System Dynamics: Unconstrained Growth and Decay

CS 313 – Computational Science

1st Sem., S.Y. 2022 - 2023

Introduction

Some situations where the rate at which an amount is changing is proportional to the amount present:

- Population of people, deer or bacteria
- When money is compounded continuously, the rate of change of the amount is also proportional to the amount present.
- For a radioactive element, the amount of radioactivity decays at a rate proportional to the amount present.
- Also, the concentration of a chemical pollutant decays at a rate proportional to the concentration of pollutant present.

Rate of Change

The average velocity, or average rate of change of position with respect to time, is the change in position (Δs) over the change in time (Δt) and incorporates average speed as well as direction by its sign.

Definition Suppose $s(t)$ is the position of an object at time t, where $a \leq t \leq b$. Then the **change in time**, Δt, is $\Delta t = b - a$; and the **change in position**, Δs, is $\Delta s = s(b) - s(a)$. Moreover, the **average velocity**, or the **average rate of change of s with respect to t**, of the object from time $a = b - \Delta t$ to time b is

$$\textbf{average velocity} = \frac{\textbf{change in position}}{\textbf{change in time}} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(b) - s(a)}{b - a} = \frac{s(b) - s(b - \Delta t)}{\Delta t}$$

Rate of Change

Illustration:

Rate of Change

Definition The **instantaneous velocity**, or the **instantaneous rate of change of s with respect to t**, at $t = b$ is the number the average velocity, $\dfrac{s(b) - s(b - \Delta t)}{\Delta t}$, approaches as Δt comes closer and closer to 0 (provided the ratio approaches a number). In this case, the **derivative of $y = s(t)$ with respect to t** at $t = b$, written $s'(b)$ or $\left.\dfrac{dy}{dt}\right|_{t=b}$, is the instantaneous velocity at $t = b$. In general, the **derivative of $y = s(t)$ with respect to t** is written as $s'(t)$, or $\dfrac{dy}{dt}$, or dy/dt.

A function, such as y = s(t), can represent many things other than position.

- Q(t) might represent a quantity (mass) of radioactive carbon-14 at time t, and the instantaneous rate of change of Q with respect to t, Q′(t) = dQ/dt, is the instantaneous rate of decay.
- P(t) might symbolize a population at time t, so that P′(t) = dP/dt, is the rate of change of the population with respect to t.

Differential Equations

Suppose we have a population in which no individuals arrive or depart;

- *the only change in the population comes from births and deaths.*
- *no constraints, such as competition for food or a predator, exist on growth of the population.*

When no limiting factor exists, we have the Malthusian model for unconstrained population growth,

- *the rate of change of the population is directly proportional (∝) to the number of individuals in the population.*
- *If P represents the population and t represents time, then we have the following proportion:*

$$\boxed{\frac{dP}{dt} \propto P}$$

Differential Equations

For a positive growth, the larger the population, the greater the change in population.

$$\boxed{\frac{dP}{dt} = rP}$$

The constant r is the **growth rate**, or **instantaneous growth rate**, or **continuous growth rate**, while dP/dt is **the rate of change of the population.**

Differential Equations

Definitions A **differential equation** is an equation that contains one or more derivatives. An **initial condition** is the value of the dependent variable when the independent variable is zero. A **solution** to a differential equation is a function that satisfies the equation and initial condition(s).

Suppose we started with a bacterial population of size 100, an instantaneous growth rate of 10% = 0.10, and time measured in hours. Thus, we have

$$\boxed{\frac{dP}{dt} = 0.10P}$$

with P_0 = 100. The equation $\frac{dP}{dt} = 0.10P$ with the initial condition P_0 = 100 is a differential equation because it contains a derivative.

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A solution to this differential equation is a function, P(t), whose derivative is 0.10P(t), with P(0) = 100.

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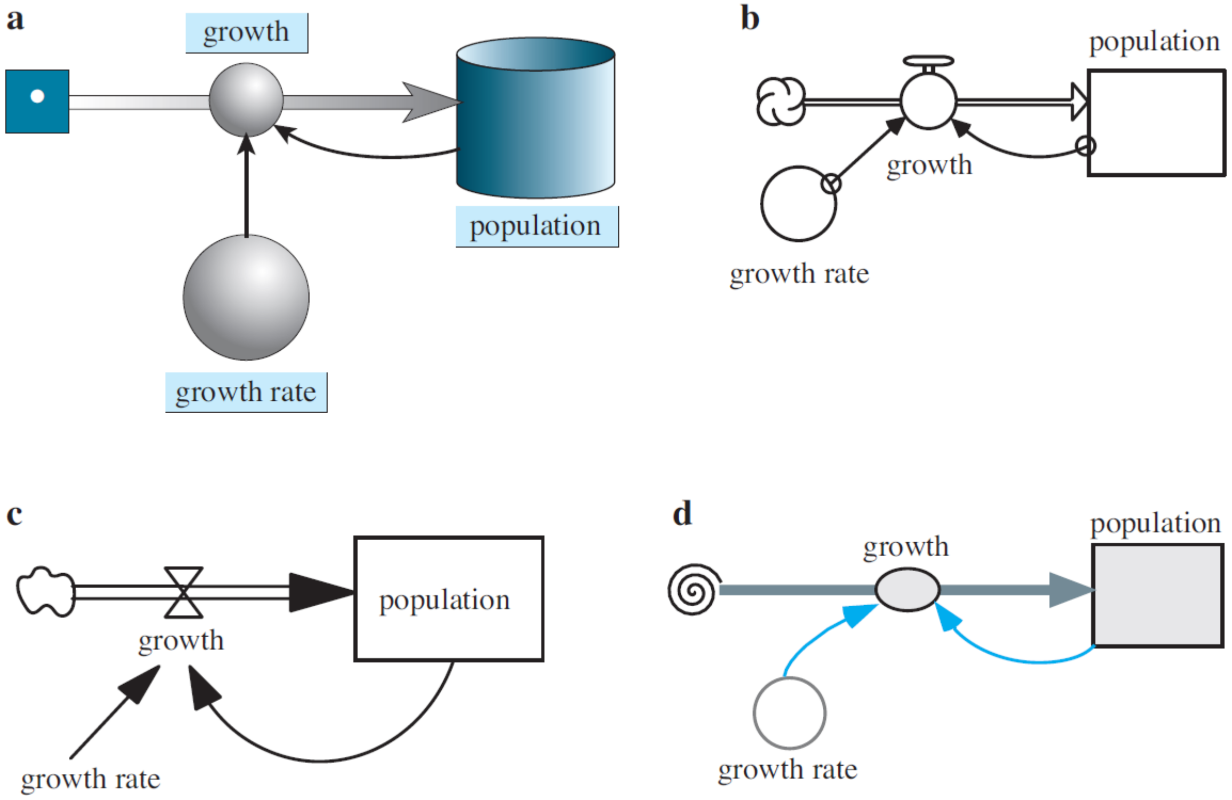
Difference Equation



Diagrams of population models where growth is proportional to population: (a) Berkeley Madonna (b) STELLA (c) Vensim PLE (d) Text's format

Each diagram depicts the situation with *population* indicating *P*, *growth_rate* representing *r*, and *growth* meaning *dP/dt*.

Difference Equation

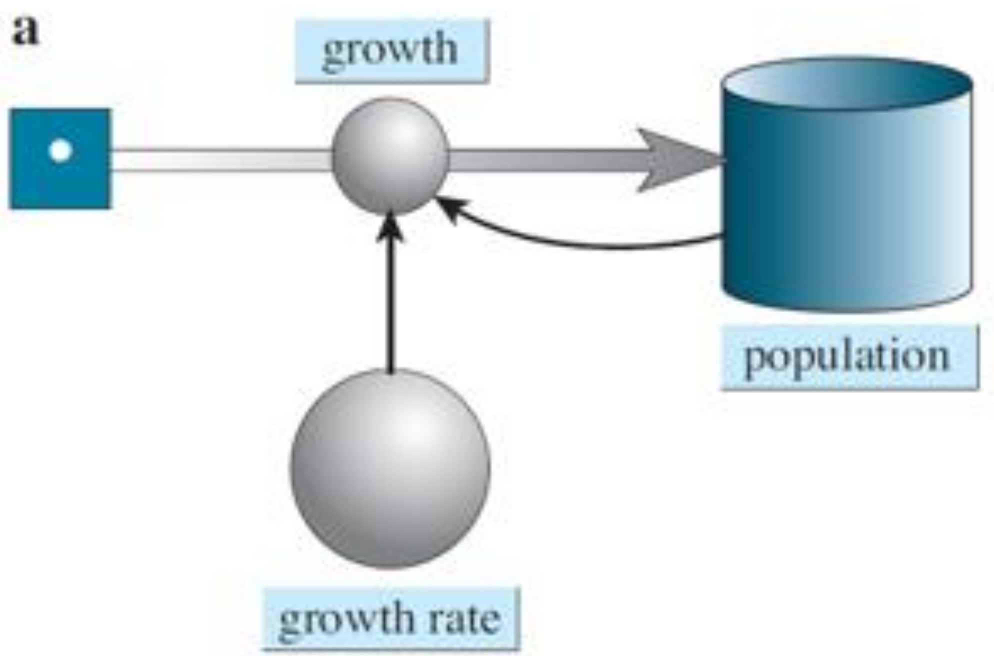


Diagram of population model where growth is proportional to population using Berkeley Madonna

- A **stock** (**box variable**, or **reservoir**), such as *population*, accumulates with time.
- By contrast, a **converter** (**variable-auxiliary/constant**, or **formula**), such as *growth_rate*, does not accumulate but stores an equation or a constant.
- The growth is the additional number of organisms that join the population. Thus, a **flow** (**rate**), such as *growth*, is an activity that changes the magnitude of a stock – and represents a derivative. Because both population and growth rate are necessary to determine the growth, we have **arrows** (**connectors**, or **arcs**) from these quantities to the flow indicator.

Difference Equation

- For a simulation with a system dynamics tool or a program we write, we consider time advancing in small, incremental steps.

- For time, t, and length of a time step, Δt, the previous time is $t - \Delta t$.

- A system dynamics tool might call the change in time dt, DT, or something else instead of Δt.

Finite Difference Equation

$$population(t) = population(t - \Delta t) + (growth) * \Delta t$$

$$(\text{new population}) = (\text{old population}) + (\text{change in population})$$

or

$$population(t) = population(t - \Delta t) + \Delta population$$

- This equation, called a **finite difference equation**, indicates that the population at one time step is the population at the previous time step plus the change in population over that time interval:

Finite Difference Equation

Definition A **finite difference equation** is of the following form:

(new value) = (old value) + (change in value)

Such an equation is a discrete approximation to a differential equation.

Approximation of the derivative *dP/dt* when solving for *growth*:

$$growth = \frac{\Delta population}{\Delta t} = \frac{population(t) - population(t - \Delta t)}{\Delta t}$$

Quick Review Question

Consider the differential equation $dQ/dt = -0.0004Q$, with $Q_0 = 200$.

a. Using delta notation, give a finite difference equation corresponding to the differential equation.

b. At time $t = 9.0$ s, give the time at the previous time step, where $\Delta t = 0.5$ s.

c. If $Q(t - \Delta t) = 199.32$ and $Q(t) = 199.28$, give ΔQ.

Example

Table of Estimated Populations, Where the Initial Population is 100, the Continuous Growth Rate is 10% per Hour, and the Time Step is 0.005 h

t	$population(t)$	=	$population(t - \Delta t)$	+	$(growth)$	*	Δt
0.000	100.000000						
0.005	100.050000	=	100.000000	+	10.000000	*	0.005
0.010	100.100025	=	100.050000	+	10.005000	*	0.005
0.015	100.150075	=	100.100025	+	10.010003	*	0.005
0.020	100.200150	=	100.150075	+	10.015008	*	0.005
0.025	100.250250	=	100.200150	+	10.020015	*	0.005
0.030	100.300375	=	100.250250	+	10.025025	*	0.005
0.035	100.350525	=	100.300375	+	10.030038	*	0.005
0.040	100.400701	=	100.350525	+	10.035053	*	0.005

$$
\begin{aligned}
population(0.030) &= population(0.025) + (growth \text{ at time } 0.025\text{ h}) * \Delta t \\
&= 100.250250 + 10.025025 * 0.005 \\
&= 100.250250 + 0.050125 \\
&= 100.300375
\end{aligned}
$$

Review Question

Evaluate *population*(0.045), the population at the next time interval after the end of Table to six decimal places.

t	$population(t)$	=	$population(t - \Delta t)$	+	$(growth)$	*	Δt
0.000	100.000000						
0.005	100.050000	=	100.000000	+	10.000000	*	0.005
0.010	100.100025	=	100.050000	+	10.005000	*	0.005
0.015	100.150075	=	100.100025	+	10.010003	*	0.005
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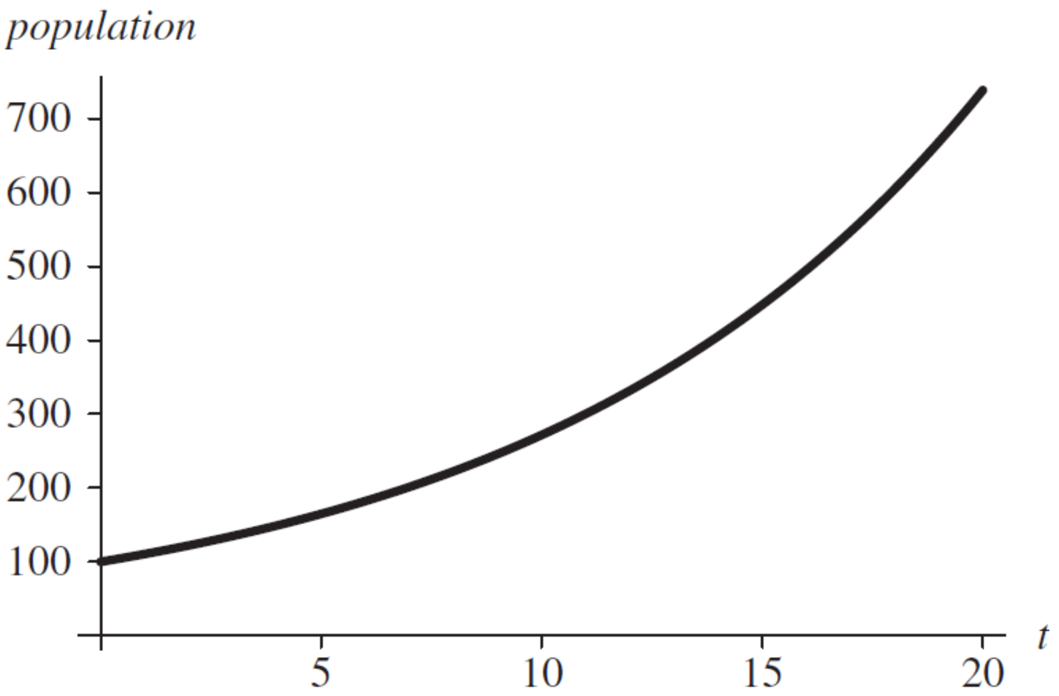
0.035	100.350525	=	100.300375	+	10.030038	*	0.005
0.040	100.400701	=	100.350525	+	10.035053	*	0.005

Example

Table of Estimated Growths and Populations, Reported on the Hour, Where the Initial Population is 100, the Growth Rate is 10%, and the Time Step is 0.005 h

t (h)	*growth*	*population*
0.000	10.00	100.00
1.000	11.05	110.51
2.000	12.21	122.13
3.000	13.50	134.98
4.000	14.92	149.17
5.000	16.49	164.85
6.000	18.22	182.18
7.000	20.13	201.34
8.000	22.25	222.51
9.000	24.59	245.90
10.000	27.18	271.76
11.000	30.03	300.33
12.000	33.19	331.91
13.000	36.68	366.81
14.000	40.54	405.38
15.000	44.80	448.00
16.000	49.51	495.11
17.000	54.72	547.16
18.000	60.47	604.69
19.000	66.83	668.27
20.000		738.54

Rule of Thumb Although the simulation takes longer because of more computation, it is usually more accurate to use a small step size (Δt), say, 0.01 or less.



Graph of population versus time (hours) for the data in Table

Simulation Program

Algorithm 1 Algorithm for simulation of unconstrained growth

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize length of time step Δt
$numIterations \leftarrow simulationLength/\Delta t$
for i going from 1 through *numIterations* do the following:
    $growth \leftarrow growthRate * population$
    $population \leftarrow population + growth * \Delta t$
    $t \leftarrow i * \Delta t$
    display t, *growth*, and *population*

For example, if the simulation length is 10 h and Δt is 0.25 h, then the number of loop iterations is *numIterations* = 10/0.25 = 40.

There is a loop index (i) that goes from 1 through *numIterations*. Inside the loop, time *t* is calculate as the product of *i* and *Δt*.

For example, if Δt is 0.25 h, during the irst iteration, the index i becomes 1 and the time is 1 * Δt = 0.25 h. On loop iteration i = 8, the time gets the value 8 * Δt = 8 * 0.25 h = 4.00 h.

Simulation Program

Algorithm 2 Alternative algorithm to Algorithm 1 for simulation of unconstrained growth that does not display *growth*

initialize *simulationLength*
initialize *population*
initialize *growthRate*
initialize Δt
growthRatePerStep $\leftarrow$ ***growthRate*** ***** $\boldsymbol{\Delta t}$
numIterations $\leftarrow$ *simulationLength*/Δt
for i going from 1 through *numIterations* do the following:
    population $\leftarrow$ ***population*** **+** ***growthRatePerStep*** ***** ***population***
    $t \leftarrow i * \Delta t$
    display t and *population*

Analytical Solution: Explanation with Derivatives

The preceding model can be solved analytically for unconstrained growth, which is the differential equation $dP/dt = 0.10P$ with initial condition $P_0 = 100$, as follows:

$$P = 100\, e^{0.10t}$$

The differential equation $dP/dt = 0.10P$ for P can be solved analytically by finding a function whose derivative is 0.10 times the function itself.

The only functions that are their own derivative are exponential functions of the following form:

$$f(t) = ke^{t}, \text{ where } k \text{ is a constant}$$

Analytical Solution: Explanation with Derivatives

Example)

The derivative of $5e^t$ is $5e^t$. To obtain a factor of 0.10 through use of the chain rule, we have the general solution

$$P = ke^{0.10t}$$

If $P = 5e^{0.10t}$, then

$$\frac{dP}{dt} = \frac{d(5e^{0.10t})}{dt} = 5\frac{d(e^{0.10t})}{dt} = 5(0.10e^{0.10t}) = 0.10(5e^{0.10t}) = 0.10P$$

Completion of the Analytical Solution

Thus, the general solution to $dP/dt = 0.10P$ is $P = ke^{0.10t}$ for a constant k.

A particular value for k can be determined by using the initial condition $P_0 = 100$, thus, a particular solution of the form

$$P = ke^{0.10t}$$

Substituting 0 for t and 100 for P, we get

$$100 = ke^{0.10(0)} = ke^{0} = k(1) = k$$

The constant is the initial population.

Completion of the Analytical Solution

In general, the solution to

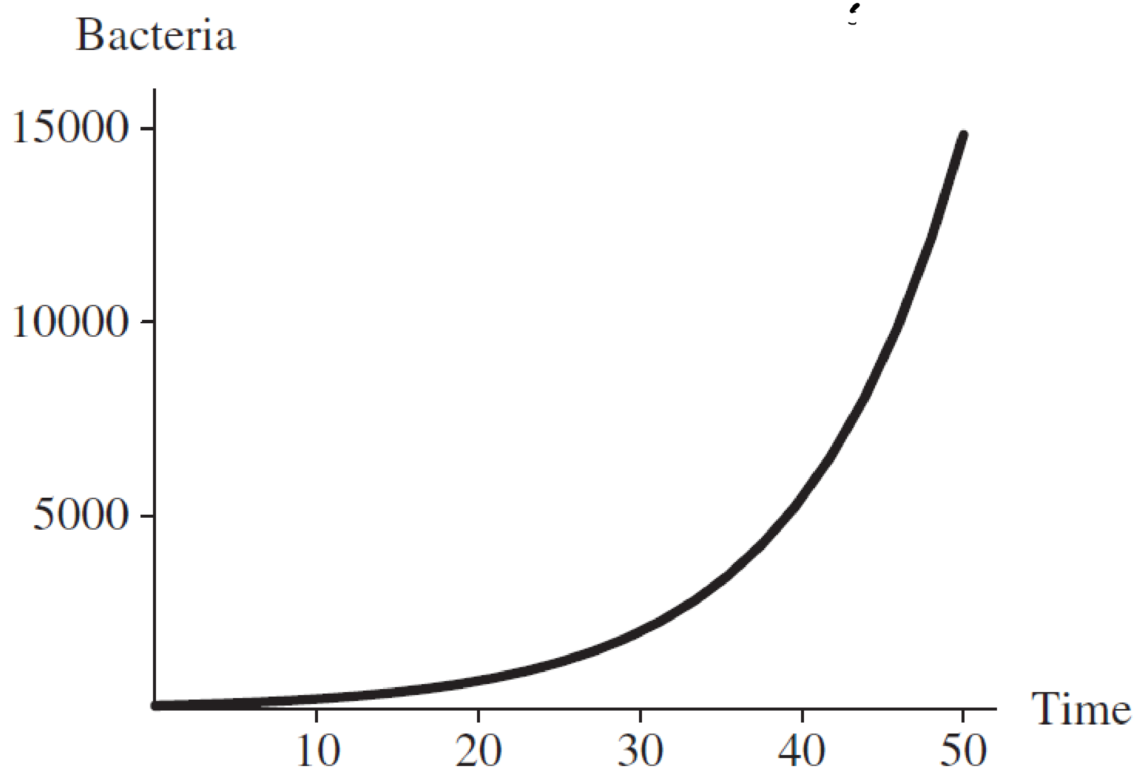
$$\frac{dP}{dt} = rP \text{ with initial population } P_0$$

is

$$P = P_0 e^{rt}$$

Example

For $P = 100e^{0.10t}$



Bacterial population with a continuous growth rate of 10% per hour and an initial population of 100 bacteria

Review Question

Give the solution of the differential equation

$$\frac{dP}{dt} = 0.03P \text{, where } P_0 = 57$$

Model Refinement

The model can be further refined by having separate parameters for birth rate and death rate instead of combined growth rate:

$$growth\_rate = birth\_rate - death\_rate$$

Unconstrained Decay

The rate of change of the mass of a radioactive substance is proportional to the mass of the substance, and the constant of proportionality is negative. Thus, the mass decays with time.

Unconstrained Decay

Example) The constant of proportionality for radioactive carbon-14 is approximately -0.000120968. The continuous decay rate is about 0.0120968% per year, and the differential equation is as follows, where Q is the quantity (mass) of carbon-14:

$$\frac{dQ}{dt} = -0.000120968Q$$

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$$\frac{dQ}{dt} = -0.000120968Q$$

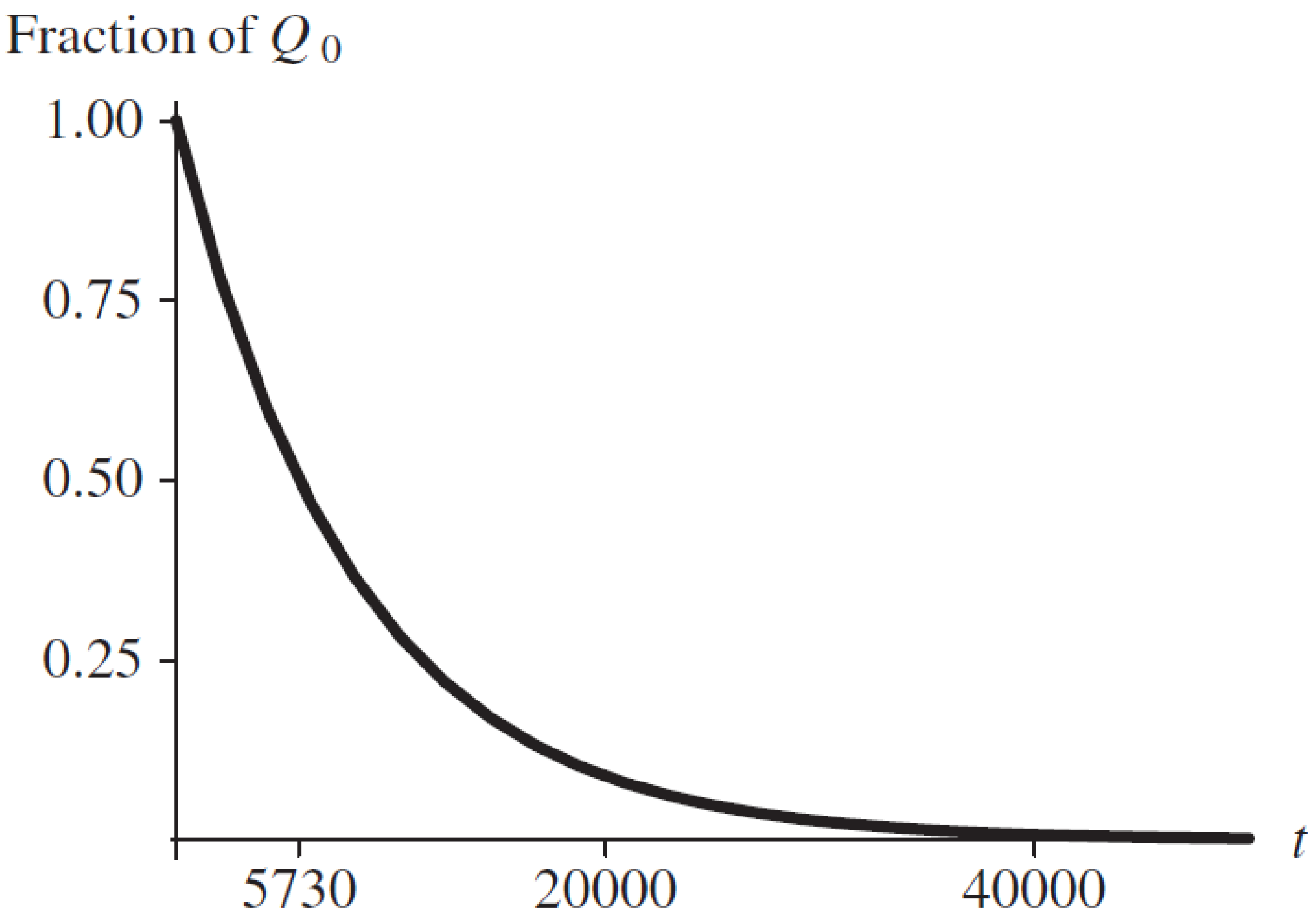
The analytical solution to this equation is

$$Q = Q_0 e^{-0.000120968t}$$

After 10,000 yr, only about 29.8% of the original quantity of carbon-14 remains, as the following shows:

$$Q = Q_0 e^{-0.000120968(10,000)} = 0.298292Q_0$$

Unconstrained Decay



Exponential decay of radioactive carbon-14 as a fraction of the initial quantity Q_0, with time (t) in years

Unconstrained Decay

- **Carbon dating** uses the amount of carbon-14 in an object to estimate the age of an object.
- All living organisms accumulate small quantities of carbon-14, but accumulation stops when the organism dies.
- For example, we can compare the proportion of carbon-14 in living bone to that in the bone of a mummy and estimate the age of the mummy using the model.

Unconstrained Decay

Carbon dating: Example) Suppose the proportion of carbon-14 in a mummy is only about 20% of that in a living human. To estimate the age of the mummy, we use the preceding model with the information that Q = 0.20Q0. Substituting into the analytical model, we have

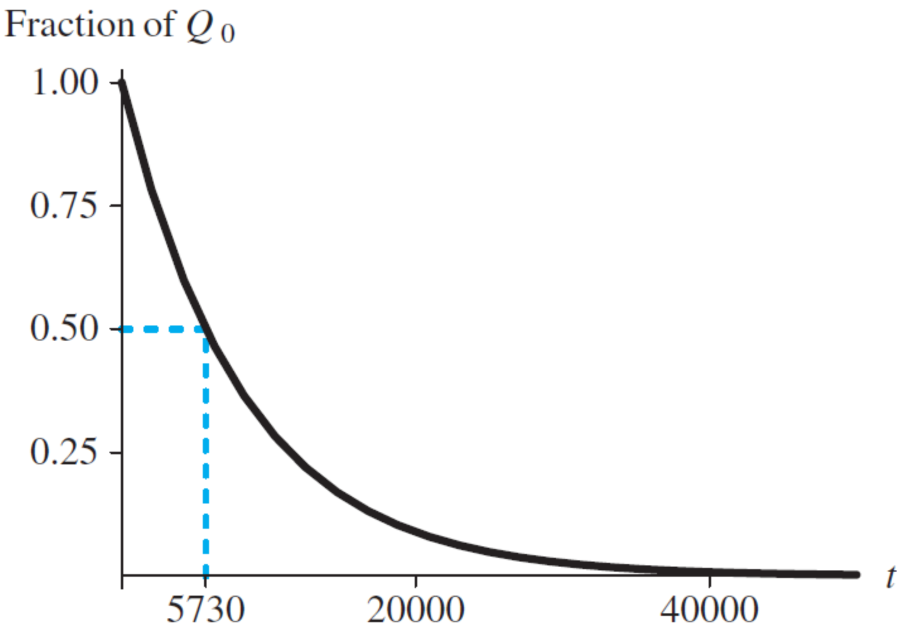
$$0.20Q_0 = Q_0 e^{-0.000120968t}$$

$$\ln(0.20) = \ln(e^{-0.000120968t}) = -0.000120968t$$

$$t = \ln(0.20)/(-0.000120968) \approx 13{,}305 \text{ yr}$$

Unconstrained Decay

The **half-life** is the period of time that it takes for the substance to decay to half of its original amount. The given Figure illustrates that the half-life of radioactive carbon-14 is about 5730 yr.



The half-life of radioactive carbon-14 indicated as 5730 yr

Review Question

Radium-226 has a continuous decay rate of about 0.0427869% per year. Determine its half-life in whole years.

Exercises

1. **a.** For an initial population of 100 bacteria and a continuous growth rate of 10% per hour, determine the number of bacteria at the end of one week.

   b. How long will it take the population to double?

2. **a.** Suppose the initial population of a certain animal is 15,000 and its continuous growth rate is 2% per year. Determine the population at the end of 20 yr.

   b. Suppose we are performing a simulation of the population using a step size of 0.083 yr. Determine the growth and the population at the end of the first three time steps.

Exercises

a. What proportion of the original quantity of carbon-14 is left after 30,000 yr?
b. If 60% is left, how old is the item?

References

Shiflet, Angela B. and Shiflet, George W. 2014. Introduction to Computational Science: Modeling and Simulation for the Sciences, 2nd Edition, Princeton University Press.

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