpendicular from D onto BC. The point K is chosen on AB so that $KH \parallel AC$. Let M be the midpoint of AK. Prove that $MC = MB + BH$.

Problem 1.46 (5713525138856417722)**.** A cyclic quadrilateral $ABCD$ has circumcircle Γ, and $AB + BC = AD + DC$. Let E be the midpoint of arc BCD, and $F(\neq C)$ be the antipode of A wrt Γ. Let I, J, K be the incenter of $\triangle ABC$, the A-excenter of $\triangle ABC$, the incenter of $\triangle BCD$, respectively. Suppose that a point P satisfies $\triangle BIC \stackrel{+}{\sim} \triangle KPJ$. Prove that EK and PF intersect on Γ.

Problem 1.47 (522990139281725)**.** For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \ldots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \ldots)$ is a p-sequence, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for

$n \geqslant 0$. (a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n? (b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

Problem 1.48 (457324036151847)**.** Let O and H be the circumcenter and the orthocenter, respectively, of an acute triangle ABC. Points D and E are chosen from sides AB and AC, respectively, such that A, D, O, E are concyclic. Let P be a point on the circumcircle of triangle ABC. The line passing P and parallel to OD intersects AB at point X, while the line passing P and parallel to OE intersects AC at Y. Suppose that the perpendicular bisector of $\overline{HP}$ does not coincide with XY, but intersect XY at Q, and that points A, Q lies on the different sides of DE. Prove that $\angle EQD = \angle BAC$.

Problem 1.49 (3435532350205377704)**.** Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.

Problem 1.50 (6978535805224432571)**.** The Fibonacci numbers $F_0, F_1, F_2, \dots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \dots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Problem 1.51 (623590906176957)**.** The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 1.52 (7997372712267182584)**.** Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Problem 1.53 (625002281186392279)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Problem 1.54 (8972547734710795566)**.** Let incircle (I) of triangle ABC touch the sides BC, CA, AB at D, E, F respectively. Let (O) be the circumcircle of ABC. Ray EF meets (O) at M. Tangents at M and A of (O) meet at S. Tangents at B and C of (O) meet at T. Line TI meets OA at J. Prove that $\angle ASJ = \angle IST$.

Problem 1.55 (8569243655022492300)**.** Given a $\triangle ABC$ and a point P. Let O, D, E, F be the circumcenter of $\triangle ABC, \triangle BPC, \triangle CPA, \triangle APB$, respectively and let T be the intersection of BC with EF. Prove that the reflection of O in EF lies on the perpendicular from D to PT.

Problem 1.56 (2918584823978789760)**.** A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Problem 1.57 (7553717274310387624)**.** Let ABC be a triangle with incentre I and circumcircle ω. The incircle of the triangle ABC touches the sides BC, CA and AB at D, E and F, respectively. The circumcircle of triangle ADI crosses ω again at P, and the lines PE and PF cross ω again at $X$and Y, respectively. Prove that the lines AI, BX and CY are concurrent.

Problem 1.58 (3192129869376364982)**.** Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

Problem 1.59 (596902679696332)**.** Find all positive integers $n \geqslant 2$ for which there exist n real numbers $a_1 < \cdots < a_n$ and a real number $r > 0$ such that the $\frac{1}{2}n(n-1)$ differences $a_j - a_i$ for $1 \leqslant i < j \leqslant n$ are equal, in some order, to the numbers $r^1, r^2, \ldots, r^{\frac{1}{2}n(n-1)}$.

Problem 1.60 (258585206260584)**.** Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Problem 1.61 (4948608980214807448)**.** Let ABC be a scalene triangle with circumcenter O and orthocenter H. Let AYZ be another triangle sharing the vertex A such that its circumcenter is H and its orthocenter is O. Show that if Z is on BC, then A, H, O, Y are concyclic.

Problem 1.62 (6842650432632160)**.** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 1.63 (2749225075653830789)**.** In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and every quadrilateral Q_i has three corners of one color and one corner of the other color.

Problem 1.64 (6558910862034852540)**.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2$$

Problem 1.65 (3866807698726339637)**.** Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbors - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Problem 1.66 (967014444176640)**.** Let $m, n \geqslant 2$ be integers, let X be a set with n elements, and let $X_1, X_2, \ldots, X_m$ be pairwise distinct non-empty, not necessary disjoint subset of X. A function $f\colon X \to \{1, 2, \ldots, n+1\}$ is called nice if there exists an index k such that

$$\sum_{x \in X_k} f(x) > \sum_{x \in X_i} f(x) \quad \text{for all } i \neq k.$$

Prove that the number of nice functions is at least n^n.

Problem 1.67 (8534263250311217423)**.** In acute triangle $\triangle ABC$, $\angle A > \angle B > \angle C$. $\triangle AC_1B$ and $\triangle CB_1A$ are isosceles triangles such that $\triangle AC_1B \overset{+}{\sim} \triangle CB_1A$. Let lines BB_1, CC_1 intersects at T. Prove that if all points mentioned above are distinct, $\angle ATC$ isn't a right angle.

Problem 1.68 (1293772592063302344)**.** In non-isosceles acute $\triangle ABC$, AP, BQ, CR is the height of the triangle. A_1 is the midpoint of BC, AA_1 intersects QR at K, QR intersects a straight line that crosses A and is parallel to BC at point D, the line connecting the midpoint of AH and K intersects DA_1 at A_2. Similarly define B_2, C_2. $\triangle A_2B_2C_2$ is known to be non-degenerate, and its circumscribed circle is ω. Prove that: there are circles $\odot A'$, $\odot B'$, $\odot C'$ tangent to and INSIDE ω satisfying: (1) $\odot A'$ is tangent to AB and AC, $\odot B'$ is tangent to BC and BA, and $\odot C'$ is tangent to CA and CB. (2) A', B', C' are different and collinear.

Problem 1.69 (682786464566571)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.70 (727078403801409)**.** Let ABC be a triangle with incenter I and circumcircle Ω. A point X on Ω which is different from A satisfies $AI = XI$. The incircle touches AC and AB at E, F, respectively. Let M_a, M_b, M_c be the midpoints of sides BC, CA, AB, respectively. Let T be the intersection of the lines M_bF and M_cE. Suppose that AT intersects Ω again at a point S.

Prove that X, M_a, S, T are concyclic.

Problem 1.71 (537574018594693)**.** Let ABC be a triangle with O as its circumcenter. A circle Γ tangents OB, OC at B, C, respectively. Let D be a point on Γ other than B with $CB = CD$, E be the second intersection of DO and Γ, and F be the second intersection of EA and Γ. Let X be a point on the line AC so that $XB \perp BD$. Show that one half of $\angle ADF$ is equal to one of $\angle BDX$ and $\angle BXD$.

Problem 1.72 (5261846980754565299)**.** Let A, B, C be the midpoints of the three sides $B'C', C'A', A'B'$ of the triangle $A'B'C'$ respectively. Let P be a point inside ΔABC, and AP, BP, CP intersect with BC, CA, AB at P_a, P_b, P_c, respectively. Lines P_aP_b, P_aP_c intersect with $B'C'$ at R_b, R_c respectively, lines P_bP_c, P_bP_a intersect with $C'A'$ at S_c, S_a respectively. and lines P_cP_a, P_cP_b intersect with $A'B'$ at T_a, T_b, respectively. Given that S_c, S_a, T_a, T_b are all on a circle centered at O.

Show that $OR_b = OR_c$.

Problem 1.73 (5066939379306191291)**.** Let ABC be an acute triangle with circumcenter O and circumcircle Ω. Choose points D, E from sides AB, AC, respectively, and let ℓ be the line passing through A and perpendicular to DE. Let ℓ intersect the circumcircle of triangle ADE and Ω again at points P, Q, respectively. Let N be the intersection of OQ

and BC, S be the intersection of OP and DE, and W be the orthocenter of triangle SAO.

Prove that the points S, N, O, W are concyclic.

Problem 1.74 (402654566950359). Let a_1, a_2, ... be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 1.75 (132497611943266). Suppose that a, b, c, d are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 1.76 (8895719454292056765). Given a non-right triangle ABC with $BC > AC > AB$. Two points $P_1 \neq P_2$ on the plane satisfy that, for $i = 1, 2$, if AP_i, BP_i and CP_i intersect the circumcircle of the triangle ABC at D_i, E_i, and F_i, respectively, then $D_iE_i \perp D_iF_i$ and $D_iE_i = D_iF_i \neq 0$. Let the line P_1P_2 intersects the circumcircle of ABC at Q_1 and Q_2. The Simson lines of Q_1, Q_2 with respect to ABC intersect at W.

Prove that W lies on the nine-point circle of ABC.

Problem 1.77 (1580707630770476037). Two triangles intersect to form seven finite disjoint regions, six of which are triangles with area 1. The last region is a hexagon with area A. Compute the minimum possible value of A.

Problem 1.78 (876239022447910). Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

Problem 1.79 (723258861624579). Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Problem 1.80 (733773583946080). AB and AC are tangents to a circle ω with center O at B, C respectively. Point P is a variable point on minor arc BC. The tangent at P to ω meets AB, AC at D, E respectively. AO meets BP, CP at U, V respectively. The line through P perpendicular to AB intersects DV at M, and the line through P perpendicular to AC intersects EU at N. Prove that as P varies, MN passes through a fixed point.

Problem 1.81 (5867489266334805897). Let $ABCDE$ be a pentagon inscribed in a circle Ω. A line parallel to the segment BC intersects AB and AC at points S and T, respectively. Let X be the intersection of the line BE and DS, and Y be the intersection of the line CE and DT.

Prove that, if the line AD is tangent to the circle $\odot(DXY)$, then the line AE is tangent to the circle $\odot(EXY)$.

Problem 1.82 (3906812380515301028). Given a triangle $\triangle ABC$. Denote its incenter and orthocenter by I, H, respectively. If there is a point K with

$$AH + AK = BH + BK = CH + CK$$

Show that H, I, K are collinear.

Problem 1.83 (290912955085727393). Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

Problem 1.84 (6193947856984766386). Let $ABCD$ be a cyclic quadrilateral. Assume that the points Q, A, B, P are collinear in this order, in such a way that the line AC is tangent to the circle ADQ, and the line BD is tangent to the circle BCP. Let M and N be the midpoints of segments BC and AD, respectively. Prove that the following three lines are concurrent: line CD, the tangent of circle ANQ at point A, and the tangent to circle BMP at point B.

Problem 1.85 (215375559035207). ABC is an isosceles triangle, with $AB = AC$. D is a moving point such that $AD \parallel BC$, $BD > CD$. Moving point E is on the arc of BC in circumcircle of ABC not containing A, such that $EB < EC$. Ray BC contains point F with $\angle ADE = \angle DFE$. If ray FD intersects ray BA at X, and intersects ray CA at Y, prove that $\angle XEY$ is a fixed angle.

Problem 1.86 (2252133047011954512). On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

Problem 1.87 (161342796381450). For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

Problem 1.88 (796349431725149)**.** An acute, non-isosceles triangle ABC is inscribed in a circle with centre O. A line go through O and midpoint I of BC intersects AB, AC at E, F respectively. Let D, G be reflections to A over O and circumcentre of (AEF), respectively. Let K be the reflection of O over circumcentre of (OBC). $a)$ Prove that D, G, K are collinear. $b)$ Let M, N are points on KB, KC that $IM \perp AC$, $IN \perp AB$. The midperpendiculars of IK intersects MN at H. Assume that IH intersects AB, AC at P, Q respectively. Prove that the circumcircle of $\triangle APQ$ intersects (O) the second time at a point on AI.

Problem 1.89 (5514383858686655851)**.** Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

Problem 1.90 (7494618588207758150)**.** An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \cdots + 2018$?

Problem 1.91 (3813623497653179264)**.** The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

Problem 1.92 (70043882336455)**.** Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is not marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Problem 1.93 (2134021625648303394)**.** The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily distinct) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$. Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

Problem 1.94 (6654677204410680146)**.** In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

Problem 1.95 (4892352754475215646)**.** We say that a set S of integers is rootiful if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 1.96 (318208660266829737)**.** Let ABC be an acute-angled triangle with $AB \neq AC$, and let I and O be its incenter and circumcenter, respectively. Let the incircle touch BC, CA and AB at D, E and F, respectively. Assume that the line through I parallel to EF, the line through D parallel toAO, and the altitude from A are concurrent. Prove that the concurrency point is the orthocenter of the triangle ABC.

Problem 1.97 (8799177804774743019)**.** In each square of a garden shaped like a 2022×2022 board, there is initially a tree of height 0. A gardener and a lumberjack alternate turns playing the following game, with the gardener taking the first turn: The gardener chooses a square in the garden. Each tree on that square and all the surrounding squares (of which there are at most eight) then becomes one unit taller. The lumberjack then chooses four different squares on the board. Each tree of positive height on those squares then becomes one unit shorter. We say that a tree is majestic if its height is at least 10^6. Determine the largest K such that the gardener can ensure there are eventually K majestic trees on the board, no matter how the lumberjack plays.

Problem 1.98 (436681276656848)**.** For the quadrilateral $ABCD$, let AC and BD intersect at E, AB and CD intersect at F, and AD and BC intersect at G. Additionally, let W, X, Y, and Z be the points of symmetry to E with respect to AB, BC, CD, and DA respectively. Prove that one of the intersection points of $\odot(FWY)$ and $\odot(GXZ)$ lies on the line FG.

Problem 1.99 (528504335909385)**.** Given a triangle $\triangle ABC$ whose incenter is I and A-excenter is J. A' is point so that AA' is a diameter of $\odot(\triangle ABC)$. Define H_1, H_2 to be the orthocenters of $\triangle BIA'$ and $\triangle CJA'$. Show that $H_1 H_2 \parallel BC$

Problem 1.100 (80567267310692)**.** Let n be a positive integer. Given is a subset A of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from A such that $c + 2a > 3b$.

Problem 1.101 (660403976209529)**.** A number is called Norwegian if it has three distinct positive divisors whose sum is equal to 2022. Determine the smallest Norwegian number. (Note: The total number of positive divisors of a Norwegian number is allowed to be larger than 3.)

Problem 1.102 (208441124738479)**.** Let $f : \{1, 2, 3, \ldots\} \to \{2, 3, \ldots\}$ be a function such that $f(m + n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

Problem 1.103 (931951248564234)**.** Let $n > 3$ be a positive integer. Suppose that n children are arranged in a circle, and n coins are distributed between them (some children may have no coins). At every step, a child with at least 2 coins may give 1 coin to each of their immediate neighbors on the right and left. Determine all initial distributions of the coins from which it is possible that, after a finite number of steps, each child has exactly one coin.

Problem 1.104 (852531542088551). Given a triangle ABC for which $\angle BAC \neq 90°$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

Problem 1.105 (239934686230450). Let triangle$ABC(AB < AC)$ with incenter I circumscribed in $\odot O$. Let M, N be midpoint of arc $\widehat{BAC}$ and $\widehat{BC}$, respectively. D lies on $\odot O$ so that $AD//BC$, and E is tangency point of A-excircle of $\triangle ABC$. Point F is in $\triangle ABC$ so that $FI//BC$ and $\angle BAF = \angle EAC$. Extend NF to meet $\odot O$ at G, and extend AG to meet line IF at L. Let line AF and DI meet at K. Proof that $ML \perp NK$.

Problem 1.106 (5897111412933990257). Let ABC be a triangle with circumcircle Γ, and points E and F are chosen from sides CA, AB, respectively. Let the circumcircle of triangle AEF and Γ intersect again at point X. Let the circumcircles of triangle ABE and ACF intersect again at point K. Line AK intersect with Γ again at point M other than A, and N be the reflection point of M with respect to line BC. Let XN intersect with Γ again at point S other that X.

Prove that SM is parallel to BC.

Problem 1.107 (591652153716935). Let M be the midpoint of BC of triangle ABC. The circle with diameter BC, ω, meets AB, AC at D, E respectively. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PAC, \angle PCA = \angle PAB$, and $2PM \cdot DE = BC^2$. Point X lies outside ω such that $XM \parallel AP$, and $\frac{XB}{XC} = \frac{AB}{AC}$. Prove that $\angle BXC + \angle BAC = 90°$.

Problem 1.108 (3159161448000677570). Let $a > 1$ be a positive integer and $d > 1$ be a positive integer coprime to a. Let $x_1 = 1$, and for $k \geq 1$, define

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \text{ does not divide } x_k \\ x_k/a & \text{if } a \text{ divides } x_k \end{cases}$$

Find, in terms of a and d, the greatest positive integer n for which there exists an index k such that x_k is divisible by a^n.

Problem 1.109 (819328919046836). Which positive integers n make the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \frac{n^2(n-1)}{4}$$

true?

Problem 1.110 (797215984506934). Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Problem 1.111 (37921131297270). You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leq r \leq 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

Problem 1.112 (11684474669717623345)**.** Let I, O, ω, Ω be the incenter, circumcenter, the incircle, and the circumcircle, respectively, of a scalene triangle ABC. The incircle ω is tangent to side BC at point D. Let S be the point on the circumcircle Ω such that AS, OI, BC are concurrent. Let H be the orthocenter of triangle BIC. Point T lies on Ω such that $\angle ATI$ is a right angle. Prove that the points D, T, H, S are concyclic.

Problem 1.113 (719467452801051)**.** Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

Problem 1.114 (3417358984411200361)**.** Let ABC be a triangle with circumcircle Ω, circumcenter O and orthocenter H. Let S lie on Ω and P lie on BC such that $\angle ASP = 90^\circ$, line SH intersects the circumcircle of $\triangle APS$ at $X \neq S$. Suppose OP intersects CA, AB at Q, R, respectively, QY, RZ are the altitude of $\triangle AQR$. Prove that X, Y, Z are collinear.

Problem 1.115 (409146991986056)**.** For each prime p, construct a graph G_p on $\{1, 2, \ldots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2 + 1 - n)(n^2 + 1 - m)$. Prove that G_p is disconnected for infinitely many p

Problem 1.116 (908587245178389)**.** Let I be the incenter of triangle ABC, and ℓ be the perpendicular bisector of AI. Suppose that P is on the circumcircle of triangle ABC, and line AP and ℓ intersect at point Q. Point R is on ℓ such that $\angle IPR = 90^\circ$.Suppose that line IQ and the midsegment of ABC that is parallel to BC intersect at M. Show that $\angle AMR = 90^\circ$

(Note: In a triangle, a line connecting two midpoints is called a midsegment.)

Problem 1.117 (669395675904242)**.** Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 1.118 (8811824418974048155)**.** $ABCDE$ is a cyclic pentagon, with circumcentre O. $AB = AE = CD$. I midpoint of BC. J midpoint of DE. F is the orthocentre of $\triangle ABE$, and G the centroid of $\triangle AIJ.CE$ intersects BD at H, OG intersects FH at M. Show that $AM \perp CD$.

Problem 1.119 (6306108494297192985)**.** Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Problem 1.120 (3245291910836201005)**.** Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

Problem 1.121 (1222382895728709073). Given a triangle ABC, a circle Ω is tangent to AB, AC at B, C, respectively. Point D is the midpoint of AC, O is the circumcenter of triangle ABC. A circle Γ passing through A, C intersects the minor arc BC on Ω at P, and intersects AB at Q. It is known that the midpoint R of minor arc PQ satisfies that $CR \perp AB$. Ray PQ intersects line AC at L, M is the midpoint of AL, N is the midpoint of DR, and X is the projection of M onto ON. Prove that the circumcircle of triangle DNX passes through the center of Γ.

Problem 1.122 (2003233604438068678). Given a triangle ABC and a point O on a plane. Let Γ be the circumcircle of ABC. Suppose that CO intersects with AB at D, and BO and CA intersect at E. Moreover, suppose that AO intersects with Γ at A, F. Let I be the other intersection of Γ and the circumcircle of ADE, and Y be the other intersection of BE and the circumcircle of CEI, and Z be the other intersection of CD and the circumcircle of BDI. Let T be the intersection of the two tangents of Γ at B, C, respectively. Lastly, suppose that TF intersects with Γ again at U, and the reflection of U w.r.t. BC is G.

Show that F, I, G, O, Y, Z are concyclic.

Problem 1.123 (448881061747528). A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \cdots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

Problem 1.124 (7220404010846068686). Let ABC be a acute, non-isosceles triangle. D, E, F are the midpoints of sides AB, BC, AC, resp. Denote by (O), (O') the circumcircle and Euler circle of ABC. An arbitrary point P lies inside triangle DEF and DP, EP, FP intersect (O') at D', E', F', resp. Point A' is the point such that D' is the midpoint of AA'. Points B', C' are defined similarly. a. Prove that if $PO = PO'$ then $O \in (A'B'C')$; b. Point A' is mirrored by OD, its image is X. Y, Z are created in the same manner. H is the orthocenter of ABC and XH, YH, ZH intersect BC, AC, AB at M, N, L resp. Prove that M, N, L are collinear.

Problem 1.125 (5873161915777778529). In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

Problem 1.126 (16776483958513). Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 1.127 (844684477828422). Let point H be the orthocenter of a scalene triangle ABC. Line AH intersects with the circumcircle Ω of triangle ABC again at point P. Line BH, CH meets with AC, AB at point E and F, respectively. Let PE, PF meet Ω again at point Q, R, respectively. Point Y lies on Ω so that lines AY, QR and EF are concurrent. Prove that PY bisects EF.

Problem 1.128 (8152181601565653036)**.** Let D be a point on segment PQ. Let ω be a fixed circle passing through D, and let A be a variable point on ω. Let X be the intersection of the tangent to the circumcircle of $\triangle ADP$ at P and the tangent to the circumcircle of $\triangle ADQ$ at Q. Show that as A varies, X lies on a fixed line.

Problem 1.129 (4389998719836463980)**.** Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

Problem 1.130 (8255863576892581507)**.** Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

Problem 1.131 (9153191064326230951)**.** Let scalene triangle ABC have altitudes AD, BE, CF and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines PE at $X \neq P$ and PF at $Y \neq P$. Prove that $XY \parallel BC$.

Problem 1.132 (4095301988849693)**.** In a cyclic convex hexagon $ABCDEF$, AB and DC intersect at G, AF and DE intersect at H. Let M, N be the circumcenters of BCG and EFH, respectively. Prove that the BE, CF and MN are concurrent.

Problem 1.133 (282712203118607)**.** Let ABC be an acute-angled triangle with $AC > AB$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC, and AB at W, X, and Y, respectively. The circumcircles of triangles AXY and ABC intersect again at $Z \neq A$. Prove that if $W \neq D$ and $OW = OD$, then DZ is tangent to the circle AXY.

Problem 1.134 (1891712635906763103)**.** Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle BMC$ so that $\angle KBC = 90°$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.

Problem 1.135 (6302540840099076878)**.** Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

Problem 1.136 (8866273454792491736)**.** Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x - y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

Problem 1.137 (227919487650283)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Ω. Let M be the midpoint of side BC. Point D is chosen from the minor arc BC on Γ such that $\angle BAD = \angle MAC$. Let E be a point on Γ such that DE is perpendicular to AM, and F be a point on line BC such that DF is perpendicular to

BC. Lines HF and AM intersect at point N, and point R is the reflection point of H with respect to N.

Prove that $\angle AER + \angle DFR = 180°$.

Problem 1.138 (493493847475466779)**.** Let ABC be a triangle and let H be the orthogonal projection of A on the line BC. Let K be a point on the segment AH such that $AH = 3KH$. Let O be the circumcenter of triangle ABC and let M and N be the midpoints of sides AC and AB respectively. The lines KO and MN meet at a point Z and the perpendicular at Z to OK meets lines AB, AC at X and Y respectively. Show that $\angle XKY = \angle CKB$.

Problem 1.139 (423911944927735)**.** In acute $\triangle ABC$, O is the circumcenter, I is the incenter. The incircle touches BC, CA, AB at D, E, F. And the points K, M, N are the midpoints of BC, CA, AB respectively.

a) Prove that the lines passing through D, E, F in parallel with IK, IM, IN respectively are concurrent.

b) Points T, P, Q are the middle points of the major arc BC, CA, AB on $\odot ABC$. Prove that the lines passing through D, E, F in parallel with IT, IP, IQ respectively are concurrent.

Problem 1.140 (5835156231907738776)**.** Given triangle ABC with A-excenter I_A, the foot of the perpendicular from I_A to BC is D. Let the midpoint of segment I_AD be M, T lies on arc BC(not containing A) satisfying $\angle BAT = \angle DAC$, I_AT intersects the circumcircle of ABC at $S \neq T$. If SM and BC intersect at X, the perpendicular bisector of AD intersects AC, AB at Y, Z respectively, prove that AX, BY, CZ are concurrent.

Problem 1.141 (275429739915708)**.** Consider a 100×100 square unit lattice $\mathbf{L}$ (hence $\mathbf{L}$ has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in $\mathbf{L}$ and every point in $\mathbf{L}$ is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

Problem 1.142 (443006607452241)**.** Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Problem 1.143 (8700998965901287095)**.** Let ABC be an acute triangle with circumcircle ω. Let P be a variable point on the arc BC of ω not containing A. Squares $BPDE$ and $PCFG$ are constructed such that A, D, E lie on the same side of line BP and A, F, G lie on the same side of line CP. Let H be the intersection of lines DE and FG. Show that as P varies, H lies on a fixed circle.

Problem 1.144 (8609709793627283757)**.** Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Problem 1.145 (1837105952530316058)**.** Let $k \geq 2$ be an integer. Find the smallest integer $n \geq k+1$ with the property that there exists a set of n distinct real numbers such that each of its elements can be written as a sum of k other distinct elements of the set.

Problem 1.146 (1965233157265405983)**.** Given a triangle $\triangle ABC$. Denote its incircle and circumcircle by ω, Ω, respectively. Assume that ω tangents the sides AB, AC at F, E, respectively. Then, let the intersections of line EF and Ω to be P, Q. Let M to be the mid-point of BC. Take a point R on the circumcircle of $\triangle MPQ$, say Γ, such that $MR \perp EF$. Prove that the line AR, ω and Γ intersect at one point.

Problem 1.147 (5299971832672937326). Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $\triangle ANK, \triangle BKL, \triangle CLM, \triangle DMN$.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

Problem 1.148 (1613309914397651478). Let $ABCD$ be a convex quadrilateral with $\angle B < \angle A < 90^o$. Let I be the midpoint of AB and S the intersection of AD and BC. Let R be a variable point inside the triangle SAB such that $\angle ASR = \angle BSR$. On the straight lines AR, BR , take the points E, F, respectively so that BE, AF are parallel to RS. Suppose that EF intersects the circumcircle of triangle SAB at points H, K. On the segment AB, take points M, N such that $\angle AHM = \angle BHI$, $\angle BKN = \angle AKI$.

a) Prove that the center J of the circumcircle of triangle SMN lies on a fixed line.

b) On BE, AF , take the points P, Q respectively so that CP is parallel to SE and DQ is parallel to SF. The lines SE, SF intersect the circle (SAB), respectively, at U, V. Let G be the intersection of AU and BV. Prove that the median of vertex G of the triangle GPQ always passes through a fixed point .

Problem 1.149 (6276002868513182227). Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 1.150 (4308913658510445082). Let $ABCD$ be a convex quadrilateral, the incenters of $\triangle ABC$ and $\triangle ADC$ are I, J, respectively. It is known that AC, BD, IJ concurrent at a point P. The line perpendicular to BD through P intersects with the outer angle bisector of $\angle BAD$ and the outer angle bisector $\angle BCD$ at E, F, respectively. Show that $PE = PF$.

Problem 1.151 (8851048763094130212). Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines AC, DT, and BK are concurrent.

Problem 1.152 (5998250511147866097). Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Problem 1.153 (6783316811528119504). Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Problem 1.154 (1440964279096111130). Let a be a positive integer. We say that a positive integer b is a-good if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a-good, but $b+2$ is not a-good. Prove that $b + 1$ is prime.

Problem 1.155 (3859961452154270883). A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

Problem 1.156 (7500559455615129254)**.** For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

Problem 1.157 (5990443173263547430)**.** Given a fixed circle (O) and two fixed points B, C on that circle, let A be a moving point on (O) such that $\triangle ABC$ is acute and scalene. Let I be the midpoint of BC and let AD, BE, CF be the three heights of $\triangle ABC$. In two rays $\overrightarrow{FA}, \overrightarrow{EA}$, we pick respectively M, N such that $FM = CE, EN = BF$. Let L be the intersection of MN and EF, and let $G \neq L$ be the second intersection of (LEN) and (LFM).

a) Show that the circle (MNG) always goes through a fixed point.

b) Let AD intersects (O) at $K \neq A$. In the tangent line through D of (DKI), we pick P, Q such that $GP \parallel AB, GQ \parallel AC$. Let T be the center of (GPQ). Show that GT always goes through a fixed point.

Problem 1.158 (8053761138620448460)**.** Let ABC be a scalene triangle, and points O and H be its circumcenter and orthocenter, respectively. Point P lies inside triangle AHO and satisfies $\angle AHP = \angle POA$. Let M be the midpoint of segment $\overline{OP}$. Suppose that BM and CM intersect with the circumcircle of triangle ABC again at X and Y, respectively.

Prove that line XY passes through the circumcenter of triangle APO.

Problem 1.159 (8330669807899443473)**.** Let ABC be an acute scalene triangle, and let A_1, B_1, C_1 be the feet of the altitudes from A, B, C. Let A_2 be the intersection of the tangents to the circle ABC at B, C and define B_2, C_2 similarly. Let A_2A_1 intersect the circle $A_2B_2C_2$ again at A_3 and define B_3, C_3 similarly. Show that the circles AA_1A_3, BB_1B_3, and CC_1C_3 all have two common points, X_1 and X_2 which both lie on the Euler line of the triangle ABC.

Problem 1.160 (518384374486289)**.** Let O be the center of the equilateral triangle ABC. Pick two points P_1 and P_2 other than B, O, C on the circle $\odot(BOC)$ so that on this circle B, P_1, P_2, O, C are placed in this order. Extensions of BP_1 and CP_1 intersects respectively with side CA and AB at points R and S. Line AP_1 and RS intersects at point Q_1. Analogously point Q_2 is defined. Let $\odot(OP_1Q_1)$ and $\odot(OP_2Q_2)$ meet again at point U other than O.

Prove that $2\,\angle Q_2UQ_1 + \angle Q_2OQ_1 = 360°$.

Remark. $\odot(XYZ)$ denotes the circumcircle of triangle XYZ.

Problem 1.161 (15595788767204175)**.** Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem 1.162 (9026100911884959358)**.** Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Problem 1.163 (857598260795435)**.** Let $ABCD$ be a rhombus with center O. P is a point lying on the side AB. Let I, J, and L be the incenters of triangles PCD, PAD, and PBC, respectively. Let H and K be orthocenters of triangles PLB and PJA, respectively.

Prove that $OI \perp HK$.

Problem 1.164 (569685816807741)**.** Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

Problem 1.165 (607556370102952)**.** Let Ω be the circumcircle of an acute triangle ABC. Points D, E, F are the midpoints of the inferior arcs BC, CA, AB, respectively, on Ω. Let G be the antipode of D in Ω. Let X be the intersection of lines GE and AB, while Y the intersection of lines FG and CA. Let the circumcenters of triangles BEX and CFY be points S and T, respectively. Prove that D, S, T are collinear.

Problem 1.166 (8417327567048605288)**.** Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 1.167 (120381541018683)**.** Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x \in F} 1/x \neq r$ for all finite subsets F of S.

Problem 1.168 (3353450172272500341)**.** Let $ABCD$ be a cyclic quadrilateral. Let DA and BC intersect at E and let AB and CD intersect at F. Assume that A, E, F all lie on the same side of BD. Let P be on segment DA such that $\angle CPD = \angle CBP$, and let Q be on segment CD such that $\angle DQA = \angle QBA$. Let AC and PQ meet at X. Prove that, if $EX = EP$, then EF is perpendicular to AC.

Problem 1.169 (7948249970111159954)**.** A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 < \ldots < t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C.$$

Problem 1.170 (1121095467606378762)**.** Let $\Gamma, \Gamma_1, \Gamma_2$ be mutually tangent circles. The three circles are also tangent to a line l. Let Γ, Γ_1 be tangent to each other at B_1, Γ, Γ_2 be tangent to each other at B_2, Γ_1, Γ_2 be tangent to each other at C. $\Gamma, \Gamma_1, \Gamma_2$ are tangent to l at A, A_1, A_2 respectively, where A is between A_1, A_2. Let $D_1 = A_1C \cap A_2B_2, D_2 = A_2C \cap A_1B_1$. Prove that D_1D_2 is parallel to l.

Problem 1.171 (6919176010062551987)**.** Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q\leq n, p,q \text{ primes}} (p+q)$$

Problem 1.172 (3923745101517032298)**.** Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.

Problem 1.173 (728988632553727)**.** Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

Problem 1.174 (119129720704350)**.** Let H be the orthocenter of a given triangle ABC. Let BH and AC meet at a point E, and CH and AB meet at F. Suppose that X is a point on the line BC. Also suppose that the circumcircle of triangle BEX and the line AB intersect again at Y, and the circumcircle of triangle CFX and the line AC intersect again at Z. Show that the circumcircle of triangle AYZ is tangent to the line AH.

Problem 1.175 (8782897210450267045)**.** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Problem 1.176 (6025085618534905645)**.** Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_B O_D}$.

Problem 1.177 (296367141382799)**.** Given a triangle $\triangle ABC$ with orthocenter H. On its circumcenter, choose an arbitrary point P (other than A, B, C) and let M be the mid-point of HP. Now, we find three points D, E, F on the line BC, CA, AB, respectively, such that $AP \parallel HD, BP \parallel HE, CP \parallel HF$. Show that D, E, F, M are colinear.

Problem 1.178 (633974672407561)**.** Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers with the property that

$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$

for all positive integers n. Show that $a_{2022} \leq 1$.

Problem 1.179 (47893544380608)**.** Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \dots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \dots, n\}$ that are red. Prove that there exists a positive integer $a \in \{1, 2, \dots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \dots, N$.

Problem 1.180 (639126468624733)**.** Let $ABCDEF$ be a hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that segments BD and DF intersect CE at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to CE.

Problem 1.181 (8528437132500966626)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Problem 1.182 (712971117639738)**.** Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x + y + z)P(x, y, z) + (xy + yz + zx)Q(x, y, z) + xyzR(x, y, z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i + j + k \geq n$.

Problem 1.183 (8690567757444826166)**.** A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Problem 1.184 (822921222405372)**.** Let $n \geq 3$ be a fixed integer. There are $m \geq n + 1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n + 1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

Problem 1.185 (9103148252094553273)**.** The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a path from X to Y is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called diverse if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Problem 1.186 (117986541208663)**.** Given a triangle ABC. D is a moving point on the edge BC. Point E and Point F are on the edge AB and AC, respectively, such that $BE = CD$ and $CF = BD$. The circumcircle of $\triangle BDE$ and $\triangle CDF$ intersects at another point P other than D. Prove that there exists a fixed point Q, such that the length of QP is constant.

Problem 1.187 (9154783649392 50)**.** Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 1.188 (183354438240037)**.** Let I, O, H, and Ω be the incenter, circumcenter, orthocenter, and the circumcircle of the triangle ABC, respectively. Assume that line AI intersects with Ω again at point $M \neq A$, line IH and BC meets at point D, and line MD intersects with Ω again at point $E \neq M$. Prove that line OI is tangent to the circumcircle of triangle IHE.

Problem 1.189 (651490142085731)**.** Let I be the incenter of triangle ABC, and let ω be its incircle. Let E and F be the points of tangency of ω with CA and AB, respectively. Let X and Y be the intersections of the circumcircle of BIC and ω. Take a point T on BC such that $\angle AIT$ is a right angle. Let G be the intersection of EF and BC, and let Z be the intersection of XY and AT. Prove that AZ, ZG, and AI form an isosceles triangle.

Problem 1.190 (6020628633767269011)**.** Let $ABCDE$ be a regular pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that as P varies, R lies on a fixed line.

Problem 1.191 (8402748184217471405)**.** In ΔABC, $AD \perp BC$ at D. E, F lie on line AB, such that $BD = BE = BF$. Let I, J be the incenter and A-excenter. Prove that there exist two points P, Q on the circumcircle of ΔABC , such that $PB = QC$, and $\Delta PEI \sim \Delta QFJ$.

Problem 1.192 (14270621317473499943)**.** Let ABC be a triangle with circumcenter O and orthocenter H such that OH is parallel to BC. Let AH intersects again with the circumcircle of ABC at X, and let XB, XC intersect with OH at Y, Z, respectively. If the projections of Y, Z to AB, AC are P, Q, respectively, show that PQ bisects BC.

Problem 1.193 (302438226120877)**.** Given triangle ABC. Let $BPCQ$ be a parallelogram (P is not on BC). Let U be the intersection of CA and BP, V be the intersection of AB and CP, X be the intersection of CA and the circumcircle of triangle ABQ distinct from A, and Y be the intersection of AB and the circumcircle of triangle ACQ distinct from A. Prove that $\overline{BU} = \overline{CV}$ if and only if the lines AQ, BX, and CY are concurrent.

Problem 1.194 (210358073900610)**.** Let triangle ABC have altitudes BE and CF which meet at H. The reflection of A over BC is A'. Let (ABC) meet $(AA'E)$ at P and $(AA'F)$ at Q. Let BC meet PQ at R. Prove that $EF \parallel HR$.

Problem 1.195 (5363953658134647103)**.** Let ABC be a triangle with incenter I. The line through I, perpendicular to AI, intersects the circumcircle of ABC at points P and Q. It turns out there exists a point T on the side BC such that $AB + BT = AC + CT$ and $AT^2 = AB \cdot AC$. Determine all possible values of the ratio IP/IQ.

Problem 1.196 (6209707374283278028)**.** Let ABC be a triangle and D be a point inside triangle ABC. Γ is the circumcircle of triangle ABC, and DB, DC meet Γ again at E, F , respectively. Γ_1, Γ_2 are the circumcircles of triangle ADE and ADF respectively. Assume X is on Γ_2 such that BX is tangent to Γ_2. Let BX meets Γ again at Z. Prove that the line CZ is tangent to Γ_1 .

Problem 1.197 (308110166188097)**.** Let A, B be two fixed points on the unit circle ω, satisfying $\sqrt{2} < AB < 2$. Let P be a point that can move on the unit circle, and it can move to anywhere on the unit circle satisfying $\triangle ABP$ is acute and $AP > AB > BP$. Let H be the orthocenter of $\triangle ABP$ and S be a point on the minor arc AP satisfying $SH = AH$. Let T be a point on the minor arc AB satisfying $TB || AP$. Let $ST \cap BP = Q$.

Show that (recall P varies) the circle with diameter HQ passes through a fixed point.

Problem 1.198 (3838489129977355762)**.** Two triangles ABC and $A'B'C'$ are on the plane. It is known that each side length of triangle ABC is not less than a, and each side length of triangle $A'B'C'$ is not less than a'. Prove that we can always choose two points in the two triangles respectively such that the distance between them is not less than $\sqrt{\dfrac{a^2+a'^2}{3}}$.

Problem 1.199 (6566259136811987209)**.** Let Ω be the A-excircle of triangle ABC, and suppose that Ω is tangent to lines BC, CA, and AB at points D, E, and F, respectively. Let M be the midpoint of segment EF. Two more points P and Q are on Ω such that EP and FQ are both parallel to DM. Let BP meet CQ at point X. Prove that the line AM is the angle bisector of $\angle XAD$.

Problem 1.200 (8639636622304457736)**.** Let $\triangle ABC$ be a triangle, and let S and T be the midpoints of the sides BC and CA, respectively. Suppose M is the midpoint of the segment ST and the circle ω through A, M and T meets the line AB again at N. The tangents of ω at M and N meet at P. Prove that P lies on BC if and only if the triangle ABC is isosceles with apex at A.

Problem 1.201 (1872712387771032593)**.** Let H be the orthocenter of triangle ABC, and AD, BE, CF be the three altitudes of triangle ABC. Let G be the orthogonal projection of D onto EF, and DD' be the diameter of the circumcircle of triangle DEF. Line AG and the circumcircle of triangle ABC intersect again at point X. Let Y be the intersection of GD' and BC, while Z be the intersection of AD' and GH. Prove that X, Y, and Z are collinear.

Problem 1.202 (7017112574129036660)**.** Let ABC be a triangle with $AB < AC$, and let I_a be its A-excenter. Let D be the projection of I_a to BC. Let X be the intersection of AI_a and BC, and let Y, Z be the points on AC, AB, respectively, such that X, Y, Z are on a line perpendicular to AI_a. Let the circumcircle of AYZ intersect AI_a again at U. Suppose that the tangent of the circumcircle of ABC at A intersects BC at T, and the segment TU intersects the circumcircle of ABC at V. Show that $\angle BAV = \angle DAC$.

Problem 1.203 (7268978143074030034)**.** Given two circles ω_1 and ω_2 where ω_2 is inside ω_1. Show that there exists a point P such that for any line ℓ not passing through P, if ℓ intersects circle ω_1 at A, B and ℓ intersects circle ω_2 at C, D, where A, C, D, B lie on ℓ in this order, then $\angle APC = \angle BPD$.

Problem 1.204 (684771433215596)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 1.205 (240654526717277)**.** Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Problem 1.206 (17901140622253914451)**.** Given a triangle $\triangle ABC$ and a point O. X is a point on the ray $\overrightarrow{AC}$. Let X' be a point on the ray $\overrightarrow{BA}$ so that $\overline{AX} = \overline{AX_1}$ and A lies in the segment $\overline{BX_1}$. Then, on the ray $\overrightarrow{BC}$, choose X_2 with $\overline{X_1X_2} \parallel \overline{OC}$.

Prove that when X moves on the ray $\overrightarrow{AC}$, the locus of circumcenter of $\triangle BX_1X_2$ is a part of a line.

Problem 1.207 (221552874820768)**.** The incircle of a scalene triangle ABC touches the sides BC, CA, and AB at points D, E, and F, respectively. Triangles APE and AQF are constructed outside the triangle so that

$$AP = PE, AQ = QF, \angle APE = \angle ACB, \text{ and } \angle AQF = \angle ABC.$$

Let M be the midpoint of BC. Find $\angle QMP$ in terms of the angles of the triangle ABC.

Problem 1.208 (8670333331361701457)**.** Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problem 1.209 (13360308368399041368)**.** Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if area (BCD) + area (ADE) = area (ABD) + area (ABP).

Problem 1.210 (2265193939454652363)**.** A circle ω with radius 1 is given. A collection T of triangles is called good, if the following conditions hold: each triangle from T is inscribed in ω; no two triangles from T have a common interior point. Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

Problem 1.211 (2139114147569608698)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 1.212 (233559801569582)**.** Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

Problem 1.213 (571373387028298)**.** Let ABC be a triangle with $\angle BAC > 90^\circ$, and let O be its circumcenter and ω be its circumcircle. The tangent line of ω at A intersects the tangent line of ω at B and C respectively at point P and Q. Let D, E be the feet of the altitudes from P, Q onto BC, respectively. F, G are two points on $\overline{PQ}$ different from A, so that A, F, B, E and A, G, C, D are both concyclic. Let M be the midpoint of $\overline{DE}$. Prove that DF, OM, EG are concurrent.