matate

PONTE A ENTRENAR

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§1 Problemas

Problem 1.1 (5713525138564177722)**.** A cyclic quadrilateral $ABCD$ has circumcircle Γ, and $AB + BC = AD + DC$. Let E be the midpoint of arc BCD, and $F(\neq C)$ be the antipode of A wrt Γ. Let I, J, K be the incenter of $\triangle ABC$, the A-excenter of $\triangle ABC$, the incenter of $\triangle BCD$, respectively. Suppose that a point P satisfies $\triangle BIC \overset{+}{\sim} \triangle KPJ$. Prove that EK and PF intersect on Γ.

Problem 1.2 (7500559455615129254)**.** For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

Problem 1.3 (579228243242060)**.** Let $ABCD$ be a parallelogram. A line through C crosses the side AB at an interior point X, and the line AD at Y. The tangents of the circle AXY at X and Y, respectively, cross at T. Prove that the circumcircles of triangles ABD and TXY intersect at two points, one lying on the line AT and the other one lying on the line CT.

Problem 1.4 (12937725920633023344)**.** In non-isosceles acute $\triangle ABC$, AP, BQ, CR is the height of the triangle. A_1 is the midpoint of BC, AA_1 intersects QR at K, QR intersects a straight line that crosses A and is parallel to BC at point D, the line connecting the midpoint of AH and K intersects DA_1 at A_2. Similarly define B_2, C_2. $\triangle A_2B_2C_2$ is known to be non-degenerate, and its circumscribed circle is ω. Prove that: there are circles $\odot A'$, $\odot B'$, $\odot C'$ tangent to and INSIDE ω satisfying: (1) $\odot A'$ is tangent to AB and AC, $\odot B'$ is tangent to BC and BA, and $\odot C'$ is tangent to CA and CB. (2) A', B', C' are different and collinear.

Problem 1.5 (3245291910836201005)**.** Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

Problem 1.6 (6612845742708555351)**.** Cyclic quadrilateral $ABCD$ has circumcircle (O). Points M and N are the midpoints of BC and CD, and E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Let EN meet FM at P. Denote S as the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

Problem 1.7 (227919487650283)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Ω. Let M be the midpoint of side BC. Point D is chosen from the minor arc BC on Γ such that $\angle BAD = \angle MAC$. Let E be a point on Γ such that DE is perpendicular to AM, and F be a point on line BC such that DF is perpendicular to BC. Lines HF and AM intersect at point N, and point R is the reflection point of H with respect to N.

Prove that $\angle AER + \angle DFR = 180°$.

Problem 1.8 (165465510156789)**.** Let Ω be the circumcircle of an isosceles trapezoid $ABCD$, in which AD is parallel to BC. Let X be the reflection point of D with respect to BC. Point Q is on the arc BC of Ω that does not contain A. Let P be the intersection of DQ and BC. A point E satisfies that EQ is parallel to PX, and EQ bisects $\angle BEC$. Prove that EQ also bisects $\angle AEP$.

Problem 1.9 (132497611943266)**.** Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 1.10 (3866807698726339637)**.** Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbors - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Problem 1.11 (7550072974614174968)**.** Let $n \geqslant 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[0, 1]$. Let $s = x_1 + x_2 + \ldots + x_n$, and assume that $s \geqslant 3$. Prove that there exist integers i and j with $1 \leqslant i < j \leqslant n$ such that

$$2^{j-i} x_i x_j > 2^{s-3}.$$

Problem 1.12 (7220404010846068686)**.** Let ABC be a acute, non-isosceles triangle. D, E, F are the midpoints of sides AB, BC, AC, resp. Denote by (O), (O') the circumcircle and Euler circle of ABC. An arbitrary point P lies inside triangle DEF and DP, EP, FP intersect (O') at D', E', F', resp. Point A' is the point such that D' is the midpoint of AA'. Points B', C' are defined similarly. a. Prove that if $PO = PO'$ then $O \in (A'B'C')$; b. Point A' is mirrored by OD, its image is X. Y, Z are created in the same manner. H is the orthocenter of ABC and XH, YH, ZH intersect BC, AC, AB at M, N, L resp. Prove that M, N, L are collinear.

Problem 1.13 (2918584823978789760)**.** A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Problem 1.14 (660403976209529)**.** A number is called Norwegian if it has three distinct positive divisors whose sum is equal to 2022. Determine the smallest Norwegian number. (Note: The total number of positive divisors of a Norwegian number is allowed to be larger than 3.)

Problem 1.15 (952584318797289)**.** Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.