Generation of Random Numbers

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Outline:

« Generation of random numbers is too important to be left by chance » :
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- Pseudo Random Numbers
- Quasi Random Numbers
- True Random numbers

Generation of non-uniform variates

Introduction
Random Number Generation Basics

- ✓ Generate a sequence of numbers uniformly distributed between 0 and 1
- ✓ This property helps : Transforming this uniform distribution to… any kind of distributions!

What is an random number ?

- Is 7 a random number ?
  - 7 is a prime number, but there is no such thing as single random number
- A random number comes with its friends…
  - A set of numbers that have “no statistical relation” with the other numbers in the sequence
- In a Uniform distribution of random numbers the range retained is [0,1]. Within this range, every number has the same chance of turning up.
  - 0.0000001 is just as likely as 0.5

Random number vs Pseudo random numbers

- **True Random Numbers** (TRN) have no defined sequence or formulation. Thus, for any *n* random numbers, each appears with equal probability. They come from : Radioactive decay, Thermal noise, Cosmic ray arrival… (They could be interesting for cryptography…)
- If we restrict ourselves to computer algorithms generating numbers that (tries) to have no statistical correlations, we call them **Pseudo-Random Numbers.**
- Pseudo-random numbers have an advantage for numerical experiments, they are **reproducible.**
- Not being able to reproduce experiments is what differs a Science from Pseudo-science (and a scientist from a pseudo-scientist).

Standard system libraries – not for Science

- **Pseudo-Random Numbers** produced by a basic pseudo-random generator (PRNG) are interesting for the operating system, for hardware purposes, for game development, for prototyping, but not for scientific experiments.
- Standard C Library :
  - See "man rand" on your Unix environment, Windows or Mac documentation
  - Rather poor pseudo-random number generator with small period
  - Often results in 16-bit integers from 0 up to RAND_MAX (32767)
  - Some have better performance with periods up to 2^{32}, but they are still very week. (only 4 billions possibilities).

Initialization of a PRNG

- Pseudo-Random Numbers Generator algorithms have some state that can be initialized.
- For basic algorithms, the sate is only the last generated number. With old generators the state is often called a seed, and the process of initializing is called seeding.
- For fine generators, the state can be complex and up to a few kilobytes (6KB for the Mersenne Twister for instance) and we speak about "PRNG status initialization".
- We can set this state using the initialization methods given by the generator API like srand(), srand48()…
  - But why would you want to do this?

Initialization of a PRNG (continued)

There are 2 main reasons to setup PRNG statuses

- **For Science and for debugging**,
  - You need a deterministic an repeatable process (How do you debug if you program is driven by TRNs – at each run you have a different program behavior…)
- **To run different independent experiments :**
  - Since we need a deterministic PRNG, the default (same) initialization state will always generates the same sequence of random numbers and the same program behavior (euh… not really random isn't it ?).
  - Common solutions:
    - **Run loops of experiments without re-initializing the generator between two experiments (need a period long enough).**
    - **Call the initialization method for each independent experiment with 'independent' statuses (complex to determine – used for parallel computing with care).**

Ex: Generating
the same image at each run

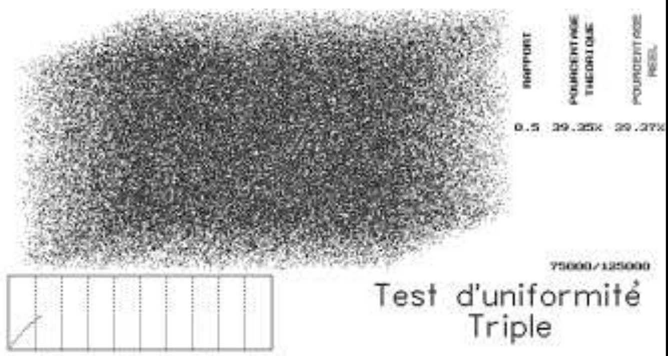
Part I

Deterministic Generation

(1) Pseudo Random Numbers
(2) Quasi Random Numbers

We need a uniform reproducible Pseudo-Random Number Generator (PRNG)

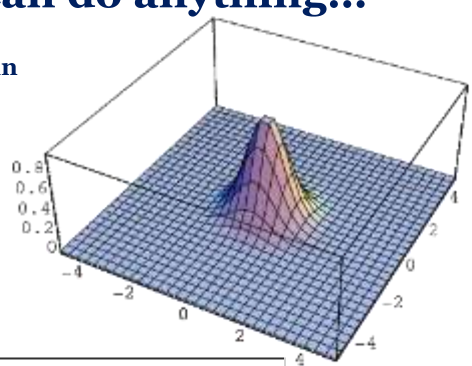
- How do we test for randomness ?
  - Statistical tests
  - Empirical tests
- How do we test for Uniformity ? Spectral tests…

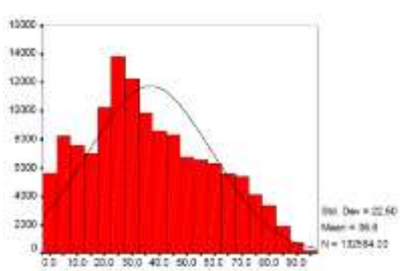


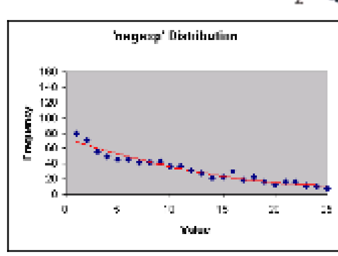
With uniformity we can do anything…

***« God does not play dice…»* A. Einstein**

1. Anamorphosis (inverting a distribution law) x = F⁻¹ (RandomNumber)
2. Rejection Method
3. Reproducing Histograms







Linear Congruential Generators (LCGs)

- Based on a linear recurrence formula
- For instance: $x_n = (5x_{n-1} + 1)\bmod 16$
- With $x_0 = 5$ we get:

$$x_1 = (5.(5) + 1)\bmod 16 = 26 \bmod 16 = 10$$

- The 32 first pseudo-random numbers generated are:

10, 3, 0, 1, 6, 15, 12, 13, 2, 11, 8, 9, 14, 7, 4, 5, 10, 3, 0, 1, 6, 15, 12, 13, 2, 11, 8, 9, 14, 7, 4, 5.

How do we get real numbers between 0 and 1 ?

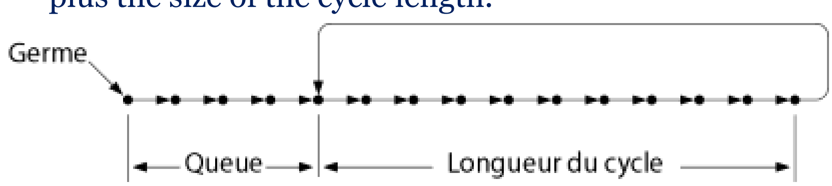
- We divide the obtained number by the maximum value.
- In this toy case, 16 is the modulus (maximum) we get numbers between [0..1[
- For the LCG presented we get: 0.625, 0.1875, 0, 0.0625, 0.375, 0.9375, 0.75, 0.8125, 0.125, 0.6875, 0.5, 0.5625, 0.875, 0.4375, 0.25, 0.3125, 0.625, 0.1875, 0, 0.0625, 0.375, 0.9375, 0.75, 0.8125, 0.125, 0.6875, 0.5, 0.5625, 0.875, 0.4375, 0.25, 0.3125

LCGs main characteristics

- When the generator is known, it is possible to reproduce sequences from the *xo* initial value.
- We deterministically produce a sequence that mimicks randomness
- Reproducibility is the essence of Science
- This value is used to initialise a generator to a different state, thus providing a new sequence. In this simple case of generator the value is called a **seed.**
- In the basic exemple, we have a cyclic repetition of the 16 first numbers. The length of the cycle (also called period) is 16

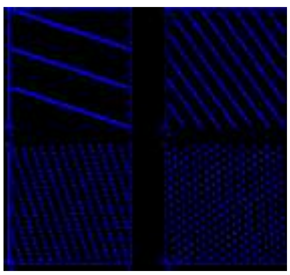
Some particularities…

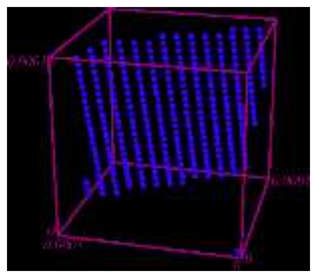
- Depending on the initial state, the cycle can be found after an initial queue or warming period.
- In this case the maximum number of different pseudorandom values is equal to the size of the queue plus the size of the cycle length.

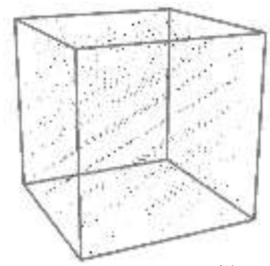


Known problems with LCGs

- **They have to be avoided for scientific applications !**
- Their mathematical stucture presents weaknesses that prevent them to be successful for spectral tests (possible bad uniformity in more than 1 dimension)
- LCGs are fast and can be used for game and other software applications







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Tausworthe Generators

- Proposed by Tausworthe in 1965
- Application in cryptography
- Generation of long random streams
- Random sequences of binary numbers that can be divided in substrings of a given size

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Tausworthe Generators

- The recurrence formula is given by:

$$b_n = c_{q-1}b_{n-1} \oplus c_{q-2}b_{n-2} \oplus c_{q-3}b_{n-3} \oplus \ldots \oplus c_0 b_{n-q}$$

Where c_i and b_i are binary variables

- When this kind of generator uses the last q bits of a sequence. It is named an AutoRegressive sequence of order q or AR(q).
- An AR(q) generator can have a maximum period of $2^{q-1.}$

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Tausworthe Generators

- Supposing that we have a « Delay » operator D such as $Db_n=b_{n+1}$ then:

$$D^q b(i-q) = c_{q-1}D^{q-1}b(i-q) + c_{q-2}D^{q-2}b(i-q) + \ldots + c_0 b(i-q) \bmod 2$$

where

$$D^q - c_{q-1}D^{q-1} - c_{q-2}D_{q-2} - \ldots - c_0 = 0 \bmod 2$$

or

$$D^q + c_{q-1}D^{q-1} + c_{q-2}D_{q-2} + \ldots + c_0 = 0 \bmod 2$$

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Tausworthe Generators

- Such an operator is a polynom named **caracteristic polynom**, and this becomes more readable by replace D by x :

$$x^q + c_{q-1}x^{q-1} + c_{q-2}x^{q-2} + \ldots + c_0$$

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Tausworthe Generators

- The generator period depends on this characteristic polynomial
- For a polynom of order q its maxium period is equal to 2^{q-1}.
- The polynom giving the maximum period is the named a **primitive polynomial**.

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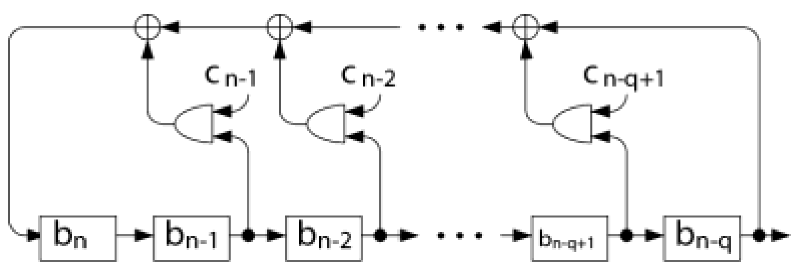
Tausworthe Generators

- Tausworthe sequences can be easily generated with shift registers.



General polynom of order q

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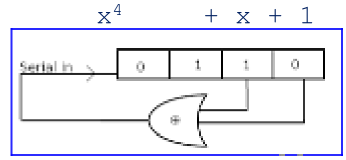
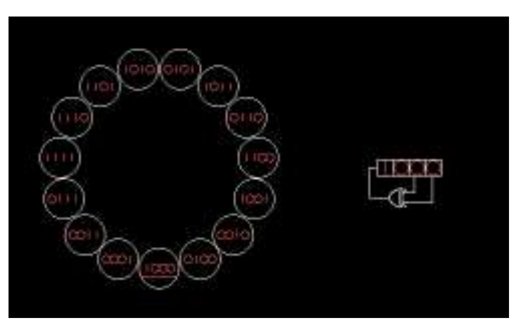
This leads to what we call: LFSR for Linear Feed Back Shift Register generators

$x^4 + x + 1$



This (wikipaedia) animated GIF presents the functioning of a 4-bits Linear Feed Back Shift Register (Fibonacci like) with its complete state diagram.

$x^4 + x + 1$



A simple XOR gate is used for the feedback and the bit is reinjected on the left after a rigtht shift of the register.

The maximum sequence is obtained with all the possible values except 0 (2^4-1 states)
(See Wikipedia for more details of this short tutorial)

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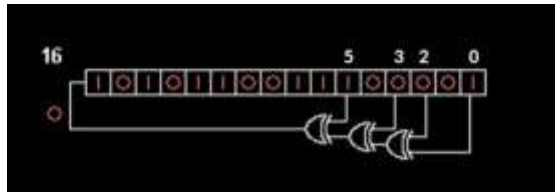
Principle of characteristic polynomials in LFSR

- Example of LFSR of Fibonacci kind with 3 lags on 16 bits
- The 4 feedback tap numbers in white correspond to a **primitive polynomial** selected to maximize the number of states: 65535 states (excluding the all-zeroes state.



$$x^{16} + x^5 + x^3 + x^2 + 1$$

- The hexadecimal state "ACE1 hex" shown on this image will be followed by "5670 hex"

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Tausworthe Generators

- Given an AR(q) sequence, Tausworthe proposes the construction of x_n numbers of l bits
- The b_n bit sequence is split in successive groups of 's' bits
- The first l bits of each group are given by:

$$x_n = 0.b_{sn} b_{sn+1} b_{sn+2} b_{sn+3} \dots b_{sn+l-1}$$

or

$$x_n = \sum_{j=1}^{l} 2^{-j} b_{sn+j-1}$$

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Tausworthe Generators : properties

- *s*, is a constant s ≥ *l*
  - This ensures that the generated numbers do not have overlapping bits.
- *s* is prime to 2^q-1
  - This ensures that the numbers of *l* bits are drawn within an integer period.
- The *l* bits numbers generated by the preceeding equations have the following properties:
  - The average of the sequence of numbers is: ½
  - The variance is of: 1/12
  - The coorrelation of the whole serie is: 0

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Tausworthe Generators : an example

- Considering the following primitive polynomial:

$$x^7+x^3+1$$

- Then we have:

$$b_{n+7} \oplus b_{n+3} \oplus b_n = 0, \mathrm{n} = 0,1,2,\ldots$$

where

$$b_{n+7} = b_{n+3} \oplus b_n, \mathrm{n} = 0,1,2,\ldots$$

substituting n *by* n − 7,

$$b_n = b_{n-4} \oplus b_{n-7}, \mathrm{n} = 7,8,9\ldots$$

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Tausworthe Generators : an example

- With $b_0=b_1=\ldots=b_6=1$, we obtain the following sequence of bits:

$$
\begin{aligned}
b_7 &= b_3 \oplus b_0 = 1 \oplus 1 = 0 \\
b_8 &= b_4 \oplus b_1 = 1 \oplus 1 = 0 \\
b_9 &= b_5 \oplus b_2 = 1 \oplus 1 = 0 \\
b_{10} &= b_6 \oplus b_3 = 1 \oplus 1 = 0 \\
b_{11} &= b_7 \oplus b_4 = 0 \oplus 1 = 1 \\
&\ldots
\end{aligned}
$$

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Tausworthe Generators : an example

- The complete sequence is:

  1111111 0000111 0111100 1011001 0010000

  0010001 0011000 1011101 0110110 0000110

  0110101 0011100 1111011 0100001 0101011

  1110100 1010001 0111111 1000011 1000000

- The first 7 bits constitute the seed of this generator.
- This sequence is repeating itself every 127 bits.
- $2^7-1=127$, the x^7+x^3+1 is a **prime polynomial**.

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