Differential Calculus - Rate of Change

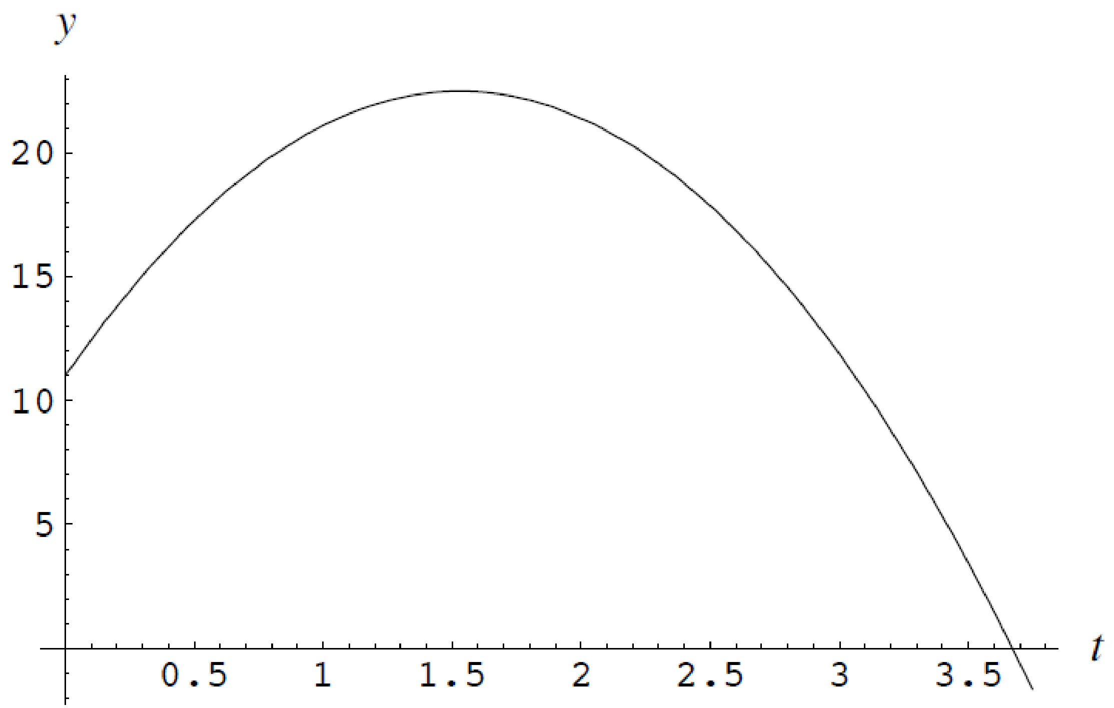
CS 313 – Computational Science

1st Sem., S.Y. 2022 - 2023

Rate of Change

- The rate of change of position with respect to time is the velocity.
- Instantaneous velocity is the rate of change of position for a time interval which is very small (almost zero).
- We can approximate the velocity at any particular time t if we know the heights at times shortly before and after t and compute the average velocity over that time period.
- The instantaneous rate of change is also called the <u>derivative</u>.

Velocity [Example: Tossing a ball from a bridge]



Height (y) in meters versus time (t) in seconds of a ball thrown straight up from a bridge

Time (*t*) in seconds	**Height (*y*) in meters**
0.00	11.0000
0.25	14.4438
0.50	17.2750
0.75	19.4938
1.00	21.1000
1.25	22.0938
1.50	22.4750
1.75	22.2438
2.00	21.4000
2.25	19.9437
2.50	17.8750
2.75	15.1937
3.00	11.9000
3.25	7.9938
3.50	3.4750
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Table of times and heights for a ball thrown straight up from a bridge

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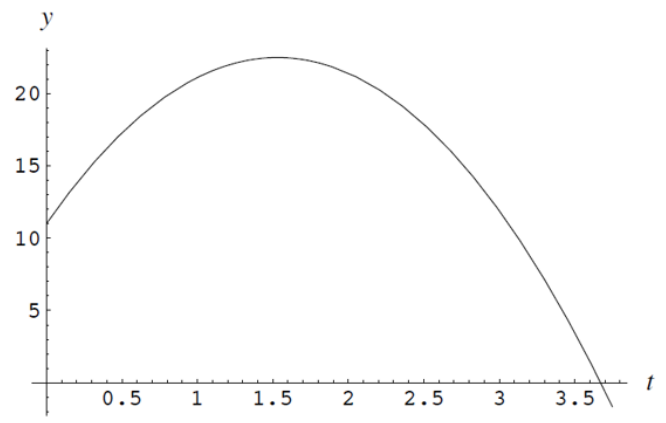

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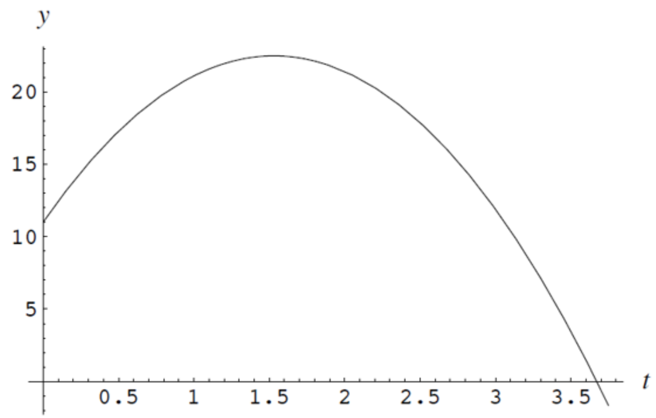
Notes:

- windless day
- the ball, after reaching its highest point, falls and eventually landed in the water.
- The ball's height above the water (*y*) is a function (*s*) of time (*t*), written as ***y = s(t)***.
- The height of the ball at water level is *y* = 0 *m*, and a negative value of y indicates that the ball is under water.

Approximated values

a. The height of the bridge – 10 m
b. The maximum height of the ball – 22.4 m
c. When the ball reaches its maximum height – 1.5s
d. When the ball hits the water – 3.7s

Velocity [Example: Tossing a ball from a bridge]



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The **average velocity** is the is the rate of the change in height, or position, to the change in time.

$$\text{average velocity from 0 to 1 seconds} = \frac{s(1)-s(0)}{1.00-0.00} = \frac{21.1000-11.0000}{1.00} = 10.1 \text{ m/sec}$$

Definition Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. The **average velocity**, or the **average rate of change of** s **with respect to** t, of the object from time a to time b is

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(b)-s(a)}{b-a}$$

Average Velocity

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Approximating the velocity of the ball at the instant t = 1 sec by finding the mean of the average velocities during the first second (10.1 m/sec) and the next second (0.3 m/sec), as follows:

$$\text{approximation of velocity at } t = 1 \text{ sec} = \frac{10.1 + 0.3}{2} = 5.2 \text{ m/sec}$$

The average velocity between times on either side of t = 1 sec can be equivalently evaluated by knowing the heights for times as close to t = 1 sec as possible.

Such as in this case, for t = 0.75 sec and t = 1.25 sec.

Average Velocity

Definition Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. Then the **change in time**, Δt, is $\Delta t = b - a$; and the **change in position**, Δs, is $\Delta s = s(b) - s(a)$. Moreover, the **average velocity**, or the **average rate of change of** s **with respect to** t, of the object from time a to time $b = a + \Delta t$ is

$$\textbf{average velocity} = \frac{\textbf{change in position}}{\textbf{change in time}}$$

$$= \frac{\Delta s}{\Delta t}$$

$$= \frac{s(b) - s(a)}{b - a}$$

$$= \frac{s(a + \Delta t) - s(a)}{\Delta t}$$

Instantaneous Velocity

To obtain the instantaneous velocity at t = 1 sec – that is, the *exact velocity* of the ball precisely one second after it starts to move, we determine the average velocity with changes in time, Δt, closer and closer to 0.

This method can also be referred to as **taking the limit of the average velocities as Δt approaches 0,**

$$\textbf{instantaneous velocity at 1 sec} = \lim_{\Delta t \to 0} \frac{s(1+\Delta t) - s(1)}{\Delta t}$$

In the quotient, Δt can be positive or negative, but not zero.

Concept Suppose that as x approaches some number c, $f(x)$ approaches a number L. We say the **limit of $f(x)$ as x approaches c is L**, and we write

$$\lim_{x \to c} f(x) = L$$

Definition The **instantaneous velocity**, or the **instantaneous rate of change of s with respect to t**, at $t = a$ is

$$\textbf{instantaneous velocity at } a \textbf{ sec} = \lim_{\Delta t \to 0} \frac{s(a+\Delta t) - s(a)}{\Delta t}$$

the limit of the average velocity from $t = a$ to $t = a + \Delta t$ as Δt approaches 0, provided the limit exists.

Instantaneous Velocity

Table 2.3.2 Average velocities between $(1, s(1)) = (1, 21.1)$ and $(1 + \Delta t, s(1 + \Delta t))$

Δt	$s(1 + \Delta t)$	$\dfrac{s(1+\Delta t) - s(1)}{\Delta t}$	Δt	$s(1 + \Delta t)$	$\dfrac{s(1+\Delta t) - s(1)}{\Delta t}$
0.10	21.571	**4.710**	-0.10	20.531	**5.690**
0.09	21.528	**4.759**	-0.09	20.592	**5.641**
0.08	21.485	**4.808**	-0.08	20.653	**5.592**
0.07	21.440	**4.857**	-0.07	20.712	**5.543**
0.06	21.394	**4.906**	-0.06	20.770	**5.494**
0.05	21.348	**4.955**	-0.05	20.828	**5.445**
0.04	21.300	**5.004**	-0.04	20.884	**5.396**
0.03	21.252	**5.053**	-0.03	20.940	**5.347**
0.02	21.202	**5.102**	-0.02	20.994	**5.298**
0.01	21.152	**5.151**	-0.01	21.048	**5.249**

The tables shows additional values of *s(t)*, for *t* close to 1 along with the average velocities between each such time and $t = 1$. The left table has values of Δt starting at 0.10 and decreasing to 0.01 along with columns for the corresponding s(1 + Δt) and the average velocity,

$$\frac{[s(1+\Delta t) - s(1)]}{\Delta t}.$$

The right table has negative values of Δt from -0.10 to a value closer to 0. Observing the third columns for both sides, the average velocities appear to be converging to 5.20.

In fact,

$$\lim_{\Delta t \to 0} \frac{s(1+\Delta t) - s(1)}{\Delta t} = 5.20$$

so that the instantaneous velocity of this ball at $t = 1$ sec is 5.20 m/sec.

Derivative

In general, when the limit exists,

$$\lim_{\Delta t \to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$$

is the derivative of s with respect to t at a. Two notations for the derivative of *y* = *s*(*t*) with respect to *t* :

$$\frac{dy}{dt}, \qquad s'(t).$$

For the derivative of *s* at *t* = *1*, which in this case is 5.20, the derivative is written as

$$\left.\frac{dy}{dt}\right|_{t=1} \text{ or } s'(1) = 5.20 \text{ m/sec.}$$

Definition The **derivative of $y = s(t)$ with respect to t** at $t = a$ is the instantaneous rate of change of s with respect to t at a:

$$s'(a) = \left.\frac{dy}{dt}\right|_{t=a} = \lim_{\Delta t \to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$$

provided the limit exists. If the derivative of s exists at a, we say the function is **differentiable** at a.

Differential Calculus

Differentiation Rules

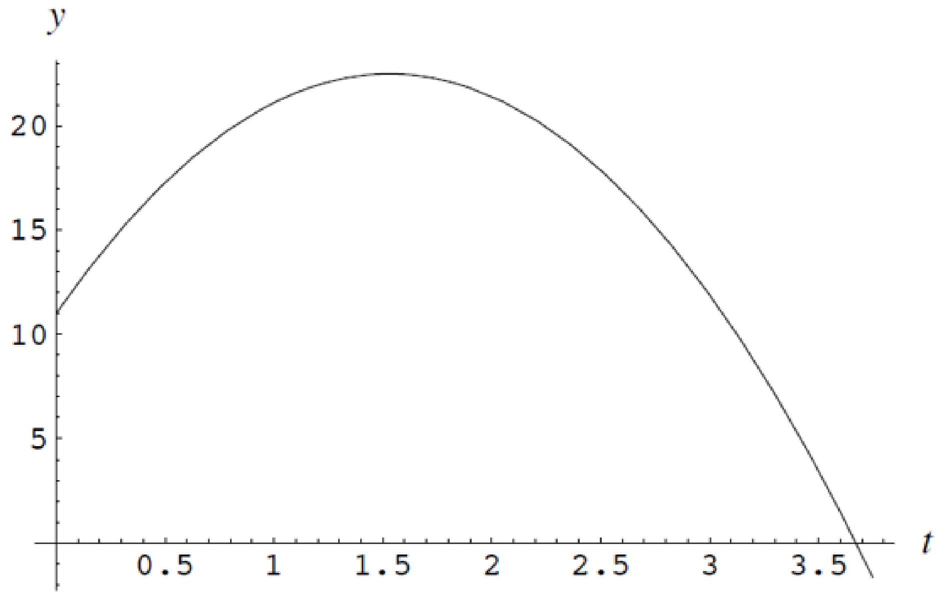
1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' + uv'}{v^2}$
5. $\frac{d}{dx}[v] = 0$
6. $\frac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}\left[|u|\right] = \frac{u}{|u|}(u')$
9. $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot u'$
10. $\frac{d}{dx}\left[e^u\right] = e^u \cdot u'$
11. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \cdot u'$
12. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \cdot u'$

Differential Calculus

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	e^x	e^x
$\ln(x)$	$1/x$	$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$	$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\mathrm{cosec}^2(x)$	$\sec(x)$	$\sec(x)\tan(x)$
$\mathrm{cosec}(x)$	$-\mathrm{cosec}(x)\cot(x)$	$\tan^{-1}(x)$	$1/(1+x^2)$
$\sin^{-1}(x)$	$1/\sqrt{1-x^2}$ for $\|x\|<1$	$\cos^{-1}(x)$	$-1/\sqrt{1-x^2}$ for $\|x\|<1$
$\sinh(x)$	$\cosh(x)$	$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\mathrm{sech}^2(x)$	$\coth(x)$	$-\mathrm{cosech}^2(x)$
$\mathrm{sech}(x)$	$-\mathrm{sech}(x)\tanh(x)$	$\mathrm{cosech}(x)$	$-\mathrm{cosech}(x)\coth(x)$
$\sinh^{-1}(x)$	$1/\sqrt{x^2+1}$	$\cosh^{-1}(x)$	$1/\sqrt{x^2-1}$ for $x>1$
$\tanh^{-1}(x)$	$1/(1-x^2)$ for $\|x\|<1$	$\coth^{-1}(x)$	$-1/(x^2-1)$ for $\|x\|>1$

Slope of Tangent Line

Consider graphically the instantaneous velocity, or the derivative s'(t). Suppose we again wish to examine the velocity of the ball at time t = 1 sec. If we keep zooming in on the graph of the height of the ball y = s(t) at time t = 1 sec, the appearance of these graphs is increasingly linear.



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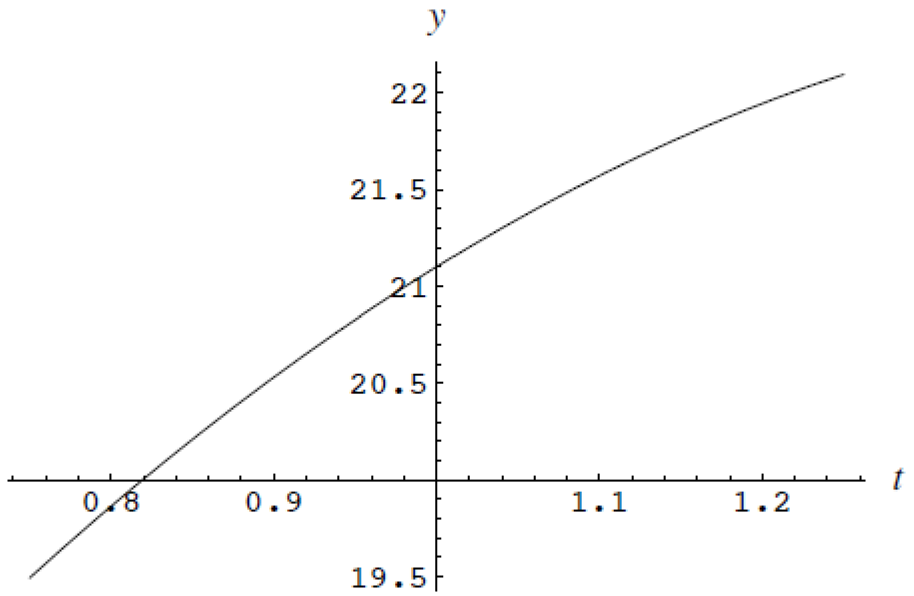

graph of $y = s(t)$ from $t = 0.75$ to 1.25 sec

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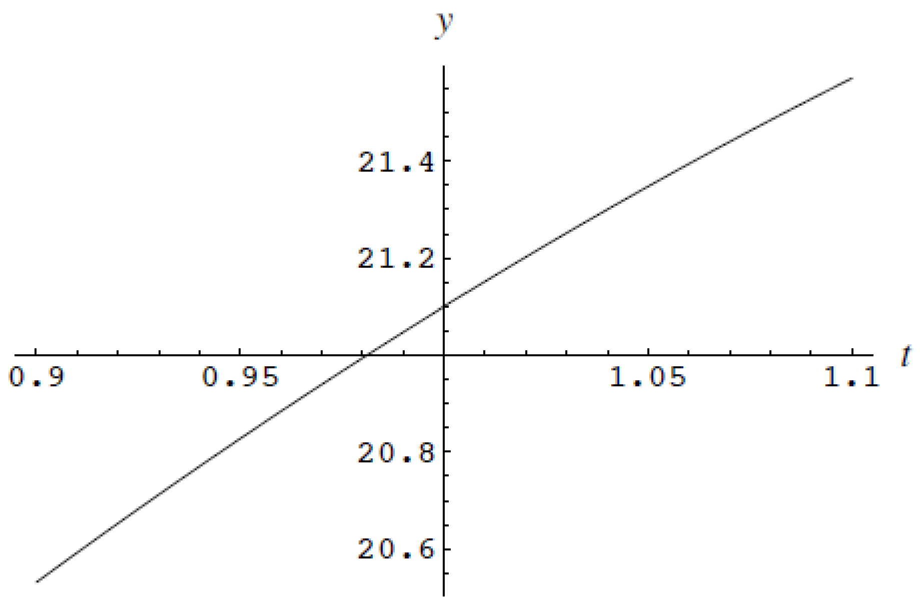
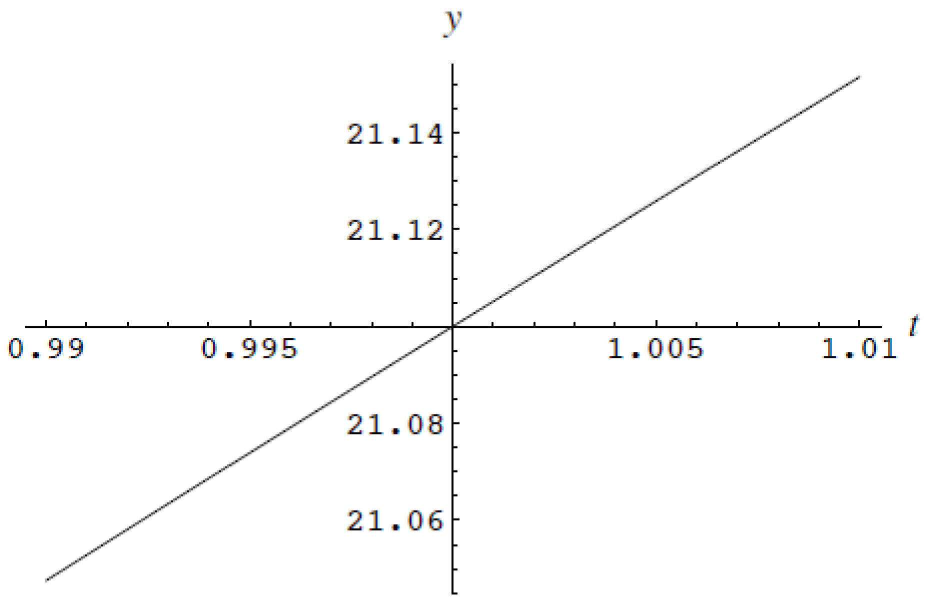


graph of $y = s(t)$ from $t = 0.9$ to 1.1 sec



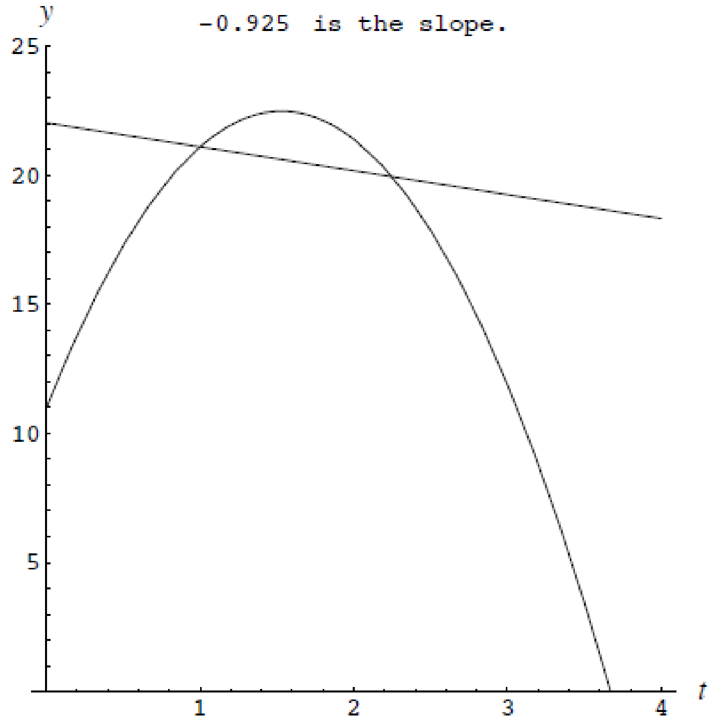
graph of $y = s(t)$ from $t = 0.99$ to 1.11 sec

Slope of Tangent Line

In general, the slope of the tangent line to a curve at a point is the derivative of the function at that point.

Definition The **slope** of a non-vertical line through two distinct points (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1) / (x_2 - x_1)$.

The following graphs illustrate that the secant line through t = 1 and t = 1 + Δt approaches the tangent line as Δt gets smaller.

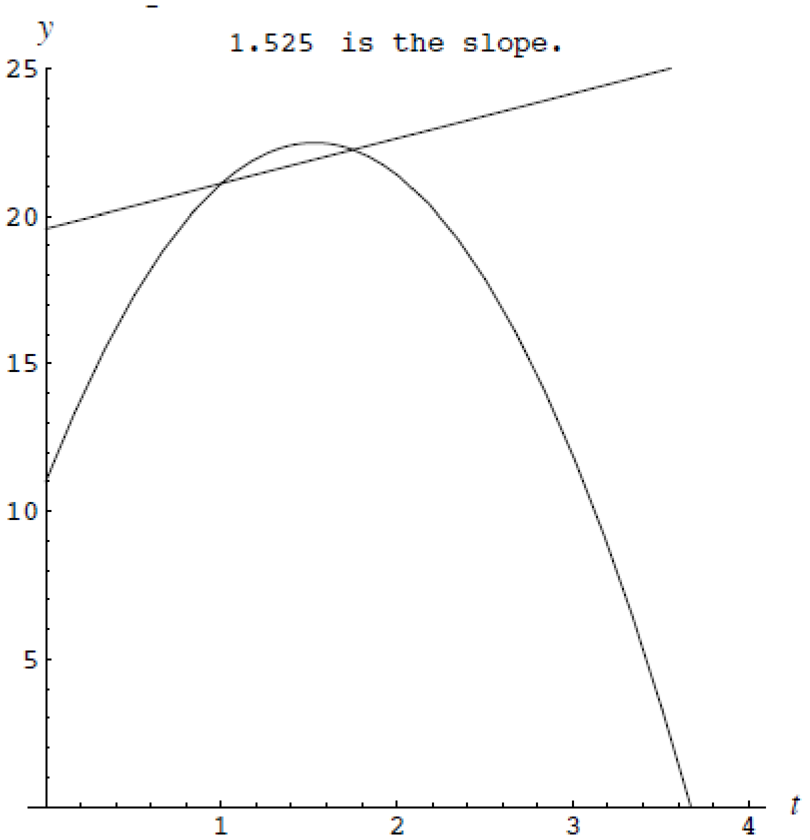


Secant line through (1, 21.1) and (2.25, 19.9437) with Δt = 1.25 sec and slope of -0.925 m/sec

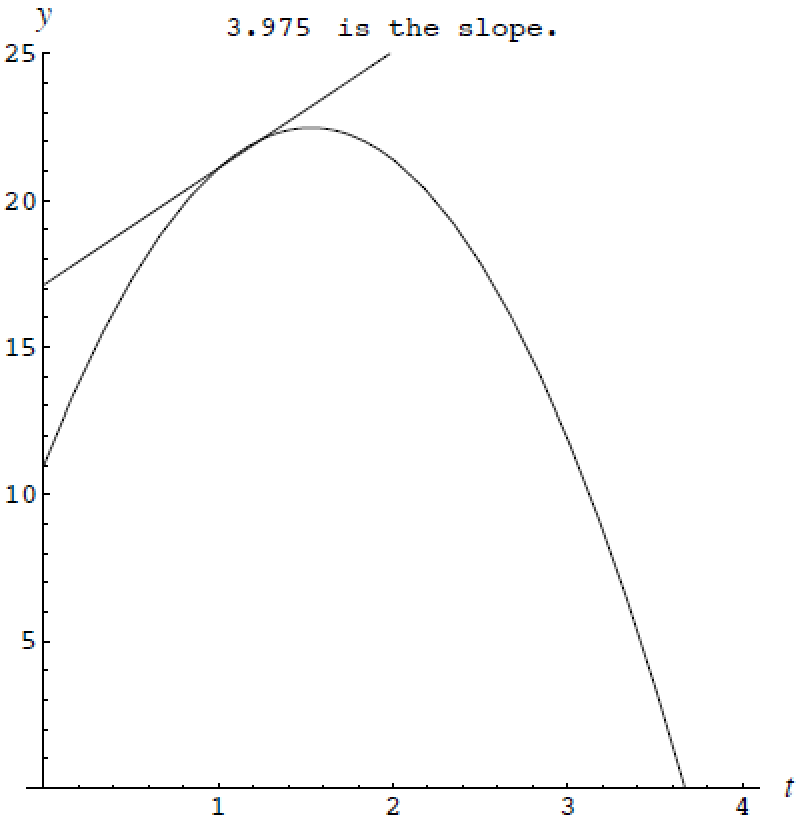
The change in time is $\Delta t = 1.25$, and the slope of the secant line is as follows:

$$\frac{19.9437 - 21.1}{1.25} = -0.925$$

Slope of Tangent Line



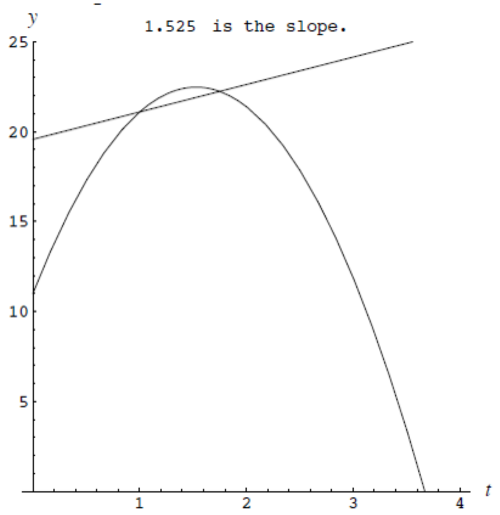
Secant line through (1, 21.1) and (1,75, 22.2438) with $\Delta t = 0.75$ sec and slope of 1.525 m/sec



Secant line through (1, 21.1) and (1.25, 22.0938) with $\Delta t = 0.25$ sec and slope of 3.975 m/sec

In general, the slope of the tangent line to a curve at a point is the derivative of the function at that point.

Slope of Tangent Line



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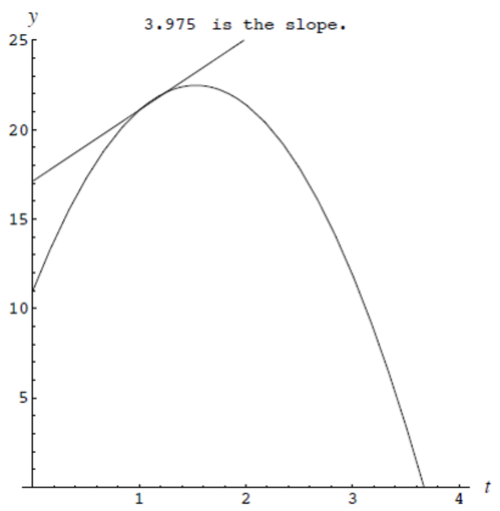

Secant line through (1, 21.1) and (1.25, 22.0938) with $\Delta t = 0.25$ sec and slope of 3.975 m/sec

- The secant lines approach the tangent line as Δt goes to 0.
- The slopes of given graphs' secant lines approach the slope of the tangent line at t = 1.
- The slopes of the secant lines through (1, 21.1) and (1 + Δt, s(1 + Δt)) are average velocities, and the limit as Δt approaches 0 is the instantaneous velocity, or the derivative, of the function s at 1.
- The derivative at a point is the slope of the tangent line to the curve at that point. We also call this slope the slope of the curve at that point.

Concept Geometrically, the **derivative** at a point is the slope of the tangent line to the curve at that point.

Differential Equations

A differential equation is an equation that contains a derivative.

For example, if *y* is a position of a ball above water at time *t*, then the rate of change, or derivative, of *y* with respect to *t* is the velocity. Suppose the velocity function is $v(t) = dy/dt = s'(t) = -9.8t + 15$, and the initial position, or initial condition, is $y_0 = s(0) = 11$. Thus, we have the following differential equation with initial condition:

$$dy/dt = -9.8t + 15 \text{ and } y_0 = 11$$

or

$$s'(t) = -9.8t + 15 \text{ and } s(0) = 11$$

Definitions A **differential equation** is an equation that contains one or more derivatives. An **initial condition** is the value of the dependent variable when the independent variable is zero. A **solution** to a differential equation is a function that satisfies the equation and initial condition(s).

Second Derivative