习题一：

19. 总体 $X \sim N(\mu, 4)$. 样本均值 $\bar{X} \sim N(\mu, \frac{4}{n})$，$Z \triangleq \frac{\bar{X}-\mu}{2/\sqrt{n}} \sim N(0,1)$

(2) $$E|Z| = \int_{-\infty}^{+\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} x \cdot e^{-\frac{x^2}{2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \left(-e^{-\frac{x^2}{2}} \Big|_0^{+\infty} \right) = \sqrt{\frac{2}{\pi}}$$

$$E(|\bar{X}-\mu|) = \frac{2}{\sqrt{n}} E\left(\left| \frac{\bar{X}-\mu}{2/\sqrt{n}} \right| \right) = \frac{2}{\sqrt{n}} \cdot \sqrt{\frac{2}{\pi}} \le 0.1 \Rightarrow n \ge \cdots$$

22. 总体 $X \sim N(\mu, \sigma^2)$，样本方差 S^2，$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

(1) $$ES^2 = \frac{\sigma^2}{n-1} E\left(\frac{(n-1)S^2}{\sigma^2}\right) = \cdots$$

$$DS^2 = \frac{\sigma^4}{(n-1)^2} D\left(\frac{(n-1)S^2}{\sigma^2}\right) = \cdots$$

(2) $n=16$.

$$P\left(\frac{S^2}{\sigma^2} \le 2.04\right) = P\left(\frac{15 S^2}{\sigma^2} \le 15 \times 2.04\right) = \cdots$$

26. 总体 $X \sim N(\mu, \sigma^2)$.

样本：$X_1, \cdots, X_n$，　样本均值 $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$，

样本二阶中心矩：$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$X_{n+1} \sim N(\mu, \sigma^2)$ 且与 $X_1, \cdots, X_n$ 独立.

$$T = \frac{X_{n+1} - \bar{X}}{S_n} \sqrt{\frac{n-1}{n+1}}$$

$X_{n+1} \sim N(\mu, \sigma^2)$，$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$，且 X_{n+1} 与 $\bar{X}$ 独立.

$$X_{n+1} - \bar{X} \sim N\left(0, \frac{n+1}{n}\sigma^2\right), \quad \frac{X_{n+1} - \bar{X}}{\sqrt{\frac{n+1}{n}}\,\sigma} \sim N(0,1)$$

又：$\frac{nS_n^2}{\sigma^2} \sim \chi^2(n-1)$ 且与 $X_{n+1} - \bar{X}$ 独立.

从而
$$T = \frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}} = \frac{\dfrac{X_{n+1} - \bar{X}}{\sqrt{\frac{n+1}{n}}\,\sigma}}{\sqrt{\dfrac{nS_n^2}{\sigma^2}\Big/(n-1)}} \sim t(n-1)$$

28. 总体 $X \sim N(0,1)$，样本 $X_1, \cdots, X_5$.

$X_1+X_2 \sim N(0,2)$，$X_3^2+X_4^2+X_5^2 \sim \chi^2(3)$，且相互独立

$$\Rightarrow \frac{\frac{X_1+X_2}{\sqrt{2}}}{\sqrt{(X_3^2+X_4^2+X_5^2)/3}} = \sqrt{\frac{3}{2}}\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2+X_5^2}} \sim t(3)$$

又 $\because$ $-(X_1+X_2) \sim N(0,2)$

$$\frac{-\frac{X_1+X_2}{\sqrt{2}}}{\sqrt{(X_3^2+X_4^2+X_5^2)/3}} = -\sqrt{\frac{3}{2}}\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2+X_5^2}} \sim t(3)$$

$\therefore$ $c = \pm\sqrt{\frac{3}{2}}$

29. $X_1, \cdots, X_m$，$Y_1, \cdots, Y_n$ 相互独立.

$X_i \sim N(a, \sigma^2)$. $Y_j \sim N(b, \sigma^2)$. $i=1,\cdots,m$，$j=1,\cdots,n$

$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i \sim N(a, \frac{\sigma^2}{m})$，$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N(b, \frac{\sigma^2}{n})$

$S_1^2 = \frac{1}{m}\sum_{i=1}^{n}(X_i-\bar{X})^2 \Rightarrow \frac{mS_1^2}{\sigma^2} \sim \chi^2(m-1)$

$S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 \Rightarrow \frac{nS_2^2}{\sigma^2} \sim \chi^2(n-1)$

$$T = \frac{\alpha(\bar{X}-a)+\beta(\bar{Y}-b)}{\sqrt{\frac{mS_1^2+nS_2^2}{m+n-2}}\cdot\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}}$$

$\bar{X}-a \sim N(0, \frac{\sigma^2}{m})$，$\bar{Y}-b \sim N(0, \frac{\sigma^2}{n})$，且相互独立.

$\xi \triangleq \alpha(\bar{X}-a)+\beta(\bar{Y}-b) \sim N\left(0, \left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)\sigma^2\right)$

$\frac{mS_1^2}{\sigma^2} \sim \chi^2(m-1)$，$\frac{nS_2^2}{\sigma^2} \sim \chi^2(n-1)$，且相互独立

$\eta \triangleq \frac{mS_1^2+nS_2^2}{\sigma^2} \sim \chi^2(m+n-2)$

且 ξ 与 η 相互独立 $\Rightarrow$ $T = \dfrac{\xi\Big/\left(\sigma\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}\right)}{\sqrt{\eta/(m+n-2)}} \sim t(m+n-2)$

31. $X_1,\cdots,X_{2n}$ 为总体 $X\sim N(\mu,\sigma^2)$ 的一样本. $\bar{X}=\frac{1}{2n}\sum_{i=1}^{2n}X_i$

$Y=\sum_{i=1}^{n}(X_i+X_{n+i}-2\bar{X})^2$, 求 EY.

设 $\xi_i=X_i+X_{n+i}$, $i=1,2,\cdots n$, $\xi_i\sim N(2\mu,2\sigma^2)$.

则 $\xi_1,\cdots,\xi_n$ 可看作来自总体 $\xi\sim N(2\mu,2\sigma^2)$ 的一个样本,

且其样本均值 $\bar{\xi}=\frac{1}{n}\sum_{i=1}^{n}(X_i+X_{n+i})=2\bar{X}$

$$Y=\sum_{i=1}^{n}(\xi_i-\bar{\xi})^2$$

又∵ $\dfrac{\sum_{i=1}^{n}(\xi_i-\bar{\xi})^2}{2\sigma^2}\sim\chi^2(n-1)$, $E\left[\dfrac{\sum_{i=1}^{n}(\xi_i-\bar{\xi})^2}{2\sigma^2}\right]=(n-1)$

从而 $EY=2(n-1)\sigma^2$.