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Probability and Applied Statistics

Formula Sheet #2

Chapter 3

Poisson Distribution:

A random variable Y is said to have a Poisson probability distribution if and only if

$$p(y) = \frac{\lambda^y}{y!}e^{-\lambda}, \qquad y = 0, 1, 2, \ldots, \lambda > 0.$$

If Y is a random variable possessing a Poisson distribution with parameter λ, then

$$\mu = E(Y) = \lambda \ and \ \sigma^2 = V(Y) = \lambda.$$

By definition,

$$E(Y) = \sum_y yp(y) = \sum_{y=0}^{\infty} y\frac{\lambda^y e^{-\lambda}}{y!}.$$

Tchebysheff's Theorem:

Tchebysheff's Theorem – Let Y be a random variable with mean μ and finite variance σ^2. Then, for any constant k > 0,

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad or \quad P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

<u>Chapter 4</u>

The Probability Distribution for a Continuous Random Variable:

Let Y denote any random variable. The distribution function of Y, denoted by F(y), is such that $F(y) = P(Y \leq y) for - \infty < y < \infty$.

Properties of a Distribution Function:

If F(y) is a distribution function, then

1. $F(-\infty) \equiv \lim_{y \to -\infty} F(y) = 0.$
2. $F(\infty) \equiv \lim_{y \to \infty} F(y) = 1.$
3. $F(y)\ is\ a\ nondecreasing\ function\ of\ y. [If\ y_1\ and\ y_2\ are\ any\ values\ such\ that\ y_1 < y_2, then\ F(y_1) \leq F(y_2).]$

A random variable Y with distribution function F(y) is said to be continuous if F(y) is continuous for $-\infty < y < \infty$.

Let F(y) be the distribution function for a continuous random variable Y. Then f(y) given by

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

whenever the derivative exists, is called the probability density function for the random variable Y.

It follows that F(y) can be written as

$$F(y) = \int_{-\infty}^{y} f(t)dt.$$

Properties of a Density Function – If f(y) is a density function for a continuous random variable, then

1. $f(y) \geq 0$ $for\ all\ y, -\infty < y < \infty$.
2. $\int_{-\infty}^{\infty} f(y)dy = 1$.

If the random variable Y has density function f(y) and a < b, then the probability that Y falls in the interval [a, b] is

$$P(a \leq Y \leq b) = \int_{a}^{b} f(y)dy.$$

Expected Value for Continuous Random Variables:

The expected value of a continuous random variable Y is

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy,$$

provided that the integral exists.

Let g(Y) be a function of Y, then the expected value of g(Y) is given by

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy,$$

provided that the integral exists.

Let c be a constant and let $g(Y), g_1(Y), g_2(Y), \ldots, g_k(Y)$ be functions of a continuous random variable Y. Then the following results hold:

1. $E(c) = c$.
2. $E[cg(Y)] = cE[g(Y)]$.

3. $E[g_1(Y) + g_2(Y) + \ldots + g_k(Y)] = E[g_1(Y)] + E[g_2(Y)] + \ldots + E[g_k(Y)]$

The Uniform Probability Distribution:

If $\theta_1 < \theta_2$, a random variable Y is said to have a continuous uniform probability distribution on the interval (θ_1, θ_2) if and only if the density function of Y is

$$f(y) = \begin{cases} \dfrac{1}{\theta_1 - \theta_2}, \theta_1 \le y \le \theta_2. \\ 0, \qquad elsewhere \end{cases}$$

If $\theta_1 < \theta_2$ and Y is a random variable uniformly distributed on the interval (θ_1, θ_2), then

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2} \; and \; \sigma^2 = V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$$

The Gamma Probability Distribution:

A random variable Y is said to have a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if and only if the density function of Y is

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, 0 \le y \le \infty \\ 0, \qquad elsewhere \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

If Y has a gamma distribution with parameters α and β, then

$$\mu = E(Y) = \alpha\beta \; and \; \sigma^2 = V(Y) = \alpha\beta^2$$

By definition,

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy = \int_{0}^{\infty} y\left(\frac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}\right)dy \text{ and } V(Y) = E[Y^2] - [E(Y)]^2$$

A random variable Y is said to have an exponential distribution with parameter $\beta > 0$ if and only if the density function of Y is

$$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & 0 \le y \le \infty \\ 0 & elsewhere \end{cases}$$

If Y is an exponential random variable with parameter β, then

$$\mu = E(Y) = \beta \text{ and } \sigma^2 = V(Y) = \beta^2$$

with $\alpha = 1$

Chapter 5

Bivariate and Multivariate Probability Distributions:

Let Y_1 and Y_2 be discrete random variables. The joint probability function for Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2), \quad -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

If Y_1 and Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$, then

1. $p(y_1, y_2) \ge 0$ for all y_1, y_2
2. $\sum_{y_1,y_2} p(y_1, y_2) = 1$, where the sum is over the values (y_1, y_2) that are assigned nonzero probabilities.

For any random variables Y_1 and Y_2, the joint distribution function $F(y_1, y_2)$ is

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2), \qquad -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

Let Y_1 and Y_2 be continuous random variables with joint distribution function $F(y_1, y_2)$. If there exists a nonnegative function $f(y_1, y_2)$, such that

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_2 dt_1$$

for all $-\infty < y_1 < \infty, -\infty < y_2 < \infty$, then Y_1 and Y_2 are said to be jointly continuous random variables. The function $f(y_1, y_2)$ is called the joint probability density function

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If Y_1 and Y_2 are random variables with joint distribution function $F(y_1, y_2)$, then

1. $F(-\infty, -\infty) = F(-\infty, y_2) = F(y_1, -\infty) = 0.$
2. $F(\infty, \infty) = 1.$
3. If $y_1^* \ge y_1$ and $y_2^* \ge y_2$, then $F(y_1^*, y_2^*) - F(y_1^*, y_2) - F(y_1, y_2^*) + F(y_1, y_2) \ge 0$

If Y_1 and Y_2 are jointly continuous random variables with a joint density function given by $f(y_1, y_2)$, then

1. $f(y_1, y_2) \ge 0$ for all y_1, y_2
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1.$

Marginal and Conditional Probability Distributions:

Let Y_1 and Y_2 be jointly discrete random variables with probability function $p(y_1, y_2)$. Then the marginal probability functions of Y_1 and Y_2, respectively, are given by

$$p_1(y_1) = \sum_{all\ y_2} p(y_1, y_2) \quad and \quad p_2(y_2) \neq= \sum_{all\ y_1} p(y_1, y_2)$$

Let Y_1 and Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$. Then the marginal density functions of Y_1 and Y_2, respectively, are given by

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \quad and \quad f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$$

If Y_1 and Y_2 are jointly discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then the conditional discrete probability function of Y_1 and Y_2 is

$$p(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p(y_1, y_2)}{p_2(y_2)}$$

provided that $p_2(y_2) > 0$.

If Y_1 and Y_2 are jointly continuous random variables with joint density function $f(y_1, y_2)$, then the conditional distribution of Y_1 given $Y_2 = y_2$ is

$$F(y_1|y_2) = P(Y_1 \leq y_1|Y_2 = y_2).$$

Let Y_1 and Y_2 be jointly continuous random variables with join density $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$ respectively. For any y_2 such that $f_2(y_2) > 0$, the conditional density of Y_1 given $Y_2 = y_2$ is given by

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$$

And, for y_1 such that $f_1(y_1) > 0$, the conditional density of Y_2 given $Y_1 = y_1$ is given by

$$f(y_2|y_1) = \frac{f(y_1, y_2)}{f_1(y_1)}$$

Independent Random Variables:

Let Y_1 have distribution function $F_1(y_1)$, Y_2 have distribution function $F_2(y_2)$, and Y_1 and Y_2 have joint distribution function $F(y_1, y_2)$. Then Y_1 and Y_2 are said to be independent if and only if

$$\boxed{F(y_1, y_2) = F_1(y_1)F_2(y_2)}$$

For every pair of real numbers (y_1, y_2).

If Y_1 and Y_2 are not independent, they are said to be dependent.

If Y_1 and Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$\boxed{p(y_1, y_2) = p_1(y_1)p_2(y_2)}$$

For all pairs of real numbers (y_1, y_2).

If Y_1 and Y_2 are continuous random variables with joint density function $f(y_1, y_2)$ and marginal density functions $f_1(y_1)$ and $f_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$\boxed{f(y_1, y_2) = f_1(y_1)f_2(y_2)}$$

For all pairs of real numbers (y_1, y_2).

Let Y_1 and Y_2 have a joint density $f(y_1, y_2)$ that is positive if and only if $a \leq y_1 \leq b$ and $c \leq y_2 \leq d$, for constants a, b, c, and d; and $f(y_1, y_2) = 0$ otherwise. Then Y_1 and Y_2 are independent random variables if and only if

$$\boxed{f(y_1, y_2) = g(y_1)h(y_2)}$$

where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone.