Discrete Structures 2 [1]

A Study Guide for Students of Sorsogon State University - Bulan Campus[2]

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Sorsogon State University - Bulan Campus

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Preface

> *"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."*
>
> – John von Neumann

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Boolean Algebra

1.1 Introduction

Circuits in computers are made up of millions of tiny switches that can be in one of two states: on or off. These switches are controlled by electrical signals that represent logical values. The behavior of these switches can be described using a mathematical system called Boolean Algebra. **Boolean Algebra** is a branch of mathematics that deals with logical values and operations on these values. It is widely used in computer science and engineering to design and analyze digital circuits. Computers uses the binary number system, which has only two digits: 0 and 1 which means "low voltage" and "high volt" respectively. These digits correspond to the logical values FALSE and TRUE, respectively.

1.2 History of Boolean Algebra

George Boole was an English mathematician and logician who lived in the 19th century. He was born in 1815 and died in 1864. Boole is best known for his work in the field of logic, which laid the foundation for modern computer science.

Boole's most famous work is his book *The Laws of Thought*, which was published in 1854. In this book, Boole introduced the concept of Boolean Algebra, which is a mathematical system for dealing with logical values. Boolean Algebra is based on the idea that logical values can be represented as either TRUE or FALSE.

In 1938, **Claude Shannon** showed that the two-valued Boolean Algebra could be used to describe the operation of electrical switches. This discovery laid the foundation for the design of digital circuits and computers.

1.3 Fundamental Operations

The three fundamental operations of Boolean Algebra are:

- **AND** - The AND operation takes two or more inputs and produces a 1 output only if all inputs are 1.
- **OR** - The OR operation takes two or more inputs and produces a 1 output if at least one input is 1.
- **NOT** - The NOT operation takes a single input and produces the opposite value. If the input is 1, the output is 0, and vice versa.

	Formal Logic	Set Theory	Boolean Algebra
Variables	p, q, r, ...	A, B, C, ...	x, y, z, ...
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $-$	$\bullet$, +, '
Special Elements	F, T	$\emptyset$, U	0, 1

Table 1.1: Comparison of Formal Logic, Set Theory, and Boolean Algebra

Table 1.1 shows a comparison of the notation used in formal logic, set theory, and Boolean Algebra. In formal logic, variables are represented by letters such as p, q, r, etc., and the logical operations are represented by symbols such as $\wedge$, $\vee$, and $\neg$. In set theory, variables are represented by capital letters such as A, B, C, etc., and the set operations are represented by symbols such as $\cap$, $\cup$, and $-$. In Boolean Algebra, variables are represented by letters such as x, y, z, etc., and the Boolean operations are represented by symbols such as $\cdot$, $+$, and $'$. The special elements in each system are F and T in formal logic, $\emptyset$ and U in set theory, and 0 and 1 in Boolean Algebra.

Though the notation used in each system is different, the underlying concepts are the same. For example, the AND operation in Boolean Algebra is similar to the logical conjunction operation in formal logic, where the output is TRUE only if all inputs are TRUE. Similarly, the OR operation in Boolean Algebra is similar to the logical disjunction operation in formal logic, where the output is TRUE if at least one input is TRUE. Compared to set theory, the AND operation in Boolean Algebra is similar to the intersection operation, where the output is the set of elements that are common to all input sets.

1.3.1 AND Operation

The AND operation is denoted by the symbol $\cdot$ or juxtaposition. The output of the AND operation is 1 only if all inputs are 1. In Boolean Algebra, the AND operation is represented by the multiplication symbol $\cdot$ or by juxtaposition. **Juxtaposition** is the act of placing two or more things side by side or close together.

Input 1	Input 2	Output
0	0	0
0	1	0
1	0	0
1	1	1

Table 1.2: Truth Table for the AND Operation

Table 1.2 shows the truth table for the AND operation. The output is 1 only if both inputs are 1; otherwise, the output is 0.

Suppose we have the variables x and y, and we want to represent the AND operation between them. We can write this as $x \cdot y$ or xy via juxtaposition. The output of this operation is 1 only if both x and y are 1.

Input 1	**Input 2**	**Output**
x	y	xy
0	0	0
0	1	0
1	0	0
1	1	1

Table 1.3: Truth Table for the AND Operation with Variables

Table 1.3 shows the truth table for the AND operation with variables x and y. The output is 1 only if both x and y are 1; otherwise, the output is 0.

Exercise

1. Consider the three input AND operation xyz. Write the truth table for this operation and determine the output for each combination of inputs.

1.3.2 OR Operation

The OR operation is denoted by the symbol $+$. The output of the OR operation is 1 if at least one input is 1. In Boolean Algebra, the OR operation is represented by the addition symbol $+$.

Input 1	**Input 2**	**Output**
0	0	0
0	1	1
1	0	1
1	1	1

Table 1.4: Truth Table for the OR Operation

Table 1.4 shows the truth table for the OR operation. The output is 1 if at least one input is 1; otherwise, the output is 0.

Suppose we have the variables x and y, and we want to represent the OR operation between them. We can write this as $x + y$. The output of this operation is 1 if at least one of x and y is 1.

Input 1	**Input 2**	**Output**
x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

Table 1.5: Truth Table for the OR Operation with Variables

Table 1.5 shows the truth table for the OR operation with variables x and y. The output is 1 if at least one of x and y is 1; otherwise, the output is 0.

Exercise
Write the truth table for the following OR operations:

1. $f(x, y, z) = x + y + z$
2. $f(x, y, z) = (x + y)z$
3. $f(x, y, z) = x + yz$

1.3.3 NOT Operation

The NOT operation is denoted by the symbol $'$. The output of the NOT operation is the opposite of the input. If the input is 1, the output is 0, and vice versa. In Boolean Algebra, the NOT operation is represented by the prime symbol $'$ or by an overline.

Input	**Output**
0	1
1	0

Table 1.6: Truth Table for the NOT Operation

Table 1.6 shows the truth table for the NOT operation. The output is the opposite of the input. If the input is 1, the output is 0, and vice versa.

Suppose we have the variable x, and we want to represent the NOT operation on it. We can write this as x' or $\overline{x}$. The output of this operation is the opposite or complement of x.

Input	**Output**
x	x'
0	1
1	0

Table 1.7: Truth Table for the NOT Operation with Variables

Table 1.7 shows the truth table for the NOT operation with variable x. The output is the opposite of x. If x is 1, the output is 0; if x is 0, the output is 1.

Exercise
Write the truth table for the following NOT operations:

1. $f(x) = (x')'$
2. $f(x, y) = (x + y)'$
3. $f(x, y) = (x \cdot y)'$
4. $f(x, y, z) = (x + yz)'$

1.4 Other Operations

In addition to the fundamental operations of AND, OR, and NOT, there are several other operations in Boolean Algebra that are commonly used. These operations include:

- **XOR** - The XOR operation takes two inputs and produces a 1 output if the inputs are different.
- **NAND** - The NAND operation is the complement of the AND operation. The output of the NAND operation is 0 only if all inputs are 1.

- **NOR** - The NOR operation is the complement of the OR operation. The output of the NOR operation is 0 if at least one input is 1.
- **XNOR** - The XNOR operation is the complement of the XOR operation. The output of the XNOR operation is 1 if the inputs are the same.

1.4.1 XOR Operation

The XOR operation is denoted by the symbol $\oplus$. The output of the XOR operation is 1 if the inputs are different. In Boolean Algebra, the XOR operation is represented by the symbol $\oplus$. If the XOR operation takes more than two inputs, it is called the **parity function**. A parity function is a function that determines whether the number of inputs that are 1 is even or odd.

Input 1	**Input 2**	**Output**
0	0	0
0	1	1
1	0	1
1	1	0

Table 1.8: Truth Table for the XOR Operation

Table 1.8 shows the truth table for the XOR operation. The output is 1 if the inputs are different; otherwise, the output is 0.

Suppose we have the variables x and y, and we want to represent the XOR operation between them. We can write this as $x \oplus y$. The output of this operation is 1 if x and y are different.

Input 1	**Input 2**	**Output**
x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Table 1.9: Truth Table for the XOR Operation with Variables

Table 1.9 shows the truth table for the XOR operation with variables x and y. The output is 1 if x and y are different; otherwise, the output is 0.

Exercise
Write the truth table for the following XOR operations:

1. $f(x, y, z) = (x \oplus y)z$
2. $f(x, y, z) = x \oplus yz$

1.4.2 NAND Operation

The NAND operation is simply the complement of the AND operation. The output of the NAND operation is 0 only if all inputs are 1. In Boolean Algebra, the NAND operation is represented by putting an overline or $'$ over the AND operation.

Table 1.10 shows the truth table for the NAND operation with variables x and y. The output is 0 only if both x and y are 1; otherwise, the output is 1.

Input 1	**Input 2**	**Output**
x	y	$(xy)'$
0	0	1
0	1	1
1	0	1
1	1	0

Table 1.10: Truth Table for the NAND Operation with Variables

1.4.3 NOR Operation

The NOR operation is simply the complement of the OR operation. The output of the NOR operation is 0 if at least one input is 1. In Boolean Algebra, the NOR operation is represented by putting an overline or $'$ over the OR operation.

Input 1	**Input 2**	**Output**
x	y	$(x+y)'$
0	0	1
0	1	0
1	0	0
1	1	0

Table 1.11: Truth Table for the NOR Operation with Variables

Table 1.11 shows the truth table for the NOR operation with variables x and y. The output is 0 if at least one of x and y is 1; otherwise, the output is 1.

1.4.4 XNOR Operation

The XNOR operation is simply the complement of the XOR operation. The output of the XNOR operation is 1 if the inputs are the same. In Boolean Algebra, the XNOR operation is represented by putting an overline or $'$ over the XOR operation.

Input 1	**Input 2**	**Output**
x	y	$(x \oplus y)'$
0	0	1
0	1	0
1	0	0
1	1	1

Table 1.12: Truth Table for the XNOR Operation with Variables

Table 1.12 shows the truth table for the XNOR operation with variables x and y. The output is 1 if x and y are the same; otherwise, the output is 0.

1.5 Tautology and Fallacy

In Boolean Algebra, a **tautology** is a statement that is always TRUE, regardless of the values of its variables. A **fallacy** is a statement that is always FALSE, regardless of the values of its variables.

Table 1.13 shows examples of tautologies and fallacies. The expression $x + x'$ is a tautology because it is always TRUE, regardless of the value of x. The expression xx' is a fallacy because

x	x'	$x + x'$	xx'
0	1	1	0
1	0	1	0

Table 1.13: Examples of Tautologies and Fallacies

it is always FALSE, regardless of the value of x.

1.6 Boolean Functions

A **Boolean function** is a function that takes one or more Boolean variables as input and produces a Boolean output. Boolean functions are used to represent logical operations in Boolean Algebra. The output of a Boolean function is determined by the values of its input variables and the logical operations applied to them. It is also known as a **switching function**. Boolean variables are variables that can take on one of two values: 0 or 1. In Boolean Algebra, variables are typically denoted by letters such as x, y, z, etc. The values of these variables represent logical values: 0 corresponds to FALSE, and 1 corresponds to TRUE.

A Boolean function is a function in the form $f : B^n \to B$, where $B = \{0, 1\}$ is the set of Boolean values, and n is the number of input variables and is called the **arity** of the function.

A **literal** is a variable or its complement. For example, x and x' are literals. In the boolean function $f = (x + yz) + x'$, there are three variables: x, y, and z. The literals in this function are x, y, z, and x' which are the variables and their complements.

Exercises
Write the truth table for the following Boolean functions:

1. $f(x, y) = [xy + (x + y)']'$
2. $f(x, y) = (x + y) \oplus (xy)'$
3. $f(x, y, z) = x(y + z')$
4. $f(x, y, z) = (x + y)(y + z)(z + x)$
5. $f(x, y, z) = x \oplus y \oplus z$

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