习题二.

12. 总体 $X \sim f(x)=\frac{1}{2\theta}e^{-\frac{|x|}{\theta}}$, $\quad -\infty<x<+\infty$, $\quad \theta>0$.

(1) θ 的矩估计量

$EX=0$

$$EX^2=\int_{-\infty}^{+\infty}\frac{x^2}{2\theta}e^{-\frac{|x|}{\theta}}dx=\frac{1}{\theta}\int_0^{+\infty}x^2e^{-\frac{x}{\theta}}dx$$

令 $\frac{x}{\theta}=t$

$$=\theta^2\int_0^{+\infty}t^2e^{-t}dt=2\theta^2$$

令 $2\hat{\theta}^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2$, 得 θ 的矩估计量 $\hat{\theta}_1=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}X_i^2}$

或: $$E|X|=\int_{-\infty}^{+\infty}\frac{|x|}{2\theta}e^{-\frac{|x|}{\theta}}dx=\frac{1}{\theta}\int_0^{+\infty}xe^{-\frac{x}{\theta}}dx$$

令 $\frac{x}{\theta}=t$

$$=\theta\int_0^{+\infty}te^{-t}dt=\theta$$

令 $\theta=\frac{1}{n}\sum_{i=1}^{n}|X_i|$, 得 θ 的矩估计量 $\hat{\theta}_2=\frac{1}{n}\sum_{i=1}^{n}|X_i|$

14. 总体 $X\sim U(\theta_1,\theta_1+\theta_2)$, $\quad \theta_1\in R$, $\theta_2>0$.

(1) θ_1,θ_2 的矩估计量.

$\because X\sim U(\theta_1,\theta_1+\theta_2)$

$$EX=\frac{2\theta_1+\theta_2}{2}$$

$$DX=\frac{\theta_2^2}{12}$$

令 $$\begin{cases}\frac{2\theta_1+\theta_2}{2}=\overline{X}\\ \frac{\theta_2^2}{12}=S_n^2\end{cases},\qquad S_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

得 θ_1,θ_2 的矩估计量:

$$\hat{\theta}_1=\overline{X}-\sqrt{3}S_n,\qquad \hat{\theta}_2=2\sqrt{3}S_n$$

(2) 似然函数

$$L(\theta_1,\theta_2)=\begin{cases}\frac{1}{\theta_2^n}, & x_i\in[\theta_1,\theta_1+\theta_2],\ i=1,2,\cdots,n\\ 0, & \text{其它}\end{cases}$$

当 $x_i\in[\theta_1,\theta_1+\theta_2]$，$i=1,2,\cdots,n$，

即当 $\theta_1\le x_{(1)}$，$x_{(n)}\le\theta_1+\theta_2$ 时，$L(\theta_1,\theta_2)=\frac{1}{\theta_2^n}$ 关于 θ_2 为递减函数，θ_2 取越小，$L(\theta_1,\theta_2)$ 越大。

$\therefore$ 当 $\theta_1=x_{(1)}$，$\theta_2=x_{(n)}-x_{(1)}$ 时，$L(\theta_1,\theta_2)$ 取到最大值

$\Rightarrow$ θ_1,θ_2 的极大似然估计量：

$\hat\theta_1=X_{(1)}$，$\hat\theta_2=X_{(n)}-X_{(1)}$

15. 总体 X 的分布列

X	0	1	2	3
P	θ^2	$2\theta(1-\theta)$	θ^2	$1-2\theta$

$0<\theta<\frac{1}{2}$.

样本观测值：3, 1, 3, 0, 3, 1, 2, 3.

求 θ 的矩估计值和极大似然估计值.

解：(1) $EX=0\times\theta^2+1\times2\theta(1-\theta)+2\times\theta^2+3(1-2\theta)=3-4\theta$

令 $3-4\theta=\bar X$ 得 θ 的矩估计量：$\hat\theta_1=\frac{3-\bar X}{4}$.

θ 的矩估计值 $\hat\theta_1=\frac{3-\bar x}{4}=\frac{1}{4}$，其中 $\bar x=\frac{1}{8}(3+1+3+0+3+1+2+3)=2$.

(2) 样本观测值中有1个取值为0，2个取值为1，1个取值为2，4个取值为3.

故似然函数

$$L(\theta)=[\theta^2]^1\cdot[2\theta(1-\theta)]^2[\theta^2]^1[1-2\theta]^4=2^2\theta^6(1-\theta)^2(1-2\theta)^4$$

$$\ln L(\theta)=6\ln\theta+\ln4+2\ln(1-\theta)+4\ln(1-2\theta)$$

$$\frac{d\ln L(\theta)}{d\theta}=\frac{6}{\theta}-\frac{2}{1-\theta}-\frac{8}{1-2\theta}\hat=0 \quad 得\quad 12\theta^2-14\theta+3=0.$$

故 θ 的极大似然估计值 $\hat\theta_2=\frac{7-\sqrt{13}}{12}$.　（$\theta=\frac{7+\sqrt{13}}{12}$ 舍掉）

16、总体 $X \sim U[\theta-\frac{1}{2}, \theta+\frac{1}{2}]$.

(1) 证 θ 的极大似然估计不唯一.

似然函数

$$L(\theta)=\begin{cases}1, & x_i\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}],\ i=1,\cdots,n \\ 0, & 其它\end{cases}$$

当 $x_i\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}]$, $i=1,\cdots,n$,

即当 $\theta-\frac{1}{2}\le x_{(1)}\le x_{(n)}\le\theta+\frac{1}{2}$ 时, $L(\theta)$ 均取最大值1.

∴ 任取 $\hat{\theta}\in[X_{(n)}-\frac{1}{2}, X_{(1)}+\frac{1}{2}]$ 均为 θ 的极大似然估计.

(2) $\alpha=?$, $\hat{\theta}=\alpha X_{(n)}+(1-\alpha)X_{(1)}$ 为 θ 的无偏估计.

$$X\sim f(x)=\begin{cases}1, & x\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}] \\ 0, & 其它\end{cases}$$

$$F(x)=\begin{cases}0, & x<\theta-\frac{1}{2} \\ x-\theta+\frac{1}{2}, & x\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}] \\ 1, & x>\theta+\frac{1}{2}\end{cases}$$

$$f_{X_{(n)}}(x)=n[F(x)]^{n-1}f(x)$$
$$=\begin{cases}n(x-\theta+\frac{1}{2})^{n-1}, & x\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}] \\ 0, & 其它\end{cases}$$

$$EX_{(n)}=\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} nx(x-\theta+\frac{1}{2})^{n-1}\,dx$$

令 $x-\theta+\frac{1}{2}=t$

$$=\int_0^1 n(\theta-\frac{1}{2}+t)t^{n-1}\,dt$$
$$=(\theta-\frac{1}{2})t^n\Big|_0^1+\frac{n}{n+1}t^{n+1}\Big|_0^1=\theta-\frac{1}{2}+\frac{n}{n+1}$$

$$f_{X_{(1)}}(x)=n[1-F(x)]^{n-1}f(x)$$
$$=\begin{cases}n(\theta+\frac{1}{2}-x)^{n-1}, & x\in[\theta-\frac{1}{2}, \theta+\frac{1}{2}] \\ 0, & 其它\end{cases}$$

$$E X_{(1)} = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} n x (\theta+\tfrac{1}{2}-x)^{n-1} dx$$

令 $\theta+\frac{1}{2}-x=t$

$$= \int_0^1 n(\theta+\tfrac{1}{2}-t) t^{n-1} dt$$

$$= (\theta+\tfrac{1}{2}) t^n \Big|_0^1 - \frac{n}{n+1} t^{n+1} \Big|_0^1 = \theta+\frac{1}{2}-\frac{n}{n+1}$$

$$E[\alpha X_{(n)} + (1-\alpha) X_{(1)}]$$

$$= \alpha(\theta-\tfrac{1}{2}) + \frac{\alpha n}{n+1} + (1-\alpha)(\theta+\tfrac{1}{2}) - \frac{(1-\alpha)n}{n+1} = \theta$$

$\Rightarrow \alpha = \frac{1}{2}$.

24. 总体 $X \sim U[0,\theta]$，样本 $X_1, X_2, \cdots, X_n$.

(1) 证：$\hat{\theta}_1 = \frac{n+1}{n} X_{(n)}$，$\hat{\theta}_2 = (n+1) X_{(1)}$ 都是 θ 的无偏估计.

(2). 上述哪个估计方差更小？

解：

$$X \sim f(x) = \begin{cases} \frac{1}{\theta}, & 0<x<\theta, \\ 0, & 其它 \end{cases}, \quad F(x) = \begin{cases} 0, & x\le 0 \\ \frac{x}{\theta}, & 0<x<\theta \\ 1, & x\ge\theta \end{cases}$$

$$f_{X_{(1)}}(x) = n[1-F(x)]^{n-1} f(x) = \begin{cases} \frac{n(\theta-x)^{n-1}}{\theta^n}, & 0<x<\theta \\ 0, & 其它 \end{cases}$$

$$f_{X_{(n)}}(x) = n[F(x)]^{n-1} f(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & 0<x<\theta \\ 0, & 其它. \end{cases}$$

(1)

对 $\hat{\theta}_2$：

$$E X_{(1)} = \int_0^\theta x \cdot \frac{n(\theta-x)^{n-1}}{\theta^n} dx$$

令 $\theta-x=t$

$$= \int_0^\theta (\theta-t) \frac{n t^{n-1}}{\theta^n} dt = \int_0^\theta \frac{n t^{n-1}}{\theta^{n-1}} dt - \int_0^\theta \frac{n t^n}{\theta^n} dt$$

$$= \frac{1}{\theta^{n-1}} t^n \Big|_0^\theta - \frac{n}{n+1} \frac{t^{n+1}}{\theta^n} \Big|_0^\theta = \frac{\theta}{n+1}$$

$\therefore E[\hat{\theta}_2] = E[(n+1) X_{(1)}] = \theta$

故 $\hat{\theta}_2$ 为 θ 的无偏估计.

$$E[X_{(1)}^2] = \int_0^\theta x^2 \cdot \frac{n(\theta-x)^{n-1}}{\theta^n} dx \overset{令\ \theta-x=t}{=\!=} \int_0^\theta (\theta-t)^2 \frac{n t^{n-1}}{\theta^n} dt$$

$$= \int_0^\theta (\theta^2 - 2\theta t + t^2) \frac{n t^{n-1}}{\theta^n} dt$$

$$= \frac{1}{\theta^{n-2}} t^n \Big|_0^\theta - \frac{2}{\theta^{n-1}} \cdot \frac{n}{n+1} t^{n+1} \Big|_0^\theta + \frac{1}{\theta^n} \frac{n}{n+2} t^{n+2} \Big|_0^\theta$$

$$= \left(1 - \frac{2n}{n+1} + \frac{n}{n+2}\right)\theta^2 = \cdots = \frac{2\theta^2}{(n+1)(n+2)}$$

$$D X_{(1)} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \frac{n}{(n+1)^2(n+2)} \theta^2$$

$$D(\hat{\theta}_2) = D[(n+1) X_{(n)}] = \frac{n}{n+2}\theta^2$$

(2) 对 $\hat{\theta}_1$:

$$E X_{(n)} = \int_0^\theta x \cdot \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \cdot \frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$$

$\because E(\hat{\theta}_1) = E[\frac{n+1}{n} X_{(n)}] = \theta$ 故 $\hat{\theta}_1$ 为 θ 的无偏估计.

$$E[X_{(n)}^2] = \int_0^\theta x^2 \cdot \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \cdot \frac{x^{n+2}}{n+2}\Big|_0^\theta = \frac{n}{n+2}\theta^2$$

$$D X_{(n)} = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n}{(n+1)^2(n+2)}\theta^2$$

$$D(\hat{\theta}_1) = D[\frac{n+1}{n} X_{(n)}] = \frac{(n+1)^2}{n^2} \cdot \frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{1}{n(n+2)}\theta^2$$

当 $n > 1$ 时, $D(\hat{\theta}_1) < D(\hat{\theta}_2)$

28. 总体 $X \sim N(\mu, \sigma^2)$. μ 已知, σ^2 未知.

2个独立样本: $X_1, \cdots, X_m$ 和 $Y_1, \cdots, Y_n$.

$$\bar{X} = \frac{1}{m}\sum_{i=1}^m X_i, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$$

$$T_1 = \sum_{i=1}^m (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2,$$

$$T_2 = \sum_{i=1}^m (X_i - \mu)^2 + \sum_{i=1}^n (Y_i - \mu)^2.$$

(1) 求 $\frac{T_i}{\sigma^2}$ 的分布，$(i=1,2)$

由题意：$\frac{\sum_{i=1}^{m}(X_i-\bar{X})^2}{\sigma^2}\sim\chi^2(m-1)$，$\frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma^2}\sim\chi^2(n-1)$，且相互独立，

由 χ^2 分布的可加性

$$\frac{T_1}{\sigma^2}=\frac{\sum_{i=1}^{m}(X_i-\bar{X})^2}{\sigma^2}+\frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma^2}\sim\chi^2(m+n-2).$$

另：$\frac{\sum_{i=1}^{m}(X_i-\mu)^2}{\sigma^2}\sim\chi^2(m)$，$\frac{\sum_{i=1}^{n}(Y_i-\mu)^2}{\sigma^2}\sim\chi^2(n)$，且相互独立。

$$\frac{T_2}{\sigma^2}=\frac{\sum_{i=1}^{m}(X_i-\mu)^2}{\sigma^2}+\frac{\sum_{i=1}^{n}(Y_i-\mu)^2}{\sigma^2}\sim\chi^2(m+n).$$

(2) $T_i^*=C_iT_i$，求 C_i 使 T_i^* 为 σ^2 的无偏估计 $(i=1,2)$.

$$ET_1^*=E[C_1T_1]=C_1\sigma^2E[\frac{T_1}{\sigma^2}]=C_1(m+n-2)\sigma^2=\sigma^2$$

$$\therefore C_1=\frac{1}{m+n-2}$$

$$ET_2^*=E[C_2T_2]=C_2\sigma^2E[\frac{T_2}{\sigma^2}]=C_1(m+n)\sigma^2=\sigma^2$$

$$\therefore C_2=\frac{1}{m+n}$$

(3) 当 μ 已知时，T_1^*，T_2^* 哪个估计 σ^2 较优

T_1^*，T_2^* 均为 σ^2 的无偏估计

$$D(T_1^*)=D(C_1T_1)=C_1^2\sigma^4D(\frac{T_1}{\sigma^2})=\frac{\sigma^4}{(m+n-2)^2}\cdot2(m+n-2)=\frac{2\sigma^4}{m+n-2}$$

$$D(T_2^*)=D(C_2T_2)=C_2^2\sigma^4D(\frac{T_2}{\sigma^2})=\frac{\sigma^4}{(m+n)^2}\cdot2(m+n)=\frac{2\sigma^4}{m+n}$$

$\because D(T_2^*)<D(T_1^*)$. 故 T_2^* 较优

46. 总体 $X \sim N(\mu, \sigma^2)$. 样本 $X_1, \cdots, X_{2n}$.

$\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\quad T = \sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=n+1}^{2n}(X_i - \mu)^2$

(1) $\frac{T}{\sigma^2}$ 的分布.

$\because \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$, $\quad \frac{\sum_{i=n+1}^{2n}(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$, 且相互独立

$$\Rightarrow \frac{T}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} + \frac{\sum_{i=n+1}^{2n}(X_i - \mu)^2}{\sigma^2} \sim \chi^2(2n-1).$$

(2) 当 μ 已知, 基于 T 构造 σ^2 的置信水平为 $1-\alpha$ 的置信区间.

$\because \quad \frac{T}{\sigma^2} \sim \chi^2(2n-1)$.

$$P\left(\chi^2_{1-\frac{\alpha}{2}}(2n-1) < \frac{T}{\sigma^2} < \chi^2_{\frac{\alpha}{2}}(2n-1)\right) = 1-\alpha$$

$$P\left(\frac{T}{\chi^2_{\frac{\alpha}{2}}(2n-1)} < \sigma^2 < \frac{T}{\chi^2_{1-\frac{\alpha}{2}}(2n-1)}\right) = 1-\alpha$$

$\therefore$ σ^2 的置信水平为 $1-\alpha$ 的置信区间为:

$$\left(\frac{T}{\chi^2_{\frac{\alpha}{2}}(2n-1)}, \quad \frac{T}{\chi^2_{1-\frac{\alpha}{2}}(2n-1)}\right)$$

1 总体 $X \sim N(\mu_1, \sigma^2)$，样本 $X_1, \cdots, X_{n_1}$，$\overline{X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$

$Y \sim N(\mu_2, \sigma^2)$，$Y_1, \cdots, Y_{n_2}$，$\overline{Y} = \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i$

两样本相互独立.

$S_1^{*2} = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i - \overline{X})^2$，$S_2^{*2} = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(Y_i - \overline{Y})^2$

① 证：对任常数 a, b $(a+b=1)$，$T = aS_1^{*2} + bS_2^{*2}$ 都为 σ^2 的无偏估计；

② 并确定 a, b 的值，使 DT 达到最小.

解：$\because \frac{(n_1-1)S_1^{*2}}{\sigma^2} \sim \chi^2(n_1-1)$，$\frac{(n_2-1)S_2^{*2}}{\sigma^2} \sim \chi^2(n_2-1)$

$$ES_1^{*2} = \frac{\sigma^2}{n_1-1} E\left(\frac{(n_1-1)S_1^{*2}}{\sigma^2}\right) = \frac{\sigma^2}{n_1-1}(n_1-1) = \sigma^2$$

$$DS_1^{*2} = \frac{\sigma^4}{(n_1-1)^2} D\left(\frac{(n_1-1)S_1^{*2}}{\sigma^2}\right) = \frac{\sigma^4}{(n_1-1)^2} 2(n_1-1) = \frac{2\sigma^4}{n_1-1}$$

同理：$ES_2^{*2} = \sigma^2$，$DS_2^{*2} = \frac{2\sigma^4}{n_2-1}$

① $ET = E(aS_1^{*2} + bS_2^{*2}) = (a+b)\sigma^2 = \sigma^2$ $(a+b=1)$

② $\because S_1^{*2}$ 与 S_2^{*2} 相互独立.

$$DT = D(aS_1^{*2} + bS_2^{*2}) = a^2 D(S_1^{*2}) + (1-a)^2 D(S_2^{*2})$$

$$= \left[\frac{a^2}{n_1-1} + \frac{(1-a)^2}{n_2-1}\right] \cdot 2\sigma^4$$

令 $f(a) = \frac{a^2}{n_1-1} + \frac{(1-a)^2}{n_2-1}$

$f'(a) = \frac{2a}{n_1-1} + \frac{2(a-1)}{n_2-1} \triangleq 0$　　$f''(a) = \frac{2}{n_1-1} + \frac{2}{n_2-1} > 0$

$\frac{a(n_2-1) + (a-1)(n_1-1)}{(n_1-1)(n_2-1)} = 0$　　$a(n_1+n_2-2) = n_1-1$

$$\Rightarrow a = \frac{n_1-1}{n_1+n_2-2}$$

$\therefore$ 当 $a = \dfrac{n_1 - 1}{n_1 + n_2 - 2}$，$b = \dfrac{n_2 - 1}{n_1 + n_2 - 2}$ 时，DT 取到最小值.