

Données de réseaux : modèles probabilistes

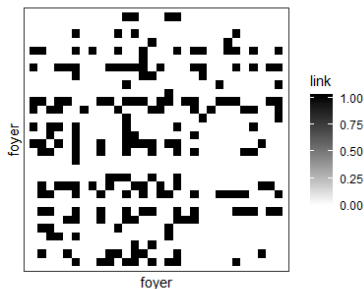
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Formation Réseaux MIRES / ReSodiv
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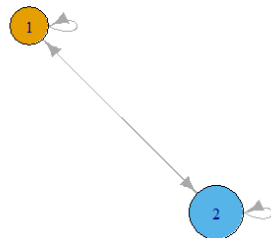
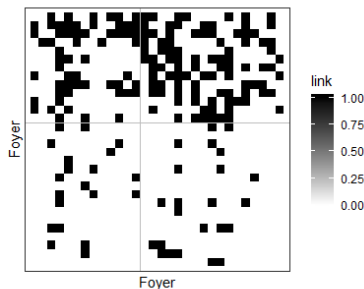
Qu'attendre des SBM ?

Outils automatiques pour faire des groupes de sommets qui ont le même rôle dans le réseau



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A propos de l'aspect “probabiliste”

- ▶ *Hypothèse* : réseau observé Y_{ij} réalisation d'un phénomène aléatoire
- ▶ *Modèle* : forme particulière de ce phénomène
- ▶ *Aléa* : représente le fait que l'observateur n'est pas capable de prédire Y_{ij}
 - ▶ Exemple de l'expérience du Pile ou Face
 - ▶ Si je connaissais tous les paramètres physiques de la pièce et du lancer, je serais capable de prédire l'issue du jeu
 - ▶ En tant qu'observateur, je ne peux pas prédire : expérience aléatoire

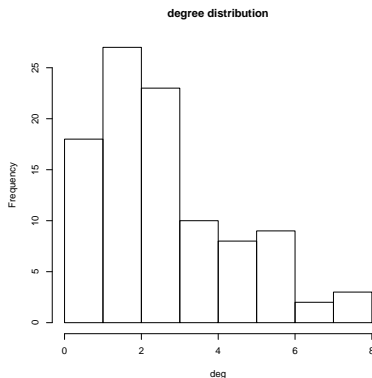
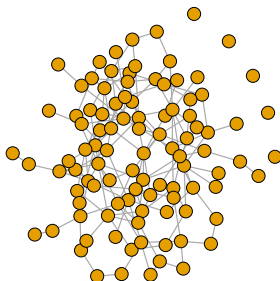
A first random graph model for network : null model

- ▶ Erdős-Rényi (1959) Model for n nodes
- ▶ Let $(y_{ij})_{i,j=1\dots n}$ be an adjacency matrix (i.e. representing a simple network) with $y_{ij} \in \{0, 1\}$
- ▶ ER assumes that y_{ij} is the realisation of :

$$\forall 1 \leq i, j \leq n, \quad Y_{ij} \overset{i.i.d.}{\sim} \text{Bern}(p),$$

where Bern is the Bernoulli distribution and $p \in [0, 1]$ a probability for a link to exist.

A first random graph model for network : null model



Limitations of an ER graph to describe real networks

- ▶ Degree distribution too concentrated, no high degree nodes,
- ▶ All nodes are equivalent (no nestedness...),
- ▶ No modularity.

Stochastic Block Model (SBM)

Latent block model (LBM)

Stochastic Block Model

Nowicki, & Snijders (2001)

Let (y_{ij}) be an adjacency matrix such that $y_{ij} \in \{0, 1\}$. y_{ij} is the realisation of the following processus.

Latent variables

- ▶ The nodes $i = 1, \dots, n$ are partitionned into K clusters
- ▶ $Z_i = k$ if node i belongs to cluster (block) k
- ▶ Z_i independant variables

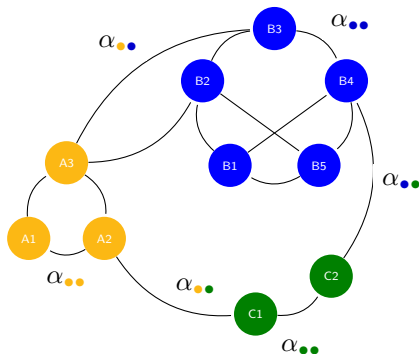
$$\mathbb{P}(Z_i = k) = \pi_k$$

Conditionally to $(Z_i)_{i=1, \dots, n}$

(Y_{ij}) independant and

$$Y_{ij} | Z_i, Z_j \sim \text{Bern}(\alpha_{Z_i, Z_j}) \quad \Leftrightarrow \quad P(Y_{ij} = 1 | Z_i = k, Z_j = \ell) = \alpha_{k\ell}$$

Stochastic Block Model : illustration



Parameters

Let n nodes divided into 3 clusters

- ▶ $\mathcal{K} = \{\bullet, \bullet, \bullet\}$ clusters
- ▶ $\pi_{\bullet} = \mathbb{P}(i \in \bullet), \bullet \in \mathcal{K}, i = 1, \dots, n$
- ▶ $\alpha_{\bullet\bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \pi), \quad \forall \bullet \in \mathcal{K},$$

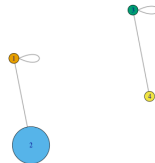
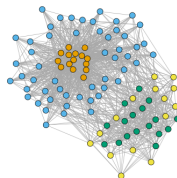
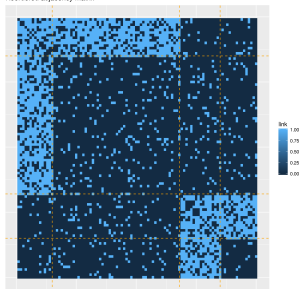
$$Y_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{B}(\alpha_{\bullet\bullet})$$

SBM : A great generative model

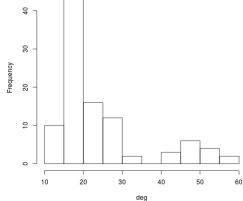
- ▶ Generative model : easy to simulate
- ▶ No a priori on the type of structure
- ▶ Combination of modularity, nestedness, etc...

Networks with hubs generated by SBM

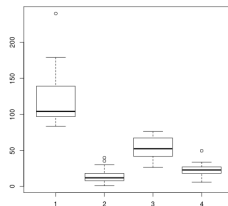
Reordered adjacency matrix



Histogram of degrees

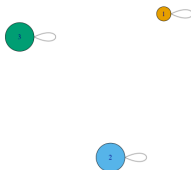
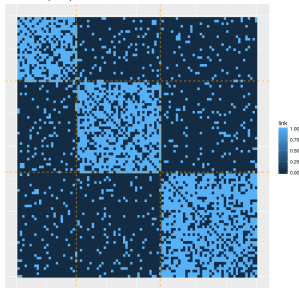


Betweenness by block

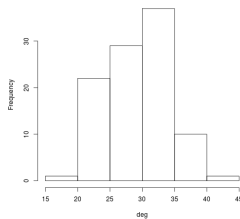


Community network generated by SBM

Reordered adjacency matrix

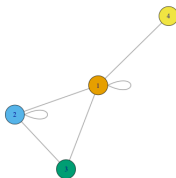
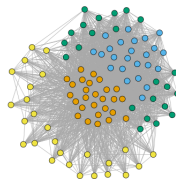
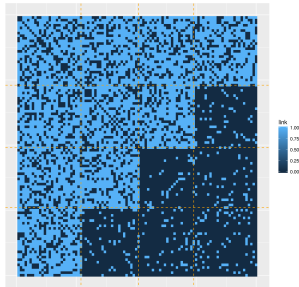


Histogram of degrees

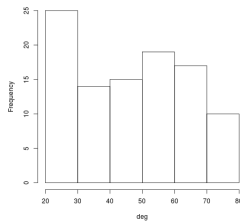


Nestedness generated by SBM

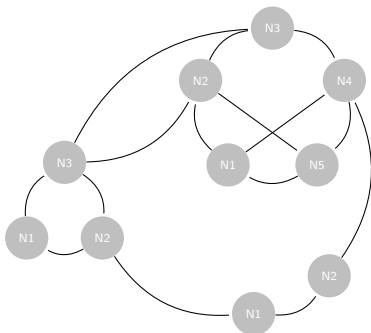
Reordered adjacency matrix



Histogram of degrees



Inférence statistique



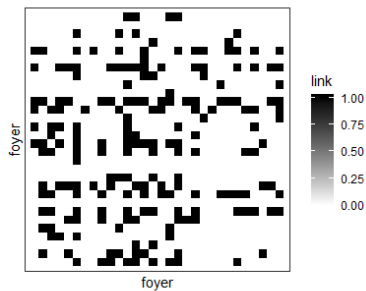
Inférence

- ▶ $\mathcal{K} = \{\bullet, \bullet, \bullet\}$, $\text{card}(\mathcal{K})$ known
 - ▶ $\pi_{\bullet} = ?$,
 - ▶ $\alpha_{\bullet, \bullet} = ?$
 - ▶ $Z_1, \dots, Z_n = ?$
- ▶ $\text{card}(\mathcal{K}) ?$

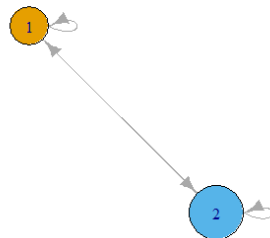
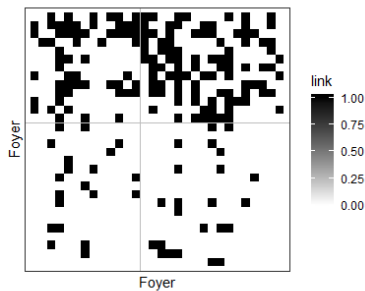
Nowicki, & Snijders (2001), Daudin et al. (2008)

R package : blockmodels.

Exemple du Vanuatu



Exemple du Vanuatu



Stochastic Block Model (SBM)

Latent block model (LBM)

Some possible extensions

Probabilistic model for binary bipartite networks

Let Y_{ij} be a bi-partite network. Individuals in row and cols are not the same.

Latent variables : bi-clustering

- ▶ Nodes $i = 1, \dots, n_1$ partitionned into K_1 clusters, nodes $j = 1, \dots, n_2$ partitionned into K_2 clusters
- ▶
$$\begin{aligned} Z_i^1 &= k && \text{if node } i \text{ belongs to cluster (block) } k \\ Z_j^2 &= \ell && \text{if node } j \text{ belongs to cluster (block) } \ell \end{aligned}$$
- ▶ Z_i^1, Z_j^2 independent variables

$$\mathbb{P}(Z_i^1 = k) = \pi_k^1, \quad \mathbb{P}(Z_j^2 = \ell) = \pi_\ell^2$$

Probabilistic model for binary bipartite networks

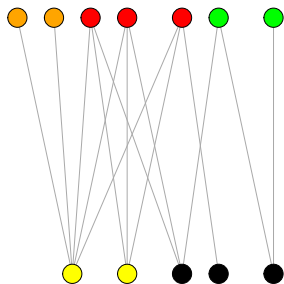
Conditionally to $(Z_i^1)_{i=1,\dots,n_1}, (Z_j^2)_{j=1,\dots,n_2}\dots$

(Y_{ij}) independent and

$$Y_{ij}|Z_i^1, Z_j^2 \sim \text{Bern}(\alpha_{Z_i^1, Z_j^2}) \Leftrightarrow \mathbb{P}(Y_{ij} = 1 | Z_i^1 = k, Z_j^2 = \ell) = \alpha_{k\ell}$$

Govaert & Nadif (2008)

Latent Block Model : illustration



Latent Block Model

- ▶ n_1 row nodes $\mathcal{K}_1 = \{\bullet, \bullet, \bullet\}$ classes
- ▶ $\pi_{\bullet}^1 = \mathbb{P}(i \in \bullet), \bullet \in \mathcal{K}_1, i = 1, \dots, n$
- ▶ n_2 column nodes $\mathcal{K}_2 = \{\bullet, \bullet\}$ classes
- ▶ $\pi_{\bullet}^2 = \mathbb{P}(j \in \bullet), \bullet \in \mathcal{K}_2, j = 1, \dots, m$
- ▶ $\alpha_{\bullet, \bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$Z_i^1 = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \pi^1), \quad \forall \bullet \in \mathcal{Q}_1,$$

$$Z_j^2 = \mathbf{1}_{\{j \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \pi^2), \quad \forall \bullet \in \mathcal{Q}_2,$$

$$Y_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \text{Bern}(\alpha_{\bullet, \bullet})$$

Valued-edge networks

Values-edges networks

Information on edges can be something different from presence/absence.
It can be :

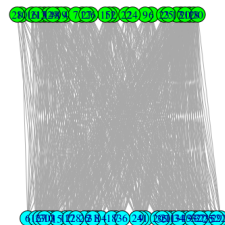
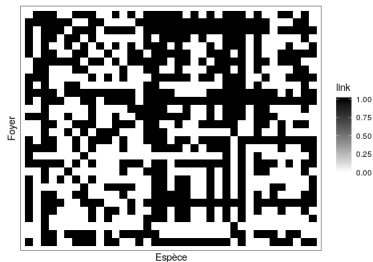
1. a count of the number of observed interactions,
2. a quantity interpreted as the interaction strength,

Natural extensions of SBM and LBM

1. Poisson distribution : $Y_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{P}(\lambda_{\bullet\bullet})$,
2. Gaussian distribution : $Y_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{N}(\mu_{\bullet\bullet}, \sigma^2)$,
[Mariadassou et al. \(2010\)](#)
3. More generally,

$$Y_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{F}(\theta_{\bullet\bullet})$$

Exemple sur Vanuatu Foyers/espèces cultivées



Exemple sur Vanuatu Foyers/espèces cultivées

