System Dynamics: Constrained Growth

CS 313 – Computational Science

1st Sem., S.Y. 2024-2025

Introduction

An animal introduced to a new environment will often reproduce at a very high rate.

- Eurasian perch, called ruffe (*Gymnocephalus cernuus*) introduced to Lake Superior from an ocean-going ship's ballast
- Ruffe is a meal of last resort for most predators (preyed upon by very few species of larger fish and then only if other prey are scarce)
- Eurasian perch has little to no value as a fishery
- Formidable competitor
- More adaptable in their dietary choices
- Tolerate wide ranges of temperature and pH
- Prolific breeders

Introduction

- When introduced to Loch Lomond, Scotland, ruffe populations increased exponentially and decimated the eggs of local salmon (Adams 1998).
- Populations of native North American fish like yellow perch, perch-trout, and emerald shiners have all declined since ruffe were introduced (McLean 1993).
- Ruffe either predate on their competitors eggs or decrease their food resources.
- Endemic populations increase rapidly at first, but they eventually encounter resistance from the environment – competitors, predators, limited resources, and disease.
- This maximum population size that the environment can support indefinitely is termed the **carrying capacity**.

Carrying Capacity

In Unconstrained Growth and Decay, we considered a population growing without constraints, such as competition for limited resources. For such a population, P, with instantaneous growth rate, r, the rate of change of the population has the following differential equation model:

$$\boxed{\frac{dP}{dt} = rP}$$

With initial population P_0, the analytical solution is $P = P_0 e^{rt}$. We also developed the following finite difference equation for the change in P from one time to the next:

$$\boxed{\begin{aligned} \Delta P &= P(t) - P(t - \Delta t) \\ &= (r\,P(t - \Delta t))\,\Delta t \end{aligned}}$$

Carrying Capacity

As indicated in the introduction, no confined population can grow without bound. Competition for food, shelter, and other resources eventually limits the possible growth.

- For example, suppose a deer refuge can support at most 1000 deer. We say that the carrying capacity (M) for the deer in the refuge is 1000.

Definition The **carrying capacity** for an organism in an area is the maximum number of organisms that the area can support.

Quick Review Question

Cycling back to Step 2 of the modeling process, this question begins refinement of the population model to accommodate descriptions of population growth from the "Introduction" of this module.

a. Determine any additional variable and its units.

b. Consider the relationship between the number of individuals *(P)* and carrying capacity *(M)* as time *(t)* increases. List all the statements below that apply to the situation where the population is much smaller than the carrying capacity.

A. P appears to grow almost proportionally to t.

Quick Review Question

B. P appears to grow almost without bound.
C. P appears to grow faster and faster.
D. P appears to grow more and more slowly.
E. P appears to decline faster and faster.
F. P appears to decline more and more slowly.
G. P appears to grow almost linearly with slope M.
H. P is appears to be approaching M asymptotically.
I. P appears to grow exponentially.
J. dP/dt appears to be almost proportional to P.
K. dP/dt appears to be almost zero.
L. The birth rate is about the same as the death rate.
M. The birth rate is much greater than the death rate.
N. The birth rate is much less than the death rate.

Quick Review Question

c. List all the choices from Part b that apply to the situation where the population is close to but less than the carrying capacity.

d. List all the choices from Part b that apply to the situation where the population is close to but greater than the carrying capacity.

Revised Model

- In the revised model, for an initial population much lower than the carrying capacity, we want the population to increase in approximately the same exponential fashion as in unconstrained model.
- As the population size get closer and closer to the carrying capacity, we need to dampen the growth more and more.
- Near the carrying capacity, the number of deaths should be almost equal to the number of births, so that the population remains roughly constant.
- To accomplish this dampening of growth, we could compute the number of deaths as a changing fraction of the number of births, which we model as rP.

Revised Model

- When the population is very small, we want the fraction to be almost zero, indicating that few individuals are dying.
- When the population is close to the carrying capacity, the fraction should be almost 1 = 100%.
- For population larger than the carrying capacity, the fraction should be even larger so that the population decrease in size through deaths.
- Such a fraction is P/M.

Revised Model

- For example, if the population P is 10 and the carrying capacity M is 1000, then $P/M = 10/1000 = 0.01 = 1\%$.
- For a population $P = 995$, close to the carrying capacity, $P/M = 995/1000 = 0.995 = 99.5\%$.
- For the excessive $P = 1400$, $P/M = 1400/1000 = 1.400 = 140\%$.

Revised Model

Thus, we can model the instantaneous rate of change of the number of deaths (D) as a fraction P/M times the instantaneous rate of change of the number of births (r), as the following differential equation indicates:

$$\boxed{\frac{dD}{dt} = \left(r\frac{P}{M}\right)P}$$

Revised Model

The differential equation for the instantaneous rate of change of the population subtracts this value from the instantaneous rate of change of the number of births, as follows:

$$\frac{dP}{dt} = \underbrace{(rP)}_{births} - \underbrace{\left(r\frac{P}{M}\right)P}_{deaths}$$

or

$$\frac{dP}{dt} = r\left(1 - \frac{P}{M}\right)P$$

Revised Model

For the discrete simulation, where $P(t-1)$ is the population estimate at time t – *1*, the number of deaths from t – *1* to time t is:

$$\Delta D = \left(r \frac{P(t-1)}{M} \right) P(t-1) \quad \text{for } \Delta t = 1$$

In general, we approximate the number of deaths from time $(t - \Delta t)$ to time t by multiplying the corresponding value by Δt, as follows:

$$\Delta D = \left(r \frac{P(t-\Delta t)}{M} \right) P(t-\Delta t)\Delta t$$

Revised Model

where, $P(t - \Delta t)$ is the population estimate at $(t - \Delta t)$. Thus, the change in population from time $(t - \Delta t)$ to time t is the difference of the number of births and the number of deaths over that period:

$$\Delta P = \text{births} - \text{deaths}$$

$$\Delta P = \underbrace{\left(rP(t-\Delta t)\right)\Delta t}_{\text{births}} - \underbrace{\left(r\frac{P(t-\Delta t)}{M}\right)P(t-\Delta t)\Delta t}_{\text{deaths}}$$

$$= (r\Delta t)\left(1 - \frac{P(t-\Delta t)}{M}\right)P(t-\Delta t)$$

or

$$\Delta P = k\left(1 - \frac{P(t-\Delta t)}{M}\right)P(t-\Delta t), \text{ where } k = r\Delta t$$

Revised Model

The Differential equation and Difference equation are called **logistic equations.**

$$\frac{dP}{dt} = r\left(1 - \frac{P}{M}\right)P$$

$$\Delta P = k\left(1 - \frac{P(t-\Delta t)}{M}\right)P(t-\Delta t), \text{ where } k = r\Delta t$$

Revised Model

Figure 2.3.1 displays the S-shaped curve characteristic of a logistic equation, where the initial population is less than the carrying capacity of 1000.

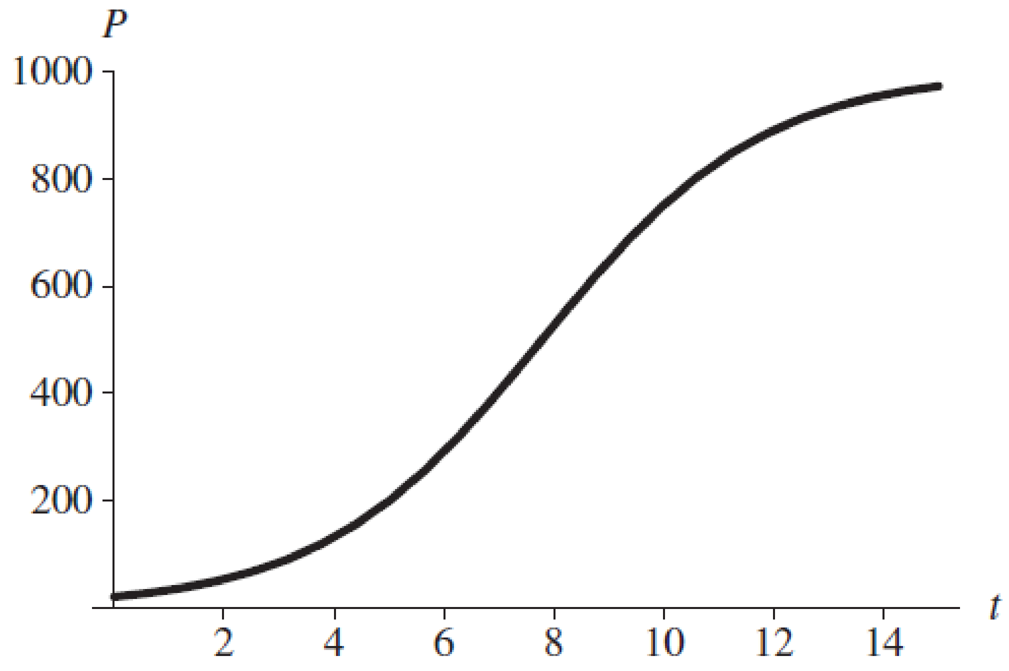

Figure 2.3.1 Graph of logistic equation, where initial population is 20, carrying capacity is 1000, and instantaneous rate of change of births is 50%, with time (t) in years

Revised Model

Figure 2.3.2 shows how the population decreases to the carrying capacity when the initial population is 1500.

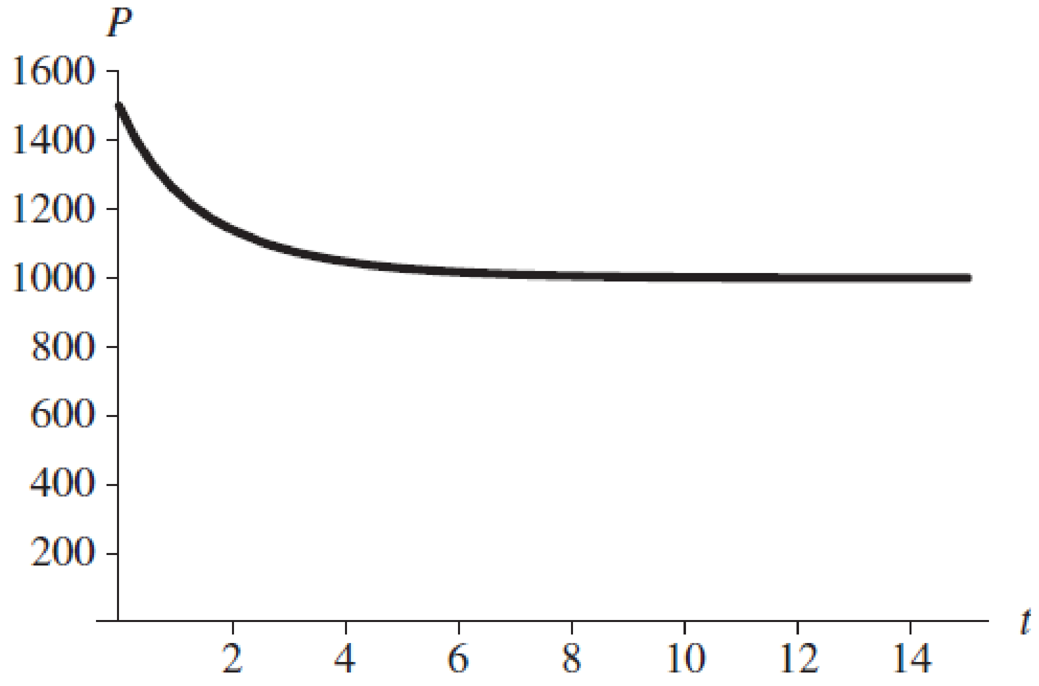

Figure 2.3.2 Graph of logistic equation, where initial population is 1500, carrying capacity is 1000, and instantaneous rate of change of births is 50%, with time (t) in years

Revised Model

To compute for the population P at time t:

$$P(t) = P(t - \Delta t) + \Delta P$$

or

$$\boxed{P(t) = P(t - \Delta t) + k\left(1 - \frac{P(t - \Delta t)}{M}\right)P(t - \Delta t)}$$

Revised Model

Thus, the model appears to match observations from the "Introduction" qualitatively. To verify a particular model, we should estimate parameters, such as birth rate, and compare the results of the model to real data.

Review Question

a. Complete the difference equation to model constrained growth of a population P with respect to time t over a time step of 0.1 units, given that the population at time $t - \Delta t$ is $p \leq 1000$, the carrying capacity is 1000, the instantaneous rate of change of births is 105%, and the initial population is 20.

$\Delta P =$ ____(____ ____ ______)$(p)(0.1)$

b. What is the maximum population?

c. Suppose the population at time $t = 5$ yr is 600 individuals. What is the population, rounded to the nearest integer, at time 5.1 yr?

Equilibrium and Stability

- The logistic equation with carrying capacity $M = 1000$ has an interesting property.
- If the initial population is less than 1000, the population increases to a limit of 1000.
- If the initial population is greater than 1000, the population decreases to the limit of 1000.
- If the initial population is 1000, we see that $P/M = 1000/1000 = 1$ and $dP/dt = r(1 - 1)P = 0$. $\Delta P = 0$.

Equilibrium and Stability

- A population starting at the carrying capacity remains there.
- We say that M = 1000 is an equilibrium size for the population because the population remains steady at that value of $P(t) = P(t - \Delta t) = 1000$ for all $t > 0$.

Definitions An **equilibrium solution** for a differential equation is a solution where the derivative is always zero. An **equilibrium solution** for a difference equation is a solution where the change is always zero.

Equilibrium and Stability

- Even if an initial positive population does not equal the carrying capacity $M = 1000$, eventually, the population size tends to that value.
- We say that the solution $P = 1000$ to the logistic equation (1) or (2) is **stable**.
- By contrast, for a positive carrying capacity, the solution $P = 0$ is **unstable**.
- If the initial population is close to but not equal to zero, the population does not tend to that solution over time.
- For the logistic equation, any displacement of the initial population from the carrying capacity exhibits the limiting behavior of Figure 2.3.1 or 2.3.2.

Equilibrium and Stability

- In general, we say that a solution is stable if for a small displacement from the solution, *P* tends to the solution.

Definition Suppose that q is an equilibrium solution for a differential equation dP/dt or a difference equation ΔP. The solution q is **stable** if there is an interval (a, b) containing q, such that if the initial population $P(0)$ is in that interval, then

1. $P(t)$ is finite for all $t > 0$;
2. As time, t, becomes larger and larger, $P(t)$ approaches q.

The solution q is **unstable** if no such interval exists.

Exercises

1. A fish population in a lake follows the logistic growth model. The carrying capacity of the lake is 5,000 fish, and the population grows at a rate of 0.6 per year. If the initial population is 400 fish, find the population after 4 years.
2. A certain animal population grows logistically with a carrying capacity of 10,000 and an intrinsic growth rate of 0.8 per year. However, a constant harvesting rate of 500 animals per year is applied. If the initial population is 2,000, find the population after 5 years.
3. A city with a population of 1 million people grows according to a logistic model with a carrying capacity of 4 million people. The growth rate is 0.04 per year. What will the population be after 10 years?