Differential Models for Viral Dispersion in Public Facilities

Introduction

The continued threat of COVID-19 has undoubtedly caused tremendous impacts on people's lifestyles and commotion. Yet despite the combined actions from the society to curtail the infection, the cyclic waxes and wanes of the infection situation seem to allude to us the immanent necessity to coexist with the disease. Of particular relevance are public policies that open up public facilities central to people's social activities such as food centers and malls. Therefore, it would be of interest to understand and model the process of viral dispersion in these places, and to derive the basic principles on the evaluation of viral exposure.

Section 1

Compartmental models for viral dispersion in well-ventilated facilities

To commence with a mathematical model, it is intuitive to visualize a public facility in a coordinate system, and to derive suitable equations which determine how the state of viral dispersion at each location within the facility evolve with time. Nonetheless, such formulations inevitably invoke some PDEs, which are complicated to solve and which may not necessarily yield more information than what we wanted. Instead, for well-ventilated public facilities, it is natural to assume that the viral particles are evenly distributed at different locations within the facility. This means that whatever matters concerns only the boundaries, and the facility itself can be treated as one entire compartment. Such compartmental models can then be formulated with ODEs that are much simpler than PDEs.

1.1. Basic mathematical description of compartmental models

To delineate our model, let $u(t)$ denote the total viral count within the facility at time t. Suppose that the area of the facility is A, so that the viral concentration per unit area is $c(t) = u(t)/A$. When the facility is populated, some new viral particles will be continuously expectorated by the infected people, and some existent viral particles will escape into the environment. The rate of creation of new viral particles would be a function of the number of infected people in the facility, which is in turn proportional to the level of crowdedness $\alpha(t)$ of the facility times the area A. We denote this rate for creation as $F\big(A\alpha(t)\big)$ at time t. The dispersion of existent particles into the environment is naturally a function of the concentration difference between the facility and its environment. To simplify the problem, the environment can be assumed to have infinite ability to absorb and disperse viral particles, so that its viral concentration is always zero. This means that the dispersion function depends entirely on $c(t)$, which we denote as $G\big(c(t)\big)$. The rate of variation for $u(t)$ must be a balance between rate of creation and rate of dispersion, such that

$$\frac{u(t)}{dt} = F\big(A\alpha(t)\big) - G\big(c(t)\big).$$

The approach is then to decide on the functional forms of $F\big(A\alpha(t)\big)$ and $G\big(c(t)\big)$, and then solve the ODE based on some suitable initial conditions.

1.2. Models for constant crowd and time-invariant viral creation

The simplest case is where the public facility retains the same level of crowdedness over time, which is an adequate description of the situation in peak hours. Since $\alpha(t)$ becomes a constant, we denote it here simply as α. Let the rate of expectoration of for new viral particles per infected person be λ, and let the prevalence of the disease in the local community be ω, we have $F(A\alpha) = \lambda\omega A\alpha = AC$ for $C = \lambda\omega\alpha$. Suppose also that the layout of the facility has perimeter L. The dispersion function can then be easily modeled as a diffusion process, such that $G\big(c(t)\big) = \kappa L c(t) = \kappa L \frac{u(t)}{A}$, where κ represents the diffusivity of the viral particles in air. The ODE then becomes

$$\frac{u(t)}{dt} = AC - \frac{L}{A}\kappa u(t) \Leftrightarrow \frac{u(t)}{dt} + \frac{L}{A}\kappa u(t) = AC.$$

The solution of this ODE is then

$$\frac{u(t)}{dt}e^{\frac{L\kappa}{A}t} + \frac{L}{A}\kappa u(t)e^{\frac{L\kappa}{A}t} = ACe^{\frac{L\kappa}{A}t}$$

$$\Rightarrow \frac{d}{dt}\left(u(t)e^{\frac{L\kappa}{A}t}\right) = ACe^{\frac{L\kappa}{A}t}$$

$$\Rightarrow u(t)e^{\frac{L\kappa}{A}t} = \frac{A^2C}{L\kappa}e^{\frac{L\kappa}{A}t} + B$$

$$\Rightarrow u(t) = \frac{A^2C}{L\kappa} + Be^{-\frac{L\kappa}{A}t}$$

for some constant B. Let u_0 denote the initial viral count at $t = 0$, such that $B = u_0 - \frac{A^2C}{L\kappa}$. We eventually have

$$u(t) = \frac{A^2C}{L\kappa} + \left(u_0 - \frac{A^2C}{L\kappa}\right)e^{-\frac{L\kappa}{A}t}, C = \lambda\omega\alpha.$$

Since $L, A, \kappa > 0$, as $t \to \infty$, $u(t)$ reaches a steady state u_∞, where $u_\infty = \frac{A^2C}{L\kappa}$. The steady state for viral concentration is then $c_\infty = \frac{AC}{L\kappa} = \frac{A\lambda\omega\alpha}{L\kappa}$, which directly measures an individual's viral exposure per unit time.

The properties of this steady state can be used by both individuals and policy makers to compare the levels of riskiness of different public facilities in the same community. Since λ and κ is an inherent property of the virus, and ω can be treated as invariant within the community, we have $c_\infty \propto \frac{A}{L}\alpha$. One noteworthy consequence is that for two public facilities of the same level of crowdedness, a more spacious facility tends to result in more viral exposure, because if a facility's perimeter increases on the order of k, the area usually increases on the order of k^2, and the ratio $\frac{A}{L}$ and therefore c_∞ increases on the order of k. For example, even if a policy maker loosens restrictions to all food centers within the same community, an individual could still risk more viral exposure from a less crowded but central food center than from a more crowded but small food center.

Section 2

Heat equation for poorly ventilated facilities in 1-dimensional formulation

Although compartmental models offer parsimonious descriptions on the overall exposure risk for well-ventilated facilities, such models do not account for the fact that within the same facility, viral concentration may differ in different locations, where some spots may be particularly prone to viral accumulation. These considerations matter when air circulation within the facility is poor, and invoke more sophisticated models in PDEs to study viral concentrations at different locations within the facility.

2.1. Derivation of 1-dimensional heat equation

Let $c(x,t)$ be the viral concentration for point x at time t, which must be twice-differentiable over x and once-differentiable over t, and be continuous both within the facility and with the environment it contacts. and let $\alpha(x)$ be the rate of creation of new viral particles per unit length, which may vary with different locations but is assumed to be independent of time. We divide the entire x-axis into small intervals of width δx, and consider the diffusion process for each interval on the boundaries x and $x+\delta x$. Since rate of diffusion is proportional to concentration gradient, the viral particles diffuse out at boundary x with rate $\kappa c_x(x,t)$, and at boundary $x+\delta x$ with rate $-\kappa c_x(x+\delta x,t)$. Simultaneously, new viral particles are created at a rate $\int_x^{x+\delta x}\alpha(x)dx$ in $[x,x+\delta x]$, and the overall rate of variation of viral count is $\int_x^{x+\delta x} c_t(x,t)dx$. A balance of these three factors yields

$$\int_x^{x+\delta x} c_t(x,t)dx = \int_x^{x+\delta x}\alpha(x)dx + \kappa c_x(x+\delta x,t) - \kappa c_x(x,t).$$

We then let $\delta x \to 0$, divide both sides by δx, and take the limits, such that

$$\frac{\int_x^{x+\delta x} c_t(x,t)dx}{\delta x} = \frac{\int_x^{x+\delta x}\alpha(x)dx}{\delta x} + \kappa\frac{c_x(x+\delta x,t)-c_x(x,t)}{\delta x}$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\int_x^{x+\delta x} c_t(x,t)dx}{\delta x} = \lim_{\delta x\to 0}\frac{\int_x^{x+\delta x}\alpha(x)dx}{\delta x} + \kappa\lim_{\delta x\to 0}\frac{c_x(x+\delta x,t)-c_x(x,t)}{\delta x}$$

$$\Rightarrow c_t(x,t) = \alpha(x) + \kappa c_{xx}(x,t),$$

which is the heat equation with $\alpha(x)$ as the source function.

2.2. Steady state equations for 1-dimensional heat equation

In the heat equation above, the additional source term makes analytical solution of the PDE much harder. Nonetheless, the most important consideration for us is the steady state equation $c_\infty(x)$ of the system, where $c_t(x,t)=0$ and thus

$$\alpha(x) + \kappa c_{xx}(x,t) = 0$$

$$\Rightarrow c_{\infty,xx}(x) = -\frac{\alpha(x)}{\kappa},$$

which can then be solved provided $\alpha(x)$ as well as suitable boundary conditions.

2.3. Numerical methods for 1-dimensional steady states

While steady state equations reduce to 2nd-order ODEs that are not hard to solve, analytical solutions may still be too troublesome when $\alpha(x)$ is complicated or has many discontinuity points over x. Therefore, possibility of numerical methods should also be discussed for such cases.

To commence, let $u_{\delta x}(x,t) \approx c(x,t)\delta x$ denote the number of viral particles within interval $(x, x+\delta x)$. We can then unpack the heat equation derived above, and consider some length interval δx and some time interval δt, such that

$$c(x,t+\delta t) - c(x,t) \approx \alpha(x)\delta t + \kappa\delta t c_{xx}(x,t)$$

$$\Rightarrow c(x,t+\delta t)\delta x - c(x,t)\delta x \approx \alpha(x)\delta x\delta t + \kappa\delta t\big(c_x(x+\delta x,t) - c_x(x,t)\big)$$

$$\Rightarrow u_{\delta x}(x,t+\delta t)\delta x - u_{\delta x}(x,t)\delta x$$

$$\approx \alpha(x)\delta x^2\delta t + \kappa\delta t\big(c(x+\delta x,t) - c(x,t) - c(x,t) + c(x-\delta x,t)\big)$$

$$\approx \alpha(x)\delta x^2\delta t - \kappa\delta t\big(2c(x,t) - c(x+\delta x,t) - c(x-\delta x,t)\big)$$

$$\Rightarrow u_{\delta x}(x,t+\delta t)\delta x^2 - u_{\delta x}(x,t)\delta x^2$$

$$\approx \alpha(x)\delta x^3\delta t - \kappa\delta t\big(2u_{\delta x}(x,t) - u_{\delta x}(x+\delta x,t) - u_{\delta x}(x-\delta x,t)\big)$$

$$\approx \alpha\delta x^3\delta t - 2\kappa\delta t\big(u_{\delta x}(x,t) - \bar{u}_{\delta x}(x,t)\big).$$

Here, $\bar{u}_{\delta x}(x,t) = \frac{u_{\delta x}(x+\delta x,t) - u_{\delta x}(x-\delta x,t)}{2}$, which can be viewed as the mean number of viral particles of the two intervals adjacent to the interval $(x, x+\delta x)$. In practice, it is always possible to choose a distance scale, where δx becomes one. Suppose also that we choose an appropriate time scale so that $\delta t = 1$. The equation above becomes

$$u_1(x,t+1) = u_1(x,t) + \alpha(x) - 2\kappa\big(u_1(x,t) - \bar{u}_1(x,t)\big).$$

This leads naturally to the protocol below to simulate the evolution of viral dynamics in discrete arrays.

1. Partition the facility of length L into intervals of unit lengths, and label these intervals with indices 1 to L.
2. Prepare an array $(u_n(t))_{n=1}^{N}$ to update the viral counts in each interval, with values initialized at $t = 0$. Set parameters α and κ.
3. Iterate the system by the rule $u_n(t+1) = u_n(t) + \alpha - \kappa\big(2u_n(t) - u_{n-1}(t) - u_{n+1}(t)\big)$. Care must be taken at the boundaries, where the environment always has zero viral counts, and walls do not permit any diffusion to take place.
4. Stop iteration when a steady state is reached.

2.4. Case studies for common real scenarios

As all the theoretical and numerical concerns have been addressed, this section discusses the analytical and numerical solutions for some scenarios commonly encountered in the real world, and compares the consistency between analytical and numerical results.

Example 1: Facility with no wall under uniform viral creation

When the public facility has evenly distributed activity zones over all location, viral creation can be assumed to be independent of location, and be denoted as the constant α. The steady state equation can then be easily solved as

$$c_{\infty,xx}(x) = -\frac{\alpha}{\kappa}$$

$$\Rightarrow c_{\infty,x}(x) = -\frac{\alpha}{\kappa}x + A$$

$$\Rightarrow c_{\infty}(x) = -\frac{\alpha}{2\kappa}x^2 + Ax + B.$$

Suppose that the facility is located in the interval $[0, L]$. In the absence of walls, the facility contacts the environment at both $x = 0$ and $x = L$, so that the boundary conditions are $c_{\infty}(0) = c_{\infty}(L) = 0$. This leads to

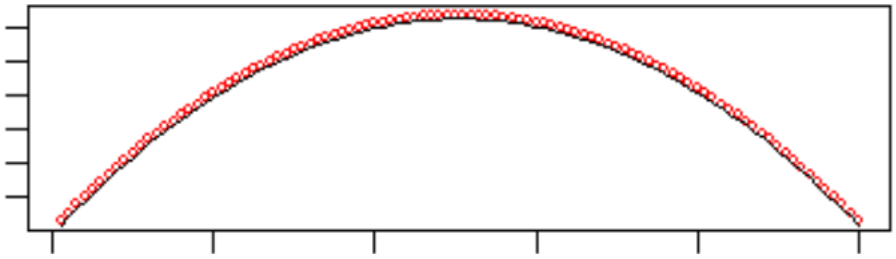
$$A = \frac{\alpha L}{2\kappa}, B = 0$$

$$\Rightarrow c_{\infty}(x) = -\frac{\alpha}{2\kappa}(x^2 - Lx),$$

which prescribes a quadratic curve. The numerical method outlined in the previous section is also performed for $L = 100$. The combined analytical and numerical results are demonstrated in Plot 1 as the black solid line and red dots respectively, which shows close correspondence between the two methods.

Plot 1: Steady state for facility without walls and constant source function

$L = 100$



Example 2: Facility with one-sided wall under uniform viral creation

Consider another public facility which contacts the environment only on one side at $x = L$, while at $x = 0$ there is an impermeable wall. Whereas one boundary condition is clearly $c_{\infty}(L) = 0$, the other boundary condition requires more consideration. One trick is to use the fact that at steady state, the total viral count between any interval must remain the same over time. Therefore, consider a partition of the interval $[0, L]$ into two subintervals $[0, s]$ and $[s, L]$. For subinterval $[0, s]$, viral escape must occur exclusively over $x = s$, where the rate of escape of viral particles is $-\kappa c_{\infty,x}(s)$. Simultaneously, the overall rate of creation of new viral particles is αs, which for steady state must satisfy $\alpha s + \kappa c_{\infty,x}(s) = 0$ for all $s \in [0, L]$, so that the boundary condition becomes $c_{\infty,x}(x) = -\frac{\alpha}{\kappa}x$. The solution to these two boundary conditions is then

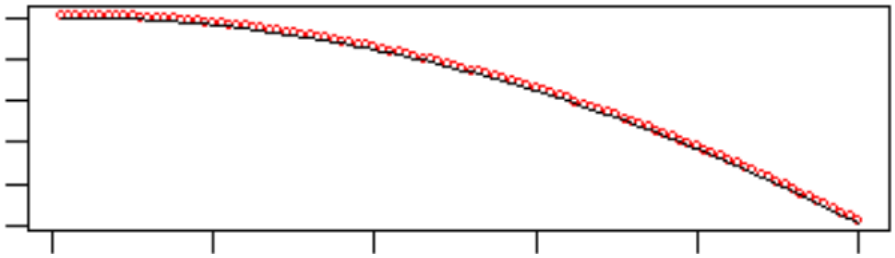
$$-\frac{\alpha}{2\kappa}L^2 + AL + B = 0, -\frac{\alpha}{\kappa}x + A = -\frac{\alpha}{\kappa}x$$

$$\Rightarrow A = 0, B = \frac{\alpha L^2}{2\kappa}$$

$$\Rightarrow c_{\infty}(x) = \frac{\alpha}{2\kappa}(L^2 - x^2)$$

as demonstrated in Plot 2.

Plot 2: Steady state for facility with left-sided wall and constant source function
$L = 100$



Example 3: Facility with piecewise constant source function

In real scenarios, it is common for the source function α to vanish at some locations of the facility where no crowd occurs, and assume a constant value over locations where the crowd is present. In such cases, $\alpha(x)$ is a piecewise constant function of x, and the steady state equations for different locations would differ. For example, consider a public facility between $[0, L]$, where $\alpha(x)$ takes the form

$$\alpha(x) = \begin{cases} \hat{\alpha}, x \in [0.2L, 0.4L] \cup [0.6L, 0.8L] \\ 0, otherwise \end{cases},$$

which is typical for sales events where booths are located in discrete sections and crowd concentration occurs exclusively near the booth areas. The interval $[0, L]$ is thus partitioned into 5 subintervals with 5 steady state equations where

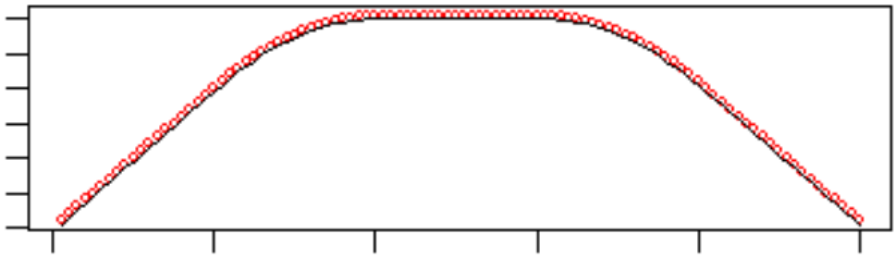
$$c_\infty(x) = \begin{cases} A_1 x + B_1, x \in [0, 0.2L] \\ -\frac{\hat{\alpha}}{2\kappa} x^2 + A_2 x + B_2, x \in [0.2L, 0.4L] \\ A_3 x + B_3, x \in [0.4L, 0.6L] \\ -\frac{\hat{\alpha}}{2\kappa} x^2 + A_4 x + B_4, x \in [0.6L, 0.8L] \\ A_5 x + B_5, x \in [0.8L, L] \end{cases},$$

which has 4 continuity constraints as well as at 4 differentiability constraints $x = 0.2L, 0.4L, 0.6L, 0.8L$. With 2 additional boundary conditions at $x = 0$ and $x = L$, we have 10 constraints for 10 unknowns, which is solvable but complicated. To illustrate, we take the boundary conditions to be $c_\infty(0) = c_\infty(L) = 0$, and exploit the symmetries in the equations to simplify the solutions. We intuitively note that the facility's layout and the source function is symmetric about $x = L/2$, such that $c_\infty(x) = c_\infty(L - x)$ and $c_{\infty,x}(x) = -c_{\infty,x}(L - x)$. Therefore, we have $A_1 = -A_5$ and $A_3 = 0$. After some basic algebraic manipulations on the boundary and continuity conditions, we eventually have the solution

$$c_\infty(x) = \begin{cases} 0.2\frac{L\hat{\alpha}}{\kappa} x, x \in [0, 0.2L] \\ -\frac{\hat{\alpha}}{\kappa}\left(\frac{1}{2}x^2 - 0.4Lx + 0.02L^2\right), x \in [0.2L, 0.4L] \\ 0.06\frac{L^2\hat{\alpha}}{\kappa}, x \in [0.4L, 0.6L] \\ -\frac{\hat{\alpha}}{\kappa}\left(\frac{1}{2}x^2 - 0.6Lx + 0.12L^2\right), x \in [0.6L, 0.8L] \\ \frac{\hat{\alpha}}{\kappa}(0.2L^2 - 0.2Lx), x \in [0.8L, L] \end{cases}$$

as demonstrated in Plot 3.

Plot 3: Steady state for facility with no walls
and piecewise constant source function
$L = 100$



Section 3

Heat equation for poorly ventilated facilities in 2-dimensional formulation

From 1-dimensional heat equation, it is intuitive to explore the possibility to model 2-dimensional cases with a heat equation as well. These possibilities, as well as the nature, uniqueness and characteristics for common steady states, will be discussed in detail.

3.1. Derivation of 2-dimensional heat equation

For 2-dimensional cases, we consider a small rectangular area $R = [x, x + \delta x] \times [y, y + \delta y]$. Similarly, the rate of variation of viral concentration is a balance of the rate of viral creation within the area and the rate of viral dispersion from the four sides of the rectangle, where

$$\int_y^{y+\delta y}\int_x^{x+\delta x} c_t(x,y,t)dxdy = \int_y^{y+\delta y}\int_x^{x+\delta x} \alpha(x,y)dxdy$$

$$-\kappa\left[\int_y^{y+\delta y}\big(c_x(x,y,t) - c_x(x+\delta x,y,t)\big)dy + \int_x^{x+\delta x}\Big(c_y(x,y,t) - c_y(x,y+\delta y,t)\Big)\,dx\right].$$

Divide both sides by $\delta x \delta y$, with $\delta x \to 0$ and $\delta y \to 0$ sequentially. We then have

$$\lim_{\delta x,\delta y\to 0} \frac{\int_y^{y+\delta y}\int_x^{x+\delta x} c_t(x,y,t)dxdy}{\delta x\delta y} = c_t(x,y,t),$$

$$\lim_{\delta x,\delta y\to 0} \frac{\int_y^{y+\delta y}\int_x^{x+\delta x} \alpha(x,y)dxdy}{\delta x\delta y} = \alpha(x,y),$$

$$\lim_{\delta x,\delta y\to 0} \frac{\int_y^{y+\delta y}\big(c_x(x,y,t) - c_x(x+\delta x,y,t)\big)dy}{\delta x\delta y}$$

$$= -\lim_{\delta y\to 0} \frac{\int_y^{y+\delta y} c_{xx}(x,y,t)dy}{\delta y} = -c_{xx}(x,y,t),$$

$$\lim_{\delta x,\delta y\to 0} \frac{\int_x^{x+\delta x}\Big(c_y(x,y,t) - c_y(x,y+\delta y,t)\Big)\,dx}{\delta x\delta y}$$

$$= -\lim_{\delta x\to 0} \frac{\int_x^{x+\delta x} c_{yy}(x,y,t)dx}{\delta x} = -c_{yy}(x,y,t),$$

such that

$$c_t(x,y,t) = \alpha(x,y) + \kappa\left(c_{xx}(x,y,t) + c_{yy}(x,y,t)\right),$$

which eventually becomes

$$c_t = \alpha + \kappa\nabla^2 c.$$

This is simply the 2-dimensional heat equation with the additional source function for viral creation. The Laplacian of concentration c then represents the difference between the mean viral concentration around x and the viral concentration at x.

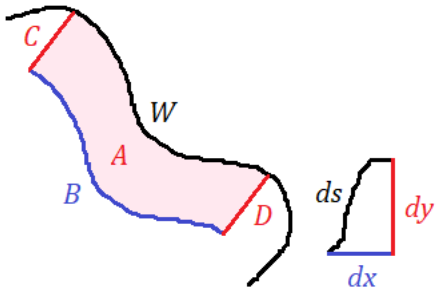
3.2. Uniqueness theorem for steady state as a Poisson's equation

The steady state equation with $c_t = 0$ reduces to

$$\nabla^2 c = -\alpha/\kappa = -\hat{\alpha},$$

which is a Poisson's equation. To explore the nature of the boundary conditions, consider a facility S as a closed subset in $\mathbb{R}^2$. Over some sections of the boundaries are impermeable walls, whose coordinates form the set ∂S_1. Other sections of the boundaries contact the external environment over the set of points ∂S_2. The union $\partial S_1 \cup \partial S_2 = \partial S$ then completely encloses the facility. In real applications, we need $c_\infty(x,y)$ to be continuous within S and with the boundary, and twice-differentiable everywhere over S. Under the same assumption for the environment as an infinite reservoir, the boundary conditions for all $(x,y) \in \partial S_2$ must be $c_\infty(x,y) = 0$, which are Dirichlet in nature. For wall boundaries, consider a section of wall W with arbitrary shape, and translate an arbitrary subset $W' \subseteq W$ to form B, as well as an enclosed shape $W'CBD$ with area A. For steady state, the rate of creation of new viral particles within area A is $\iint_{W'CBD} \alpha dA$, which must balance the dispersion of viral particles out of $W'CBD$ at boundaries B, C and D. A schematic representation is demonstrated in Plot 4.

Plot 4: Schematic demonstration
to derive boundary conditions at walls



To compute how viral particles traverse these boundaries, consider an infinitesimal section ds, which is well-approximated by $ds^2 = dx^2 + dy^2$. Rate of dispersion of viral particles out of ds can then be approximated by the respective rates over dx and dy, which are $-\kappa c_{\infty,y} dx$ and $\kappa c_{\infty,x} dy$. Let $\boldsymbol{n}$ be the unit normal vector for the section ds. The overall diffusion rate over B can be expressed as

$$-\kappa \int_B \left(c_{\infty,y} dx - c_{\infty,x} dy\right) = -\kappa \int_B \left(c_{\infty,y} \frac{dx}{ds} - \kappa c_{\infty,x} \frac{dy}{ds}\right) ds$$

$$= -\kappa \int_B \begin{pmatrix} c_{\infty,y} \\ c_{\infty,x} \end{pmatrix} \cdot \begin{pmatrix} dx/ds \\ -dy/ds \end{pmatrix} ds = -\kappa \int_B \nabla c_\infty \cdot \boldsymbol{n} ds.$$

Therefore, the steady state equation over the entire area can be described as

$$\iint_{W'CBD} \alpha dA = -\kappa\left(\int_B \nabla c_\infty \cdot \boldsymbol{n} ds + \int_C \nabla c_\infty \cdot \boldsymbol{n} ds + \int_D \nabla c_\infty \cdot \boldsymbol{n} ds\right).$$

We then let B approach W over the lines C and D, such that $B \to W$ and $C, D \to \emptyset$. Clearly, we have $A \to 0$ and $\iint_{W'CBD} \alpha dA \to 0$, as well as $\int_C \nabla c_\infty \cdot \boldsymbol{n} ds \to 0$ and $\int_D \nabla c_\infty \cdot \boldsymbol{n} ds \to 0$, such that $\int_B \nabla c_\infty \cdot \boldsymbol{n} ds \to \int_{W'} \nabla c_\infty \cdot \boldsymbol{n} ds = 0$ by continuity of c_∞, for all subsets W'. This is only possible if we have $\nabla c_\infty(x, y) \cdot \boldsymbol{n} = 0$ for all $(x, y) \in W$, which would then be the Neumann boundary conditions applicable to all walls in ∂S_1. Combined with the Dirichlet boundary conditions over ∂S_2, the overall boundary condition for S is mixed. Therefore, by uniqueness theorem of Poisson's equation, all facilities with arbitrary walls and contacts with the environment, must have a unique steady state provided the source function.

This result is very important for us to study the steady state distribution of viral particles within the facility. Suppose that we have obtained a unique solution c for the equation $\nabla^2 c = -\alpha/\kappa$, and would like to explore what happens with another set of parameters α' and κ' scaled from α and κ by some constant. By linearity of the Laplace operator, we must have $\nabla^2\left(\frac{\kappa\alpha'}{\alpha\kappa'}c\right) = \frac{\kappa\alpha'}{\alpha\kappa'}\nabla^2 c = -\alpha'/\kappa'$ with $c' = \frac{\kappa\alpha'}{\alpha\kappa'}c$ as the new unique solution to the new Poisson's equation $\nabla^2 c' = -\alpha'/\kappa'$, which is simply a scaled version of c. This means that the comparative tendency between different locations to accumulate viral particles is independent to the scales used for α and κ, which shall drastically simplify our procedures to obtain analytical and numerical solutions for the Poisson's equation.

Nonetheless, unlike the heat equation in the 1-dimensional case, it is usually inadmissible to write a simple and universal formula as the analytical solution for most boundary conditions. Instead, solutions will have to be discussed on a case-to-case basis, and frequently involve complicated integrals on trigonometric series. Therefore, the sections below shall discuss the analytical solutions for commonly encountered scenarios, followed by a brief discussion on relevant numerical methods. An exhaustive exploration for all types of boundary conditions is beyond the scope of this write-up.

3.3. Steady state solutions for common scenarios

Example 1: Circular facility without walls under uniform viral creation

For steady state equations, the simplest case is a circular facility with radius R that is not bounded by any walls, so that viral particles may disperse into the environment at all points on the circumference. We center the facility at the origin, and prescribe the Dirichlet boundary conditions $c(x, y) = 0, \forall x^2 + y^2 = R^2$.

Since the boundary conditions are Dirichlet, the steady state solution must be unique for each $\hat{\alpha}$, such that it is adequate to simply work out the correct functional form by trial and error. Upon inspection, $\nabla^2 c_\infty = c_{\infty,xx} + c_{\infty,yy}$ reduces to a constant, which means that we can try a solution where c_∞ is quadratic on both x and y. Denote this solution as $c_\infty = Ax^2 + By^2 + Cxy + Dx + Ey + F$. Despite the complications of unknown terms, we can exploit the symmetries in viral distribution to simplify the solution. Intuitively, for a circular layout with uniform rate of creation of viral particles at all locations, we should have $c_\infty(x, y) = c_\infty(-x, y) = c_\infty(x, -y) = c_\infty(-x, -y)$, so that the terms with C, D and E vanish. Furthermore, viral distribution should remain the same on the x and y-directions, such that $A = B$, and therefore $c_\infty = A(x^2 + y^2) + E$. Subsitution into the Poisson's equation and the boundary conditions yields

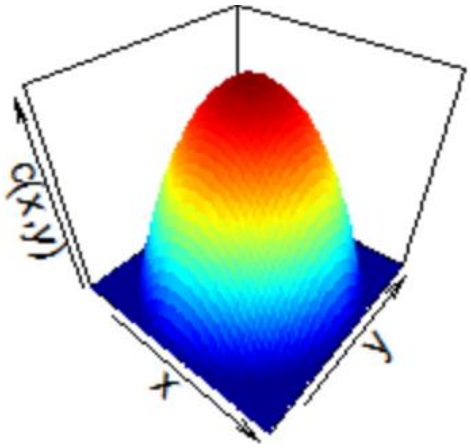
$$\begin{cases} \nabla^2[A(x^2 + y^2) + E] = 4A = -\hat{\alpha} \\ AR^2 + E = 0 \end{cases} \Rightarrow A = -\frac{\hat{\alpha}}{4}, E = \frac{\hat{\alpha}R^2}{4},$$

such that the solution is

$$c_\infty = -\frac{\hat{\alpha}}{4}(x^2 + y^2) + \frac{\hat{\alpha}R^2}{4}.$$

The surface map of the steady state is demonstrated in Plot 5.

Plot 5: Steady state for circular facility without walls and constant source function
Resolution: 51 × 51



Example 2: Square facilities without walls under uniform viral creation

A more complicated case involves square facilities without walls, where the boundary conditions are Dirichlet of the form $c(0,y) = c(L,y) = c(x,0) = c(x,L) = 0, \forall x, y \in [0,L]$. Clearly, the simple quadratic form in the circular case would not work, since no quadratic function may have zero value over a square boundary. Instead, we must resort to the overall solution to the Poisson's equation for rectangular area and zero Dirichlet boundary conditions, which is readily available and applicable to the current case as

$$c_\infty(x,y) = \hat{\alpha}\int_0^L \int_0^L G(x,y,\xi,\eta)d\eta d\xi,$$

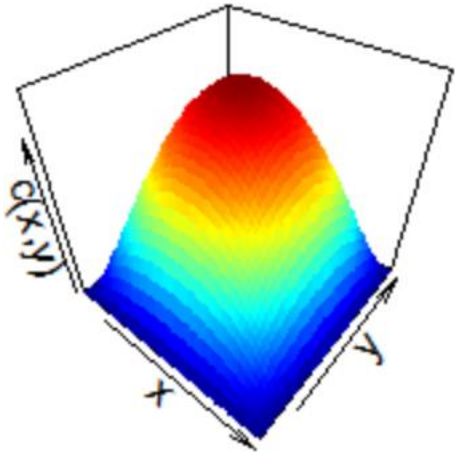
where $G(x,y,\xi,\eta)$ is the Green's function with

$$G(x,y,\xi,\eta) = 2\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)}{n\pi\sinh(n\pi)} H_n(y,\eta),$$

$$H_n(y,\eta) = \begin{cases} \sinh\left(\frac{n\pi\eta}{L}\right)\sinh\left(\frac{n\pi}{L}(L-y)\right), L \geq y > \eta \geq 0 \\ \sinh\left(\frac{n\pi y}{L}\right)\sinh\left(\frac{n\pi}{L}(L-\eta)\right), L \geq \eta > y \geq 0 \end{cases}.$$

Subsequent numerical computations can be done on the Green's function and double integral until convergence, which obtains the surface map in Plot 6.

Plot 6: Steady state for square facility
without walls and constant source function
Resolution: 50 × 50



Example 3: Square facility without walls with constant crowd at selected zones

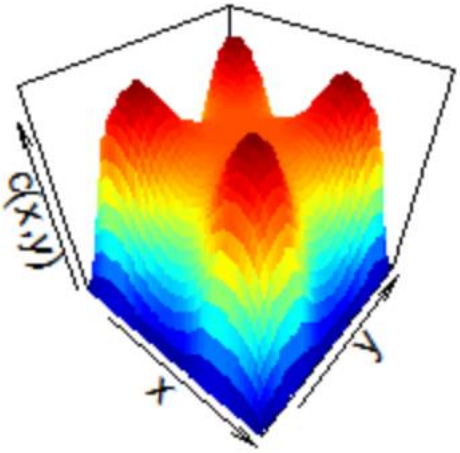
Generally, solution by Green's function is applicable to arbitrary source functions in the Poisson's equation. Suppose now that we have a public facility with some zones as corridor with no crowd in activities, and other zones as activity zones. In this case, $\hat{\alpha}$ becomes a function of x and y, which vanishes at locations where no crowd is present. The same solution with Green's function can be applied with $\hat{\alpha}(x,y)$ moved into the double integral, where

$$c_\infty(x,y) = \int_0^L \int_0^L \hat{\alpha}(\xi,\eta) G(x,y,\xi,\eta) d\eta d\xi.$$

To demonstrate, suppose that the facility has a vertical and horizontal central corridor with width $L/3$, and the crowd collects evenly at other locations. We then specify $\hat{\alpha}(x,y)$ as a piecewise constant function where

$$\hat{\alpha}(x,y) = \begin{cases} 1, x,y \in [0,L/3] \cup [2L/3,L] \\ 0, otherwise \end{cases}.$$

Plot 7: Steady state for square facility
without walls and piecewise constant source function
Resolution: 50 × 50



3.4. General notions on numerical method for 2-dimensional Poisson's equation

With protocols established for 1-dimensional scenarios, extension of numerical methods to 2-dimensional cases is straightforward. As we similarly unpack the 2-dimensional heat equation, we obtain

$$c_t = \alpha + \kappa\nabla^2 c$$

$$\Rightarrow c(x,y,t+\delta t) - c(x,y,t) \approx \alpha\delta t + \kappa\delta t\left(c_{xx}(x,y,t) + c_{yy}(x,y,t)\right)$$

$$\Rightarrow \delta A\big(c(x,y,t+\delta t) - c(x,y,t)\big)$$

$$\approx \alpha\delta A\delta t + \kappa\delta t\left[\delta y\big(c_x(x+\delta x,y,t) - c_x(x,y,t)\big) + \delta x\left(c_y(x,y,+\delta y,t) - c_y(x,y,t)\right)\right]$$

$$\Rightarrow \delta A^2\big(c(x,y,t+\delta t) - c(x,y,t)\big)$$

$$\approx \alpha\delta A^2\delta t + \kappa\delta t\big[\delta y^2\big(c(x+\delta x,y,t) + c(x-\delta x,y,t) - 2c(x,y,t)\big) + \delta x^2\big(c(x,y+\delta y,t) + c(x,y-\delta y,t) - 2c(x,y,t)\big)\big]$$

$$\Rightarrow \delta A^2\big(u_{\delta A}(x,y,t+\delta t) - u_{\delta A}(x,y,t)\big)$$

$$\approx \alpha\delta A^3\delta t + \kappa\delta t\big[\delta y^2\big(u_{\delta A}(x+\delta x,y,t) + u_{\delta A}(x-\delta x,y,t) - 2u_{\delta A}(x,y,t)\big) + \delta x^2\big(u_{\delta A}(x,y+\delta y,t) + u_{\delta A}(x,y-\delta y,t) - 2u_{\delta A}(x,y,t)\big)\big],$$

where $u_{\delta A}(x,y,t)$ denotes the total viral count within the area δA. Suppose that we still choose $\delta x, \delta y, \delta t = 1$, and therefore $\delta A = 1$, and denote the mean viral count for the intervals above, below, left and right to the interval of consideration as $\bar{u}_{\delta A}(x,y,t)$. The above-mentioned equation is reduced to

$$u_1(x,y,t+1) = u_1(x,y,t) + \alpha - 4\kappa\big(u_1(x,y,t) - \bar{u}_1(x,y,t)\big),$$

such that the same protocol delineated in the 1-dimensional case applies equally with a 2-dimensional array.

Conclusion

Thus far, this small write-up has provided a preliminary discussion on a number of applicable methods to model viral dispersion in public facilities. For well-ventilated facilities, we present compartmental models as a simple approach to compute and compare the process of viral dispersion. For poorly ventilated facilities, we show that the mechanisms of viral dispersion can be modeled by heat equations appended with a source function, where steady state equations are obtained as we set the time derivatives to zero. For 1-dimensional cases, steady state equations for constant or piecewise constant source functions can be readily solved by a concatenation of quadratic equations, which apply to many real scenarios. For 2-dimensional cases, steady state equations are Poisson's equations with respect to their relevant source functions, where Green's functions are needed except for a few special cases. Future explorations could be done to solve the steady state equations for viral dispersion in more complicated facilities, or to extend the methods discussed into 3-dimensional cases. In addition, it would also be value to consider scenarios that involve stochastic differential equations with stochastic source functions, and discuss how their statistical properties evolve with time.

References

[1] **First boundary value problem for Poisson's equation**. Retrieved from: http://eqworld.ipmnet.ru/en/solutions/lpde/lpde302.pdf.