Riemannian Geometry Project: Chern-Weil homomorphism and the (Generalized) Gauss-Bonnet Theorem

Sophus Valentin Willumsgaard

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0.1 Introduction

De Rham's theorem gives a starting point to connecting the topological data of a manifold with its geometric data. An example of this is given by the Chern-Weil homomorphism, which defines characteristic classes of vector bundles based on a connection on the vector bundle.

Using this one can prove the Gauss-Bonnet theorem and its generalization to $2n$ dimensional manifolds.

The main reference for this project is Appendix C in [MS74] Einstein notation is used.

1 Characteristic classes from curvature

1.1 Connections for a complex vector bundles

We first recall the definition of a connection for a general vector bundle. We will discuss complex connections on complex vector bundles, but the case of real vector bundles can be recovered by replacing $\mathbb{C}$ with $\mathbb{R}$.

Let M be a smooth manifold, and let $E \to M$ be a complex smooth vector bundle. Note that all definitions make sense for real vector bundles if we replace $\mathbb{C}$ with $\mathbb{R}$. Let $\tau^*_{\mathbb{C}} = \hom_{\mathbb{R}}(\tau, \mathbb{C})$, where τ is the tangent bundle on M.

Definition 1 (Connection on a complex vector bundle)**.** *A complex connection or covariant derivative on E is a $\mathbb{C}$-linear map*

$$\nabla : C^\infty(E) \to C^\infty(\tau^*_{\mathbb{C}} \otimes E),$$

which satisfy the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f\nabla(s)$$

for every $f \in C^\infty$ and $s \in C^\infty(E)$.

If E is trivial with a global frame (s_i), we can describe the connection by the value of $\nabla(s_i)$, since ∇ is $\mathbb{C}$-linear and we have a rule which gives the value for a section scaled by a function.

Lemma 2. *A connection on a trivial bundle is determined by the values $\nabla(s_i)$. The values $\nabla(s_i)$ can be compiled in a matrix of complex 1-forms $[\omega^j_{i,E}]$ so $\nabla(s_i) = \omega^j_{i,E} \otimes s_j$.*

we omit the vector bundle from the notation, when the given bundle is clear. Given a map $\varphi : N \to M$ and a vector bundle $E \to M$ with a connection ∇, there is a unique connection $f^*\nabla$ on the induced bundle φ^*E such that the following square commutes

$$\begin{array}{ccc} C^\infty(E) & \xrightarrow{\nabla} & C^\infty(\tau_{\mathbb{C}} \otimes \varphi^* E) \\ \downarrow & & \downarrow \\ C^\infty(\varphi^* E) & \xrightarrow{\nabla'} & C^\infty(\tau_{\mathbb{C}} \otimes \varphi^* E). \end{array}$$

1.2 Curvature for a vector bundle with connection

We can define another derivative on $\tau^*_{\mathbb{C}} \otimes E$, since we know how to derive each part individually.

Lemma 3. *There is a unique $\mathbb{C}$-linear mapping*

$$\hat{\nabla} : C^\infty(\tau^*_{\mathbb{C}} \otimes E) \to C^\infty(\Lambda^2 \, \tau^*_{\mathbb{C}} \otimes E),$$

Which satisfy the Leibniz formula

$$\hat{\nabla}(\theta \otimes s) = d\theta \otimes s - \theta \wedge \nabla(s)$$

for every $\theta \in \tau^*_{\mathbb{C}}$ and $s \in C^\infty(E)$.

The composition $K = \hat{\nabla} \circ \nabla$ is $C^\infty(M;\mathbb{C})$-linear, so the value $K(s)(x)$ for $s \in C^\infty(E)$ and $x \in M$ only depends on $s(x)$. As such we can describe K as a map of vector bundles $E \to \Lambda^2 \tau^*_{\mathbb{C}} \otimes E$. If we have a complex vector bundle with a global frame (s_i), we can get a coordinate expression of K:

$$\begin{aligned} K(s_i) = \hat{\nabla} \circ \nabla(s_i) = \hat{\nabla}(\omega_i^j \otimes s_j) &= \\ d\omega_i^j \otimes s_j - \omega_i^j \wedge \nabla(s_j) &= \\ d\omega_i^j \otimes s_j - \omega_i^j \wedge \omega_j^k \otimes s_k &= \\ (d\omega_i^j - \omega_i^k \wedge \omega_k^j) \otimes s_j&. \end{aligned}$$

We can then as we did for ∇ give a matrix of 2-forms $\Omega_E = [\Omega^j_{i,E}]$, which determine the curvature tensor

$$\Omega^j_{i,E} = d\omega_i^j - \omega_i^k \wedge \omega_k^j.$$

If we use matrix notation we get

$$\Omega_E = d\omega - \omega \wedge \omega.$$

As we did for ω, we will omit the vector bundle from the notation, when the bundle is clear. If we had used real coefficients instead of complex coefficients and the Levi-Civita connection on the tangent bundle, we would have gained the regular Riemannian curvature tensor, as a map $\tau \to \bigwedge^2 \tau \otimes \tau$.

1.3 Transformation law for curvature

Given a complex vector bundle along with a connection, we have shown how we locally obtain a matrix of complex 2-forms. We would like to combine these into a global closed $(2n)$-form, so it will give a element of complex de Rham cohomology. However the 2-forms $[\Omega_i^j]$ depends on a frame (s_i), so we have to find out how they transform under coordinate change, and which expressions of $[\Omega_i^j]$ are invariant under these changes, so we get a geometric object.

Let $(s_i), (s'_i)$ be two different frames with transition matrix $f = [f_i^j]$ such that $s_i = f_i^j s'_j$. Then we have

$$\Omega_i^j \otimes s_j = K(s_i) = K(f_i^k s'_k) = f_i^k K(s'_k) = f_i^k \Omega'^a_k \otimes s'_a = f_i^k \Omega'^a_k (f^{-1})^j_a \otimes s_j.$$

Writing in matrix notation we have $\Omega = f\Omega' f^{-1}$, so a geometric construction on Ω will have be invariant under conjugation of an invertible matrix. This leads us to invariant polynomials.

1.4 Invariant polynomials

Motivated by the previous section, we define invariant polynomials.

Definition 4 (Invariant Polynomial)**.** *An invariant polynomial as a function*

$$P : M_n(\mathbb{C}) \to \mathbb{C}$$

which can be expressed as a complex polynomial in the entries of the matrix and satisfies $P(XY) = P(YX)$ *or equivalently* $P(TXT^{-1}) = P(X)$ *for an invertible matrix* T*.*

We will classify these invariant polynomials using the theory of symmetric polynomials. Recall that eigenvalues of a endomorphism does not depend on a basis, so the set of eigenvalues is conjugation-invariant, and therefore any symmetric polynomial in the eigenvalues is an invariant polynomial.

Lemma 5. *Any invariant polynomial can be expressed as a polynomial function on* $\sigma_1, \cdots, \sigma_n$*, where* σ_i *is the i'th elementary symmetric function evaluated on the eigenvalues.*

Proof. Let P be an invariant polynomial. Given $A \in M_n(\mathbb{C})$, we can choose B so that BAB^{-1} is an upper triangular matrix. Furthermore if we conjugate by $\mathrm{diag}(1/\epsilon, 1/\epsilon, ..., 1/\epsilon^n)$, and using continuity of P, we can make the non-diagonal entries arbitrary small, without changing the diagonal elements.

It follows that $P(BAB^{-1})$ is a symmetric polynomial in the diagonal values, which is the eigenvalues of A. It follows from the fundamental theorem of symmetric polynomials that P is a polynomial in $\sigma_1, ..., \sigma_n$. $\square$

Recall that σ_1 and σ_n correspond to the sum and product of the eigenvalues respectively.

1.5 Characteristic classes from curvature

Assume we have a complex vector bundle $E \to M$ with a connection ∇. Since the even differential forms is an commutative $\mathbb{C}$-algebra, we can evaluate $P(\Omega)$ in a local frame, to get a sum of even forms. Since a coordinate change acts on Ω by conjugation, $P(\Omega)$ will not depend on coordinates, and will agree on intersections, and as such give a global sum of even complex forms $P(\Omega)$. Using the Bianchi identity we can prove that it is closed.

Theorem 6. $P(\Omega)$ *is closed.*

Proof. [MS74] $\square$

$P(\Omega)$ then gives an element in the de Rham cohomology ring $[P(\Omega)]$.

Theorem 7. $[P(\Omega)]$ *does not depend on the choice of connection.*

Proof Sketch. The main idea is that the connections on a vector bundle form a convex set, ie. if ∇_0, ∇_1 are connections, then $t\nabla_0 + (1-t)\nabla_1$ is a connection for every $t \in \mathbb{R}$. The proof then uses the homotopy invariance of De Rham cohomology. See [MS74, p. 298] for the full proof.

$\square$

This means that we can forget about the connection in the construction. In conclusion, given an invariant polynomial with n variables P, we have a homomorphism

$$\psi : \mathrm{Vect}^n_{\mathbb{C}}(M) \to H^*(M; \mathbb{C})$$

From the set of isomorphism classes of complex vector bundles of rank n to de Rham cohomology, given by sending a vector bundle $E \to M$ to $[P(\Omega)]$ for any connection on the vector bundle. We will now show that this map is well behaved with induced bundles.

Lemma 8. *Given a complex bundle $E \to M$, a map $\varphi : N \to M$ and an invariant polynomial P, we have $\varphi^* P(\Omega_E) = P(\Omega_{\varphi^* E})$ for the induced connection on $\varphi^* E$. In particular, $\varphi^*[P(\Omega_E)] = [P(\Omega_{\varphi^* E})]$.*

Proof. Given a local frame (s_i) for E, we have $(s_i \circ \varphi)$ is a frame for $\varphi^* E$. From the definition of the induced connection we have

$$\omega^j_{i,\varphi^* E} \otimes (s_j \circ \varphi) = \nabla_{\varphi^* E}(s_i \circ \varphi) = \nabla_E(s_i) \circ \varphi = (\omega^j_{i,E} \otimes s_j) \circ \varphi = \varphi^* \omega^j_{i,E} \otimes (s_j \circ \varphi)$$

so we have $\omega^j_{i,\varphi^* E} = \varphi^* \omega^j_{i,E}$. By the formula for Ω based on ω, we get that $\Omega^j_{i,\varphi^* E} = \varphi^* \Omega^j_{i,E}$. Since φ^* is a ring homomorphism we have

$$\varphi^* P(\Omega_E) = P(\varphi^* \Omega_E) = P(\Omega_{\varphi^* E}).$$

□

Since we have shown that ψ is compatible with induced bundles, it is a complex characteristic class. This is already a nice result, since it takes geometric data as the curvature of a vector bundle, and turns it into topological data.

There is also a strong theory of characteristic classes, which we will use to show results such as the Gauss Bonnet theorem. Note that this construction could have be done for real bundles and we would have gotten a map

$$\mathrm{Vect}^n_{\mathbb{R}}(M) \to H^*(M; \mathbb{R}).$$

2 Characteristic classes for Riemannian manifolds

2.1 Pfaffian

We will now return to the realm of Riemannian geometry. In the previous chapter, we did not have any preferences on our basis, so we only looked at polynomials, which were invariant for any basis. However if we instead have a Riemannian manifold, then we can restrict to only looking at orthonormal frames, and the linear endomorphisms taking orthonormal frames to orthonormal frames, which is $O(n)$. We will first analyze what changes in the linear algebra.

Let ∇ be a metric connection on a Riemannian manifold M.

Lemma 1. *Let $\theta_1, \ldots, \theta_n$ be a orthonormal frame for a neighborhood of the tangent bundle of a Riemannian manifold M and ∇ be a metric connection on the tangent bundle. Then the matrix $[\omega_{ij}]$ is skew symmetric.*

Proof. since the frame is orthonormal, $\langle \theta_i, \theta_j \rangle$ is a constant function. We then have

$$\begin{aligned} 0 = \nabla_X \langle \theta_i, \theta_j \rangle &= \langle \nabla_X \theta_i, \theta_j \rangle + \langle \theta_i, \nabla_X \theta_j \rangle \\ &= \langle \omega_i^k(X) \otimes \theta_k, \theta_j \rangle + \langle \omega_j^k(X) \otimes \theta_k, \theta_i \rangle \\ &= \omega_i^j(X) + \omega_j^i(X) \end{aligned}$$

Showing the matrix is skew-symmetric. ∎

Note that this implies that Ω is skew-symmetric, since

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = -d\omega_j^i - \omega_k^i \wedge \omega_j^k = -d\omega_j^i + \omega_j^k \wedge \omega_k^i = -\Omega_j^i$$

This means that if we want to build 2-forms out of Ω, we only have to look at skew symmetric matrices invariant under conjugation by orthogonal matrices. The Pfaffian gives a closed 2-form which can not be obtained from an invariant polynomial.

Lemma 2. *There exist a unique up to sign polynomial with integer coefficients, which assigns to each $2n \times 2n$ skew-symmetrix matrix A over a commutative ring, a ring element $Pf(A)$ whose square is the determinant of A. Furthermore*

$$Pf(BAB^t) = Pf(A)\det(B)$$

for any $2n \times 2n$ matrix B.

Given an oriented manifold of dimension $2n$, if we locally give an oriented orthonormal frame $(\theta_1, ..., \theta_n)$, we have a local 2-form $Pf(\Omega)$. Given another oriented orthonormal frame $(\tilde{\theta}_1, ..., \tilde{\theta}_n)$ we have $\tilde{\Omega} = B\Omega B^t$ for B a special orthogonal matrix. So

$$Pf(\tilde{\Omega}) = Pf(B\Omega B^t) = Pf(\Omega)\det(B) = Pf(\Omega).$$

As such $Pf(\Omega)$ does not depend on a specific orthonormal basis, so we can build it to a global form. Similar to the case of invariant polynomials this gives a closed form, which further gives an element of the de Rham cohomology ring.

2.2 Generalized Gauss-Bonnet theorem

We have now shown, that given a Riemannian manifold of dimension $2n$, we can get an element of the De Rham cohomology ring $Pf(\Omega)$, which square is the determinant of Ω.

Since the determinant is equal to the product of eigenvalues, it is the same as $\sigma_{2n}(\Omega)$, which is a characteristic class described earlier. Corollary C.10 says that $\sigma_{2n}(\Omega)$ is equal to $(2\pi)^{2n} p_n$, where p_n is the n'th Pontryagin class.

Corollary 15.8 says that p_n is the square of the Euler class in $H^*(M;\mathbb{R})$, so we have $Pf(\Omega) = \pm(2\pi)^n e$. It can be checked that the sign is $+$, so we get

$$Pf(\Omega) = (2\pi)^n e.$$

This is a very significant result, since it holds for any Riemannian manifold, but e does not depend on the Riemannian structure.

It is also significant, since the Euler class has the property, that when paired with the fundamental class, it gives the Euler characteristic [MS74, p. 130]. Since the pairing can be given by the integral, we have

$$\int_M Pf(\Omega) = \int_M (2\pi)^n e = (2\pi)^n e[M]$$

where $e[M]$ is the Euler characteristic for any Riemannian manifold M.

In the special case $n = 1$, we have

$$Pf\left(\begin{bmatrix} 0 & \Omega_{12} \\ -\Omega_{12} & 0 \end{bmatrix}\right) = \Omega_{12}$$

Note that Ω_{12} is exactly the Gaussian curvature $K d\theta_1 \wedge d\theta_2$ [Tu17, Theorem 12.3], so in this case we get

$$\int_M K dA = 2\pi e[M]$$

showing the classical Gauss-Bonnet theorem for oriented Riemannian manifolds of dimension 2.

3 References

[MS74] John Milnor and Jim Stasheff. *Characteristic Classes.* Annals of mathematics studies. Princeton University Press, 1974.

[Tu17] Loring W. Tu. *Differential Geometry.* 2017.