Sensitivity Analysis for OPM-based Valuation in Matrix Form

Introduction

Within the realm of equity valuation, the Black-Scholes formula forms the backbone of OPM-based valuation methods, especially for companies with complicated capital structures. Given the breakpoints for a company's capital structure and its equity valuation, the Black-Scholes formula calculates how value distributes to shares of different priorities in different tranches, which then enables the computation of the price per share, or PPS, for each share type. Nonetheless, it is well-established that outputs from such models are sensitive to the Black-Scholes parameters, where it is essential to simulate different business situations to locate any substantial discrepancies in the outputs produced. Despite the ubiquity of such practices, it is still of interest to provide some theoretical derivations on the nature and properties of model sensitivities, which shall be the topic of discussion for this small write-up. We will commence with a recapitulation of the essential elements of the Black-Scholes formula and the existent OPM procedures, followed by a formal mathematical derivation for model sensitivities, which would then be applied to two hypothetical case scenarios.

Section 1. Black-Scholes formula and OPM-based valuation

In the landscape of private equities and venture capitals, investment frequently involves privately held companies with shares of different priorities as determined from a variety of customized contracts, which form a capital structure that is complicated to value. To provide a standardized valuation procedure, the OPM-based approach aims to reduce such capital complexities into a sequence of tranches with well-demarcated breakpoints, onto which the Black-Scholes formula could value each tranche as a call option with the equity valuation of the company as stock price, and the tranche's breakpoint as the strike price. Incremental values are computed from the option values obtained from each tranche, which are then allocated to different share types in accordance with their ratios in each tranche. This allocation eventually enables the computation for the total values of all share types, as well as their respective PPSs.

For a tranche with breakpoint b, let S denote the equity valuation for the company, and let τ, r and σ denote the time to liquidity event, the risk-free rate, and the volatility measure respectively. The existent Black-Scholes formula computes the value V of a call option as

$$V = \Phi(d_1)S - \Phi(d_2)be^{-r\tau}$$

with

$$d_1 = \frac{\ln(S/b)}{\sigma\sqrt{\tau}} + \frac{r\sqrt{\tau}}{\sigma} + \frac{\sigma\sqrt{\tau}}{2},$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln(S/b)}{\sigma\sqrt{\tau}} + \frac{r\sqrt{\tau}}{\sigma} - \frac{\sigma\sqrt{\tau}}{2},$$

where Φ denotes the cumulative distribution function of a standard normal distribution. For an isolated V, the measures for model sensitivities are well-established in the form of Black-Scholes Greeks, which are the partial derivatives of option value over different Black-Scholes parameters. However, this simplicity does not extend easily to the above-mentioned capital structures, because the incremental value for each tranche depends both on the option value of the tranche itself as well as the tranche next to it, and each share type usually receives allocation from different tranches. This means that the sensitivities for PPS involves rather linear combinations of the sensitivities for each tranche, such that the sensitivities of each share type cannot be computed separately. These naturally invite the use of matrices for simultaneous sensitivity computations, which shall be the focus of the subsequent section.

Section 2. Reformulation of valuation process in matrix form

As mentioned, matrix represents a viable tool for simultaneous analysis of different share types, where we can use the techniques for matrix differentiation to compute derivative vectors for PPSs of all share types over any parameter. Before we commence with the actual derivations, we reiterate the inputs and outputs for the OPM-based valuation process, and introduce the relevant notations. For the set of inputs, we have

- a list of the breakpoints for all tranches, denoted by the vector $\mathbf{b}$,
- an allocation matrix for different share types in different tranches, denoted by the matrix $\mathbf{A}$, where $\mathrm{A}_{i,j}$ is the proportion of the ith share type allocated in tranche j, and
- a list of the fully diluted shares, or FDS, for each share type, which shall be represented in the diagonal matrix $\mathbf{F}$.

We explicitly omit the calculation for the breakpoints and the allocation matrix, since they are determined in the analysis of the company's capital structure, which is irrelevant to the mathematics of the OPM itself. For the output, we have

- a list of the prices per share for each share type, denoted by the vector $\mathbf{p}$.

The inputs and outputs are connected by the application of Black-Scholes formula separately to each breakpoint. We set the combined evaluation for all breakpoints as a vector-to-matrix map denoted by $\mathbf{V}(\mathbf{b})$, with $\mathrm{V}(\mathbf{b})_{i,i}$ as the option value for the ith tranche, and $\mathrm{V}(\mathbf{b})_{i,j} = 0$ for all $i \neq j$. We also construct a shifted matrix $\mathbf{V}^*(\mathbf{b})$, with $\mathrm{V}^*(\mathbf{b})_{i,i} = \mathrm{V}(\mathbf{b})_{i+1,i+1}$ and all other elements as well as the bottom right element zero. The other scalar inputs are the standard inputs for the Black-Scholes formula, which are the equity value S, the time to liquidity event τ, the risk-free rate r, and the volatility σ, which are shared by all tranches.

With all the inputs and outputs set out, we can then recapitulate each step in the OPM valuation process, and express the results in matrix form as tabulated below.

Procedure	Output
Calculate the option values for all tranches.	$\mathbf{V}(\mathbf{b})$
Compute the incremental values for all tranches.	$\mathbf{V}(\mathbf{b}) - \mathbf{V}^*(\mathbf{b})$
Allocate the incremental values by the allocation matrix.	$\mathbf{A}[\mathbf{V}(\mathbf{b}) - \mathbf{V}^*(\mathbf{b})]$
Compute the total values for all share types.*	$\mathbf{A}[\mathbf{V}(\mathbf{b}) - \mathbf{V}^*(\mathbf{b})]\mathbf{1}$
Compute the price per share, or PPS, for all share types.**	$\mathbf{p} = \mathbf{F}^{-1}\mathbf{A}[\mathbf{V}(\mathbf{b}) - \mathbf{V}^*(\mathbf{b})]\mathbf{1}$
*: $\mathbf{1}$ denotes a vector of ones.	
**: For dividend holders, this step is not required.	

Therefore, the eventual output, the PPS vector, is expressed as

$$\mathbf{p} = \mathbf{F}^{-1}\mathbf{A}[\mathbf{V}(\mathbf{b}) - \mathbf{V}^*(\mathbf{b})]\mathbf{1},$$

where $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{b}$ are independent of the Black-Scholes parameters. This means that the partial derivative for any Black-Scholes parameter ω can be expressed as

$$\frac{\partial \mathbf{p}}{\partial \omega} = \mathbf{F}^{-1}\mathbf{A}\left[\frac{\partial \mathbf{V}(\mathbf{b})}{\partial \omega} - \frac{\partial \mathbf{V}^{*}(\mathbf{b})}{\partial \omega}\right]\mathbf{1}.$$

We recall that the Black-Scholes Greeks for the call option value V with strike price X are

$$\Delta = \frac{\partial V}{\partial S} = \Phi(d_1),$$

$$\nu = \frac{\partial V}{\partial \sigma} = S\sqrt{\tau}\phi(d_1) = S\sqrt{\tau}\Delta,$$

$$\rho = \frac{\partial V}{\partial r} = X\tau e^{-r\tau}\Phi(d_2),$$

where ϕ is the probability density function for the standard normal distribution. Substitution into the matrix formulations yields the simultaneous derivatives of PPSs to the relevant parameters as

$$\frac{\partial \mathbf{p}}{\partial S} = \mathbf{F}^{-1}\mathbf{A}[\boldsymbol{\Delta} - \boldsymbol{\Delta}^{*}]\mathbf{1}, \Delta_{i,i} = \Phi\left(d_{1,i}\right), \Delta_{i,i}^{*} = \Delta_{i+1,i+1},$$

$$\frac{\partial \mathbf{p}}{\partial \sigma} = \mathbf{F}^{-1}\mathbf{A}[\mathbf{N} - \mathbf{N}^{*}]\mathbf{1}, \mathrm{N}_{i,i} = S\sqrt{\tau}\phi\left(d_{1,i}\right), \mathrm{N}_{i,i}^{*} = \mathrm{N}_{i+1,i+1},$$

$$\frac{\partial \mathbf{p}}{\partial r} = \mathbf{F}^{-1}\mathbf{A}[\mathbf{P} - \mathbf{P}^{*}]\mathbf{1}, \mathrm{P}_{i,i} = \mathrm{b}_i\tau e^{-r\tau}\Phi\left(d_{2,i}\right), \mathrm{P}_{i,i}^{*} = \mathrm{P}_{i+1,i+1}.$$

The linearity of the system also means that we can conveniently conduct sensitivity analysis for any investment position with a mixture of different share types. Suppose that $\mathbf{w}$ denotes the proportion vector for all share types in an investment position, the mean share price $\bar{\pi}$ and its partial derivatives can be obtained easily as

$$\bar{\pi} = \mathbf{w}^{T}\mathbf{p} = \mathbf{w}^{T}\mathbf{F}^{-1}\mathbf{A}[\mathbf{V}(\mathbf{b}) - \mathbf{V}^{*}(\mathbf{b})]\mathbf{1},$$

$$\frac{\partial \bar{\pi}}{\partial \omega} = \mathbf{w}^{T}\frac{\partial \mathbf{p}}{\partial \omega} = \mathbf{w}^{T}\mathbf{F}^{-1}\mathbf{A}\left[\frac{\partial \mathbf{V}(\mathbf{b})}{\partial \omega} - \frac{\partial \mathbf{V}^{*}(\mathbf{b})}{\partial \omega}\right]\mathbf{1}$$

for any Black-Scholes parameter ω.

Section 3. Demonstrations for hypothetical case scenarios

Upon derivation of the mathematical details, we shall demonstrate how these results can be used to direct valuation decisions. Since detailed capital structure is not accessible for privately held companies, we shall use hypothetical cases in published manuals, which are adequate to demonstrate the basic principles. One example comes from a hypothetical case Cotopaxi Tech in an article from the CPA journal. Another example comes from a hypothetical case ComplexCo in a manual from Mercer Capital.

3.1. Case 1: Cotopaxi Tech

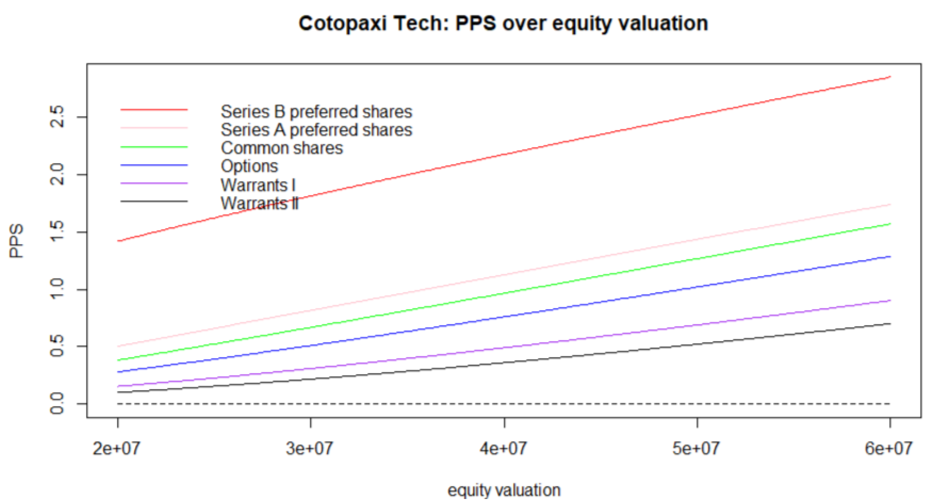
The capital structure of Cotopaxi Tech includes Series B preferred shares, Series A preferred dividends, Series A preferred shares, as well as common shares, options and two classes of warrants. The details for the breakpoints, the allocation matrix and the fully diluted share counts can be found in the link provided in the reference, and for the purpose of this write-up shall be omitted. We commence with a basic sanity check to ensure that our matrix-formulated approach yields the correct answer as shown in Table 3.1.1. The Black-Scholes parameters are set to $S = 40000000$, $\tau = 3$, $r = 0.02$ and $\sigma = 0.8$.

Table 3.1.1: Manual and matrix model results for Cotopaxi Tech

Share type	PPS from manual	PPS from matrix model
Series B preferred shares	\$2.17	\$2.175
Series A preferred dividends	N/A	N/A
Series A preferred shares	\$1.13	\$1.128
Common shares	\$0.96	\$0.966
Options	\$0.76	\$0.760
Warrants I	\$0.49	\$0.491
Warrants II	\$0.36	\$0.360

The PPSs from the two approaches correspond well except small discrepancies due to the precisions used. We then proceed to compute the trends of PPS estimates over equity valuation and volatility. Suppose that an analyst states that the estimated equity value for the company is not a point estimate at \$40000000, but instead lies between \$20000000 to \$60000000. We then plot the PPS trend for each share type over the estimated equity value, which is display in Plot 3.1.1.

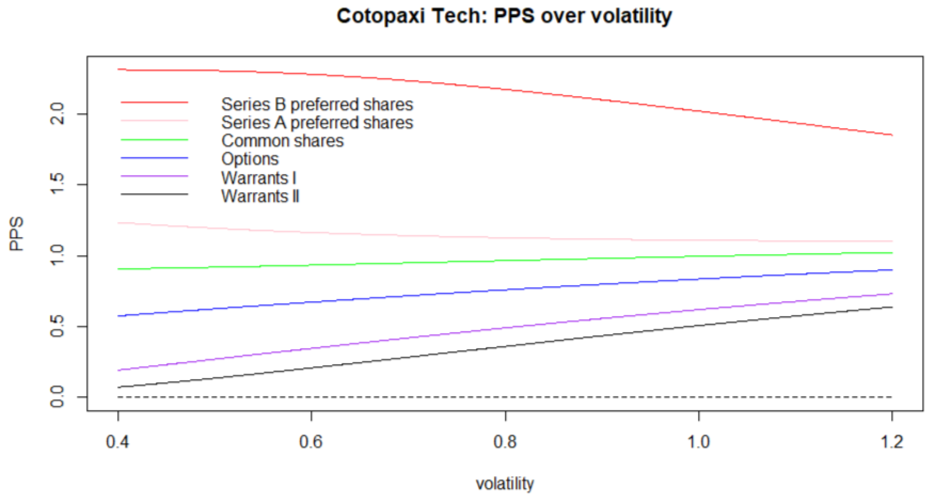
Plot 3.1.1: Cotopaxi Tech's PPS trends over different equity values



From the plot, we can see an almost linear relationship between PPS and equity valuation, which holds consistently for shares of all priorities. In addition, more senior shares are more valuable, and the ratio of PPSs from any pair of share types remains stable. This means that both the absolute and the comparative values of all share types do not experience abrupt variations, which is a desirable feature amid potential inaccuracies in the estimation of equity value.

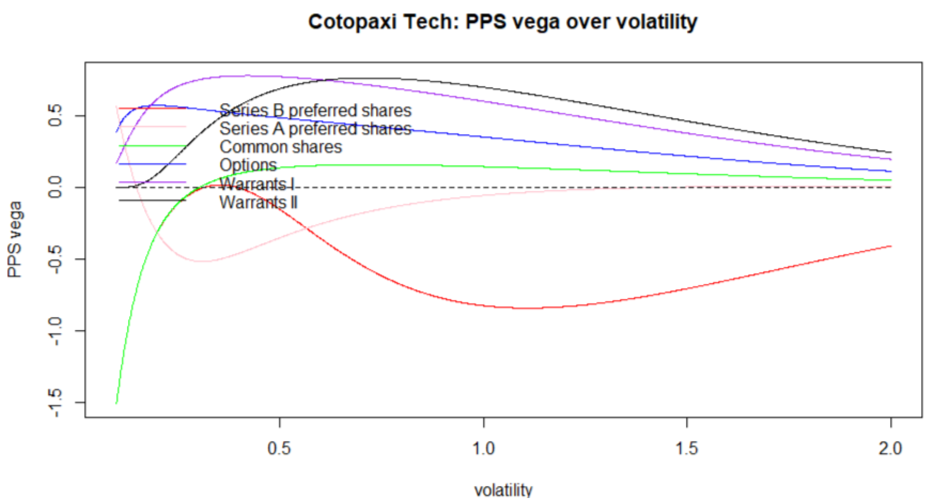
Next, we consider the case when an analyst cannot pinpoint the volatility parameter, but instead puts the volatility estimate as between 0.4 and 1.2. The plot for the PPS trend over this domain of volatilities is displayed in Plot 3.1.2.

Plot 3.1.2: Cotopaxi Tech's PPS trends over different volatilities



For the case of volatilities, we can see that different share types display different extents and directions of sensitivity. Shares of intermediate priorities, such as the Series A preferred shares and the common shares, have PPS trends that are relatively insensitive to variations in volatility. More senior shares suffer from decreased PPSs from increased volatility, whereas more junior shares obtain increased PPSs from increased volatility. Subsequently, Plot 3.1.3 shows the volatility derivatives of PPSs over a wide domain of volatilities.

Plot 3.1.3: Cotopaxi Tech's $\partial p/\partial\sigma$ trends over different volatilities



For Series B preferred shares, the volatility derivatives have sharply increased absolute values for volatilities below 0.2 and between 0.5 and 2.0, which indicates that the estimated PPS for Series B preferred shares is relatively insensitive for volatilities between 0.2 and 0.5, but sensitive otherwise. In contrast, the estimated PPS for Series A preferred shares are sensitive over low to medium volatilities, and relatively insensitive for volatilities over 0.5. Common shares are only sensitive for volatilities below 0.2, whereas options, Warrants I and Warrants II show similar shapes of volatility derivatives, with the most sensitive point around volatilities 0.1, 0.4 and 0.7 respectively.

3.2. Case 2: ComplexCo

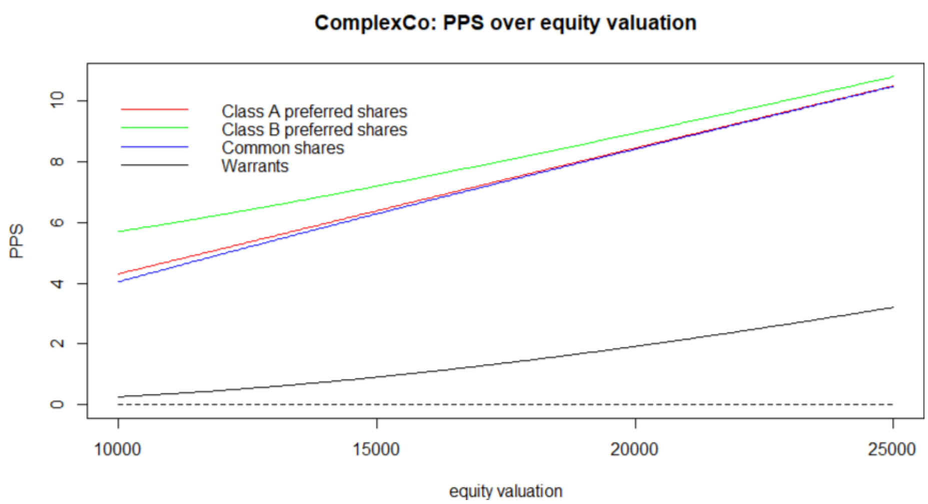
The capital structure of ComplexCo consists of Class A and B preferred shares with *pari passu* priorities, as well as common shares and options with residual claims, the details of which can be found in the link provided in the reference as well. The Black-Scholes parameters in this case are $S = 17500$, $\tau = 4$, $r = 0.015$ and $\sigma = 0.35$. The results of the sanity check show close correspondence between the results from the manual and that from the matrix model, as shown in Table 3.2.1.

Table 3.2.1: Manual and matrix model results for ComplexCo

Share type	PPS from manual	PPS from matrix model
Class A preferred shares	\$7.42	\$7.417
Class B preferred shares	\$8.05	\$8.044
Common shares	\$7.35	\$7.353
Warrants	\$1.38	\$1.377

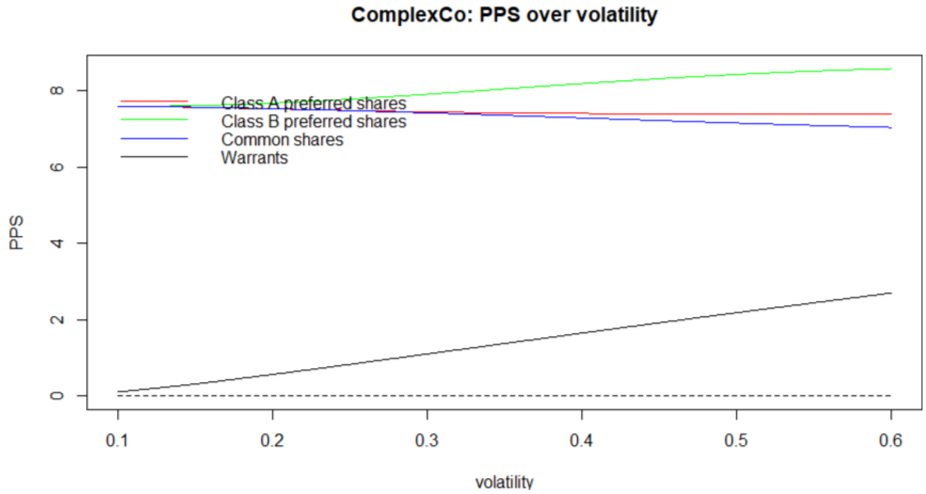
Similar to Case 1, suppose that the analyst is unsure of the exact equity value for the company, but claims a valuation between \$10000 and \$25000. The results for sensitivities over equity value is displayed in Plot 3.2.1. In this case, as equity value increases, the PPS deviation of warrants increases with respect to the other three share types, whereas the PPSs for Class A and B preferred shares as well as the common shares approach each other.

Plot 3.2.1: ComplexCo's PPS trends over different equity values



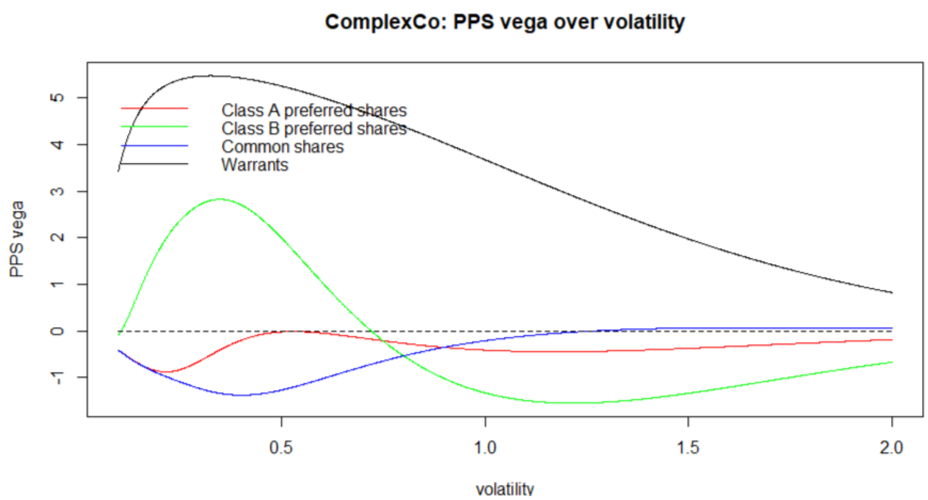
Next, suppose that the analyst claims a volatility estimate between 0.1 and 0.6. The trends are then displayed in Plot 3.2.2.

Plot 3.2.2: ComplexCo's PPS trends over different volatilities



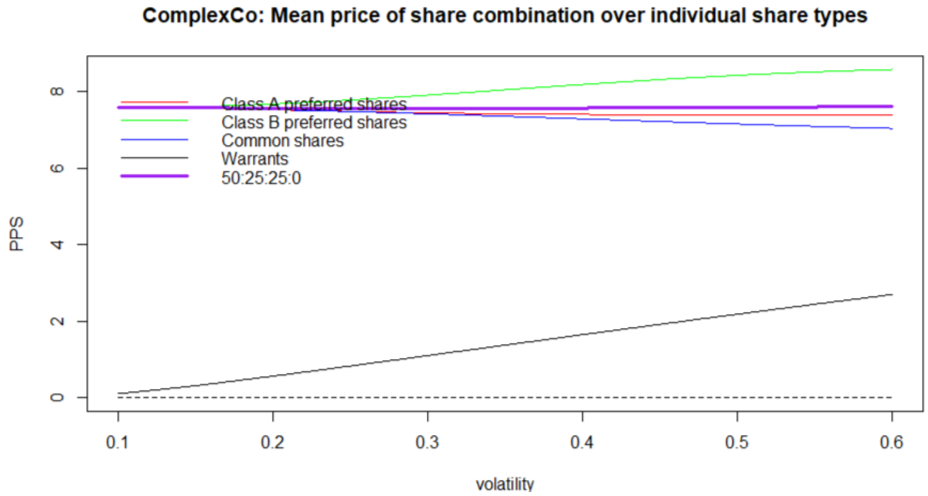
Clearly, we can see that the dynamics of PPSs over volatility is more complicated than Case 1. The PPS for warrants is clearly the most sensitive to variations in volatility, followed by Class B preferred shares. Class A preferred shares and common shares are relatively insensitive to variations in volatility. Plot 3.2.3 provides a detailed view of the relevant volatility derivatives. Generally, PPS of warrants is consistently more sensitive than all other share types, and the PPS of Class B preferred shares is mostly sensitive over volatilities between 0.2 and 0.6. In contrast, the PPSs for Class A preferred shares and common shares are much less sensitive.

Plot 3.2.3: ComplexCo's $\partial p/\partial\sigma$ trends over different volatilities

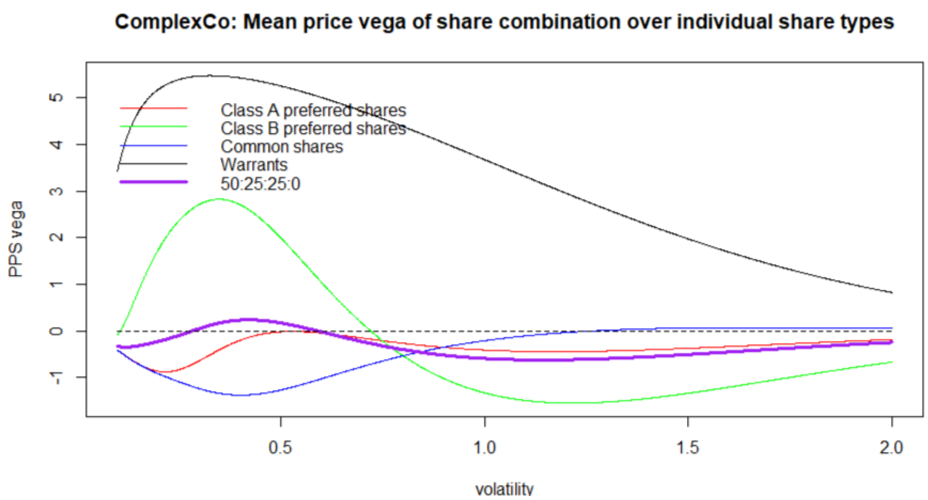


Eventually, suppose that a particular investor's position consists of 50% Class A preferred shares, 25% Class B preferred shares and 25% common shares. A comparative view of the mean share price and its volatility derivative over individual share types is displayed in Plot 3.2.4 and 3.2.5, which show that the combination achieves mean share prices similar to the PPSs for Class A and B preferred shares as well as the common shares, with low sensitivity maintained over a wide domain of volatilities.

Plot 3.2.4: Comparative plot of ComplexCo's mean price of share combination over individual share types



Plot 3.2.5: Comparative plot of ComplexCo's volatility derivative of share combination over individual share types



Conclusion

This small write-up has reiterated the basic OPM procedures into matrix form. Compared to common approaches to simulate different scenarios, the matrix representation provides a succinct and holistic view of the entire valuation process, as well as desirable mathematical properties to compute PPSs and relevant partial derivatives to conveniently obtain trends on the sensitivity of PPS estimates to variations in Black-Scholes parameters. Based on such results, one could compute any linear combination of the PPSs or their partial derivatives to assess how sensitive an investor's overall position is to different parameter values. Nonetheless, since the matrix representation is mathematically equivalent to a typical OPM-based valuation, what this write-up concerns is essentially a reformulation, instead of a renovation, of the Black-Scholes model, such that all model assumptions must be equally checked before its transition into an investment decision.

References

1. **Valuing Securities Using the Option Pricing Method**. Available at: https://www.cpajournal.com/2020/09/09/valuing-securities-using-the-option-pricing-method.
2. **A Layperson's Guide to the Option Pricing Model**. Available at: https://mercercapital.com/content/uploads/Article_Mercer-Capital-Guide-Option-Pricing-Model.pdf.