Probabilistic models for networks

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Network data

Networks can account for

- Ecological network : Food web , Co-existence networks, Host-parasite interactions, Plant-pollinator interactions,
- Social network
- Inventory datasets
- ...

Networks may be or not bipartite: Interactions between nodes belonging to the same or to different functional group(s).

Terminology

A network consists in:

- nodes/vertices which represent individuals / species /ships which may interact or not,
- links/edges/connections which stand for an interaction between a pair of nodes / dyads.

A network may be

- directed / oriented (e.g. food web...),
- symmetric / undirected (e.g. coexistence network),
- with or without loops.

This distinction only makes sense for simple networks (not bipartite).

Available data

- the network provided as:
  - an adjacency matrix (for simple network) or an incidence matrix (for bipartite network),
  - a list of pair of nodes / dyads which are linked.
- some additional covariates on nodes, dyads which can account for sampling effort.

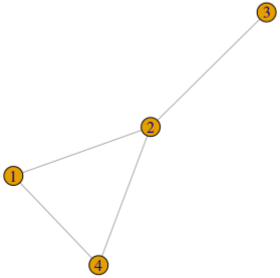
Goal

- Unraveling / describing / modeling the network topology.
- Discovering particular structure of interaction between some subsets of nodes.
- Understanding network heterogeneity.
- Not inferring the network !

Network representation and adjacency matrix

For a non-directed network

$$Y = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

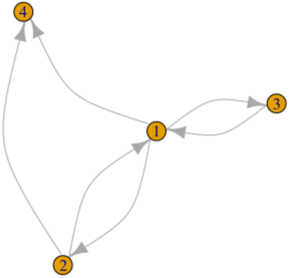


- n rows and n columns,
- symmetric matrix

Network representation and adjacency matrix

For a directed network

$$Y = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



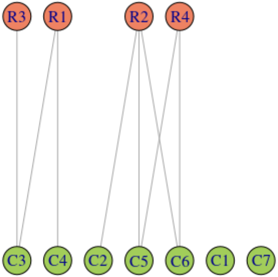
- n rows and n columns,
- symmetric matrix

Bipartite network and incidence matrix

$$Y = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

- n rows and m columns, rectangular matrix.
- corresponding adjacency matrix $(n+m) \times (n+m)$:

$$\begin{pmatrix} 0 & Y \\ Y^T & 0 \end{pmatrix}$$



Some common features studied on networks

- Degree distribution, can be viewed as a measure of heterogeneity,
- Nestedness: a network is said to be nested when its nodes that have the smallest degree, are connected to nodes with the highest degree, [Rodríguez-Gironés and Santamaría, 2006]
- Betweenness centrality: for a node, numbers of shortest paths between any pair of nodes passing through this node. [Freeman, 1978]
- Modularity: is a measure for a given partition of its tendency of favoring intra-connection over inter-connection. $\Rightarrow$ Finding the best partition with respect to modularity criterion. [Clauset et al., 2008]

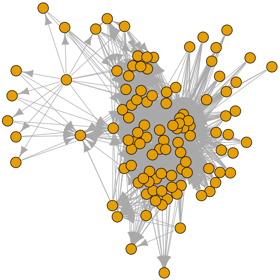
Some common features studied on networks

All this criterion shall be adapted to:

- directed network,
- bipartite network.

R packages: igraph, sna, vegan.

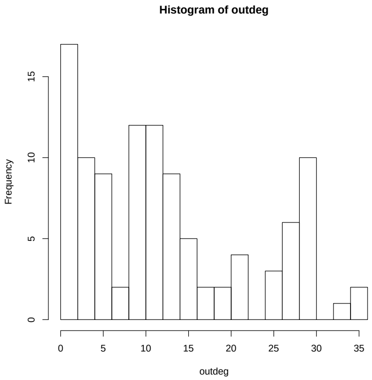
Example Chilean food web

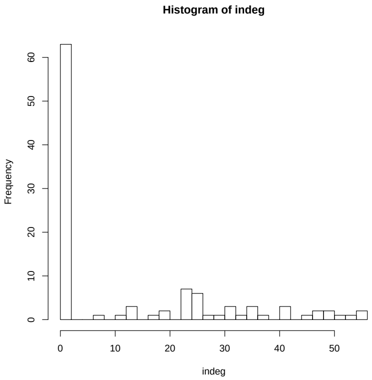


- $n = 106$ species / nodes,
- density of edges: 12.1%.

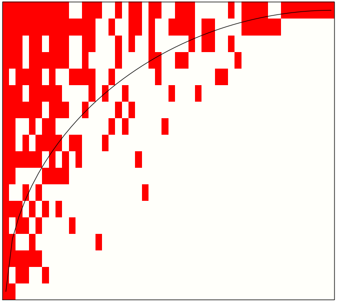
[Kéfi et al., 2016]

Degree distribution



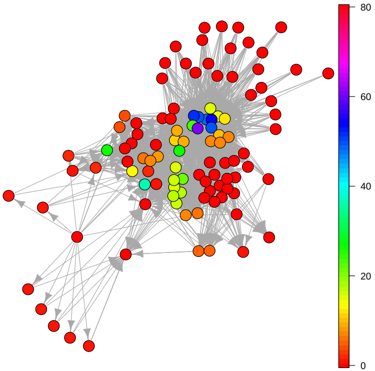


Nestedness



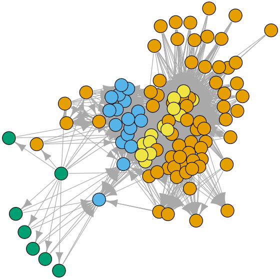
- more generally used on incidence matrices,
- significance of the nestedness index computed by random permutations of the matrix,
- this food web is found to be nested.

Betweenness



Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.000	0.000	0.000	6.604	6.929	59.570

Modularity



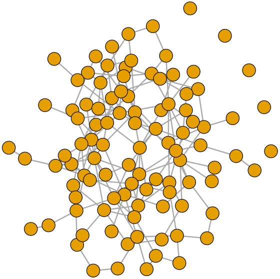
1	2	3	4
69	17	7	13
- very low modularity.

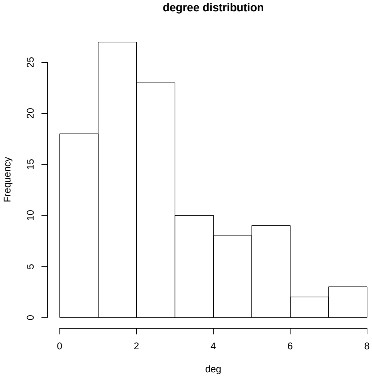
A first random graph model for network: null model

Erdős-Rényi (1959) Model for n nodes

$$\forall 1 \leq i,j \leq n, \quad Y_{ij} \overset{i.i.d.}{\sim} b(p),$$

where b is the Bernoulli distribution and $p \in [0,1]$ a probability for a link to exist.





Limitations of an ER graph to describe real networks

- Degree distribution too concentrated, no high degree nodes,
- All nodes are equivalent (no nestedness...),
- No modularity.

Stochastic Block Model

[Nowicki and Snijders, 2001] Let (Y_{ij}) be an adjacency matrix

Latent variables

- The nodes $i = 1, \dots, n$ are partitionned into K clusters
- $Z_i = k$ if node i belongs to cluster (block) k
- Z_i independant variables

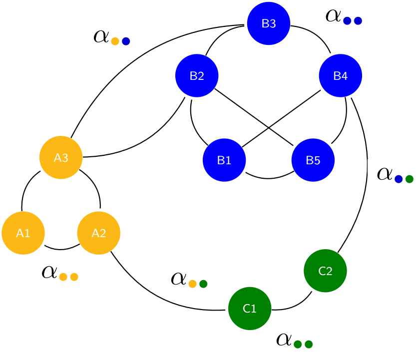
$$\mathbb{P}(Z_i = k) = \pi_k$$

Conditionally to $(Z_i)_{i=1,\dots,n}$...

(Y_{ij}) independant and

$$Y_{ij} | Z_i, Z_j \sim \mathcal{B}ern(\alpha_{Z_i,Z_j}) \quad \Leftrightarrow \quad P(Y_{ij} = 1 | Z_i = k, Z_j = \ell) = \alpha_{k\ell}$$

Stochastic Block Model : illustration



Parameters

Let n nodes divided into 3 clusters

- $\mathcal{K} = \{\textcolor{orange}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$ clusters
- $\pi_\bullet = \mathbb{P}(i \in \bullet)$, $\bullet \in \mathcal{K}, i = 1, \ldots, n$
- $\alpha_{\textcolor{orange}{\bullet}\textcolor{blue}{\bullet}} = \mathbb{P}(i \leftrightarrow j | i \in \textcolor{orange}{\bullet}, j \in \textcolor{blue}{\bullet})$

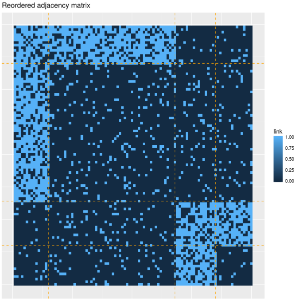
$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \pi), \quad \forall \bullet \in \mathcal{K},$$

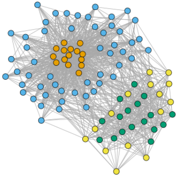
$$Y_{ij} \mid \{i \in \textcolor{orange}{\bullet}, j \in \textcolor{blue}{\bullet}\} \sim^{\text{ind}} \mathcal{B}(\alpha_{\textcolor{orange}{\bullet}\textcolor{blue}{\bullet}})$$

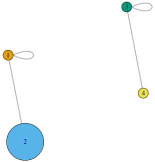
SBM : A great generative model

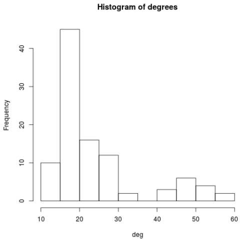
- Generative model : easy to simulate
- No a priori on the type of structure
- Combination of modularity, nestedness, etc...

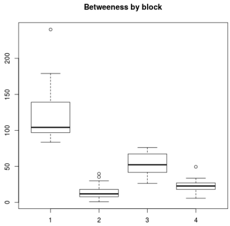
Networks with hubs generated by SBM



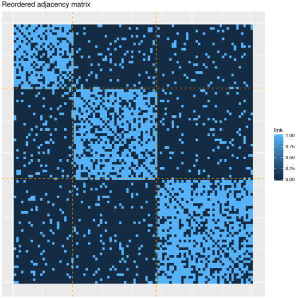


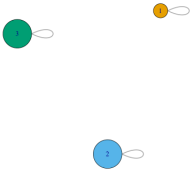


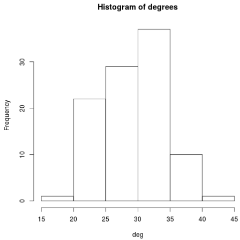




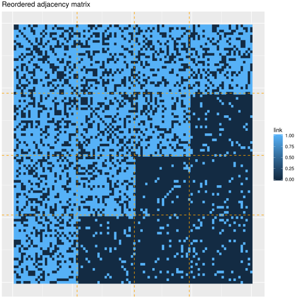
Community network generated by SBM

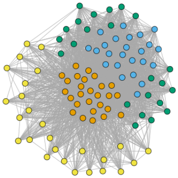


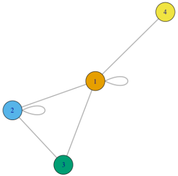


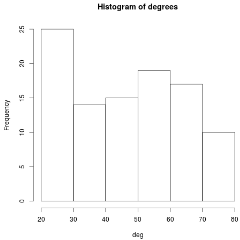


Nestedness generated by SBM

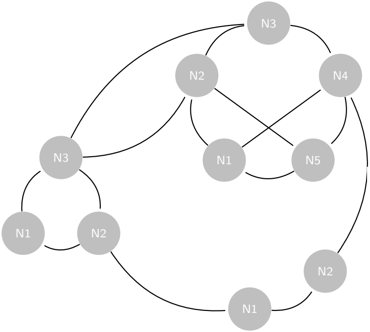








Statistical inference



Stochastic Block Model

Let *n* nodes divided into

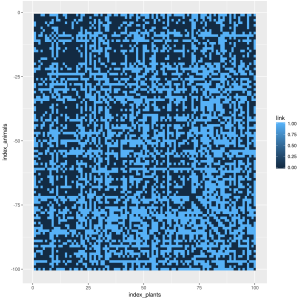
- $\mathcal{K} = \{\bullet, \bullet, \bullet\}$, card($\mathcal{K}$) known
- $\pi_\bullet = ?$,
- $\alpha_{\bullet\bullet} = ?$

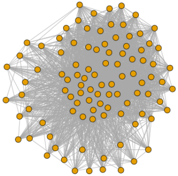
[Nowicki and Snijders, 2001], [Daudin et al., 2008]

R package: blockmodels.

Statistical inference

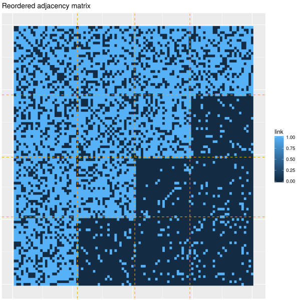
From....

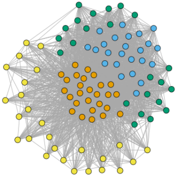




Statistical inference

... to





Statistician job

- Find the clusters
- Find the number of clusters
- Practical implementation
- Theoretical results

Details given later.

Probabilistic model for binary bipartite networks

Let Y_{ij} be a bi-partite network. Individuals in row and cols are not the same.

Latent variables : bi-clustering

- Nodes $i = 1, \dots, n_1$ partitionned into K_1 clusters, nodes $j = 1, \dots, n_2$ partitionned into K_2 clusters
- $$\begin{array}{ll} Z_i^1 = k & \text{if node } i \text{ belongs to cluster (block) } k \\ Z_j^2 = \ell & \text{if node } j \text{ belongs to cluster (block) } \ell \end{array}$$
- Z_i^1, Z_j^2 independent variables

$$\mathbb{P}(Z_i^1 = k) = \pi_k^1, \quad \mathbb{P}(Z_j^2 = \ell) = \pi_\ell^2$$

Probabilistic model for binary bipartite networks

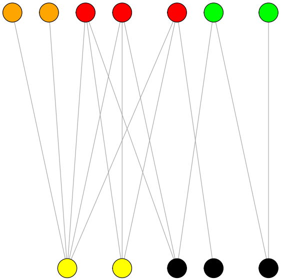
Conditionally to $(Z_i^1)_{i=1,\dots,n_1}, (Z_j^2)_{j=1,\dots,n_2} \cdots$

(Y_{ij}) independent and

$$Y_{ij}|Z_i^1, Z_j^2 \sim \mathcal{B}ern(\alpha_{Z_i^1, Z_j^2}) \quad \Leftrightarrow \quad \mathbb{P}(Y_{ij} = 1 | Z_i^1 = k, Z_j^2 = \ell) = \alpha_{k\ell}$$

[Govaert and Nadif, 2008]

Latent Block Model : illustration



Latent Block Model

- n_1 row nodes $\mathcal{K}_1 = \{\bullet, \bullet, \bullet\}$ classes
- $\pi^1_\bullet = \mathbb{P}(i \in \bullet), \bullet \in \mathcal{K}_1, i = 1, \ldots, n$
- n_2 column nodes $\mathcal{K}_2 = \{\bullet, \bullet\}$ classes
- $\pi^2_\bullet = \mathbb{P}(j \in \bullet), \bullet \in \mathcal{K}_2, j = 1, \ldots, m$
- $\alpha_{\bullet\bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$
\begin{aligned}
Z^1_i = \mathbf{1}_{\{i \in \bullet\}} &\sim^{\text{iid}} \mathcal{M}(1, \pi^1), \quad \forall \bullet \in \mathcal{Q}_1, \\
Z^2_j = \mathbf{1}_{\{j \in \bullet\}} &\sim^{\text{iid}} \mathcal{M}(1, \pi^2), \quad \forall \bullet \in \mathcal{Q}_2, \\
Y_{ij} \mid \{i \in \bullet, j \in \bullet\} &\sim^{\text{ind}} \mathcal{B}ern(\alpha_{\bullet\bullet})
\end{aligned}
$$

[Govaert and Nadif, 2008] and R package: blockmodels as well.

Valued-edge networks

Values-edges networks

Information on edges can be something different from presence/absence. It can be:

1. a count of the number of observed interactions,
2. a quantity interpreted as the interaction strength,

Natural extensions of SBM and LBM

1. Poisson distribution: $Y_{ij} \mid \{i \in \textcolor{orange}{\bullet}, j \in \textcolor{blue}{\bullet}\} \sim^{\text{ind}} \mathcal{P}(\lambda_{\textcolor{orange}{\bullet}\textcolor{blue}{\bullet}})$,
2. Gaussian distribution: $Y_{ij} \mid \{i \in \textcolor{orange}{\bullet}, j \in \textcolor{blue}{\bullet}\} \sim^{\text{ind}} \mathcal{N}(\mu_{\textcolor{orange}{\bullet}\textcolor{blue}{\bullet}}, \sigma^2)$, [Mariadassou et al., 2010]
3. More generally,

$$Y_{ij} \mid \{i \in \textcolor{orange}{\bullet}, j \in \textcolor{blue}{\bullet}\} \sim^{\text{ind}} \mathcal{F}(\theta_{\textcolor{orange}{\bullet}\textcolor{blue}{\bullet}})$$

Multiplex networks

Several kind of interactions between nodes . For instance :

- Love and friendship
- Working relations and friendship
- In ecology : mutualistic and competition

Block model for multiplex networks

$Y_{ij} \in \{0,1\}^Q = (Y^a_{ij}, Y^b_{ij}), \forall w \in \{0,1\}^2$

$$\mathbb{P}(Y^a_{ij}, Y^b_{ij} = w | Z_i = k, Z_j = \ell) = \alpha^w_{k\ell}$$

[Kéfi et al., 2016], [Barbillon et al., 2017]
In R package: blockmodels when two relations are at stake.
Remark: a particular case of multiplex network is dynamic network, [Matias and Miele, 2017].

Taking into account covariates

Sometimes covariates are available. They may be on:

- nodes,
- edges,
- both.

1. They can be used a posteriori to explain blocks inferred by SBM.
2. Extension of the SBM which takes into account covariates. Blocks are structure of interaction which is not explained by covariates !

If covariates are sampling conditions, case 2 be may more interesting.

SBM with covariates

- As before : (Y_{ij}) be an adjacency matrix
- Let $x^{ij} \in \mathbb{R}^p$ denote covariates describing the pair (i,j)

Latent variables : as before

- The nodes $i = 1, \dots, n$ are partitioned into K clusters
- Z_i independent variables

$$\mathbb{P}(Z_i = k) = \pi_k$$

Conditionally to $(Z_i)_{i=1,\dots,n}\dots$

(Y_{ij}) independent and

$$\begin{array}{lcll} Y_{ij}|Z_i, Z_j & \sim & \mathcal{B}ern(\text{logit}(\alpha_{Z_i,Z_j} + \theta \cdot x_{ij})) & \text{if binary data} \\ Y_{ij}|Z_i, Z_j & \sim & \mathcal{P}(\exp(\alpha_{Z_i,Z_j} + \theta \cdot x_{ij})) & \text{if counting data} \end{array}$$

If $K = 1$: all the connection heterogeneity is explained by the covariates.

Statistical Inference

- Selection of the number of clusters K for SBM or K_1, K_2 for LBM
- Estimation of the parameters $\pi, \boldsymbol{\theta}$ for a given number of clusters
- Clustering $\hat{\mathbf{Z}}$

Likelihood for SBM

Complete likelihood (Y) et (Z)

$$
\begin{aligned}
\ell_c(\mathbf{Y},\mathbf{Z};\theta) &= p(\mathbf{Y}|\mathbf{Z};\boldsymbol{\alpha})p(\mathbf{Z};\pi) \\
&= \prod_{i,j} f_{\alpha_{Z_i,Z_j}}(Y_{ij}) \times \prod_i \pi_{Z_i} \\
&= \prod_{i,j} \alpha_{Z_i,Z_j}^{Y_{ij}}(1-\alpha_{Z_i,Z_j})^{1-Y_{ij}} \prod_i \pi_{Z_i}
\end{aligned}
$$

Marginal likelihood (Y)

$$
\log \ell(\mathbf{Y};\theta) = \log \sum_{\mathbf{Z}\in\mathcal{Z}} \ell_c(\mathbf{Y},\mathbf{Z};\theta)\,. \tag{1}
$$

Marginal likelihood : remark

$$\log \ell(\mathbf{Y};\theta) = \log \sum_{\mathbf{Z}\in\mathcal{Z}} \ell_c(\mathbf{Y},\mathbf{Z};\theta)\,.$$

Remark

$\mathcal{Z} = \otimes_{q=0\dots Q}\{1,\dots,K_q\}^{n_q} \Rightarrow$ when K and n increase, impossible to compute.

Standard tool to maximize the likelihood when latent variables involved : EM algorithm.

From EM to variational EM

Standard EM

At iteration (t) :

- **Step E**: compute
$$Q(\theta|\theta^{(t-1)}) = \mathbb{E}_{\mathbf{Z}|\mathbf{Y},\theta^{(t-1)}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)\right]$$
- **Step M**:
$$\theta^{(t)} = \arg\max_{\theta} Q(\theta|\theta^{(t-1)})$$

Limitations of standard EM

- Step E requires the computation of $\mathbb{E}_{\mathbf{Z}|\mathbf{Y},\theta^{(t-1)}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)\right]$
- However, once conditioned by par $\mathbf{Y}$, the $\mathbf{Z}$ are not independent anymore: complex distribution if K and n big.

Variational EM : maximization of a lower bound

Idea : replace the complicated distribution $p(\cdot|\mathbf{Y};\theta) = [\mathbf{Z}|\mathbf{Y},\theta]$ by a simpler one.
Let $\mathcal{R}_{\mathbf{Y},\tau}$ be any distribution on $\mathbf{Z}$

Central identity

$$\begin{aligned}
\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\tau}) &= \log \ell(\mathbf{Y};\theta) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau}, p(\cdot|\mathbf{Y};\theta)] \quad \leq \log \ell(\mathbf{Y};\theta)\\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \sum_{\mathbf{Z}} \mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z}) \log \mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z})\\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] + \mathcal{H}(\mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z}))
\end{aligned}$$

Note that:

$$\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\tau}) = \log \ell(\mathbf{Y};\theta) \Leftrightarrow \mathcal{R}_{\mathbf{Y},\tau} = p(\cdot|\mathbf{Y};\theta)$$

Proof

By Bayes

$$
\begin{aligned}
\log \ell_c(\mathbf{Y},\mathbf{Z};\theta) &= \log p(\mathbf{Z}|\mathbf{Y};\theta) + \log \ell(\mathbf{Y};\theta)\\
\log \ell(\mathbf{Y};\theta) &= \log \ell_c(\mathbf{Y},\mathbf{Z};\theta) - \log p(\mathbf{Z}|\mathbf{Y};\theta)
\end{aligned}
$$

By integration against $\mathcal{R}_{\mathbf{Y},\tau}$:

$$
\begin{aligned}
\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell(\mathbf{Y};\theta)] &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log p(\mathbf{Z}|\mathbf{Y};\theta)]\\
\log \ell(\mathbf{Y};\theta) &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log p(\cdot|\mathbf{Y};\theta)]
\end{aligned}
$$

As a consequence:

$$
\begin{aligned}
\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\tau}) &= \log \ell(\mathbf{Y};\theta) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau}, p(\cdot|\mathbf{Y};\theta)]\\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log p(\mathbf{Z}|\mathbf{Y};\theta)]\\
&\quad -\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}\left[\log \frac{\mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z})}{p(\mathbf{Z}|\mathbf{Y};\theta)}\right]\\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log p(\mathbf{Z}|\mathbf{Y};\theta)]\\
&\quad \underbrace{-\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log \mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z})]}_{\mathcal{H}(\mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z}))} + \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log p(\mathbf{Z}|\mathbf{Y};\theta)]
\end{aligned}
$$

Variational EM

- Maximization of $\log \ell(\mathbf{Y};\theta)$ w.r.t. θ replaced by maximization of the lower bound $\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\tau})$ w.r.t. τ and θ.
- **Benefit** : we choose $\mathcal{R}_{\mathbf{Y},\tau}$ such that the maximization calculus can be done explicitly
  - In our case: mean field approximation : neglect dependencies between the (Z_i)
$$P_{\mathcal{R}_{\mathbf{Y},\tau}}(Z_i^q = k) = \tau_{ik}^q$$

Variational EM

Algorithm

At iteration (t), given the current value $(\theta^{(t-1)}, \mathcal{R}_{\mathbf{Y},\tau^{(t-1)}})$,

- **Step 1** Maximization w.r.t. τ

$$
\begin{aligned}
\tau^{(t)} &= \arg\max_{\tau \in \mathcal{T}} \mathcal{I}_{\theta^{(t-1)}}(\mathcal{R}_{\mathbf{Y},\tau}) \\
&= \arg\max_{\tau \in \mathcal{T}} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}\left[\log \ell_c(\mathbf{Y}, \mathbf{Z}; \theta^{(t-1)})\right] + \mathcal{H}\left(\mathcal{R}_{\mathbf{Y},\tau}(\mathbf{Z})\right) \\
&= \arg\max_{\tau \in \mathcal{T}} \log \ell(\mathbf{Y}; \theta^{(t-1)}) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau}, p(\cdot|\mathbf{Y}; \theta^{(t-1)})] \\
&= \arg\min_{\tau \in \mathcal{T}} \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau}, p(\cdot|\mathbf{Y}; \theta^{(t-1)})]
\end{aligned}
$$

Variational EM

Algorithm

- **Step 2** Maximization w.r.t. θ

$$
\begin{aligned}
\theta^{(t)} &= \arg\max_{\theta} \mathcal{I}_{\theta}(\mathcal{R}_{\mathbf{Y},\tau^{(t)}}) \\
&= \arg\max_{\theta} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau^{(t)}}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)\right] + \mathcal{H}\left(\mathcal{R}_{\mathbf{Y},\tau^{(t)}}(\mathbf{Z})\right) \\
&= \arg\max_{\theta} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau^{(t)}}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)\right]
\end{aligned}
$$

In practice

- Really fast
- Strongly depend on the initial values

Penalized likelihood criterion

- Selection of the number of clusters K (or K_1, K_2 in the LBM)
- Integrated Classification Likelihood (ICL) [Biernacki et al., 2000]

$$ICL(\mathcal{M}_{\mathbf{K}}) = \log \ell_c(\mathbf{Y}, \hat{\mathbf{Z}}; \hat{\theta}_{\mathbf{K}}) - \mathrm{pen}(\mathcal{M}_{\mathbf{K}}) \tag{2}$$

where

$$\hat{Z}_i^q = \underset{k \in \{1,\dots,K_q\}}{\arg\max}\ \hat{\tau}_{ik}^q. \tag{3}$$

- Integrated Complete Likelihood (ICL)

$$ICL(\mathcal{M}_{\mathbf{K}}) = \mathbb{E}_{p(\cdot|\mathbf{Y},\hat{\theta}_{\mathbf{K}})}[\log \ell_c(\mathbf{Y}, \hat{\mathbf{Z}}; \hat{\theta}_{\mathbf{K}}) - \mathrm{pen}(\mathcal{M}_{\mathbf{K}}) \tag{4}$$

Expression of the penalization

For SBM

$$pen_{\mathcal{M}} = \begin{cases} -\frac{1}{2}\left\{(K-1)\log(n) + K^2\log\left(n^2-n\right)\right\} & \text{for directed network} \\ -\frac{1}{2}\left\{\underbrace{(K-1)\log(n)}_{\text{Clust.}} + \frac{K(K+1)}{2}\log\left(\frac{n^2-n}{2}\right)\right\} & \text{for undirected network} \end{cases}$$

For LBM

$$pen_{\mathcal{M}} = -\frac{1}{2}\left\{\underbrace{(K_1-1)\log(n_1) + (K_2-1)\log(n_2)}_{\text{Bi-Clust.}} + \underbrace{(K_1K_2)\log(n_1n_2)}_{\text{Connection}}\right\}$$

Advantages of ICL

- its capacity to outline the clustering structure in networks
- Involves a trade-off between goodness of fit and model complexity
- ICL values : goodness of fit AND clustering sharpness.

Comments on the ICL versus BIC

Conjecture

$$BIC(\mathcal{M}) \quad = \quad \log \ell(\mathbf{Y}; \hat{\theta}, \mathcal{M}) - \mathrm{pen}(\mathcal{M})$$

with the same penalty

- Under this conjecture

$$\begin{aligned} ICL(\mathcal{M}) &= BIC(\mathcal{M}) + \sum_{\mathbf{Z}} p(\mathbf{Z}|\mathbf{Y}; \hat{\theta}_{\mathbf{K}}) \log p(\mathbf{Z}|\mathbf{Y}; \hat{\theta}_{\mathbf{K}}) \\ &= BIC(\mathcal{M}) - \mathcal{H}(p(\cdot|\mathbf{Y}; \theta)) \end{aligned}$$

- As a consequence, because of the entropy, ICL will encourage clustering with well-separated groups
-

$$\widehat{ICL}(\mathcal{M}) = BIC(\mathcal{M}) + \sum_{\mathbf{Z}} \mathcal{R}_{\mathbf{Y}}(\mathbf{Z}, \widehat{\tau}) \log \mathcal{R}_{\mathbf{Y}, \widehat{\tau}}(\mathbf{Z}) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y}, \widehat{\tau}}, p(\cdot|\mathbf{Y}; \widehat{\theta})] \, .$$

Algorithm in practice

- Going trough the models and initiate VEM at the same time
- Bounds on K : $\{K_{\text{min}}, \dots, K_{\text{max}}\}$

Stepwise procedure

Starting from K

- **Split** : if $K < K_{\text{max}}$
  - Maximize the likelihood (lower bound) of $\mathcal{M}_{K+1}$
  - K initializations of the VEM are proposed : split each cluster into 2 clusters
- **Merge** : If $K > K_{\text{min}}$
  - Maximize the likelihood (lower bound) of model $\mathcal{M}_{K-1}$
  - $\frac{K(K-1)}{2}$ initializations of the VEM are proposed : merging all the possible pairs of clusters

Theoretical properties for SBM

- Identifiability and a first consistency result by [Celisse et al., 2012]
- Consistency of the posterior distribution of the latent variables [Mariadassou and Matias, 2015]
- Consistency and properties of the variational estimators [Bickel et al., 2013]

Other extensions

- Time evolving networks Matias
- Multipartite networks (R-package GREMLIN, Bar-Hen, Barbillon, Donnet)
- Multilevel networks (individuals and organizations) (Chabbert-Liddell)
- Missing data in the network,

Probabilistic model for networks in a nutshell

SBM/LBM

- generative models,
- flexible,
- comprehensive models which can be linked to a lot of classical descriptors.

References I

Barbillon, P., Donnet, S., Lazega, E., and Bar-Hen, A. (2017).
Stochastic block models for multiplex networks: an application to a multilevel network of researchers.
Journal of the Royal Statistical Society: Series A (Statistics in Society), 180(1):295–314.

Bickel, P., Choi, D., Chang, X., Zhang, H., et al. (2013).
Asymptotic normality of maximum likelihood and its variational approximation for stochastic blockmodels.
The Annals of Statistics, 41(4):1922–1943.

Biernacki, C., Celeux, G., and Govaert, G. (2000).
Assessing a mixture model for clustering with the integrated completed likelihood.
IEEE transactions on pattern analysis and machine intelligence, 22(7):719–725.

References II

Celisse, A., Daudin, J.-J., and Pierre, L. (2012).
Consistency of maximum-likelihood and variational estimators in the stochastic block model.
Electronic Journal of Statistics, 6:1847–1899.

Clauset, A., Moore, C., and Newman, M. E. (2008).
Hierarchical structure and the prediction of missing links in networks.
Nature, 453(7191):98.

Daudin, J.-J., Picard, F., and Robin, S. (2008).
A mixture model for random graphs.
Statistics and computing, 18(2):173–183.

Freeman, L. C. (1978).
Centrality in social networks conceptual clarification.
Social Networks, 1(3):215 – 239.

References III

Govaert, G. and Nadif, M. (2008).

Block clustering with bernoulli mixture models: Comparison of different approaches.

Computational. Statistics and Data Analysis, 52(6):3233–3245.

Kéfi, S., Miele, V., Wieters, E. A., Navarrete, S. A., and Berlow, E. L. (2016).

How structured is the entangled bank? the surprisingly simple organization of multiplex ecological networks leads to increased persistence and resilience.

PLOS Biology, 14(8):1–21.

Mariadassou, M. and Matias, C. (2015).

Convergence of the groups posterior distribution in latent or stochastic block models.

Bernoulli, 21(1):537–573.

References IV

Mariadassou, M., Robin, S., and Vacher, C. (2010).
Uncovering latent structure in valued graphs: a variational approach.
The Annals of Applied Statistics, 4(2):715–742.

Matias, C. and Miele, V. (2017).
Statistical clustering of temporal networks through a dynamic stochastic block model.
Journal of the Royal Statistical Society: Series B (Statistical Methodology), 79(4):1119–1141.

Nowicki, K. and Snijders, T. A. B. (2001).
Estimation and prediction for stochastic blockstructures.
Journal of the American Statistical Association, 96(455):1077–1087.

Rodríguez-Gironés, M. and Santamaría, L. (2006).
Rodríguez-gironés ma, santamaría l.. a new algorithm to calculate the nestedness temperature of presence-absence matrices. j biogeogr 33: 924-935.
Journal of Biogeography, 33:924 – 935.