Minimum Mean Cycle Cancelling Algorithm

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Network Flow

flow network: a flow network denoted as (G, c, s, t) consists of a directed graph $G = (V, E)$ without multiple edges, where each edge has a non-negative capacity function $c: E(G) \to \mathbb{R}_+$. The network also includes two distinguished nodes: source s and sink t.

Each flow that satisfies all the following constraints is called a **feasible flow**.

1. $\sum_{u:(u,v)\epsilon E} f_{u,v} - \sum_{w:(v,w)\epsilon E} f_{v,w} = 0, v\epsilon V \setminus \{s, t\}$
2. $f_{u,v} \leq c_{u,v}, (u, v)\epsilon E$
3. $f_{u,v} \geq 0, (u, v)\epsilon E$

Costs – we can add to each edge it's cost, and overall, a cost function $c: E(G) \to \mathbb{R}$
There is also a function b: $V(G) \to \mathbb{R}$, that maps to each vertex v his ***supply / demand b***$(\boldsymbol{v})$. If $b(v) > 0$, $|b(v)|$ is called supply, and if $b(v) < 0$, $|b(v)|$ is called demand of v.

b-flow

b-flow (definition): given a diagraph G, with capacities u : $E(G) \rightarrow \mathbb{R}_+$, and function b: $V(G) \rightarrow \mathbb{R}$ with $\sum_{v \epsilon V(G)} b(v)$ $= 0$, a b-flow in (G , u) is a function f: $E(G) \rightarrow \mathbb{R}_+$ with $f(e) \leq u(e)$ for all $e \in E(G)$ and $\sum_{e \epsilon \delta^+_{(v)}} f(e) - \sum_{e \epsilon \delta^-_{(v)})} f(e) = b(v)$ for all $v \in E(V)$.

Thus, a b-flow with $b \equiv 0$ is a circulation. If $b(v) > 0$, $|b(v)|$ is called supply, and if $b(v) < 0$, $|b(v)|$ is called demand of v.

The minimum cost problem

Minimum cost problem: The goal is to find the cheapest possible way of sending a certain amount of flow through a flow network, from one or more sources to one or more sinks. In addition to the network graph $G = (V, E)$ and the capacities function $u: E(G) \to \mathbb{R}_+$, we are also given a **cost function** $c: E(G) \to \mathbb{R}$. Each edge has capacity and cost.

MINIMUM COST FLOW PROBLEM

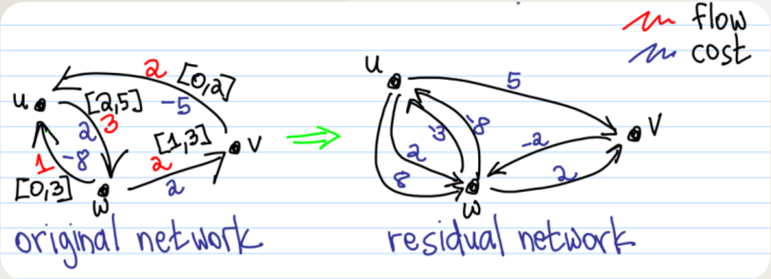
Instance:	A digraph G, capacities $u : E(G) \to \mathbb{R}_+$, numbers $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$, and weights $c : E(G) \to \mathbb{R}$.
Task:	Find a b-flow f whose cost $c(f) := \sum_{e \in E(G)} f(e)c(e)$ is minimum (or decide that none exists).

Our goal is to find a b-flow f, whose cost is minimum, or decide that it does not exist.

Residual network with respect to circulation x

Residual network with respect to circulation x: we construct the residual circulation $R_x = (V, E_x)$ as follows: for every ordered pair $u, v \in V$ we add :

1. edge (u, v) with $\hat{C}_{uv} = C_{uv}$, when $X_{uv} < U_{uv}$
2. edge (u, v) with $\hat{C}_{uv} = -C_{vu}$, when $X_{vu} > L_{vu}$



Propositions

We aim to establish an optimality criterion for b-flow to ascertain **whether its cost is minimized.**

Proposition 9.5 : Let G be a diagraph with capacities $u: E \to \mathbb{R}_+$. Let f and f' be b-flows in (G,u). Then, $g : E(\overleftrightarrow{G}) \to \mathbb{R}_+$ defined by $g(e) := \max\{0, f'(e) - f(e)\}$ and $g(\overleftarrow{e}) := \max\{0, f(e) - f'(e)\}$ for $e \in E(G)$ is a circulation in $\overleftrightarrow{G}$. Furthermore, $g(e) = 0$ for all $e \notin E(G_f)$ and $c(g) = c(f') - c(f)$.

Proposition 9.6: (Ford and Fulkerson [1962]) For any circulation f in a digraph G there is a family C of at most $|E(G)|$ circuits in G and positive numbers $h(c)$ $c \in C$ such that $f(e) = \sum_{c \epsilon C, e \epsilon E(c)} h(c)$or all $e \epsilon E(c)$.

Corollary 9.8: (Ford and Fulkerson [1962]) Let (G, u, b, c) be an instance of the MINIMUM COST FLOW PROBLEM. A b-flow f is of minimum cost if and only if there exists a feasible potential for .(G_f,c).

An Optimality Criterion

Theorem 9.7: Let (G, u, b, c) be an instance of the minimum cost flow problem. A b- flow f is of minimum cost if and only if there is no f- augmenting cycle with total negative weight (cost) in the correspondent residual graph.

<u>Proof:</u>

- Assuming there is an $f - augmenting\ cycle\ C$ with weight $\gamma < 0$,
- we can augment f along each edge on the cycle C by some $\varepsilon > 0$ and get b-flow with cost decreased by $-\gamma\varepsilon < 0$. So f is not a minimum cost flow.
- If f is not a minimum cost b-flow, there is another b-flow f' with <u>smaller</u> cost.
- We consider g as defined in proposition 9.5. Then, g is a circulation, with $c(g) < 0$.
- By proposition 9.6, there is a family C of at most $|E(G)|$ circuits in G and positive numbers $h(c)$ $c \in C$ such that $f(e) = \sum_{c\epsilon c, e\epsilon E(c)} h(c)$ or all $e\epsilon E(c)$, and h is the cost function.
- Since $g(e) = 0$ for all $e \notin E\left(G_f\right)$, all these circuits are $f - augmenting$.
- Therefore, if $c(g) < 0$, then at least one of the cycles must have negative total weight.

Karp's Algorithm

How to find the minimum mean cycle in some digraph:

DIRECTED MINIMUM MEAN CYCLE PROBLEM

Instance: A digraph G, weights $c : E(G) \to \mathbb{R}$.

Task: Find a circuit C whose mean weight $\frac{c(E(C))}{|E(C)|}$ is minimum, or decide that G is acyclic.

Theorem 7.12: Let G be a diagraph with weights c: $E(G) \to \mathbb{R}$. Let $s \in V(G)$ such that each vertex is reachable from s. For $x \in V(G)$ and $k \in \mathbb{Z}_+$.

Let $F_k(x) \coloneqq min\left\{\sum_{i=1}^{k} c\big((v_{i-1},v_i)\big) : v_0 = s, v_k = x, (v_{i-1},v) \in E(G)\ for\ all\ i\right\}$ be the minimum weight of an edge progression of length k from s to x (and ∞ if there is none). Let $\mu(G,c)$ be the minimum mean weight of a circuit in G (and $\mu(G,c) = \infty$ if G is acyclic). Then:

$$\mu(G,c) = \min_{x \in V(G)} \max_{\substack{0 \le k \le n-1 \\ F_k(x) < \infty}} \frac{F_n(x) - F_k(x)}{n-k}.$$

Pseudo-code

Let’s break down the algorithm's steps:

1. Add vertex s to graph G, and connect it to all the vertices with edges (s, x) with cost 0.
2. Set n:= $|V(G)|$. We define $F_k(x)$ is the cost of getting to vertex x from s in k steps. We will set $F_0(S)$ = 0 and $F_0(X)$ = ∞ for $x \epsilon V(G) \setminus \{s\}$ s.
3. We will go over every vertex x and for all k: $1 \leq k \leq n$ we will initialize $F_k(x)$ = ∞. Then we will calculate $F_k(x)$ as the minimum of $F_{k-1}(w) + x(w, x)$ for all vertices w that as an edge (w, x), using dynamic programming.
4. If $F_k(x)$ = ∞ for all vertices X that means that there was no cycle in G (otherwise for some vertex x $F_k(x)$ < ∞).
5. We will take for each vertex x the cycle with the minimum mean weight from all the cycles that contains x.
6. We will take the cycle with the minimum mean weight from all the cycles given from (5).

Complexity Analysis and Correctness:

(Karp, 1978) The Minimum Mean Cycle algorithm works correctly. It’s running time is $O(nm)$.
Step (3) is the primary contributor to the overall running time of the algorithm. Step (3) takes $O(nm)$ (two main “for” loops, one iterates over the vertices and the other iterates over the edges). Step (5) takes $O(n^2)$,but since we assume m>n , $O(n^2) \leq O(mn)$.

Minimum Mean Cycle Cancelling Algorithm

1. Start with some feasible b-flow. It can be found with any algorithm for the Maximum Flow Problem (Using a similar idea to the one behind Gale's theorem proof).

2. Find a cycle c in the residual graph which his mean weight is minimal. Can be implemented with Karp's algorithm for minimum mean cycle problem.

3. If c's weight is non-negative: we are done (optimality condition), and the b-flow is the maximal flow with minimum cost. Else, (c's weight is negative), augment b by
   $\gamma \coloneqq \min_{e \in E(C)} u_f(e)$ (the bottleneck).

4. Repeat those steps until there is no negative cycle in the residual graph.

Minimum Mean Cycle-Cancelling Algorithm

Input: A digraph G, capacities $u : E(G) \to \mathbb{R}_+$, numbers $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$, and weights $c : E(G) \to \mathbb{R}$.

Output: A minimum cost b-flow f.

① Find a b-flow f.

② Find a circuit C in G_f whose mean weight is minimum.
If C has nonnegative total weight (or G_f is acyclic) **then stop**.

③ Compute $\gamma := \min_{e \in E(C)} u_f(e)$. Augment f along C by γ.
Go to ②.

Lemmas and corollaries

Lemma: Let $f_1, f_2, \dots f_t$ be a sequence of b-flows such that for $i = 1, \dots t-1$ we have $\mu(f_i) < 0$ and f_{i+1} results from f_i by augmenting along C_i where C_i is a circuit of minimum weight in G_{f_i}. Then:

(a) : $\mu(f_k) < \mu(f_{k+1})$ for all k

(b) : $\mu(f_k) < \frac{n}{n-2}\mu(f_l)$ for all $k < l$ such that $c_k \cup c_l$ contains a pair of reverse edges.

Corollary 9.10. During the execution of the Minimum Mean Cycle Cancelling Algorithm, $|\mu(f)|$ decreases by at least a factor of $\frac{1}{2}$ with every $m \cdot n$ iterations.

complexity

The Minimum Mean Cycle Cancelling Algorithm runs in O($\boldsymbol{m^3n^2\, log(n)}$) time

When the costs are integral: By 9.10, we can infer that in this case, the running time is polynomial:
After $O(mnlog(n|c_{min}|))$ iterations, $\mu(f)$ is greater than $-\frac{1}{n}$. If the edge costs are integral, this implies $\mu(f) \geq 0$ and the algorithm stops .
When the costs are not necessarily integral: We will show that every $mn(\lceil log(n)\rceil + 1)$ iterations at least one edge is fixed. Therefore, there are at most $O(m^2nlog(n))$ iterations.

- Let f be the flow at some iteration, and let f' be the flow after $mn(\lceil log(n)\rceil + 1)$ iterations.
- We define cost function $c'(e) := c(e) - \mu(f')$ ($e \in E\left(G_{f'}\right)$).
- Let π be a feasible potential of $\left(G_{f'}, c'\right)$.
- So, we get $0 \leq c'_\pi(e) = c_\pi(e) - \mu(f')$,and as a result we conclude that: **(9.3)** $c_\pi(e) \geq \mu(f')$ for all $e \in \left(G_{f'}\right)$.
- Let C be the circuit of minimum mean cost in that is chosen in the algorithm to augment f.
- Since by corollary 9.10 , $\mu(f) \leq 2^{\lceil \log(n)\rceil+1}\mu(f') = 2 \cdot 2^{\lceil \log(n)\rceil}\mu(f') \leq 2n\mu(f')$
- we have, $\sum_{e\epsilon E(C)} c_\pi(e) = \sum_{e\epsilon E(C)} c(e) = \mu(f)|E(C)| \leq 2n\mu(f')|E(C)|$
- let $e \in E(C)$ with $c_\pi(e_0) \leq 2n\mu(f')$ .By 9.3 , we know that $e_0 \notin E\left(G_{f'}\right)$.

complexity

The Minimum Mean Cycle Cancelling Algorithm runs in O($m^3n^2\ log(n)$) time

Auxiliary Claim: For any b-flow f'' with $e_0 \in E(G_{f''})$ we have $\mu(f'') < \mu(f')$

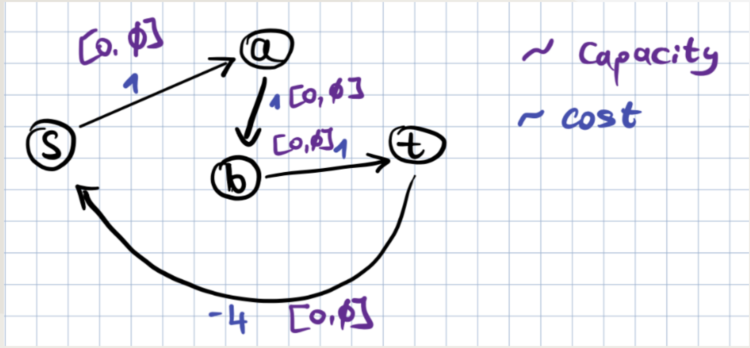
- By Lemma 9.9(a) e_0 will never be in the residual graph anymore , i.e. e_0 and $\overleftarrow{e_0}$ are fixed $mn([log(n)] + 1)$ iterations after e_0 is used in circuit C.
- let f'' be a b-flow with $e_0 \epsilon E\left(G_{f''}\right)$. We use proposition 9.5 to $f'$and f'' and get a circulation g with $g(e) = 0$ for all $e \ \notin E\left(G_{f'}\right)$ and $g(\overleftarrow{e_0}) > 0$ (because $e_0 \epsilon E\left(G_{f''}\right) \backslash E\left(G_{f'}\right)$).
- By Proposition 9.6, g can be written as the sum of flows on $f' - augmenting$ cycles. One of these circuits, for example W, must contain $\overleftarrow{e_0}$.
- By using $c_\pi(\overleftarrow{e_0}) = -c_\pi(e_0) \geq -2n\mu(f')$ and applying (9.3), to all $e \epsilon E(W) \backslash \{\overleftarrow{e_0}\}$, we now get a lower bound for the total cost of W: $c\left(E(W)\right) = \sum_{e \epsilon E(W)} c_\pi(e) \geq\ -2n\mu(f') + (n-1)\mu(f') > -n\mu(f')$.
- However, the reverse of W is an $f'' - augmenting$ cycle.
- It means that $G_{f'}$ contains a circuit whose mean cost is less than $\mu(f')$.

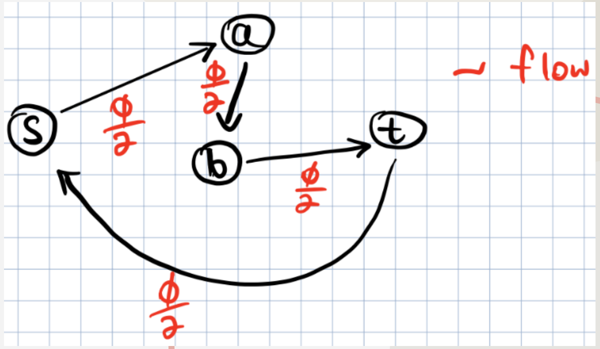
Problem with irrational costs when choosing a random negative cycle:

We will now proof that, if the capacities are irrational, the algorithm may not terminate if we don't choose the minimum mean weight circuit, but some negative circuit.

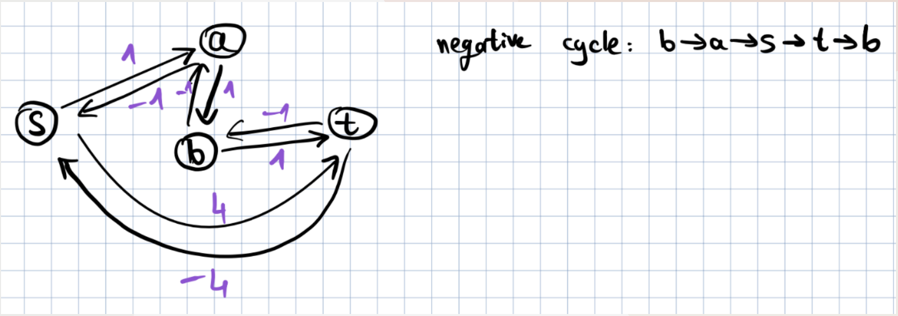
We will show an example of a case in which using an algorithm that chooses a random negative cycle rather than the minimum negative cycle causes a non-termination of the algorithm. We will use the next auxiliary claim for the example.

Auxiliary claim: It is impossible for an irrational number to be expressed as the sum of a finite number of rational numbers. However, rational numbers can be expressed as the sum of a finite number of rational numbers.





b-flow f



Residual Graph

- Then we would augment by $0 < \delta < \frac{\phi}{2}$ the cycle causing a change in the total cost.
- The golden ratio attribute states that every time we improve the cost it would be in a lower factor of the golden ratio, creating a sequence of decreases in cost $\delta_1, \delta_{2,\ldots}$ that strive to zero but never actually get to zero.
- Therefore, we can never actually fully saturate the edge, so it would never disappear from the residual graph, meaning the negative cycle will always be a negative cycle in the residual graph.
- As we know, as long as there is a negative cycle in the residual graph, the correspondent b-flow is not the optimal, so the algorithm continues to improve the target function each iteration.