Convex Optimization

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Contents

Chapter 1

Convex Sets

1.1 Affine Sets

Theorem 1.1.1 (Affine)**.**

$$x = \theta x_1 + (1-\theta)x_2.$$

Definition 1.1.1 (Affine Set)**.** Contains the line through any two distinct points in a set

Example 1.1.1. Solution of set of linear equations $x|Ax = b$ is an affine set.

Proof.

$$\begin{aligned} Ax_1 &= b \\ Ax_2 &= b \\ Ax = A(\theta x_1 + (1-\theta)x_2) &= \\ \theta Ax_1 + (1-\theta)Ax_2 \\ = \theta b + (1-\theta)b = b. \end{aligned}$$

□

Definition 1.1.2 (Line Segment)**.**

$$x = \theta x_1 + (1-\theta_2)x_2, 0 \leq \theta \leq 1.$$

Definition 1.1.3 (Convex Set)**.** Contains line segment between any two poitns in the set:

$$x1, x2 \in C, 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta)x_2 \in C.$$

Remark 1.1.1. This is analagous to a clear line path between any two points in the set

Definition 1.1.4 (Convex Combination)**.** Convex combination of $x_1, \ldots, x_n$: any point x of the form:

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k.$$

with $\theta_1 + \ldots + \theta_k = 1$, $\theta_i \geq 0$

Definition 1.1.5 (Convex Hull)**.** Set of all convex combinations of points in a set S

Remark 1.1.2. The convex hull of a set S is denoted $\mathbf{conv}S$

Think of the convex hull as the smallest convext region encompassing a set

Definition 1.1.6 (Conic (nonnegative) combination)**.** The conic combination of x_1 and x_2 : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2.$$

with $\theta_1 \geq 0, \theta_2 \geq 0$

Definition 1.1.7 (Convex Cone)**.** Set that contains all conic combinations of points in the set

1.2 Common Sets

Definition 1.2.1 (Hyperplane)**.** Set of the form $\{x | a^T x = b\} (a \neq 0)$

Remark 1.2.1. In $\mathbb{R}^2$, this is a line

Definition 1.2.2 (Halfspace)**.** set of the form $\{x | a^T x \leq b\} (a \neq 0)$

1. a is the normal vector
2. hyperplanes are affine and convex, halfspaces are convex

Definition 1.2.3 (Euclidean Ball)**.**

$$B(x_c, r) = \{x | \|x - x_c\|_2 \leq r\} = \{x_c + ru | \|u\|_2 \leq 1\}.$$

Definition 1.2.4 (Ellipsoid)**.** Set of the form:

$$\{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\}.$$

with $P \in S^n_{++}$ (i.e. P is symmetric positive definite)

Remark 1.2.2. Square roots of the eigenvalues of P are the lengths of the semi-axis of the ellipse

1.3 Norm balls and Norm Cones

Definition 1.3.1 (Norm)**.** a function that satisfies:

1. $\|x\| \geq 0 : \|x\| = 0 \iff x = 0$
2. $\|tx\| = |t|\|x\|$ for $t \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\|$

Definition 1.3.2 (Norm Ball)**.** with center x_c and radius r :

$$\{x \mid \|x - x_c\| \leq r\}.$$

Definition 1.3.3 (Norm Cone)**.**

$$\{(x, t) \mid \|x\| \leq t\}.$$

$x \in \mathbb{R}^n, t \in \mathbb{R}$

Remark 1.3.1. These are convex

Definition 1.3.4 (Polyhedra)**.** Solution set of finitely many lienar inequalities and equalities:

$$Ax \leq b, Cx = d.$$

($A \in \mathbb{R}^{m \times n}, \leq$ is componentwise inequality)

Remark 1.3.2. Think of this as an intersection of a finite number of halfspaces

Definition 1.3.5 (Positive Semidefinite Cone)**.** Notation:

1. S^n is a set up symmetric $n \times n$ matrices
2. $S^n_+ = \{X \in S^n \mid X \geq 0\}$: positive definite $n \times n$ matrices

$$X \in S^n_+ \iff z^T X z \geq 0 \text{ for all } z.$$

S^n_+ is a convex cone
$S^n_{++} = \{X \in S^n \mid X > 0\}$: positive definite $n \times n$ matrices

Example 1.3.1.

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+.$$

1.4 Operations that preserve convexity

1. Apply original definition: $\theta x_1 + (1 - \theta) x_2 \in C$
2. Show that C is obtained from exiample convex sets by operations that preserve convexity:
   (a) intersection

(b) affine functions

(c) perspective function

(d) linear fractional functions

Theorem 1.4.1 (Intersection)**.** *The intersection of any number of convex sets is convex*

Definition 1.4.1 (Affine Function)**.** Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$)

1. the image of a convex set under f is convex

$$S \in \mathbb{R}^n \text{ convex } \implies f(S) = \{f(x) \mid x \in S\} \text{ convex.}$$

2. the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \in \mathbb{R}^m \text{convex } \implies f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex.}$$

Remark 1.4.1. $f^{-1}(C)$ does not require that f is invertible, rather all points that have an image in C will be included even if they map to the same element in C

Definition 1.4.2 (Perspective function)**.** $P : \mathbb{R}^{n+1} \to R^n$:

$$P(x,t) = \frac{x}{t}, \ \mathbf{dom} P = \{(x,t) \mid t > 0\}.$$

images and inverse images of convex sets underp erspective are convex

Remark 1.4.2. This has the effect of normalizing by the last component

1.5 Generalized Inequalities

Definition 1.5.1 (Proper cone)**.** A convex cone $K \in \mathbb{R}^n$ is a proper cone if:

1. K is closed

2. K is solid (has nonempty interior)

3. K is pointed (contains no line, i.e. a ray and its negative cannot both be in the cone)

Definition 1.5.2 (Generalized inequality defined by proper cone K)**.**

$$x \leq_K y \iff y - x \in K, \ x \leq_K y \iff y - x \in \mathbf{int} K.$$

Definition 1.5.3 (Minimum Element)**.** $x \in S$ is minimum with respect to $\leq_K$ if:

$$y \in S \implies x \leq_K y.$$

Definition 1.5.4 (Minimal Element)**.** $x \in S$ is the minimal element of S with respect to $\leq_K$ if

$$y \in S, y \leq_K x \implies y = x.$$

Remark 1.5.1. Minimal implies that if you are in the set **and** comparible to the point, then you are greater than or equal to the point that is being compared

Theorem 1.5.1 (Separating Hyperplane Theorem)**.** *if C and D are disjoint convex sets, then there exists $a \neq 0, b$ such that:*

$$a^T x \leq b \text{ for } x \in C, \;\; a^T x \geq b \text{ for } x \in D.$$

the hyperplane $\{x \mid a^T x = b\}$ separates C and D

Remark 1.5.2. This is a **linear clasifier** in machine learning

Theorem 1.5.2 (Supporting Hyperplane Theorem)**.** *supporting hyperplane to a set C at boundary point x_0 :*

$$\{x \mid a^T x = a^T x_0\}.$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

Remark 1.5.3. This implies that there is a halfspace of the supporting hyperplane that contains C

1.6 Dual cones and Generalized Inequalities

Definition 1.6.1 (Dual Cone)**.** dual cone of a cone K :

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

set of all vectors that make a non-negative inner product with all vectors in K

Example 1.6.1. Few examples:

1. $K = \mathbb{R}^n_+ : K* = \mathbb{R}^n_+$
2. $K = S^n_+ : K* = S^n_+$

Chapter 2

Convex Functions

Definition 2.0.1 (Convex Function)**.** $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{com}f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y).$$

for all $x, y \in \mathbf{dom}f$, $0 \leq \theta \leq 1$

Remark 2.0.1. properties:

1. f is concave if $-f$ is convex
2. f is strictly convex if $\mathbf{dom}f$ is convex and

   $$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y).$$

   for $x, y \in \mathbf{dom}f, x \neq y, 0 < \theta < 1$

Example 2.0.1. Convex:

1. affine: $ax + b$ on $\mathbb{R}$
2. exponential: e^{ax} for $a \in \mathbb{R}$
3. powers: x^α on $\mathbb{R}_{++}$ for $\alpha \geq 1$ or $\alpha \leq 0$
4. powers of absolute values: $|x|^p$ on $\mathbb{R}$
5. negative entropy: $x \log(x)$ on R_{++}

Concave:

1. affine: $ax + b$
2. powers: x^α on R_{++} for $0 \leq \alpha \leq 1$
3. logarithm: $\log(x)$ on R_{++}

2.1 Restriction of convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$:

$$g(t) = f(x + tv), \textbf{dom } g = \{t \mid x + tv \in \textbf{dom} f\}.$$

is convex (in t) for any $x \in \textbf{dom} f, v \in \mathbb{R}^n$

This is to say that we check the convexity of f by checking convexity of functions of one variable repeatedly.

2.2 Conditions

f is **differentiabile** if $\textbf{dom} f$ is open and the gradient

$$\nabla f(x) = (\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}).$$

exists at each $x \in \textbf{dom } f$

Definition 2.2.1 (First Order Condition)**.** differentiable f w ith convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \text{ for all } x, y \in \textbf{dom} f.$$

Remark 2.2.1. This gradient term is the first order approximation of f, called a global underestimator

Definition 2.2.2 (Twice Differentiable)**.** f is twice differentiable if **dom**f is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \ i, j = 1, \ldots, n.$$

exists at each $x \in \textbf{dom} f$

Definition 2.2.3 (Second Order Conditions)**.** For twice differentable f with convex domain:

1. f is convex iff
$$\nabla^2 f(x) \geq 0 \text{ for all } x \in \textbf{dom} f.$$
2. if $\nabla^2 f(x) > 0$ for all $x \in \textbf{dom} f$ then f is strictly convex

Example 2.2.1. Examples of second order conditions being met:

1. quadratic function: $f(x) = (\frac{1}{2}) x^T P x + q^T x + r$ (with $P \in S^n$)
$$\nabla f(x) = Px + q, \ \nabla^2 f(x) = P.$$
convex if $P \geq 0$

2. Least Squares Objective: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^T(Ax - b), \nabla^2 f(x) = 2A^T A.$$

convex for any A

3. Quadratic over linear: $f(x, y) = \frac{x^2}{y}$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \geq 0.$$

2.3 Epigraph and Sublevel Set

Definition 2.3.1 (Alpha-Sublevel Set)**.** α-sublevel set of $f : \mathbb{R}^n \to R$:

$$C_\alpha = \{x \in \textbf{dom} f \mid f(x) \leq \alpha\}.$$

sublevel sets of convex functions are convex (converse is false)

Definition 2.3.2 (Epigraph)**.** Epigraph of $f : \mathbb{R}^n \to \mathbb{R}$

$$\textbf{epi} f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \textbf{dom} f, f(x) \leq t\}.$$

f is convex iff $\textbf{epi} f$ is a convex set

2.4 Preservation of Convexity

1. **Nonnegative multiple:** αf is convex if f is convex and $\alpha \geq 0$
2. **Sum:** $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)
3. **Composition with affine function:** $f(Ax + b)$ is convex if f is convex
4. **Pointwise Maximum:** if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex
5. **Composition of Scalar Functions**: $g : \mathbb{R}^n \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$:

   $$f(x) = h(g(x)).$$

   f is convex if g convex, h convex, extended value extention of h is non-decreasing.
6. **Vector Composition:** composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$:

   $$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x)).$$

   f is convex if g_i convex, h convex, extended value extention of h is non-decreaseing in each argument
7. **Minimization**: if $f(x, y)$ is convex in (x, y) and C is a convex set, then

   $$g(x) = \inf_{y \in C} f(x, y).$$

   is convex

2.5 Conjugate Function

Definition 2.5.1 (Conjugate)**.** The conjugate of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom} f}(y^T x - f(x)).$$

Theorem 2.5.1 (Convexity of the Conjugate)**.** *f^* is convex, as $y^T x - f(x)$ is affine, thus we are taking the supremum with respect to x of a family of affine (convex) functions, which is inherently convex.*

2.6 Quasiconvex Functions

Definition 2.6.1 (Quasi-Convex)**.** $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $\mathbf{dom} f$ is convex and the sublevel sets:

$$S_\alpha = \{x \in \mathbf{dom} f \mid f(x) \leq \alpha\}.$$

are convex for all α

1. f is quasiconcave if $-f$ is quasiconvex

2. f is quasilinear if it is quasiconvex and quasiconcave

Remark 2.6.1. In $\mathbb{R}^n$, this has the interpretation of being monotone decreasing up to somepoint, then monotone increasing after that point

Definition 2.6.2 (Log Concave)**.** a positive function f is log-concave if $\log(f)$ is concave:

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta} \text{ for } 0 \leq \theta \leq 1.$$

f is log-convex if $\log f$ is convex

Remark 2.6.2. Almost all statistical densities are log-concave

Chapter 3

Duality

3.1 Lagrangian

We start with the standard form problem, but this time we don't restrict the objective $f_0(x)$ or the constraints, $f_i(x)$ to be convex:

Definition 3.1.1 (Lagrangian)**.** $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ with $\mathbf{dom} L = D \times \mathbb{R}^m \times \mathbb{R}^p$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x).$$

This is a weighted sum of objective and constraint functions, where λ_i is the Lagrange multiplier associated with $f_i(x) \leq 0$ and ν_i is the Lagrange multiplier associated with $h_i(x) = 0$. Think of $h_i(x)$ as a residual - 0 if the constraint is satisfied.

Definition 3.1.2 (Lagrange Dual Function)**.** $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbb{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathbb{D}} (f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)). \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

Remark 3.1.1. One interpretation of this is the optimal cost with the prices λ_i, ν_i

Corollary 3.1.0.1. *g is concave even if the original problem is non-convex*

Proof. g is the infimum over a family of affine functions, and is thus convex as proved last chapter $\square$

Theorem 3.1.1 (Lower Bound Property)**.** *if* $\lambda \geq 0$*, then* $g(\lambda, n) \leq p^*$ *where* p^* *is the optimal solution*

Proof. If is feasible and $\lambda \geq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in D} L(x, \lambda, \nu) = g(\lambda, \nu).$$

minimizing over all feasible $\tilde{x}$ gives $p^* \geq g(\lambda, \nu)$ □

3.2 Least-Norm Solution of Linear Equations

$$\text{minimize } x^T x$$
$$\text{subject to } Ax = b.$$

Dual Function:

1. Lagrangian is $L(x, \nu) x^T x + \nu^T (Ax + b)$
2. to minimize L over x, set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \implies x = -(\frac{1}{2}) A^T \nu.$$

3. plug in L to obtain g :

$$g(\nu) = L((-\frac{1}{2}) A^T \nu, \nu) = -\frac{1}{4} \nu^T A \cdot A^T \nu - b^T \nu.$$

a concave function of ν

Lower Bound Property:

$$p^* \geq -(\frac{1}{4}) \nu^T A \cdot A^T \nu - b^T \nu \text{ for all } \nu.$$

Example 3.2.1.

$$\text{minimize } f_0(x)$$
$$\text{subject to } Ax \leq b, Cx = d.$$

Dual Function:

$$\begin{aligned} g(\lambda, x) &= \inf_{x \in \mathbf{dom} f_0} (f_0(x) + (A^T \lambda + C^T \nu)^T x - b^T \lambda - d^T \nu) \\ &= -f_0^*(-A^T \lambda - C^T \nu) - b^T \lambda - d^T \nu. \end{aligned}$$

Recall that the definition of conjugate $f^*(y) = \sup_{x \in \mathbf{dom} f} (y^T x - f(x))$ Simpliefies derivation of dual if conjuage of f_0 is known

3.3 The Dual Problem

Definition 3.3.1 (Lagrange Dual Problem)**.**

$$\begin{array}{rl} \text{Maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0. \end{array}$$

- Finds best lower bound on p^* obtained from Lagrange dual function
- A convex optimization problem, optimal value d^*
- λ, ν are dual feasible if $\lambda \geq 0$, $(\lambda, \nu) \in \mathbf{dom} g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom} g$

Theorem 3.3.1 (Weak Duality)**.**

$$d^* \leq p^*.$$

- *Always holds (for convex or nonconvex)*
- *Can be used to find nontrivial lower bounds for difficult problems*

Theorem 3.3.2 (Strong Duality)**.**

$$d^* = p^*.$$

- *Does not hold in general*
- *Usually holds for convex problems*
- ***Constraint qualitifcations** guarantee strong duality*

Remark 3.3.1. $p^* - d^*$ is known as the **duality gap**

Definition 3.3.2 (Complimentary Slackness)**.** Assume strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual optimal

$$\begin{aligned} f_0(x^*) = g(\lambda^*, \nu^*) &= \inf_x (f_0(x) + \sum_{i=0}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu^* h_i(x)) \\ &\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\ &\leq f_0(x^*). \end{aligned}$$

thus, the two inequalities hold with equality

- x^* minimizes $L(x, \lambda^*, \nu^*$
- $\lambda_i^* f_i(x^*)$ for $i = 1, \ldots, m$ (known as complementary slackness):

$$\lambda_i^* > 0 \implies f_i(x^*) = 0, \qquad f_i(x^*) < 0 \implies \lambda_i^* = 0.$$

3.4 KKT Conditions

The following conditions are KKT conditions for a problem with differentiable f_i, h_i:

1. primal constraints: $f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p$
2. dual constraints: $\lambda \geq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0, i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0.$$

   If strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

The converse also holds: if $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

3.5 Perturbation and Sensitivity Analysis

Unperturbed optimization problem and its dual:

$$\begin{array}{ll} \text{mimimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

Perturbed problem and its dual:

$$\begin{array}{ll} \text{mimimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq u_i, \quad i = 1, \ldots, m \\ & h_i(x) = v_i, \quad i = 1, \ldots, p. \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) - u^T \lambda - v^T \nu \\ \text{subject to} & \lambda \geq 0 \end{array}$$

- x is primal variable; u, v are parameters
- $p^*(u, v)$ is optimal value as a function of u, v
- We want info about $p^*(u, v)$ that we can obtain from the solution of the unperturbed problem and its dual

Global Sensitivity: Assume strong duality holds for unperturbed problem, and that λ^*, ν^* are dual optimal for unperturbed problem:

Apply weak duality to perturbed problem:

$$\begin{aligned} p^*(u,v) \geq & \quad g(\lambda^*, \nu^*) - u^T\lambda^* - v^T\nu^* \\ = & \quad p^*(0,0) - u^T\lambda^* - v^T\nu^*. \end{aligned}$$

Sensitivity Interpretation:

- if λ_i^* large: p^* increases greatly if we tighten constraint i, $(u_i < 0)$
- if λ_i small: p^* does not decrease much is we loosen constraint i, $(u_i > 0)$
- if ν_i large and positive: p^* increases greatly if we take $v_i < 0$;
  if ν_i large and negative: p^* increases greatly if we take $v_i > 0$
- if ν_i small and positive: p^* does not decrease much if we take $v_i > 0$;
  if ν_i^* small and negative: p^* does not decrease much if we take $v_i < 0$

Local Sensitivity: if, in addition, $p^*(u,v)$ is differentiable at $(0,0)$, then:

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \qquad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}.$$

Proof. From global sensitivity result:

$$\begin{aligned} \frac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \searrow 0} \frac{p^*(te_i,0) - p^*(0,0)}{t} \geq -\lambda_i^* \\ \frac{\partial p^*(0,0)}{\partial u_i} = \lim_{t \nearrow 0} \frac{p^*(te_i,0) - p^*(0,0)}{t} \leq -\lambda_i^*. \end{aligned}$$

hence, equality $\square$