Branching rules and Gelfand-Tsetlin bases of the classical Lie algebras

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Contents

1 Introduction

In physical applications, we often encounter the phenomenon of symmetry breaking, in which the symmetry group G of a system is reduced to a subgroup $H \subseteq G$. An eigenstate of the system carries an irreducible representation π of the symmetry group G, and when the symmetry is reduced, π is decomposed into a direct sum of irreducible representations $\bigoplus_i \pi_i(H)$ of the smaller symmetry group. This process is known as **branching**. For finite groups (usually point groups), the branching rules can be derived from the character table. However, for infinite groups (usually Lie groups), the problem is more complicated. In the first section of this article, we will discuss the branching rules of classical Lie algebras, including the unitary algebra $\mathfrak{u}(n)_{\mathbb{C}}$, the orthogonal algebra $\mathfrak{so}(n)_{\mathbb{C}}$, and the symplectic algebra $\mathfrak{sp}(2n)_{\mathbb{C}}$.[1]

[1] Here we only consider the complexified Lie algebras, and we omit the $\mathbb{C}$ label later on.

Another important application of the branching rules is the explicit construction of the representation bases, known as the **Gelfand-Tsetlin (GT) bases**. The highest weight does not directly determine the basis vectors of the irreducible representation. Gelfand and Tsetlin found that the bases can be labeled by the representations of the **subalgebra chain**, which are obtained by the branching rules. This formalism is depicted in Fig. 1. This labeling requires two important properties:

- The branching rule is **multiplicity-free**: for each $\mathfrak{h}_i \to \mathfrak{h}_{i+1}$ in the chain, every representation of $\mathfrak{h}_{i+1}$ can only appear once.
- The subalgebra chain terminates in a Lie algebra that has only one-dimensional representations. Otherwise, the bases cannot be uniquely determined. In practice, the termination algebra is chosen as $\mathfrak{u}(1) = \mathfrak{so}(2) = \mathfrak{sp}(2)$.

As we will see in the following sections, the subalgebra chain $\mathfrak{u}(n) \to u(n-1) \to \cdots \to \mathfrak{u}(1)$ and $\mathfrak{so}(n) \to \mathfrak{so}(n-1) \to \cdots \to \mathfrak{so}(2)$ satisfy these conditions, whereas the chain $\mathfrak{sp}(2n) \to \mathfrak{sp}(2n-2) \to \cdots \to \mathfrak{sp}(2)$ does not, as the branching is not multiplicity-free. In Sec. 3.3, we will introduce the correct construction of the GT basis for $\mathfrak{sp}(2n)$.

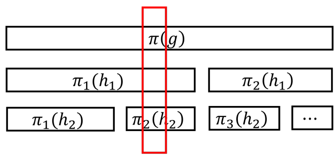

Figure 1: Graphic representation of the Gelfand-Tsetlin basis. Each box indicates an irreducible representation. For each vector in the representation space V of $\pi(g)$, it is in the representation space V_i of $\pi_i(h)$. The basis vector can then be uniquely labeled by $(\pi(g), \pi_{i_1}(h_1), \pi_{i_2}(h_2), \cdots)$.

2 Branching rules of classical Lie algebras

The branching rules of classical Lie algebras can be classified into two types:

- **Type I**: The branching rules within the same family of classical Lie algebras, such as $\mathfrak{u}(n) \to \mathfrak{u}(n-1)$, $\mathfrak{so}(n) \to \mathfrak{so}(n-1)$[2], and $\mathfrak{sp}(2n) \to \mathfrak{sp}(2n-2)$.
- **Type II**: The branching rules between different families of classical Lie algebras, such as $\mathfrak{u}(n) \to \mathfrak{so}(n)$, $\mathfrak{u}(n) \to \mathfrak{sp}(n)$ and many others.

The type-I branching rules are easier to solve, and they are important for construction the GT basis. In Sec. 2.1, we discuss the branching rule $\mathfrak{u}(n) \to \mathfrak{u}(n-1)$, and in Sec. 2.2, we discuss the branching rule $\mathfrak{so}(n) \to \mathfrak{so}(n-1)$. The branching rule $\mathfrak{sp}(2n) \to \mathfrak{sp}(2n-2)$ is discussed in Sec. 3.3.

Type-II branching rules are usually more complicated. For $\mathfrak{u}(n) \to \mathfrak{so}(n)$ and $\mathfrak{u}(n) \to \mathfrak{sp}(n)$, the branching rules can be determined by the Young tableau, which is discussed in detail in [1]. For the **R-type subalgebras**, in which the Dynkin diagram of the subalgebra is a subgraph of the **extended Dynkin diagram** of the original algebra, the branching rules can be obtained by directly deleting nodes from the Dynkin diagram, as discussed in [2].

[2]Here $B_n = \mathfrak{so}(2n+1)$ and $D_n = \mathfrak{so}(2n)$ are seen as the same family.

2.1 $\mathfrak{u}(n) \to \mathfrak{u}(n-1)$ branching rule

$\mathfrak{u}(n)$ is not a semisimple Lie algebra, since it has the Abelian ideal $\mathfrak{u}(1)$, and is isomorphic to $\mathfrak{su}(n) \oplus \mathfrak{u}(1)$. However, it provides a simple way to characterize the root system. The roots lie in a n-dimensional space, with the form $e_i - e_j, 1 \le i, j \le n$. There $n-1$ fundamental roots are $e_i - e_{i+1}, 1 \le i \le n-1$.[3]

All irreducible representations of $\mathfrak{u}(n)$ can be labeled by the **Young tableau** $[\lambda]$, satisfying the condition

$$[\lambda] = (\lambda_1, \lambda_2, \cdots, \lambda_n) \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0 \tag{1}$$

where λ_i is the number of boxes in the i-th row. The corresponding highest weight is

$$M = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \omega_i \tag{2}$$

where ω_i is the fundamental weight (the Dynkin basis). Adding a constant number of blocks to every row in the Young tableau does not change the highest weight. In practice we often choose $\lambda_n = 0$.

We consider the **tensor Young tableaux** $[\lambda]_T$, which spans the representation space of the irreducible representation $[\lambda]$. The tensor Young tableaux are constructed by filling the boxes with the numbers $1, 2, \cdots, n$, and every number can appear more than once. The requirement for the filling is that the numbers in each row must be non-decreasing, and the numbers in each column must be increasing [3]. The number n is restricted to the boxes

$$[\lambda]_{i,j} \quad \lambda_{i+1} \le j \le \lambda_i \tag{3}$$

This is illustrated in Fig. 2.

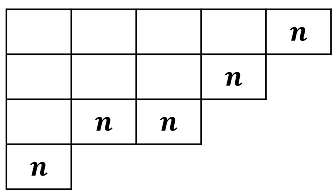

Figure 2: **Possible** locations of n in the Young tableau.

Removing n from the tensor Young tableau of $\mathfrak{u}(n)$ will result in a tensor Young tableau of $\mathfrak{u}(n-1)$. Since the irreducible representation is uniquely determined by the shape of the Young tableau, we only need to consider **the shape** $[\mu]$ after removing n. Apparently, $\mu_i \le \lambda_i$. Moreover, because the block $[\lambda]_{i,\lambda_{i+1}}$ is not filled with n, we have $\mu_i \ge \lambda_{i+1}$. Then $\mu_i \le \mu_{i+1}$, and $[\mu]$ is a valid Young tableau.

Thus, the branching rule $\mathfrak{u}(n) \to \mathfrak{u}(n-1)$ is given by

$$\begin{gathered} [\lambda_1, \lambda_2, \cdots, \lambda_n] \to [\mu_1, \mu_2, \cdots, \mu_{n-1}] \\ \lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n \end{gathered} \tag{4}$$

Eq. (4) is known as the **"interlacing condition"**. To check the multiplicity, we compare the dimension

[3] Because the roots actually lie in a $(n-1)$-dimensional hyperplane perpendicular to $e_1 + e_2 + \cdots + e_n$.

of $[\lambda]$ with the sum of dimensions of all $[\mu]$. The dimension is given by the **hook-length formula**:

$$d_{[\lambda]}(\mathfrak{u}(n)) = \prod_{i=1}^{n} \frac{1}{i!} \cdot \prod_{j<k}^{n} (\lambda_j - \lambda_k - j + k) \tag{5}$$

We can verify

$$\sum_{\lambda_i \leq \mu_i \leq \lambda_{i+1}} \prod_{j<k}^{n-1} (\mu_j - \mu_k - j + k) = \frac{1}{n!} \prod_{j<k}^{n} (\lambda_j - \lambda_k - j + k) \tag{6}$$

Therefore

$$d_{[\lambda]}(\mathfrak{u}(n)) = \sum_{[\mu]} d_{[\mu]}(\mathfrak{u}(n-1)) \tag{7}$$

where $[\mu]$ satisfies Eq. (4). Hence, the branching rule is **multiplicity-free**.

2.2 $\mathfrak{so}(n) \to \mathfrak{so}(n-1)$ branching rule

2.2.1 Review of root system of $\mathfrak{so}(n)$

The positive roots of $\mathfrak{so}(n)$ in the orthonormal basis are

$$\begin{aligned} \text{for } \mathfrak{so}(2n): &\quad e_i \pm e_j \quad 1 \leq i < j \leq n \\ \text{for } \mathfrak{so}(2n+1): &\quad e_i \pm e_j \quad 1 \leq i < j \leq n \quad \text{and} \quad \pm e_i \quad 1 \leq i \leq n \end{aligned} \tag{8}$$

The fundamental weights are

$$\begin{aligned} \text{for } \mathfrak{so}(2n): \quad \omega_i &= e_1 + e_2 + \cdots e_i \quad i = 1, 2, \cdots n-2 \\ \omega_{n-1} &= \frac{1}{2}(e_1 + e_2 + \cdots e_{n-1} - e_n) \\ \omega_n &= \frac{1}{2}(e_1 + e_2 + \cdots e_{n-1} + e_n) \\ \text{for } \mathfrak{so}(2n+1): \quad \omega_i &= e_1 + e_2 + \cdots e_i \quad i = 1, 2, \cdots n-1 \\ \omega_n &= \frac{1}{2}(e_1 + e_2 + \cdots e_{n-1} + e_n) \end{aligned} \tag{9}$$

The highest weights take the form

$$M = \sum_{i=1}^{n} m_i \omega_i \quad m_i \in \mathbb{Z}^+ \tag{10}$$

Substituting Eq. (9), we obtain

$$\begin{aligned} \text{for } \mathfrak{so}(2n): &\quad M = \sum_{i=1}^{n} \lambda_i e_i \quad 2\lambda_1, \cdots, 2\lambda_{n-1} \in \mathbb{Z}^+, 2\lambda_n \in \mathbb{Z} \\ &\quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \\ \text{for } \mathfrak{so}(2n+1): &\quad M = \sum_{i=1}^{n} \lambda_i e_i \quad 2\lambda_1, \cdots, 2\lambda_n \in \mathbb{Z}^+ \\ &\quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0 \end{aligned} \tag{11}$$

This result shows that there are two types of irreducible representations of $\mathfrak{so}(n)$, the **tensor** and the **spinor** representations. Representations with all $\lambda_i \in \mathbb{Z}$ correspond to tensor representations, which can be constructed using Young tableaux. Those with some $\lambda_i \in \mathbb{Z}+\frac{1}{2}$ correspond to spinor representations, which cannot. Therefore, the Young tableau method discussed in Sec. 2.1 cannot be used to derive the general formula for the branching rule $\mathfrak{so}(n) \to \mathfrak{so}(n-1)$. We will adopt a different method, based on the **Weyl character formula**.

2.2.2 The Weyl character formula

The Weyl character formula is

$$\chi(H) = \frac{\sum_{w\in W} \det(w) \exp(\langle w(M+\delta), H\rangle)}{\sum_{w\in W} \det(w) \exp(\langle w\delta, H\rangle)} \tag{12}$$

Here H is an element of the Cartan subalgebra, W is the Weyl group, M is the highest weight, and δ is the half of the sum of positive roots. The denominator can also be written as

$$\Delta = \sum_{w\in W} \det(w) \exp(\langle w\delta, H\rangle) = \prod_{\alpha\in\Delta^+} (e^{\langle\alpha,H\rangle/2} - e^{-\langle\alpha,H\rangle/2}) \tag{13}$$

Proofs of these formulas can be found in [4].

The Weyl group of $\mathfrak{so}(2n+1)$ is $\Sigma(n) = S_n \ltimes \mathbb{Z}_2^n$, and the Weyl group of $\mathfrak{so}(2n)$ is the subgroup $\Sigma_0(n)$ of $\Sigma(n)$, consisting of signed permutations with an even number of sign changes. [4]

2.2.3 Branching rule of $\mathfrak{so}(2n+1) \to \mathfrak{so}(2n)$

The discussion in this and the following part follows [5].

From Eq. (8), $\mathfrak{so}(2n+1)$ has n additional roots $e_1, \cdots, e_n$ compared to $\mathfrak{so}(2n)$. Then the Weyl denominator Eq. (13) has the relation:

$$\Delta_{\mathfrak{so}(2n+1)} = \Delta_{\mathfrak{so}(2n)} \cdot \prod_{i=1}^{n} (e^{H_i/2} - e^{-H_i/2}) \tag{14}$$

Here we denoted $H_i = \langle e_i, H\rangle$. Therefore

$$\begin{aligned}
&\Delta_{\mathfrak{so}(2n)}\chi_{M,\mathfrak{so}(2n+1)} \\
&= \sum_{\sigma\in S_n} \det(\sigma) \sum_{\tau\in\mathbb{Z}_2^n} \det(\tau) e^{\langle\tau\sigma(M+\delta),H\rangle} \prod_{i=1}^{n} (e^{H_i/2} - e^{-H_i/2})^{-1} \\
&= \sum_{\sigma\in S_n} \det(\sigma) \prod_{i=1}^{n} \left(\frac{e^{[\sigma(M+\delta)]_i H_i} - e^{-[\sigma(M+\delta)]_i H_i}}{e^{H_i/2} - e^{-H_i/2}} \right) \\
&= \sum_{\sigma\in S_n} \det(\sigma) \prod_{i=1}^{n} \left(\sum_{|k|<[\sigma(M+\delta')]_i} e^{kH_i} \right) \\
&= \sum_{\sigma\in S_n} \det(\sigma) \prod_{i=1}^{n} \left(\sum_{|k|<\lambda_i+n-i} e^{kH_{\sigma(i)}} \right)
\end{aligned} \tag{15}$$

Here $\delta' = \delta - \dfrac{1}{2}\sum_{i=1}^n e_i$ is the half of the sum of positive roots of $\mathfrak{so}(2n)$. The last line is obtained by the variable change $i \to \sigma(i)$.

It is apparent that this is the determinant of the matrix $M_{ij} = \sum_{|k|<\lambda_i+n-i} e^{kH_j}$, and we can subtract the $(i+1)$-th row from the i-th row:

$$
\begin{aligned}
&\Delta_{\mathfrak{so}(2n)}\chi_{M,\mathfrak{so}(2n+1)} \\
&= \sum_{\sigma\in S_n} \det(\sigma) \prod_{i=1}^n \left(\sum_{\lambda_{i+1}+n-i<|k|<\lambda_i+n-i} e^{kH_{\sigma(i)}} \right) \\
&= \sum_{\sigma\in S_n} \det(\sigma) \sum_{\lambda_{i+1}+n-i<|p_i|<\lambda_i+n-i} \left(\prod_{j=1}^n e^{p_j H_{\sigma(j)}} \right) \\
&= \sum_{\sigma\in S_n} \det(\sigma) \sum_{\lambda_{i+1}+n-i<|p_i|<\lambda_i+n-i} e^{\langle \sigma(p), H\rangle}
\end{aligned}
\tag{16}
$$

Define $\tau_i \in \{0,1\}$ by $(-1)^{\tau_i} = \mathrm{sgn}(p_i)$ for $1 \le i \le n-1$, and let $s_i = \tau_i p_i$. Choose τ_n such that $\sum_{i=1}^n \tau_i \bmod 2 = 0$, then the corresponding signed permutation $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{Z}_2^n$ lies in $\Sigma_0(n)$. Also, $s = (s_1, \dots, s_n) \in \mathbb{Z}^n$. Therefore,

$$
\begin{aligned}
&\sum_{\lambda_{i+1}+n-i<|p_i|<\lambda_i+n-i} e^{\langle \sigma(p), H\rangle} \\
&= \sum_{\tau\in\mathbb{Z}_2^n\cup\Sigma_0(n)} \sum_{\lambda_{i+1}+n-i<s_i<\lambda_i+n-i, |s_n|<\lambda_n} e^{\langle \sigma(\tau s), H\rangle} \\
&= \sum_{\tau\in\mathbb{Z}_2^n\cup\Sigma_0(n)} \sum_{\lambda_{i+1}<\mu_i<\lambda_i, |\mu_n|<\lambda_n} e^{\langle \sigma(\tau(\mu+\delta')), H\rangle}
\end{aligned}
\tag{17}
$$

Here we set $s = \mu + \delta'$. Substituting into Eq. (16), we obtain

$$
\begin{aligned}
&\Delta_{\mathfrak{so}(2n)}\chi_{M,\mathfrak{so}(2n+1)} \\
&= \sum_{w\in\Sigma_0(n)} \det(w) \sum_{\lambda_{i+1}<\mu_i<\lambda_i, |\mu_n|<\lambda_n} e^{\langle w(\mu+\delta'), H\rangle} \\
&= \sum_{\lambda_{i+1}<\mu_i<\lambda_i, |\mu_n|<\lambda_n} \Delta_{\mathfrak{so}(2n)}\chi_{\mu,\mathfrak{so}(2n)}
\end{aligned}
\tag{18}
$$

This proves the branching rule $\mathfrak{so}(2n+1) \to \mathfrak{so}(2n)$:

$$
\begin{gathered}
[\lambda_1, \lambda_2, \cdots, \lambda_n] \to [\mu_1, \mu_2, \cdots, \mu_n] \\
\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_n \ge |\mu_n|
\end{gathered}
\tag{19}
$$

Each irreducible component $[\mu]$ appears **only once** in the decomposition, hence the branching rule is **multiplicity-free**.

2.2.4 Branching rule of $\mathfrak{so}(2n) \to \mathfrak{so}(2n-1)$

In this case, the two Lie algebras have different ranks, and we utilize the $\mathfrak{u}(n) \to \mathfrak{u}(n-1)$ branching rule as an intermediate step. The whole process is similar to the previous case but more complicated, and we will only show the main steps. The details can be found in [5].

First, using Eq. (4), since the root system of $\mathfrak{u}(n-1)$ embeds naturally into that of $\mathfrak{u}(n)$, we can obtain

$$\frac{\det(e^{\alpha_i H_j})}{\prod_{i=1}^{n-1}(e^{H_i/2}-e^{-H_i/2})} = \sum_{\beta_i\in\mathbb{Z}+1/2,\alpha_{i+1}<\beta_i<\alpha_i} \det(e^{\beta_i H_j}) \tag{20}$$

Then, we observe that

$$\Delta_{\mathfrak{so}(2n)}(H_1,\cdots,H_{n-1},0) = \prod_{i=1}^{n-1}(e^{H_i/2}-e^{-H_i/2})\cdot\Delta_{\mathfrak{so}(2n-1)}(H_1,\cdots,H_{n-1}) \tag{21}$$

Thus

$$\begin{aligned}
&\Delta_{\mathfrak{so}(2n-1)}(H_1,\cdots,H_{n-1})\chi_{\lambda,\mathfrak{so}(2n)}(H_1,\cdots,H_{n-1},0)\\
=&\sum_{\tau\in\mathbb{Z}_2^n,\det\tau=1}\frac{\det(e^{[\tau(\lambda+\delta)]_i H_j})}{\prod_{i=1}^{n-1}(e^{H_i/2}-e^{-H_i/2})}\\
=&\sum_{\tau\in\mathbb{Z}_2^n,\det\tau=1}(-1)^{\kappa(\tau)}\sum_{\substack{\alpha=\mathrm{sort}(\tau(\lambda+\delta)),\beta_i\in\mathbb{Z}+1/2\\ \alpha_{i+1}<\beta_i<\alpha_i}}\det(e^{\beta_i H_j})\\
=&\sum_{\mu_i\in\mathbb{Z},|\lambda_{i+1}|<\mu_i<|\lambda_i|}\Delta_{\mathfrak{so}(2n-1)}(H_1,\cdots,H_{n-1})\chi_{\mu,\mathfrak{so}(2n-1)}(H_1,\cdots,H_{n-1})
\end{aligned} \tag{22}$$

Therefore, the branching rule $\mathfrak{so}(2n)\to\mathfrak{so}(2n-1)$ is given by the condition[4]:

$$\begin{gathered}
[\lambda_1,\lambda_2,\cdots\lambda_n]\to[\mu_1,\mu_2,\cdots\mu_{n-1}]\\
\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\lambda_{n-1}\geq\mu_{n-1}\geq|\lambda_n|
\end{gathered} \tag{23}$$

Each component $[\mu]$ still appears only once. Then the branching is **multiplicity-free**.

3 Gelfand-Tsetlin bases

3.1 GT bases of $\mathfrak{u}(n)$ and $\mathfrak{so}(n)$

As mentioned in Sec. 1, the GT bases are constructed from the irreducible representations of the subalgebra chain. For $\mathfrak{u}(n)\to u(n-1)\to\cdots\to\mathfrak{u}(1)$ and $\mathfrak{so}(n)\to\mathfrak{so}(n-1)\to\cdots\to\mathfrak{so}(2)$, the branching rules are multiplicity-free, and the bases can be uniquely labeled by the irreducible representations of the subalgebras. Specifically, they are in the form of

$$\left|\begin{matrix}
\lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{nn}\\
\lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{n-1,n-1} & \\
\cdots & \cdots & \cdots & & \\
\lambda_{11} & & & &
\end{matrix}\right| \tag{24}$$

for $\mathfrak{u}(n)$, and

[4] For the tensor representations with $\lambda_i\in\mathbb{Z}^+$. One can verify that both Eq. (19) and Eq. (23) reduces to the $\mathfrak{u}(n)\to\mathfrak{u}(n-1)$ branching rule Eq. (4).

$$\left|\begin{array}{ccccc} \lambda_{2n+1,1} & \lambda_{2n+1,2} & \cdots & \lambda_{2n+1,n-1} & \lambda_{2n+1,n} \\ \lambda_{2n,1} & \lambda_{2n,2} & \cdots & \lambda_{2n,n-1} & \lambda_{2n,n} \\ \lambda_{2n-1,1} & \lambda_{2n-1,2} & \cdots & \lambda_{2n-1,n-1} & \\ \lambda_{2n-2,1} & \lambda_{2n-2,2} & \cdots & \lambda_{2n-2,n-1} & \\ \cdots & \cdots & \cdots & & \\ \lambda_{31} & & & & \\ \lambda_{21} & & & & \end{array}\right| \tag{25}$$

for $\mathfrak{so}(2n+1)$ ($\mathfrak{so}(2n)$ is obtained by removing a line). The i-th row of the pattern is the irreducible representation of $\mathfrak{u}(n-i+1)$ or $\mathfrak{so}(2n-i+2)$. They satisfy the conditions Eq. (4), Eq. (19), and Eq. (23):

- $\mathfrak{u}(n) : \lambda_{i+1,j} \geq \lambda_{ij} \geq \lambda_{i+1,j+1}$
- $\mathfrak{so}(n) : \lambda_{i+1,j} \geq \lambda_{i,j} \geq \lambda_{i+1,j+1}$, $\lambda_{2i+1,i} \geq |\lambda_{2i,i}|$, and $\lambda_{2i-1,i-1} \geq |\lambda_{2i,i}|$.

The actions of the generators of $\mathfrak{u}(n)$ and $\mathfrak{so}(n)$ on the GT bases (denoted as ξ_Λ) can be explicitly calculated [6, 7]. For $\mathfrak{u}(n)$, we have

$$\begin{aligned} E_{k,k}\xi_\Lambda &= \left(\sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i}\right) \xi_\Lambda, \\ E_{k,k+1}\xi_\Lambda &= -\sum_{i=1}^{k} \frac{\prod_{j=1}^{k+1}(l_{ki} - l_{k+1,j})}{\prod_{j=1,j\neq i}^{k}(l_{ki} - l_{kj})} \xi_{\Lambda+\delta_{ki}} \\ E_{k+1,k}\xi_\Lambda &= \sum_{i=1}^{k} \frac{\prod_{j=1}^{k-1}(l_{ki} - l_{k-1,j})}{\prod_{j=1,j\neq i}^{k}(l_{ki} - l_{kj})} \xi_{\Lambda-\delta_{ki}} \end{aligned} \tag{26}$$

Where $l_{ij} = \lambda_{ij} - i + 1$, and $\Lambda \pm \delta_{ki}$ is obtained by replacing λ_{ki} with $\lambda_{ki} \pm 1$. Other generators are obtained by $E_{k,k+h} = [E_{k,k+1}, [E_{k+1,k+2}, [\cdots]]]$ and $E_{k+h,k} = (-1)^h [E_{k+1,k}, [E_{k+2,k+1}, [\cdots]]]$.

For $\mathfrak{so}(n)$, the result is very complex. It is discussed in [7], which we will not repeat here.

A simple example of Eq. (26) is $n = 2$, then the GT basis is $|(\lambda_{21}, \lambda_{22}), (\lambda_{11})\rangle$. We can convert this into the angular momentum basis $|jm\rangle$. Because

$$J_z|jm\rangle = \frac{1}{2}(E_{11} - E_{22})|jm\rangle = m|jm\rangle \tag{27}$$

and the definition of the highest weight implies

$$\begin{aligned} E_{11}|jj\rangle = \lambda_{21}|jj\rangle \quad E_{22}|jj\rangle = \lambda_{22}|jj\rangle \\ E_{11}|jm\rangle = \lambda_{11}|jm\rangle \end{aligned} \tag{28}$$

Moreover, $E_{11} + E_{22}$ is the identity operator **on the weight space**, with eigenvalue equal to the trace of the representation. Then $(E_{11} + E_{22})|jm\rangle = f(j)|jm\rangle$. So $E_{22}|jm\rangle = (\lambda_{21} + \lambda_{22} - \lambda_{11})|jm\rangle$.

Combining these equations, we have

$$j = \frac{1}{2}(\lambda_{21} - \lambda_{22}) \quad m = \lambda_{11} - \frac{1}{2}(\lambda_{21} + \lambda_{22}) \tag{29}$$

We can verify that $\lambda_{21} \geq \lambda_{11} \geq \lambda_{22}$ implies $-j \leq m \leq j$.

The action of the off-diagonal generators is given by

$$\begin{aligned} E_{12}|jm\rangle &= \sqrt{(j-m)(j+m+1)}|j,m+1\rangle \\ E_{21}|jm\rangle &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle \end{aligned} \tag{30}$$

Substituting Eq. (29), this is consistent with Eq. (26).

As another example, we consider $n = 3$, and the GT basis is $|(\lambda_{31}, \lambda_{32}, \lambda_{33}), (\lambda_{21}, \lambda_{22}), (\lambda_{11})\rangle$. In physics, the **isospin**, **hypercharge**, and **baryon number** operators are defined as $I_3 = \frac{1}{2}(E_{11} - E_{22})$, $Y = \frac{1}{3}(E_{11} + E_{22} - 2E_{33})$, and $B = \frac{1}{3}(E_{11} + E_{22} + E_{33})$. Using Eq. (29) and Eq. (26), the basis vector takes the form [7]:

$$|\psi\rangle = \left|(\lambda_{31}, \lambda_{32}, \lambda_{33}), (I + \frac{Y}{2} + B, -I + \frac{Y}{2} + B), (I_3 + \frac{Y}{2} + B)\right\rangle \quad \lambda_{31} + \lambda_{32} + \lambda_{33} = 3B \tag{31}$$

We usually consider $\mathfrak{su}(3)$, and choose $\lambda_{33} = 0$. Using Eq. (2), $\lambda_{31} = m_1 + m_2, \lambda_{32} = m_2$, where $m_1, m_2 \in \mathbb{Z}$ is the highest weight in the Dynkin basis. Thus $B = \frac{1}{3}(m_1 + 2m_2)$. From the interlacing condition and Eq. (31), we obtain the possible values of Y, I, I_3:

$$\begin{aligned} I &= 0, \frac{1}{2}, 1, \cdots, \frac{1}{2}(m_1 + m_2) \\ Y &= -\frac{1}{3}(2m_1 + m_2), -\frac{1}{3}(2m_1 + m_2) + 1, \cdots, \frac{1}{3}(m_1 + 2m_2) \\ I_3 &= -I, -I + 1, \cdots, I \end{aligned} \tag{32}$$

this is consistent with the standard model.

3.2 Construction of the GT bases I: Mickelsson-Zhelobenko algebra

In this and the following section, we follow [6] and discuss the general construction of the GT bases, based on the **Mickelsson-Zhelobenko algebra**, and the **(twisted) Yangian**. Because the construction is quite complicated, we will only give a brief introduction.

Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$. Its Cartan decomposition is $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{k}^+ \oplus \mathfrak{k}^-$, where $\mathfrak{k}^+$ is the positive root system of $\mathfrak{k}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra, and $R(\mathfrak{h})$ its field of fractions over the Cartan subalgebra $\mathfrak{h}$. Define $U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes R(\mathfrak{h})$. Let $J = U'(\mathfrak{g})\mathfrak{k}^+$, and its **normalizer** $\mathrm{Norm}J = \{u \in U'(\mathfrak{g}) | uJ \subseteq J\}$. The **Mickelsson-Zhelobenko (M-Z) algebra** is defined as the quotient

$$Z(\mathfrak{g}, \mathfrak{k}) = \mathrm{Norm}J/J \tag{33}$$

The algebraic structure of $Z(\mathfrak{g}, \mathfrak{k})$ is described by the **external projector**. For each $\alpha \in \mathfrak{k}^+$, define the series ($\delta$ is the half of the sum of positive roots):

$$p_\alpha = \sum_{k=0}^{\infty} e_{-\alpha}^k e_\alpha^k \frac{(-1)^k}{k! \prod_{j=1}^k (h_\alpha + \langle \delta, h_\alpha \rangle + j)} \tag{34}$$

Define the external projector as $p = \prod_{\alpha \in \mathfrak{k}^+} p_\alpha$, then we can prove

$$e_\alpha p = p e_{-\alpha} = 0 \tag{35}$$

Let β to be the positive weights in $\mathfrak{g}^+ \backslash \mathfrak{k}^+$. Then the elements $z_\beta = pe_\beta$ are generators of the M-Z algebra:

$$Z(\mathfrak{g}, \mathfrak{k}) = \left\{ \prod_\beta z_\beta^{k_\beta} \middle| k_\beta \in \mathbb{Z}^+ \right\} \tag{36}$$

Using these two important properties, we can easily construct the subspace $V^+ = \{v \in V | \mathfrak{k}^+ v = 0\}$ containing $\mathfrak{k}$-highest vectors of any vector space V:

$$V^+ = \left\{ \prod_\beta z_\beta^{k_\beta} \cdot v_0 \middle| k_\beta \in \mathbb{Z}^+ \right\} \tag{37}$$

for arbitrary $v_0 \in V^+$.

The simplest example is $Z(\mathfrak{u}(n), \mathfrak{u}(n-1))$, in which

$$\begin{aligned} p_{ij} &= \sum_{k=0}^{\infty} E_{ji}^k E_{ij}^k \frac{(-1)^k}{k!(h_i - h_j + 1) \cdots (h_i - h_j + k)} \\ p &= \prod_{i<j}^{n-1} p_{ij} \qquad z_{ni} = pE_{ni}, z_{in} = pE_{in} \end{aligned} \tag{38}$$

Denote the representation space of g with highest weight λ as $L(\lambda)$, Then

$$L(\lambda)^+ = \left\{ \prod_{i=1}^{n-1} z_{ni}^{k_i} \cdot \xi \middle| k_i \in \mathbb{Z}^+, z_{in}\xi = 0 \right\} \tag{39}$$

consists of the highest vectors of $\mathfrak{u}(n-1)$, that is, $E_{ij} L(\lambda)^+ = 0$ for $1 \le i < j \le n-1$.

Moreover, from the explicit matrix expression of z, we found that

$$\xi_\mu = \prod_{i=1}^{n-1} z_{ni}^{\lambda_i - \mu_i} \cdot \xi \tag{40}$$

has weight μ with respect to the Cartan subalgebra of $\mathfrak{u}(n-1)$. It spans the space $L(\lambda)_\mu^+$.

We require the following two conditions so that ξ_μ is non-zero:

- Because $k_i \ge 0$, $\mu_i \le \lambda_i$.
- Because $z_{ni}^{\lambda_i - \mu_i} \xi$ has weight $(\lambda_1 \cdots \lambda_{i-1}, \mu_i, \lambda_{i+1} \cdots \lambda_{n-1})$ with respect to the Cartan subalgebra of $\mathfrak{u}(n-1)$, from Eq. (1), we obtain $\lambda_{i+1} \ge \mu_i$.

Combining these two conditions, we return to the familiar interlacing condition Eq. (4). Because for each μ there is only one vector ξ_μ defined in Eq. (40), the branching is multiplicity-free.

Other M-Z algebras are given out in [6]. We change E_{ij} to $F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i}$, in which $\theta_{ij} = 1$ for the orthogonal algebras and sgn$i \cdot$ sgn$j$ for the symplectic algebra. $-n \le i, j \le n$. For $\mathfrak{so}(2n)$ and $\mathfrak{sp}(2n)$ the $i = 0$ term is omitted. Cancelling $i = \pm n$ at each time, the M-Z algebras $Z(B_n, B_{n-1}) = Z(\mathfrak{so}(2n+1), \mathfrak{so}(2n-1))$, $Z(C_n, C_{n-1}) = Z(\mathfrak{sp}(2n), \mathfrak{sp}(2n-2))$ and $Z(D_n, D_{n-1}) = Z(\mathfrak{so}(2n), \mathfrak{so}(2n-2))$ can be calculated[5].

[5]Here we returned to the Dynkin classification.

3.3 Construction of the GT bases II: Yangians and the multiplicity formula

In this section, we explore the fundamental connection between the M-Z algebra and the **Yangian** algebra. The Yangian $Y(N)$ is an infinite-dimensional Hopf algebra generated by operator-valued functions, defined by the relation

$$(u-v)[t_{ab}(u), t_{cd}(v)] = t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u) \quad 1 \leq a, b, c, d \leq N \tag{41}$$

The **twisted Yangian** $Y^{\pm}(N)$ is defined by a transformation on the Yangian. We consider $N = 2$ as an example, then the elements are:

$$s_{ab}(u) = \theta_{nb} t_{an}(u) t_{-b,-n}(-u) + \theta_{-n,b} t_{a,-n}(u) t_{-b,n}(-u) \quad a, b = \pm n \tag{42}$$

It is denoted $Y^{\pm}(2)$ for θ corresponding to the orthogonal or symplectic case, respectively.

The Yangian arises naturally from the quantization (**q-deform**) of the **universal enveloping algebra** of the Lie algebra $\mathfrak{g}$ [6, 8]. For each classical Lie algebra we may address a corresponding Yangian, as shown in Table 1.

Lie algebra	$A_n = \mathfrak{su}(n+1)$	$B_n = \mathfrak{so}(2n+1)$	$C_n = \mathfrak{sp}(2n)$	$D_n = \mathfrak{so}(2n)$
Yangian	$Y(n+1)$	$Y^+(2n+1)$	$Y^-(2n)$	$Y^+(2n)$

Table 1: The correspondence between classical Lie algebras and their Yangians.

Molev [6] established a remarkable homomorphism between the twisted Yangian and the M-Z algebra:

$$\begin{aligned} &Y^+(2) \to Z(B_n, B_{n-1}): \qquad s_{ab}(u) \to -u^{-2n} Z_{ab}(u) \\ &Y^-(2) \to Z(C_n, C_{n-1}): \qquad s_{ab}(u) \to (u + \frac{1}{2}) u^{-2n} Z_{ab}(u) \\ &Y^+(2) \to Z(D_n, D_{n-1}): \qquad s_{ab}(u) \to -2u^{-2n+2} Z_{ab}(u) \end{aligned} \tag{43}$$

Here $a, b = \pm n$, and $Z_{ab}(u)$ is a generating function whose coefficients are elements of the M-Z algebra.

The $Y^{\pm}(2)$ module $V(\lambda)^+_\mu$, homomorphic to $L(\lambda)^+_\mu$ by Eq. (43), can be decomposed into a direct sum of the irreducible representations of the Yangian, whose dimension is well known. Consequently, the dimension of $L(\lambda)^+_\mu$, which corresponds to the **multiplicity** $m(\lambda \to \mu)$ in the branching, can be naturally determined.

For the case $\mathfrak{sp}(2n) \to \mathfrak{sp}(2n-2) = C_n \to C_{n-1}$ that we are interested in, we can obtain

$$\begin{gathered} m_C(\lambda \to \mu) = \prod_{i=1}^{n} (\alpha_i - \beta_i + 1) \\ \alpha_1 = -\frac{1}{2} \quad \alpha_i = \min(\lambda_{i-1}, \mu_{i-1}) - i + \frac{1}{2} \quad \beta_i = \max(\lambda_i, \mu_i) - i + \frac{1}{2} \end{gathered} \tag{44}$$

Apparently, $m_C \neq 1$, indicating the directly constructed GT basis using the subalgebra chain $C_n \to C_{n-1} \to \cdots \to C_1$ is invalid. However, we can introduce a set of **auxiliary weights** λ'_{ij}, $1 \leq j \leq i, 1 \leq i \leq n$, which satisfies

$$\lambda_{i,j-1} \geq \lambda'_{i,j} \geq \lambda_{i,j} \quad \lambda_{i-1,j-1} \geq \lambda'_{i,j} \geq \lambda_{i-1,j} \tag{45}$$

Then the number of possible configurations of λ'_{ij} is exactly m_C. The auxiliary weights λ'_{ij} can be interpreted as the highest weight of a virtual Lie algebra $\mathfrak{sp}(2n-1)$. They provide a multiplicity-free parametrization of the basis vectors, thereby facilitating the construction of the **GT basis for** $\mathfrak{sp}(2n)$:

$$\left|\begin{array}{ccccc} \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{nn} \\ \lambda'_{n1} & \lambda'_{n2} & \cdots & \lambda'_{n,n-1} & \lambda'_{nn} \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{n-1,n-1} & \\ \lambda'_{n-1,1} & \lambda'_{n-1,2} & \cdots & \lambda'_{n-1,n-1} & \\ \cdots & \cdots & \cdots & & \\ \lambda_{11} & & & & \\ \lambda'_{11} & & & & \end{array}\right| \tag{46}$$

The branching rule at each step is multiplicity-free, ensuring that the full basis can be recursively constructed. The basis vector of $\mathfrak{sp}(2n)$ (analog of Eq. (40)) is [6]:

$$\xi_\Lambda = \prod_{k=1}^{n}\left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{ki}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{kk}-1} Z_{k,-k}(j)\right)\xi \tag{47}$$

And the matrix elements (similar to Eq. (26)) can also be calculated.

At the end of this section, we briefly conclude the three main branching rules Eq. (4), Eq. (19), Eq. (23) and Eq. (45):

- For $\mathfrak{u} \to \mathfrak{u}$, the interlacing condition is given by $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, and the multiplicity is 1.
- For $\mathfrak{so} \to \mathfrak{so}$, the interlacing condition is $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq |\mu_n|$ or $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq |\lambda_n|$, and the multiplicity is 1.
- For $\mathfrak{sp} \to \mathfrak{sp}$, an auxiliary set of parameters ν_i (corresponding to λ'_i) must be introduced to resolve the multiplicities. The branching involves a two-stage interlacing condition: $\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_n \geq \lambda_n \geq 0$ and $\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \nu_n$.

The corresponding GT bases are given in Eq. (24), Eq. (25) and Eq. (46).

4 Conclusion

In this paper, we reviewed the branching rules of the classical Lie algebras $\mathfrak{u}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$, and the associated construction of the Gelfand–Tsetlin (GT) bases via these branching structures. For each class of Lie algebra, we adopted the most elementary approach to derive the results. Alternative and more general frameworks exist, such as the **polynomial realization** method developed by Zhelobenko [9], which provides a unified perspective across different types of Lie algebras.

To address the challenges arising in non-multiplicity-free branching, particularly in the $B_n \to B_{n-1}$, $C_n \to C_{n-1}$, and $D_n \to D_{n-1}$ cases, we introduced the Mickelsson–Zhelobenko (M-Z) algebra. This algebraic framework enables the construction of explicit basis vectors in multiplicity spaces. Furthermore, we explored the connection observed by Molev [6] between the MZ algebra and the twisted Yangian $Y^{\pm}(2)$. It plays a crucial role in constructing generalized GT bases.

The GT bases yield explicit formulae for the action of the generators of Lie algebras on basis vectors, thereby providing a powerful tool for representation computations, such as in quantum field theory and nuclear physics.

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