A Geometric proof of the Recognition Principle

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Contents

1 Introduction

In this note we will give a proof of the recognition principle for $\mathbb{E}_1$,$\mathbb{E}_n$ and $\mathbb{E}_\infty$-groups. The theorem tell us that we can recognize these groups as loop spaces of connected pointed spaces, and connected pointed spaces as classifying spaces of groups. We will give a construction of the classifying space, making it geometric properties explicit.

This note is written for the course "Topics in Algebraic Topology", and it is aimed towards (and written by) people new to ∞-categories, and as such there is an emphasis on how ∞-categorical concepts are being used, and getting used to the language of ∞-categories.

2 Algebra in ∞-categories

In this section we define monoids, groups, commutative groups and modules. For a more detailed construction see [Lur17] or [Wag20].

2.1 Monoids

We will first motivate their definitions by recalling the ordinary monoids in a 1-category $\mathcal{D}$ with finite products (including a terminal object $*$).

Recall that a monoid (M, μ, i) in $\mathcal{D}$, is a object M with a unit map $i : * \to M$ and $\mu : M \times M \to M$, which is unital and associative. If we see the hom-sets as discrete spaces, the condition of being associative corresponds to having an homotopy

$$\mu \circ (id_M \times \mu) \xrightarrow{\simeq} \mu \circ (\mu \times id_M)$$

in the discrete space $\hom(M \times M \times M, M)$, since in a discrete space the homotopy can only be the identity homotopy.

A monoid in a ∞-category $\mathcal{C}$ with finite products should be defined the same way. However for an ∞-category, the mapping spaces are now animas! In particular, there can be different homotopies $\mu \circ (id_M \times \mu) \to \mu \circ (\mu \times id_M)$. As such a monoid should come with a choice of a homotopy between these maps.

However when we replace identities with homotopies in our definition, we can no longer take coherence for granted. In particular, when we multiply 4 copies $M^4 \to M$, there are 5 ways of ordering the multiplication.

$$\begin{tikzcd}
(ab)(cd) \arrow[d, "\simeq"] & & ((ab)c)d \arrow[ll, "\simeq"'] \\
a(b(cd)) \arrow[r, "\simeq"] & a((bc)d) \arrow[r, "\simeq"] & (a(bc))d \arrow[u, "\simeq"']
\end{tikzcd}$$

If our choice of homotopies should be coherent, going around the loop should be homotopic to the identity homotopy. Our associative structure should therefore include homotopies realizing these coherence conditions.

Continuing like this for multiplication of more and more elements, we get our definition should involve a lot of choice of homotopies, ensuring that our associative structure is coherent. This can be recorded in the following definition.

Definition 2.1. *Let $\mathcal{C}$ be a ∞-category with finite products (including a final object $*$). A Cartesian monoid X in $\mathcal{C}$ is a simplicial object in $\mathcal{C}$ with*
1) $X_0 \simeq *$
2) X Satisfies the Segal condition, that is the maps $e_i : [1] \to [n]$ given by

$e_i(0) = i-1, e_i(1) = i$, *induce an equivalence*

$$X_n \to \prod_{i=1}^{n} X_1$$

A monoid X with $X_1 = M$ is also called a monoid structure on M.
The category $\mathrm{Mon}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ *is the full subcategory spanned by the monoids in $\mathcal{C}$.*

In the case $\mathcal{C} = \mathrm{Cat}_\infty$, we get a monoidal ∞-category. When we have introduced ∞-operads, we will later see that a Cartesian monoid in $\mathcal{C}$ is the same as an associative- or $\mathbb{E}_1$-monoid in $\mathcal{C}$ with the Cartesian symmetric monoidal structure.

This matches our earlier definition of a monoid. We think of X_1 as the underlying object of the monoid. The map $X_0 \to X_1$, induced by the map $[1] \to [0]$ defines the unit element. We obtain the multiplication of n copies of X_1 by

$$\prod_{i=1}^{n} X_1 \xleftarrow{\sim} X_n \to X_1,$$

where the left-hand map is the inverse coming from the Segal condition, and the right-hand map comes from the map $[1] \to [n]$, sending $0 \to 0$ and $1 \to n$. All of the required homotopies come from the maps in the simplicial structures. For instance, let us consider multiplication with the unit for a Cartesian monoid in **Set**

$$X_1 \times X_0 \xrightarrow{(\mathrm{id}, s_0)} X_1 \times X_1 \xrightarrow{\mu} X_1.$$

The first step of the multiplication sends the pair (f, u), where $u \in X_1$ is the image of the unit map, to the unique simplex

$$\begin{tikzcd} 0 \arrow[d, "f"] \arrow[dr] & \\ 1 \arrow[r, "s_0(u)"'] & 2. \end{tikzcd}$$

However this is exactly $s_1 f$, and applying d_1 to finish the multiplication, we get $d_1 s_1 f = f$, showing that u acts a unit. Groups are then defined as monoids with extra structure.

Definition 2.2. *A Cartesian monoid in $\mathcal{C}$ is a Cartesian group if the shear map*

$$(\mathrm{pr}_1, *) : X_1 \times X_1 \to X_1 \times X_1$$

given by $(a, b) \to (a, a * b)$ is an equivalence.
The category $\mathrm{Grp}(\mathcal{C}) \subseteq s\mathcal{C}$ *is the full subcategory spanned by the groups in $\mathcal{C}$.*

Remark 2.3. *Unlike associativity (and commutativity in the future), making a monoid into a group does not require extra structure of homotopies, but instead require a certain map to be a equivalence. Therefore being a group is a property of a monoid and not a structure.*

Remark 2.4. *A monoid M in* An *is a group if and only if $\pi_0(M)$ is a group in* ***Set****.*

Example 2.5 (Loop space)**.** *Let (X, x) be a pointed anima. The loop space $\Omega(X, x) \subseteq \mathrm{Map}(|\Delta^1|, X)$ is the subspace of the maps, which sends the vertices to the basepoint.*
The loop space has a simplicial structure by letting $\Omega(X, x)_n \subseteq \mathrm{Map}(|\Delta^n|, X)$ be the subspace of maps sending vertices to basepoint, and letting the simplicial structure coming from $\mathrm{Map}(|-|, X) \in \mathrm{sAn}$.

We have $\mathrm{Map}(|\Delta^0|, X) \simeq *$ and $\mathrm{Map}(|\Delta^n|, X) \simeq \mathrm{Map}(|\Delta^1|, X)^n$ since the maps are determined on the spines of the simplexes. We therefore get a monoid structure given by loop concatenation.

From AlgTop1 we recall that $\pi_0(\Omega(X, x))$ is a group, showing that $\Omega(X, x)$ is a $\mathbb{E}_1$-group.

2.2 Commutative monoids and groups

The ∞-categorical version of commutative monoids is $\mathbb{E}_\infty$-monoids, which we will define here. As in the case of associativity, we need coherent homotopies encoding the commutativity. This can be encoded in a similar way to $\mathbb{E}_1$-monoids, as Δ^{op} gives us associativity, but since the order cannot be reversed in maps in Δ^{op} it does not encode swapping elements in our multiplication. Therefore we need a bigger source category.

Definition 2.6. *Let Fin_* be the category of finite pointed sets and basepoint-preserving maps. $\langle n \rangle$ denotes the object $\{*, 1, ..., n\}$ with basepoint $*$.*

We have a functor/simplicial set $\mathrm{Cut} : \Delta^{op} \to \mathrm{Fin}_*$ which on objects is given by $[n] \to \langle n \rangle$, and on morphisms by sending the opposite α^{op} of a morphism $\alpha : [m] \to [n]$ to

$$\mathrm{Cut}(\alpha^{op})(i) = \begin{cases} * & i \leq \alpha(0) \\ j & \alpha(j-1) < i \leq \alpha(j) \\ * & \alpha(n) < i \end{cases}$$

This functor can also be described as $\Delta^1/\partial\Delta^1$ as there are $n+2$ maps $\Delta^n \to \Delta^1$, n surjective maps, and 2 maps which sends everything to one of the vertices, so they both is the morphism to the basepoint on the quotient $\Delta^1/\partial\Delta^1$. It can then be checked that $\Delta^1/\partial\Delta^1$ agrees with Cut agrees on morphisms.

Definition 2.7. *Let $\mathcal{C}$ be an ∞-category with finite products. A Cartesian commutative monoid X in $\mathcal{C}$ is a functor* $\mathrm{Fin}_* \to \mathcal{C}$*, such that* $X \circ \mathrm{Cut}$ *is a Cartesian monoid.*

A commutative monoid X in $\mathcal{C}$ is a commutative group in $\mathcal{C}$, if $X \circ \mathrm{Cut}$ *is a $\mathbb{E}_1$-group.*

We write $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C})$ for the full subcategories of $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ spanned by the commutative monoids and commutative groups respectively. A commutative monoid in Cat_∞ is a symmetric monoidal ∞-category.

Example 2.8 (Cartesian symmetric monoidal category)**.** *A symmetric monoidal structure on a ∞-category $\mathcal{C}$ is called Cartesian if the following conditions are satisfied:*

1. *The unit object $1_{\mathcal{C}}$ is final.*
2. *For every pair of objects $C, D \in \mathcal{C}$, the maps*

$$C \simeq C \otimes 1_{\mathcal{C}} \leftarrow C \otimes D \to 1_{\mathcal{C}} \otimes D \simeq D$$

exhibit $C \otimes D$ as a product in $\mathcal{C}$.

By [Lur17, Cor. 2.4.1.9] if an ∞-category $\mathcal{C}$ has finite products, then it has an unique Cartesian symmetric monoidal structure up to equivalence, denoted by $C^\times$. This is the monoidal structure on An *we will use in the rest of these notes.*

2.3 ∞-Operads and modules

With ∞-operads we can give a general framework for algebraic objects in a symmetric monoidal ∞-category. Recall that a map in Fin_* is inert, if it is a bijection when restricted to the points not mapping to the basepoint. The subcategory with only the inert maps is denoted by $(\mathrm{Fin}_*)_{\mathrm{int}}$.

Definition 2.9. *A (multicolored and symmetric) ∞-operad (in anima) is a functor $p : \mathcal{O} \to \mathrm{Fin}_*$ of ∞-categories satisfying the following conditions:*

1. *Every inert in* Fin_* *has cocartesian lifts with arbitrary sources.*
2. *The cocartesian unstraightening of $\mathcal{O}$:* $(\mathrm{Fin}_*)_{\mathrm{int}} \to \mathit{Cat}_\infty$ *satisfies the Segal condition.*
3. *Let $x, y \in \mathcal{O}$ with $p(x) = \langle m \rangle$ and $p(y) = \langle n \rangle$. By (b), we can write $y \simeq (y_1, ..., y_n)$ for some $y_1 \in \mathcal{O}_1$. Then we require that the diagram*

$$\begin{tikzcd}
\hom_{\mathcal{O}}(x,y) \arrow[r, "(\rho_1,\dots,\rho_n)"] \arrow[d, "p"'] & \prod_{i=1}^n \hom_{\mathrm{Fin}_*}(x, y_i) \arrow[d, "p"] \\
\prod_{i=1}^n \hom_{\mathrm{Fin}_*}(\langle m \rangle, \langle n \rangle) \arrow[r, "(\rho_1,\dots,\rho_n)"'] & \prod_{i=1}^n \hom_{\mathrm{Fin}_*}(\langle m \rangle, \langle 1 \rangle)
\end{tikzcd}$$

The ∞-category Op_∞ is the (non full) subcategory of $Cat_\infty/\mathrm{Fin}_*$ spanned by the ∞-operads and by the functors which preserve inert morphism.

The fiber of $\langle 1\rangle$ of an ∞-operad $\mathcal{O}$ denoted $\mathcal{O}_1$ is the underlying category of the ∞-operad. The objects of this category is then the colours of the ∞-operad.

More informally an ∞-operad gives an collection of objects given by $\mathcal{O}_1$, and spaces of operations between the different objects. A single-coloured ∞-operad is then an object A along with spaces of operations $A^{\otimes n} \to A$ and with composition of these operations given by maps.

In this way an ∞-operad determines a certain structure of operations. An algebra over an ∞-operad is then an collection of objects realizing these operations.

Definition 2.10. *Given an ∞-operad $\mathcal{O}$ and a symmetric monoidal ∞-structure $\mathcal{C}^\otimes$, we have*

$$\mathrm{Alg}_{\mathcal{O}}(C^\otimes) = \mathrm{Fun}^{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{C}^\otimes)$$

This is the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$.

Example 2.11. *The best way to understand ∞-operads and their algebras is to see some examples.*

- *The algebras over the ∞-operad id : $\mathrm{Fin}_* \to \mathrm{Fin}_*$ is the commutative (or $\mathbb{E}_\infty$) monoids. $\mathrm{CMon}(\mathcal{C}^\otimes) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^\otimes)$. When $\mathcal{C}$ has the Cartesian symmetric monoidal structure, this agrees with our previous definition.*
- *Similarly, the ∞-operad $\mathbb{A}\,\mathrm{ssoc}$ gives the associative (or $\mathbb{E}_1$) monoids $\mathrm{Mon}(\mathcal{C}^\otimes) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^\otimes)$*
- *An algebra over the multicolored $\mathbb{L}\,\mathrm{Mod}$ is a pair (A, M), where A is an associative monoid and M has an action of A which has coherent homotopies making the multiplications compatible in the way $(a_1a_2)m \simeq a_1(a_2m)$. The ∞-operad $\mathbb{R}\,\mathrm{Mod}$ is defined analogously with a right action. We denote their categories of algebras by $\mathrm{RMod}(\mathcal{C}^\otimes)$ and $\mathrm{LMod}(\mathcal{C}^\otimes)$.*

Example 2.12 (Modules in discrete categories)**.** *We can recall what these notions mean in some familiar discrete categories.*
In the symmetric monoidal category $\boldsymbol{Set}^\times$ a right module is an associative monoid A with an action on a set M or a A-set.

In the symmetric monoidal category $\boldsymbol{Ab}^\otimes$ a right module is a ring R and a R-module.

We will give some properties of $\mathrm{RMod}(\mathcal{C})$. Given a monoid A, we can consider it as a free module over itself, where the action is given by multiplication. We will denote the free module by A' for a monoid A.

We have a forgetful functor $\mathrm{RMod}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$ given by forgetting the module M of the pair (A, M) . From this we define $\mathrm{RMod}_A(\mathrm{An})$ by the Cartesian square

$$\begin{tikzcd}
\mathrm{RMod}_A(\mathrm{An}) \arrow[r] \arrow[d] & \mathrm{RMod}(\mathrm{An}) \arrow[d] \\
\{A\} \arrow[r] & \mathrm{Mon}(\mathrm{An})
\end{tikzcd}$$

Where the lower map is the inclusion of A in $\mathrm{Mon}(\mathrm{An})$. In other words it is the full subcategory of $\mathrm{RMod}(\mathrm{An})$ of modules over A.

3 The Recognition Principle for $\mathbb{E}_1$-Groups.

We are now ready to prove the recognition principle for $\mathbb{E}_1$-groups. Note that the loop space defined in 2.5 is a functor $\Omega : (*/\mathrm{An})_{\geq 1} \to \mathrm{Grp}(\mathrm{An})$.

Theorem 3.1. *There is a functor B, such that Ω and B are inverses*

$$\begin{tikzcd}
\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{An}) \arrow[r, bend left, "B"] & (\mathrm{An}_*)_{\geq 1}. \arrow[l, bend left, "\Omega"]
\end{tikzcd}$$

Remark 3.2. *Both of these ∞-categories have monoidal structure coming from the Cartesian structure on* An. *Ω preserves products since it is a right adjoint, and it turns out B also preserves products. Therefore when we have shown the ∞-categories are equivalent, it upgrades to a equivalence of monoidal categories.*

To prove this, we will construct the inverse B sending a group to its classifying space. In other proofs, B is constructed by the bar construction. We will give another construction of B, which reflects the geometric properties the classifying space should have, and which fits into the larger theory of Higher Algebra.

We have a functor

$$\Theta_* : \mathrm{Mon}(\mathrm{An}) \to (\mathrm{Cat}_\infty)_*$$

sending a monoid $A \in \mathrm{Mon}(\mathrm{An})$ to $\mathrm{RMod}_A(\mathrm{An})$ with basepoint the free module A'.

Lurie define this functor in more generality in Higher Algebra and gives the construction [Lur17, Section 4.8.3].

If we take the composition of Θ_* with the core functor, which takes the maximal anima of an ∞-category, and afterwards the truncation functor $\tau_{1\geq}$ taking a pointed anima to the path component containing the basepoint, we get a functor

$$B : \mathrm{Grp}(\mathrm{An}) \to (*/\mathrm{An})_{1\geq}$$

if we restrict it to groups. This can also been seen by the diagram

$$\begin{tikzcd}
\mathrm{Grp}(\mathrm{An}) \arrow[d, hook] \arrow[rrr, bend left, "B"] & & \mathrm{An}_* \arrow[r, "\tau_{1\geq}"] & (\mathrm{An}_*)_{\geq 1} \\
\mathrm{Mon}(\mathrm{An}) \arrow[r, "\Theta_*"] & \mathrm{Cat}_\infty(\mathrm{An})_* \arrow[ur, "core"] & &
\end{tikzcd}$$

Sending a $\mathbb{E}_1$-group G to the pointed connected anima $\tau_{1\geq}(\mathrm{RMod}_G^{\cong}, G')$. We will show it is an inverse to Ω.

3.1 Groups are loop spaces

We will give a natural isomorphism $\Omega B \cong id_{\mathrm{Grp(An)}}$, so we construct a natural isomorphism $\Omega(BG, G') \cong G$ for an $\mathbb{E}_1$-group.

First we will give an alternative description of Ω as the endomorphism space of the basepoint (this is not true for based infinity categories). This can be seen since they correspond to the same subsets of

$$\mathrm{Map}(|\Delta^1|, |X|) \simeq \mathrm{Map}(\Delta^1, \mathrm{Sing}(|X|)) \simeq \mathrm{Map}(\Delta^1, X)$$

The endomorphism space only depend on the path component of the basepoint so

$$\Omega\tau_{1\geq}(\mathrm{RMod}_G^{\cong}, G') \cong \Omega(\mathrm{RMod}_G^{\cong}, G')$$

If we can show that $\Omega(\mathrm{RMod}_G, G') \cong G$, then since G is a group, all of the endomorphisms are automorphisms, and so taking the core would not change the endomorphism space

$$\Omega(\mathrm{RMod}_G^{\cong}, G') \simeq \Omega(\mathrm{RMod}_G, G') \simeq G$$

We will therefore prove that $\mathrm{End}_G(G') \simeq G$.
Note that this looks a lot like statements we have in ordinary algebra, where the endomorphisms of a free R-module is R and the same for a free G-set for a group G.

First we note that the endomorphism object has a universal property:
For every $D \in \mathrm{An}$, tensoring with G' gives a functor $L(D) = D \otimes G'$. This functor lands in RMod_G since it has the action on the right

$$(D \otimes G') \otimes G \simeq D \otimes (G' \otimes G) \to D \otimes G'$$

There is a map $\mathrm{End}_G(G') \otimes G' \to G'$ such that the composition

$$\mathrm{Map}_{\mathrm{An}}(D, \mathrm{End}_G(G')) \to \mathrm{Map}_{\mathrm{RMod}_G}(D \otimes G', \mathrm{End}_G(G') \otimes G') \to \mathrm{Map}_{\mathrm{RMod}_G}(D \otimes G', G'))$$

is an equivalence.

We will show that G with the multiplication map $G' \otimes G \to G'$ defining the module structure has this universal property.

Using Corollary 4.2.4.8 in [Lur17], tensoring with G' is left adjoint to the forgetful functor $F : \mathrm{RMod}_G \to \mathrm{An}$ with the counit map $FM \otimes G' \to M$ given by the multiplication map.
From this we get that the universal property above matches the universal property of the adjoint.

$$Map_{\mathrm{An}}(D, G) \simeq Map_{\mathrm{An}}(D, FG') \simeq Map_{\mathrm{RMod}(G)}(D \otimes G', G')$$

Showing that it has the wanted universal property. This shows that $\mathrm{End}_G(G') \simeq G$, and that they act the same way on G'. However from Lemma 4.8.5.7 in [Lur17] this is enough to also conclude, that they are equivalent as algebra objects in Anima. In conclusion we get an equivalence $\Omega(\mathrm{RMod}_G, G') \simeq G$ as $\mathbb{E}_1$-groups, given the first implication.

3.2 Loop spaces are classifying spaces of Groups

We now have to show the isomorphism $B\Omega(X, x) \cong (X, x)$. First we will find another description of $\mathrm{RMod}_{\Omega X}(\mathrm{An})$ which is easier to relate to (X, x).
Indeed this ∞-category is equivalent to $(\mathrm{Fun}(X, \mathrm{An}), y(x))$, where $y : X \to \mathrm{Fun}(X, \mathrm{An})$ is the Yoneda Embedding.
An intuition for this comes from (Un-)Straightening. From it we get an equivalence

$$\mathrm{Fun}(X, \mathrm{An}) \simeq \mathrm{LFib}(X).$$

Given a fibration $F \to E \to X$, we have an action $\Omega(X, x) \circlearrowright F$, making F into an $\Omega(X, x)$-module and as such an element of $\mathrm{RMod}_{\Omega X}(\mathrm{An})$.

To prove it we will use proposition 4.8.5.8 in [Lur17], which gives a criteria for an ∞-category to be equivalent to a module category. The proposition given below in the special case of modules in An, which is the case we need.

Theorem 3.3. *Let $\mathcal{M}$ be a ∞-category left tensored over* An*, and $M \in \mathcal{M}$. Then there exist an $\mathbb{E}_1$-monoid A such that $\mathrm{RMod}_A \simeq \mathcal{M}$, by an equivalence carrying A' to M if the following criteria is satisfied:*

1. *$\mathcal{M}$ admits geometric realizations.*
2. *The action map $\mathrm{An} \otimes \mathcal{M} \to \mathcal{M}$ preserves geometric realizations.*
3. *The functor $F : \mathrm{An} \to \mathcal{M}$ given by $F(B) = B \otimes M$ admits a right adjoint G which preserves geometric realizations.*
4. *The functor G is conservative.*

5. *For every object $N \in \mathcal{M}$ and $B \in \mathrm{An}$ the adjoint of the map*

$$F(B \otimes G(N)) \simeq B \otimes G(N) \otimes M \simeq B \otimes FG(N) \to B \otimes N$$

is an equivalence.

Corollary 3.4. *There exists a based equivalence*

$$(\mathrm{RMod}_{\Omega(X,x)}(\mathrm{An}), \Omega(X,x)) \simeq (\mathrm{Fun}(X, \mathrm{An}), \gamma(x))$$

Proof. We will prove it from Theorem 3.3. First we recall $\mathrm{Fun}(X, \mathrm{An})$ has a symmetric monoidal structure given by the objectwise monoidal structure on An. From the inclusion $\mathrm{An} \hookrightarrow \mathrm{Fun}(X, \mathrm{An})$ we get that it is left tensored over An. We have $\mathrm{Fun}(X, \mathrm{An})$ has geometric realizations by the objectwise geometric realization in An.

For the action map $\mathrm{An} \otimes \mathrm{Fun}(X, \mathrm{An}) \to \mathrm{Fun}(X, A)$, and a simplicial object $S \in \mathcal{M}$ we have

$$(|A \otimes S|)(x') = \mathrm{colim}_{\Delta^{op}}(A \times S_n)(x') \simeq A \times \mathrm{colim}_{\Delta^{op}}(S_n)(x') \simeq (A \otimes |S|)(x')$$

since the colimit is sifted, so it commutes with finite products, Showing it preserves geometric realizations.
The functor $F(B) = B \otimes y(x)$ has a right adjoint $G : \mathcal{M} \to \mathrm{An}$ given by $G(T) = \mathrm{Map}_{\mathrm{Fun}(X,\mathrm{An})}(y(x), T)$. We have to show that G is conservative.
So assume $\mathrm{Map}(y(x), T) \simeq \mathrm{Map}(y(x), T')$. This implies $\mathrm{Map}(D, T) \simeq \mathrm{Map}(D, T')$, for all $D \in \mathrm{Fun}(X, \mathrm{An})$. Indeed from the Density Theorem 5.3 every functor $D \in \mathrm{Fun}(X, \mathrm{An})$ is a colimit of representable functors $y(x')$, and since X is a connected anima, every point is connected by a path to x, and as such $y(x) \simeq y(x')$. From this we get

$$\mathrm{Map}(D, T) \simeq \lim \mathrm{Map}(y(x), T) \simeq \lim \mathrm{Map}(y(x), T') \simeq \mathrm{Map}(D, T')$$

For every $D \in \mathrm{Fun}(X, A)$, so by the Yoneda lemma, T and T' are equivalent.
Lastly we show that the map in the last criteria is adjoint to an equivalence. This map is

$$B \times \mathrm{Map}(y(x), N) \to \mathrm{Map}(y(x), B \otimes N)$$

However from the Yoneda lemma again we get that this map corresponds to

$$B \times N(x) \to (B \times N)(x)$$

which is an equivalence.

□

Summarising the situation, we have the diagram

$$\begin{tikzcd}
(Fun(X, \mathrm{An}), y(x)) \arrow[r, "\simeq"] & (\mathrm{RMod}_{\Omega X}, \Omega X') \\
(X, x) \arrow[u, "y"] & (B\Omega X, \Omega X'). \arrow[u, hook]
\end{tikzcd}$$

We want an equivalence in the bottom row. Now since X is connected, the image is also connected, so the Yoneda embedding with the equivalence, is contained in the path component of the basepoint of the core, and as such the image lands in $B\Omega X$.
Since both the Yoneda embedding and the equivalence is fully faithful, we get that the restriction is a fully faithful functor $(X, x) \to (B\Omega X, \Omega X')$. The last thing we need is the functor to be essentially surjective. However since every object in the target is connected by isomorphisms, any functor from a non-empty category is essentially surjective.

We now have the wanted equivalence between (X, x) and $(B\Omega X, \Omega X')$, and so B and Ω are inverse functors.

4 Spectra and their homotopy groups

We have now shown the Recognition Principle for $\mathbb{E}_1$-groups, and we have corresponding statements for $\mathbb{E}_n$-groups and $\mathbb{E}_\infty$-groups. However we still need to describe what $\mathbb{E}_n$-monoids are, which are more commutative than $\mathbb{E}_1$-monoids, and less than $\mathbb{E}_\infty$-monoids. To understand this we will first recall the situation in 1-categories.

4.1 Reminder of Eckman-Hilton Argument

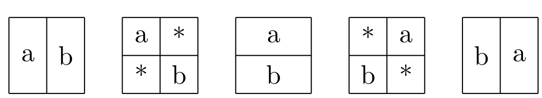
We will first give a reminder of the Eckman-Hilton Argument. In an 1-category, if we have an object with two compatible monoid structures, in the sense that the one multiplication is a homomorphism of the other multiplication structure

$$X \times X \to X,$$

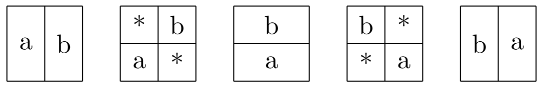
which written with elements means

$$(a \times b) \cdot (c \times d) = (a \cdot c) \times (b \cdot c).$$

In this case the two multiplications agree, and they are commutative.
We can draw a visual proof inspired by the case $\pi_2(X) = \pi(\Omega^2 X)$ where we have two operations, horizontal composition and vertical composition.



This shows that up to homotopy the operations agree, and is commutative. However if we move to ∞-categories, where we remember our choice of homotopy, we start to see some problems arise. If we take the other homotopy



There is not a 2-homotopy between these two 1-homotopies for a general X. Therefore the space of multiplications of two elements in $\Omega^2 X$ is connected, but not simply connected.
If we instead take $\Omega^3 X$, we have another dimension to move around in, and the space of composition will then be simply connected, but in general have nontrivial π_2-groups.
In conclusion we see that $\Omega^k X$ is more commutative than an $\mathbb{E}_1$-monoid, but less commutative than an $\mathbb{E}_\infty$-monoid, where the space of multiplications is contractible, which leads us to the definition of $\mathbb{E}_n$-monoids.

4.2 $\mathbb{E}_n$-monoids

From the above considerations, $\mathbb{E}_n$-monoids should have a multiplication structure corresponding to the n'th iterated loop space. This leads us to the definition of the n'th little cubes ∞-operad $\mathcal{E}_n$. Informally this operad, describes an operation, where the space of multiplications of k elements is given by the embedding of k n-cubes inside an n-cube.

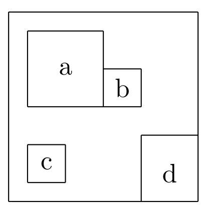


Figure 1: A multiplication of 4 elements a,b,c,d for $n = 2$.

The composition of two multiplications is given by embedding the cubes of the first operation into the other

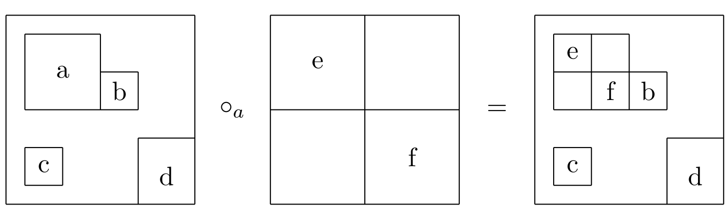


Figure 2: The composition of two multiplications

Note that this is modelled exactly as how multiplication works in $\Omega^n X$, and as such $\Omega^n X$, is the obvious example of an $\mathbb{E}_n$-monoid[1].

We give some properties of $\mathbb{E}_n$-monoids. Firstly we will look closer at the space of multiplications. First note that the space of embedding of k cubes in $[0,1]^n$ is homotopic to the space of the embedding of k points in $[0,1]^n$, by shrinking each cube to their center. This is the configuration space $Con_k([0,1]^n)$.

In the case $k=2$, we have that the configuration space is homotopy equivalent to S^{n-1}. The map to S^{n-1}, can be seen by considering the half-line between the two points and is intersection with the circle around the cube, as illustrated below in the case $k=2$.

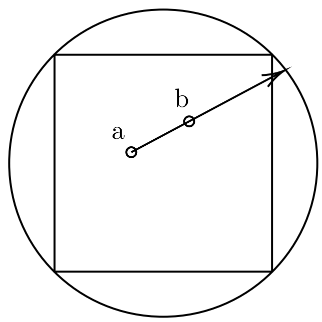


Figure 3: The map giving the homotopy equivalence between $Con_2([0,1]^2)$ and S^1.

From this we see that the space of multiplication of two elements for an $\mathbb{E}_n$-monoid is $(n-2)$-connected.

We now have two definitions of $\mathbb{E}_1$-monoids, and we will show they agree. In our first description, we have homotopies giving us associativity, but no homotopies for commutativity, and as such the space of possible multiplications of k elements, consists of a contractible component for each permutation of k elements.
However the space $Con_k([0,1])$ has a component for each permutation $\sigma \in S_k$ given by $\{f : \{k\} \to [0,1] | f(x_{\sigma(1)}) < f(x_{\sigma(2)}) < ... < f(x_{\sigma(k)}))\}$, since we cannot change the order of the points by a homotopy without moving them over each other. Furthermore each of these components are contractible, so the space of multiplications for both structures are homotopy equivalent, showing they agree.

[1] The Recognition Principle for $\mathbb{E}_n$-groups will exactly show us that every such group arises as a n'th iterated loop space. On a historical note, operads were invented to exactly describe the multiplication structure of the iterated loop space, so this is the prototypical example of why we need (∞-)operads!

Note that for an embedding f of k n-cubes in an n-cube, we can get an embedding $f \times id_{[0,1]}$ of k $(n+1)$-cubes in a $(n+1)$-cube. This gives a map from the operad $\mathbb{E}_n$ to $\mathbb{E}_{n+1}$, and as such a functor

$$\mathrm{Mon}_{\mathbb{E}_{n+1}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Op}_\infty}(\mathbb{E}_{n+1}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{Op}_\infty}(\mathbb{E}_n, \mathcal{C}) \simeq \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C})$$

Forgetting some of the commutative structure.

We can take the limit over these functors $\lim_n \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C})$. In other words, it is the category of objects with a $\mathbb{E}_n$-monoid structures for each n, which agree.

Theorem 4.1. *Given a symmetric monoidal ∞-category $\mathcal{C}$, there is an equivalence*

$$\lim_n \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$$

We will not prove the theorem formally, but we can see the space of multiplication of two elements are the same. For $\mathbb{E}_\infty$-monoids the space of multiplications is contractible. $\lim_n \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C})$ is given by the ∞-operad $\mathrm{colim}\, \mathbb{E}_n$, and as such the space of multiplications is $\mathrm{colim}\, S^n = S^\infty$, which is a contractible space, showing they agree.

Lastly, based on our earlier discussion, there is a corresponding statement of the Eckmann-Hilton argument, but as we saw earlier two monoid structures is not enough to buy full commutativity. Instead we have the statement of Dunn Additivity (Theorem 1.2.2 Derived Algebraic Geometry VI: E[k]-Algebras, Lurie)

Theorem 4.2. *For a symmetric monoidal ∞-category and $m, n \in \mathbb{N} \cup \{\infty\}$, there is an equivalence*

$$\mathrm{Mon}_{\mathbb{E}_m}(\mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C})) \simeq \mathrm{Mon}_{\mathbb{E}_{m+n}}(\mathcal{C})$$

Remark 4.3. *There is a symmetric monoidal structure on $\mathrm{Mon}_{\mathbb{E}_n}(\mathcal{C})$, so therefore we can define $\mathbb{E}_m$-monoids for this category. Also note that if either of the operations are $\mathbb{E}_\infty$, the resulting operation is also $\mathbb{E}_\infty$.*

We can recover the Eckmann-Hilton argument from Dunn Additivity. Viewing a monoid in set as a $\mathbb{E}_1$-monoid, if it can be extended to a $\mathbb{E}_2$-structure, there is a path $\hom(a * b, b * a)$. However since the space is discrete, they must agree on the nose showing that it is abelian.

4.3 Proof of Recognition Principle for $\mathbb{E}_n$-Groups

We now have the tools to prove the Recognition Principle for $\mathbb{E}_n$-groups

Theorem 4.4. *The functors*

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An}) \arrow[r, bend left, "B^n"] & (\mathrm{An}_*)_{\geq n} \arrow[l, bend left, "\Omega^n"] \end{tikzcd}$$

are inverses and monoidal.

Proof. Given we already have the proof in the $\mathbb{E}_1$ case, the n'th case follow by induction. So assume that statement is true for n. We then have a diagram

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An}) \arrow[r, bend left, "B^n"] & (\mathrm{An}_*)_{\geq n} \arrow[l, bend left, "\Omega^n"] \end{tikzcd}$$

Since the functors are monoidal we get a further equivalence

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An})) \arrow[r, bend left, "B^n"] & \mathrm{Mon}_{\mathbb{E}_1}((\mathrm{An}_*)_{\geq n}) \arrow[l, bend left, "\Omega^n"] \end{tikzcd}$$

From Dunn Additivity, the left side is equivalent to $\mathrm{Mon}_{\mathbb{E}_{n+1}}(\mathrm{An})$. The right side is equivalent to $(\mathrm{An}_*)_{n+1\geq}$, since the functors B, Ω restricts to an equivalence between $\mathrm{Mon}_{\mathbb{E}_1}((\mathrm{An})_{\geq n})$ and $(\mathrm{An}_*)_{\geq n+1}$ since B and Ω shifts homotopy groups by 1. We then have diagram of monoidal equivalences

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_{n+1}}(\mathrm{An}) \arrow[d, "\simeq"'] & (\mathrm{An}_*)_{\geq n+1} \arrow[d, bend left, "\Omega"] \\ \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An})) \arrow[r, bend left, "B^n"] & \mathrm{Mon}_{\mathbb{E}_1}((\mathrm{An}_*)_{\geq n}) \arrow[l, bend left, "\Omega^n"] \arrow[u, bend left, "B"] \end{tikzcd}$$

Taking the composition of these we get

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_{n+1}}(\mathrm{An}) \arrow[d, "\simeq"'] \arrow[r, bend left, "B^{n+1}"] & (\mathrm{An}_*)_{\geq n+1} \arrow[d, bend left, "\Omega"] \arrow[l, bend left, "\Omega^{n+1}"] \\ \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An})) \arrow[r, bend left, "B^n"] & \mathrm{Mon}_{\mathbb{E}_1}((\mathrm{An}_*)_{\geq n}) \arrow[l, bend left, "\Omega^n"] \arrow[u, bend left, "B"] \end{tikzcd}$$

showing the induction step. $\square$

4.4 Proof of the Recognition Principle for $\mathbb{E}_\infty$-groups

The last version of the Recognition principle is for commutative groups.

Theorem 4.5. *The functors*

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}) \arrow[r, bend left, "B^\infty"] & \mathrm{Sp}_{\geq 0} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

are inverses and monoidal where

$$\mathrm{Sp} = \lim(\cdots \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*)$$

and $\mathrm{Sp}_{\geq 0}$ *is the sub-∞-category consisting of spectra with no negative homotopy groups ie. the first anima is 0-connected, and each delooping is one level more connected.*

Proof. It turns out that we almost get the proof for free from the n'th case. By constructing the diagrams from that proof, but forgetting the monoid structure in the lower row we get a tower of compatible equivalences

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_3}(\mathrm{An}) \arrow[r, bend left, "B^3"] \arrow[d, "\simeq"'] & (\mathrm{An}_*)_{\geq 3} \arrow[l, bend left, "\Omega^3"] \arrow[d, bend left, "\Omega"] \\ \mathrm{Mon}_{\mathbb{E}_2}(\mathrm{An}) \arrow[r, bend left, "B^2"] \arrow[d, "\simeq"'] & (\mathrm{An}_*)_{\geq 2} \arrow[l, bend left, "\Omega^2"] \arrow[u, bend left, "B"] \arrow[d, bend left, "\Omega"] \\ \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{An}) \arrow[r, bend left, "B"] & (\mathrm{An}_*)_{\geq 1}. \arrow[l, bend left, "\Omega"] \arrow[u, bend left, "B"] \end{tikzcd}$$

which continues upwards. Since all the morphisms are compatible, we get an equivalence on the limits between $\lim_n \mathrm{Mon}_{\mathbb{E}_n}(\mathrm{An}) \simeq \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An})$ and $\mathrm{Sp}_{\geq 0}$. □

Example 4.6 (Eilenberg Maclane Space)**.** *Given an abelian group A, $B^\infty A$ is the Eilenberg-Maclane spectra*

$$K(A) = (K(A,0), K(A,1), K(A,2), ...).$$

Note that the base space $\Omega^\infty K(A) \cong K(A,0)$ is just the abelian group A.

5 Appendix: ∞-Categories and Anima

We will here define some of the general notions in ∞-categories we will use.

Definition 5.1 (Anima)**.** *An anima is an ∞-groupoid, and the ∞-category* An *is the full subcategory of* Cat_∞ *consisting of ∞-groupoids.*

Anima can also be thought of as the homotopy types of spaces/CW complexes, and as such we will use both words anima or space to denote the objects of An, depending on how we want to view it. In ∞-categories, An has a similar role to what **Set** is for 1-categories. The ∞-categorical version of the Yoneda Lemma is an example of this.

Theorem 5.2 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be an ∞-category. Given a functor $F : \mathcal{C} \to \mathrm{An}$ and an object $x \in \mathcal{C}$, the evaluation map*

$$ev_{id_x} : \mathrm{Nat}(\hom_{\mathcal{C}}(x, -), F) \to F(x)$$

is an equivalence. Furthermore we have fully faithful functors given by the Yoneda Embedding

Proof. [Lur09, Proposition 5.1.3.1] □

Theorem 5.3 (Density Theorem)**.** *Let S be a small simplicial set and let $\mathcal{C}$ be an ∞-category which admits small colimits. Composition with the Yoneda embedding y, induce a equivalence*

$$\mathrm{Fun}^L(\mathcal{P}(S), \mathcal{C}) \simeq \mathrm{Fun}(S, \mathcal{C})$$

where Fun^L *is the ∞-category of colimit preserving functors, and $\mathcal{P}(S) = Fun(S^{op}, \mathrm{An})$ is the ∞-category of presheaves. The inverse is given by left Kan extension.*

Proof. [Lur09, Theorem 5.1.5.6] □

Applying this to $id : \mathcal{P}(S) \to \mathcal{P}(S)$, we get that $id_{\mathcal{P}(S)}$ is isomorphic to the left Kan extension of the Yoneda embedding. Since Kan Extension is pointwise given by colimits, we then get that $\mathcal{P}$ is generated by colimits of the representable functors.

Theorem 5.4. *There is an equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{An}) \to \mathrm{LFib}(\mathcal{C})$$

Proof. [Lur09, Section 3.2] □

References

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