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Minimal Surfaces and Geometric Measure Theory

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Abstract

In this thesis we will develop the theory of minimal surfaces from the point of view of geometric measure theory. We will first look at minimal surfaces in 3 dimensions, and then generalize the tools and concepts to higher dimensions and more general classes of surfaces. We will develop the basics of geometric measure theory, that will lay the groundwork for introducing rectifiable varifolds, and we shall see various results on the regularity of varifolds, culminating in the main theorem of the thesis, the Allard regularity theorem.

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Chapter 1

Introduction

In this thesis, we will develop the theory of minimal surfaces from the point of view of geometric measure theory. As motivation, we will, in the introduction, quickly look at minimal surfaces in 3-dimensional space. We shall see what a minimal surface is, and see some examples.

In the main part of the thesis, we will then try and generalize the idea of a minimal surface. We shall take the procedure and tools from the 3-dimensional case, and generalize them to higher dimensions, and a broader class of surfaces. To that end the first few chapters are going to lay the groundwork, with the basics of geometric measure theory, and then the last chapters will introduce varifolds, which will be our generalized surfaces. We will define the analogous notion of a "minimal" varifold, and we shall show some regularity results about varifolds ultimately seeking to show Allard's regularity theorem.

This thesis follows various sources, but will try and give a gentler and easier introduction to the topic than for instance the canonical textbook in the field, [6], which is a bit difficult to read for those uninitiated. Hopefully this thesis will help in that aspect, by clearing up some ambiguities, and fleshing out important details where needed.

The introduction takes roots in the examples from [3], and then tries to give a quick look at the 3-dimensional minimal surfaces based on those.

The following chapter, on the Hausdorff measure and area formulae, are meant as preliminary chapters, and will introduce basic but very important concepts to be able to discuss generalized minimal surfaces, especially the Hausdorff measure will be present a.e. in the thesis. This chapters takes inspiration from [4].

The following chapters are then going to build up to varifolds and then show various results, culminating in Allard's regularity theorem. See [4], [6] and [1] amongst others.

But before all that...

1.1 Minimal surfaces in $\mathbb{R}^3$

We are interested in finding surfaces, bounded by closed curves, which minimizes their area. I.e. of all the surfaces spanning a given boundary, we seek the surface(s) with the least area of those.

Minimal surface theory lies at the intersection of many different fields of mathematics. This becomes clear when one sees the many different equivalent definitions of a minimal surface. Indeed a surface in $\mathbb{R}^3$ is said to be minimal if it satisfies one of the following equivalent definitions.

1. *Local area-minimising definition.* A surface $S \subseteq \mathbb{R}^3$ is minimal if around each point $p \in S$ there is a neighborhood which has minimal area with respect to its boundary. [3]

   This local definition is actually not equivalent to the others, as there might be surfaces with smaller area, but with the same boundary.

2. *Variational definition.* A surface $S \subseteq \mathbb{R}^3$ is minimal if and only if it a critical point of the area functional for all compactly supported variations.

   This definition shows that minimal surfaces are analogous to geodesics. [3]

3. *Mean Curvature definition.* A surface $S \subseteq \mathbb{R}^3$ is minimal if and only if its mean curvature is zero. [3]

4. *Gauss map definition*: A surface $M \subseteq R^3$ is minimal if and only if its stereographically projected Gauss map $g : M \to C \cup \{\infty\}$ is meromorphic with respect to the underlying Riemann surface structure, and M is not a piece of a sphere.[3]

See [3] for even more equivalent definitions.

We shall be most interested in the third definition when we try and generalize minimal surfaces. However for the 3-dimensional case, we shall also make use of the second definition.

Let us briefly discuss the variational definition of a minimal surface in $\mathbb{R}^3$. This brief discussion will provide a scaffold on which to build our more general theory of minimal surfaces, as the tools and process shown here will be generalizable to higher dimensions, and a wider class of surfaces.

Let $U \subseteq \mathbb{R}^3$ be open and bounded, let $u : U \to \mathbb{R}$ be C^2 and let

$$\Gamma_u := \{(x, y, u(x, y)) \mid (x, y) \in U\} \subseteq \mathbb{R}^3$$

be the graph of u. Then Γ_u is a 2-dimensional submanifold of $\mathbb{R}^3$, which for every $p = (x, y, u(x, y)) \in \Gamma_u$ has the tangent space $T_p\Gamma_u$ spanned by the vectors $(1, 0, u_x)$ and $(0, 1, u_y)$, where u_x and u_y denote the partial derivatives of u with respect to x and y respectively.

We are interested in the area of Γ_u as a submanifold, in order to be able to determine if it has least area. Now, the area of the graph Γ_u is given by

$$\begin{aligned}\operatorname{Area}(\Gamma_u) &= \int_U |(1,0,u_x)\times(0,1,u_y)|\,d\lambda \\ &= \int_U \sqrt{1+|\nabla u|^2}\,d\lambda. \end{aligned} \tag{1.1}$$

Where λ is the 1-dimensional Lebesgue measure. Now, we are going to perturb the graph by a C^2-map, that is, we consider $u' = u + t\eta$ for some $\eta \in C^2$ and small t. However, u is supposed minimal, so the area of u' should be minimized when $s = 0$. That is, u should be a critical point when varying t.

So let $\eta \in C_c^2(U)$. Then for every $t \in \mathbb{R}$, Γ_u and $\Gamma_{u+t\eta}$ have the same boundary $\partial\Gamma_u := \{(x, y, u(x, y)) \mid (x, y) \in \partial U\}$ and

$$\operatorname{Area}(\Gamma_{u+t\eta}) = \int_U \sqrt{1+|\nabla u + t\nabla\eta|^2}\,d\lambda.$$

We now differentiate with respect to t and use Greens formula to get

$$\begin{aligned}\frac{d}{dt}\operatorname{Area}(\Gamma_{u+t\eta})|_{t=0} &= \frac{d}{dt}\int_U \sqrt{1+|\nabla u + t\nabla\eta|^2}|_{t=0}\,d\lambda \\ &= \int_U \frac{d}{dt}\sqrt{1+|\nabla u + t\nabla\eta|^2}|_{t=0}\,d\lambda \\ &= \int_U \frac{1}{2\sqrt{1+|\nabla u|^2}}\frac{d}{dt}\left\langle \nabla(u+t\eta), \nabla(u+t\eta)\right\rangle|_{t=0}\,d\lambda \\ &= \int_U \frac{\langle \nabla u, \nabla\eta\rangle}{\sqrt{1+|\nabla u|^2}}\,d\lambda \\ &= -\int_U \eta \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\,d\lambda. \end{aligned}$$

We say that u is a critical point (of the area functional) if this above differential is zero at $t = 0$. If the differential is zero, then η would vanish in the last equation for all $\eta \in C_c^2(U)$, so u is a critical point if and only if

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

By the variational definition of a minimal surface, we thus see that a graph Γ_u is a minimal surface if and only if u is a critical point of the area functional.

Now, Let u be a critical point of the area functional, and $\Gamma_u \subseteq U \times \mathbb{R}$ be the graph of u. We want to show that Γ_u in fact minimizes area among all surfaces on the cylinder $U \times \mathbb{R}$ with the same boundary, $\partial\Gamma_u$.

To do that, we define N to be the unit vector

$$N := \frac{(1,0,u_x)\times(0,1,u_y)}{|(1,0,u_x)\times(0,1,u_y)|} = \frac{(1,0,u_x)\times(0,1,u_y)}{\sqrt{1+|\nabla u|^2}}.$$

By the nature of the cross product, N is orthogonal to both $(1, 0, u_x)$ and $(0, 1, u_y)$ and it is therefore the upward pointing unit normal to Γ_u.

We then define a 2-form $\omega : (U \times \mathbb{R})^2 \to \mathbb{R}$ by

$$\omega(X, Y) = \det(X, Y, N) := \det \begin{pmatrix} | & | & | \\ X & Y & N \\ | & | & | \end{pmatrix}$$

for $X, Y \in \mathbb{R}^3$. Note that ω is really just contraction by N of the standard volume formula $\tilde{\omega} = dx \wedge dy \wedge dz$. Therefore ω is the volume (or area) form of Γ_u.

Now with

$$\begin{aligned} a &:= \omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{1 + |\nabla u|^2}} \\ b &:= \omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = \frac{u_y}{\sqrt{1 + |\nabla u|^2}} \\ c &:= \omega\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{-u_x}{\sqrt{1 + |\nabla u|^2}} \end{aligned}$$

we have that

$$\begin{aligned} \omega &= a\, dx \wedge dy + b\, dx \wedge dz + c\, dy \wedge dz \\ &= \frac{dx \wedge dy - u_x\, dy \wedge dz - u_y\, dz \wedge dy}{\sqrt{1 + |\nabla u|^2}}. \end{aligned}$$

Moreover, since u is a critical point of the area functional

$$d\omega = \left[\frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1 + |\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1 + |\nabla u|^2}}\right)\right] dx \wedge dy \wedge dz = 0.$$

Hence $d\omega$ is a closed 2-form in the cylinder $U \times \mathbb{R}$, so if Σ is another smooth surface in $U \times \mathbb{R}$ with $\partial\Sigma = \partial\Gamma_u$, then Σ and Γ_u enclose an open set $O \subseteq \mathbb{R}^3$ in which $d\omega = 0$. Now, O may not be connected, and have several components, but by applying Stokes theorem in each of these components we see that

$$\int_{\Gamma_u} \omega = \int_{\Sigma} \omega.$$

On the other hand, by definition, $|\omega(X, Y)| = |\det(X, Y, N)|$ is the volume of the polyhedron spanned by X, Y and N so in particular

$$|\omega(X, Y)| \leq 1$$

for all unit vectors $X, Y \in \mathbb{R}^3$. Equality holds above when X, Y, N are orthonormal. This implies that

$$\mathrm{Area}(\Gamma_u) = \int_{\Gamma_u} \omega = \int_{\Sigma} \leq \mathrm{Area}(\Sigma)$$

which implies that Γ_u minimizes area.

Example 1.1. *Examples of minimal surfaces:*

1. *The plane $u(x, y) = 0$, which is clearly a critical point of the area functional.*

2. *The helicoid (see fig. 1.1a) $u(x, y) = \arctan(y/x)$. Indeed*

$$u_x = \frac{-y}{x^2 + y^2}, \quad \textit{and} \quad u_y = \frac{x}{x^2 + y^2}$$

Hence

$$v(x, y) := \sqrt{1 + |\nabla u|^2} = \sqrt{\frac{1 + x^2 + y^2}{x^2 + y^2}}$$

and we get that $v_x(y, x) = (4x^3 + 4xy^2 + 2x)/v(x, y)$ and $v_y(x, y) = v_x(y, x)$, therefore

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \frac{yv_x - xv_y}{v(x, y)^2} = 0$$

So the helicoid is a minimal surface.

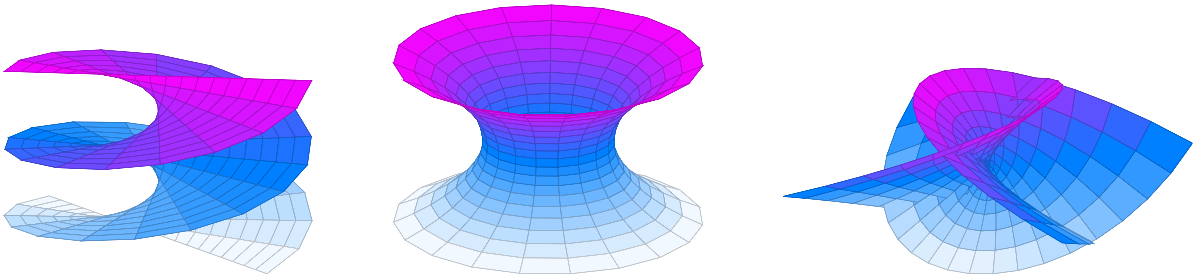
One could also have found the mean curvature of the helicoid, and found that it is zero as well. Indeed the helecoid can be parametrized as

$$r(s, t) = (s\cos(t), s\sin(t), t)$$

for $s, t \in \mathbb{R}$.

We find the partial derivatives up to order two, and get

$$\begin{aligned}
\frac{\partial r}{\partial u} &= (\cos(v), \sin(v), 0), \\
\frac{\partial r}{\partial v} &= (-u\sin(v), u\cos(v), 1) \\
\frac{\partial^2 r}{\partial u^2} &= (0, 0, 0), \\
\frac{\partial^2 r}{\partial u \partial v} &= (-\sin(v), \cos(v), 0) \\
\frac{\partial^2 r}{\partial v^2} &= (-u\cos(v), -u\sin(v), 0)
\end{aligned}$$



(a) A Helicoid (b) A Catenoid (c) Enneper's surface

Figure 1.1: Minimal Surfaces

We find the unit normal

$$N = \frac{\frac{\partial r}{\partial u} \times \frac{\partial t}{\partial v}}{\left|\frac{\partial r}{\partial u} \times \frac{\partial t}{\partial v}\right|} = \frac{1}{\sqrt{1+u^2}}(\sin(v), -\cos(v), u)$$

Then the first and second fundamental forms are

$$I(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix}, \quad \textit{and} \quad II(u,v) = \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And then the principal curvatures are the solutions to the characteristic equation

$$\det(II(u,v) - \kappa I(u,v)) = \kappa^2(1+u^2) - \frac{1}{\sqrt{1+u^2}} = 0$$

which means that the principal curvatures are $\pm 1/(1+u^2)$. *Since the mean curvature is the average of the principal curvatures, we see that the mean curvature of the helicoid is* 0.

3. *The catenoid (see fig. 1.1b)* $u(x,y) = \cosh^{-1}(\sqrt{x^2+y^2})$. *By going through the same steps as above, one sees that the catenoid is indeed a minimal surface.*

4. *Enneper's surface (see fig. 1.1c) which is parametrized by*

$$(s,t) \mapsto \left(s - \frac{s^3}{3} + st^2, -t - s^2 t + \frac{t^3}{3}, s^2 - t^2\right)$$

for $s, t \in \mathbb{R}$.

Lastly, throughout this thesis, we shall employ the folloing notation.

- λ_m will denote the m-dimensional Lebesgue measure, and we will write $\lambda = \lambda_1$.
- Given some metric space (X, d), define $B_r(x) = \{y \in X \mid d(x,y) \leq r\}$. If $X = \mathbb{R}^m$ we will sometimes denote the ball by $B_r^m(x)$ to emphasize the dimension.

- $\mathbb{P}(X)$ will denote the power set of X.
- For some open set $U \subseteq \mathbb{R}^n$ let $L^p_{loc}(U) = \{f : U \to \mathbb{R} \mid \forall K \subseteq U, K \text{ compact }, |f| \in L^p(K)\}$, i.e. the set functions that are integrable on every compact subsets of its domain
- If A is a set, $\mathring{A}$ will denote the interior of A and $\overline{A}$ will denote the closure of A.

Chapter 2

The Hausdorff Measure and area formulae

Our goal is to generalize the idea of minimal surfaces from the 3-dimensional case in the introduction. We will model our structure on [6] and [4], and deviate when necessary.

Our generalized treatment will model the 3-dimensional case closely, so we will need some of the same tools. We will do this by first describing a measure called the Hausdorff measure. This resembles the Lebesgue measure, somewhat, but it is a lot more fine-grained, and works well with measuring surfaces. We shall use this measure for various things, but the first main thing it will be used for are the area formulae. These area formulae allows us to measure the area a manifold, and later, the area of more generalized surfaces, and will help us generalize (1.1) from the introduction. We will then introduce approximate tangent spaces, that allow us to analyze functions on our generalized surfaces. We will then use these tools to, as in the introduction, perturbate our surfaces by some well-behaved map, and study the surfaces which are critical points in the resulting equation. The generalized surfaces that are stable to such perturbations will then be thought of as "minimal", and we will see that these "minimal" generalized surfaces are the ones with a generalized mean curvature identical to 0.

In the last chapter we will study generalized surfaces which are not necessarily "minimal", but which generalized mean curvature is regular, in the sense that they have a bounded L^p norm. It will turn out that such generalized surfaces are very well-behaved.

To do all of that, we will first of all look at the Hausdorff measure.

2.1 The Hausdorff Measure

In the following, let (X, d) be a metric space, and for all $m \geq 0$, let

$$\omega_m = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)}$$

where Γ is the Gamma function. We note that for all $m \in \mathbb{N}$, $\omega_m = \lambda_m(B_1^m(0))$ i.e. ω_m is the volume of the m-dimensional unit ball. We will use this number ω_m as the starting point, and then build the Hausdorff measure from it, by scaling it according to a fine cover of the set we are measuring.

Definition 2.1. *Let $\delta > 0$ be given and $A \subseteq X$. A countable collection $\{C_i\}_{i=1}^{\infty}$ of subsets of X is a δ-covering of A if*

$$A \subseteq \bigcup_{i=1}^{\infty} C_i$$

and $\operatorname{diam} C_i \leq \delta$ *for all $i \in \mathbb{N}$.*

Definition 2.2. *For any $m \geq 0$ and $\delta > 0$, we define the "size δ approximation to the m-dimensional Hausdorff measure" $\mathcal{H}_\delta^m : \mathbb{P}(X) \to [0, \infty]$ by $\mathcal{H}_\delta^m(\emptyset) = 0$, and such that for any non-empty subset $A \subseteq X$*

$$\mathcal{H}_\delta^m(A) = \omega_m \cdot \inf\left\{ \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} C_i}{2} \right)^m \middle| \{C_i\}_{i\in\mathbb{N}} \text{ is a } \delta\text{-covering of } A \right\} \tag{2.1}$$

The infimum is defined to be ∞ if no such δ-covering exists.

Since $\mathcal{H}_\delta^m$ is decreasing in δ, the following limit exists, though it might be ∞.

Definition 2.3. *We define, for any $m \geq 0$, the m-dimensional* ***Hausdorff measure****, $\mathcal{H}^m : \mathbb{P}(X) \to [0, \infty]$ by*

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^m(A) = \sup_{\delta > 0} \mathcal{H}_\delta^m(A) \tag{2.2}$$

for all $A \subseteq X$.

We note that $\mathcal{H}^0$ is the counting measure.

Theorem 2.4. *$\mathcal{H}_\delta^m$ and $\mathcal{H}^m$ are outer measures for all $m \geq 0$ and $\delta > 0$.*

Proof. We first prove that $\mathcal{H}_\delta^m$ is an outer measure. First of all, we have $\mathcal{H}_\delta^m(\emptyset) = 0$ by definition.

Secondly, let $A \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq X$. If A has no δ-covering, then there is an $i \in \mathbb{N}$ such that A_i has no

δ-covering as well, hence we clearly have

$$\mathcal{H}_\delta^m(A) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^m(A_i) = \infty.$$

If A has a δ-covering, we can assume that $\mathcal{H}_\delta^m(A_i) < \infty$ for all $i \in \mathbb{N}$. Now for $\varepsilon > 0$, we can for every $i \in \mathbb{N}$ choose a δ-covering $\{C_j^i\}_{j\in\mathbb{N}}$ of the set A_i in such a way that

$$\omega_m \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam} C_j^i}{2} \right)^m \leq \mathcal{H}_\delta^m(A_i) + \frac{\varepsilon}{2^i}.$$

But then $\bigcup_{i,j} C_j^i$ is a δ-covering of $\bigcup_{i\in\mathbb{N}} A_i$, and hence also a δ-covering of A, which implies that

$$\begin{aligned}
\mathcal{H}_\delta^m(A) &\leq \omega_m \sum_{i,j} \left(\frac{\operatorname{diam} C_j^i}{2} \right)^m \\
&\leq \sum_{i=1}^{\infty} \left(\mathcal{H}_\delta^m(A_i) + \frac{\varepsilon}{2^i} \right) \\
&\leq \left(\sum_{i=1}^{\infty} \mathcal{H}_\delta^m(A_i) \right) + \varepsilon.
\end{aligned}$$

Since $\varepsilon \to 0$ was arbitrary, the result follows.

Regarding $\mathcal{H}^m$, we clearly see that $\mathcal{H}^m(\emptyset) = 0$. Finally if $A \subseteq \bigcup_{i\in\mathbb{N}} A_i \subseteq X$ then by the first part of this proof, and the definition of the Hausdorff measure yields

$$\mathcal{H}_\delta^m(A) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^m(A_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^m(A_i).$$

So letting $\delta \to 0$ gives that $\mathcal{H}^m$ is an outer measure. ■

By the following theorem, the m-dimensional Hausdorff measure behaves like the m-dimensional Lebesgue measure on $\mathbb{R}^m$, however the Hausdorff measure is more refined. For instance, a line in 2-dimensional space would have λ_2-measure 0, but $\mathcal{H}^1$ would measure the length of the line. In fact we could take the $\mathcal{H}^m$-measure on the line for all $m \in [0, \infty]$ and find out what number we get. This is intrinsically connected with the Hausdorff dimension in which a set $A \subseteq \mathbb{R}^m$ has associated a unique number $m \in [0, \infty]$ (called the Hausdorff dimension) for which $0 < \mathcal{H}^m(A) < \infty$, and such that for all $t \neq m$ we would either have $\mathcal{H}^t(A) = 0$ or $\mathcal{H}^t(A) = \infty$. The Hausdorff dimension is also called the fractal dimension, as sets with Hausdorff dimension $m \notin \mathbb{N}$ are of interest in the theory of fractals. We will not go further into the theory of the Hausdorff dimension or fractals.

To prove that $\mathcal{H}^m = \lambda_m$ on $\mathbb{R}^m$ we will first need the following theorem, from which it follows that among all sets in $\mathbb{R}^m$ with a given diameter d, the ball with diameter d has the largest λ_m measure.

Theorem 2.5 (Isodiametric Inequality)**.** *For all sets $A \subseteq \mathbb{R}^m$ we have*

$$\lambda_m(A) \leq \omega_m \left(\frac{\operatorname{diam} A}{2}\right)^m$$

Proof. We note that if $\operatorname{diam} A = \infty$, the statement is trivial. Furthermore, $\operatorname{diam} \bar{A} = \operatorname{diam} A$, hence we can without loss of generality assume that A is compact. We will use Steiner symmetrization to prove the statement (see fig. 2.1 for a visualisation of Steiner symmetrization).

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$, and denote by $e_i = 0$ the coordinate plane spanned by $\{e_1, \ldots, e_m\} \setminus \{e_i\}$, i.e. the plane which is zero at the i'th coordinate.

Given some x in the coordinate plane $e_i = 0$ we let

$$\ell_i(x) := \{x + te_i \mid t \in \mathbb{R}\}$$

be the line perpendicular to the i'th coordinate plane, passing through x. Secondly, let π be the projection defined by

$$\pi(x + te_i) = t.$$

Furthermore let

$$\sigma_i(A, x) := \left\{x + te_i \middle| |t| \leq \frac{\lambda_1(\pi(A \cap \ell_i(x)))}{2}\right\},$$

then the Steiner symmetrization of A with respect to the i'th coordinate plane is

$$S_i(A) = \bigcup\{\sigma_i(A, x) \mid A \cap \ell_i(x) \neq \emptyset\}$$

That is we replace each $A \cap \ell_i(x)$ with a line centred at the i'th coordinate plane and with length $\lambda_1(\pi(A \cap \ell_i(x)))$.

This process gives a new compact set $S_i(A)$. Indeed, for every $\varepsilon > 0$ we can find an open set $U \subseteq \mathbb{R}$ such that $\pi(A \cap \ell_i(a)) \subseteq U$, and $\lambda_1(U) \leq \lambda_1(\pi(A \cap \ell_i(x))) + \varepsilon$. Since A is closed and $\pi^{-1}(U)$ is an open set containing the compact set $A \cap \ell_i(x)$ we see that for any sequence $\{x_j\}_{j \in \mathbb{N}}$ in the i'th coordinate plane with $x_j \to x$ we must have $A \cap \ell_i(x_j) \subseteq \pi^{-1}(U)$ for large enough j. Hence $\pi(A \cap \ell_i(x_j)) \subseteq U$ for large enough j, which implies that $\lambda_1(\pi(A \cap \ell_i(x_j))) \leq \lambda_1(\pi(A \cap \ell_i(x))) + \varepsilon$ for all large enough j. Thus $\lambda_1(\pi(A \cap \ell_i(x)))$ is an upper semi-continuous function of x when x is restricted to lie in the i'th coordinate plane, and hence $S_i(A)$ is compact when A is compact.

Now $\operatorname{diam} S_i(A) \leq \operatorname{diam} A$, so by Fubini, they have the same Lebesgue measure. Moreover, if $i \neq j$ and if A is already invariant under reflection in the j'th coordinate plane, then $S_i(A)$ is invariant under reflection in both the i'th and j'th coordinate plane by definition. Therefore we apply Steiner symmetrization successively with respect to each of the coordinate planes $e_1 = 0, \dots, e_m = 0$ and get a new set $\tilde{A}$ for which $\operatorname{diam} \tilde{A} \leq \operatorname{diam} A$, $\lambda_m(\tilde{A}) = \lambda_m(A)$, and which is invariant under the transformation $x \mapsto -x$. This implies that $\tilde{A}$ is contained in the ball with centre 0 and radius $\operatorname{diam} A/2$, from which we obtain that

$$\lambda_m(A) \leq \lambda_m(\tilde{A}) \leq \omega_m \left(\frac{\operatorname{diam} A}{2} \right)^m$$

as wanted. ■

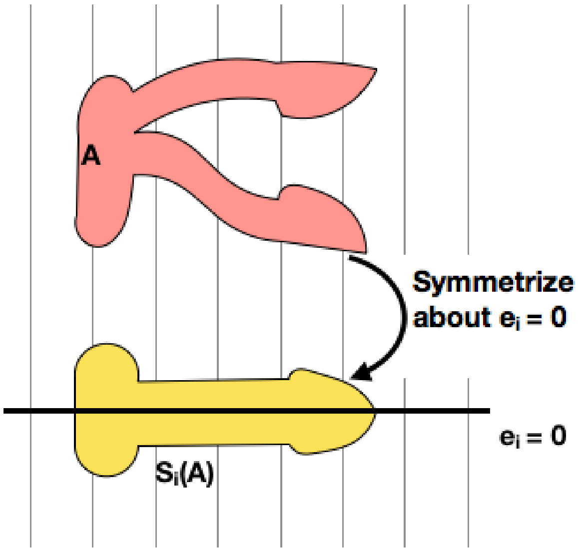


Figure 2.1: A visualisation of Steiner symmetrization

We are now ready to prove the following.

Theorem 2.6. *For all $A \subseteq \mathbb{R}^m$ and $\delta > 0$ we have*

$$\mathcal{H}^m_\delta = \mathcal{H}^m(A) = \lambda_m(A).$$

Proof. First, we want to prove that

$$\lambda_m(A) \leq \mathcal{H}^m_\delta(A)$$

for all $\delta > 0$ and $A \subseteq \mathbb{R}^m$.

So let $\delta > 0$ and $A \subseteq \mathbb{R}^m$ be given, and let $\{C_i\}_{i \in \mathbb{N}}$ be a δ-covering of A. Then

$$\begin{aligned} \lambda_m(A) &\leq \lambda_m \left(\bigcup_{i \in \mathbb{N}} C_i \right) \\ &\leq \sum_{i \in \mathbb{N}} \lambda_m(C_i) \\ &\leq \sum_{i \in \mathbb{N}} \omega_m \left(\frac{\operatorname{diam} C_i}{2} \right)^m \end{aligned}$$

where the last inequality is due to theorem 2.5. Taking the infimum over all such collections $\{C_i\}_{i\in\mathbb{N}}$ shows the wanted inequality.

Secondly, we show that

$$\mathcal{H}^m_\delta(A) \leq \lambda_m(A)$$

for all $\delta > 0$. Let $\mathcal{K}$ denote the collection of all m-dimensional open intervals I of the form $I = (a_1, b_1) \times \ldots \times (a_m, b_m)$ for $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$. Then we note that

$$\lambda_m(A) = \inf \left\{ \sum_{j=1}^{\infty} |I_j| \,\middle|\, I_1, I_2, \dots \in \mathcal{K}, A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

where $|I| = (b_1 - a_1)\ldots(b_m - a_m)$ denotes the volume of I.

Now let $I_1, I_2, \dots \in \mathcal{K}$ be any sequence of m-dimensional open intervals such that $A \subseteq \bigcup_j I_j$. Now for every bounded open set $U \subseteq \mathbb{R}^m$ and each $\delta > 0$ there exists a pairwise disjoint family of closed balls $B_1, B_2, \ldots$ such that $\bigcup_i B_i \subseteq U$, $\operatorname{diam} B_i < \delta$ for all $i \in \mathbb{N}$ and $\lambda_m(U \setminus \bigcup_i B_i) = 0$. Indeed, there exists a collection $C_1, C_2, \ldots$ of closed cubes with pairwise disjoint interiors such that $\operatorname{diam} C_i < \delta$ for all $i \in \mathbb{N}$ and $U = \bigcup_i C_i$. We write $C_i = (c_i, d_i)^m$ for $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$. Now for each $i \in \mathbb{N}$ there is a ball $B_i \subseteq \mathring{C}_i$ with $\operatorname{diam} B_i > \frac{d_i - c_i}{2}$ (i.e. half the edge length of C_i), for which

$$\lambda_m(C_i \setminus B_i) < \left(1 - \frac{\omega_m}{4^m}\right) \lambda_m(C_i)$$

and hence

$$\lambda_m\left(U \setminus \bigcup_{i\in\mathbb{N}} B_i\right) = \lambda_m\left(\bigcup_{i\in\mathbb{N}} (C_i \setminus B_i)\right) < \left(1 - \frac{\omega_m}{4^m}\right) \lambda_m(U).$$

This implies that for some large enough N we have

$$\lambda_m\left(U \setminus \bigcup_{i=1}^{N} B_i\right) \leq \left(1 - \frac{\omega_m}{4^m}\right) \lambda_m(U).$$

Since $U \setminus \bigcup_{i=1}^N B_i$ is open and $\omega_m/4^m \in (0,1]$ for all $m \geq 0$, we can repeat this argument inductively to obtain the required open balls.

So with $U = I_j$ we take such a collection of open balls $\{B_i\}_{i\in\mathbb{N}}$. Since $\mathcal{H}^m_\delta$ is absolutely continuous with respect to λ_m, (i.e. for every $A \subseteq X$ we have $\lambda_m(A) = 0 \Rightarrow \mathcal{H}^m_\delta(A) = 0$) we have that

$$\begin{aligned}
\mathcal{H}^m_\delta(I_j) &= \mathcal{H}^m_\delta\left(\bigcup_{i\in\mathbb{N}} B_i\right) \\
&\leq \sum_{i=1}^{\infty} \omega_m (\operatorname{diam} B_i)^m \\
&= \sum_{i=1}^{\infty} \lambda_m(B_i) \\
&= \lambda_m\left(\bigcup_{i\in\mathbb{N}} B_i\right) \\
&= \lambda_m(I_j) = |I_j|
\end{aligned}$$

and therefore

$$\mathcal{H}^m_\delta(A) \leq \mathcal{H}^m_\delta\left(\bigcup_{j\in\mathbb{N}} I_j\right) \leq \sum_{j\in\mathbb{N}} \mathcal{H}^m_\delta(I_j) \leq \sum_{j\in\mathbb{N}} |I_j|.$$

The second part of this proof thus follows by taking the infimum over all such collections $\{I_j\}_{j\in\mathbb{N}}$. ■

2.2 Densities

Definition 2.7. *Let (X, d) be any metric space. Then for all outer measures μ on X, $m \in [0, \infty)$, $A \subseteq X$ and $x \in X$ the upper and lower m-densities of A at the point x are given by*

$$\Theta^{*m}(\mu, A, x) = \limsup_{r\to 0^+} \frac{\mu(A \cap B_r(a))}{\omega_m r^m}$$
$$\Theta^m_*(\mu, A, x) = \liminf_{r\to 0^+} \frac{\mu(A \cap B_r(a))}{\omega_m r^m}$$

If $\Theta^{*m}(\mu, A, x) = \Theta^m_*(\mu, A, x)$ then their shared value is called the (m-dimensional) density of A at x and denoted by $\Theta^m(\mu, A, x)$.

For the following lemma, we will adopt the notation that if $B = B_r(x) \subseteq X$ is a ball, then $\widehat{B} = B_{5r}(x)$.

Lemma 2.8 (5-times covering lemma)**.** *If $\mathcal{B}$ is a collection of closed balls in X with $R = \sup\{\operatorname{diam} B \mid B \in \mathcal{B}\} < \infty$, then there is a pairwise disjoint subcollection $\mathcal{B}' \subseteq \mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B\in\mathcal{B}'} \widehat{B}.$$

In fact, an even stronger statement holds, namely, for all $B \in \mathcal{B}$ there exists a $B' \in \mathcal{B}'$ such that $B \cap B' \neq \emptyset$ and $\operatorname{diam} B' \geq \frac{1}{2} \operatorname{diam} B$ and hence $\widehat{B}' \supseteq B$.

Proof. For all $i \in \mathbb{N}$ define

$$\mathcal{B}_i := \left\{ B \in \mathcal{B} \middle| \frac{R}{2^i} < \operatorname{diam} B \leq \frac{R}{2^{i-1}} \right\}$$

Then clearly $B_i \cap B_j = \emptyset$ for all $i \neq j$, and $\mathcal{B} = \bigcup_i \mathcal{B}_i$. We now define another collection $\mathcal{B}'_i \subseteq \mathcal{B}_i$ inductively as follows.

For $i = 1$, we let $\mathcal{B}'_i$ be any maximal pairwise disjoint subcollection of $\mathcal{B}_1$. Such a collection exists by Zorn's lemma applied to $\mathcal{C} = \{\mathcal{A} \mid \mathcal{A} \text{ is a pairwise disjoint subcollection of } \mathcal{B}_1\}$, which is partially ordered by inclusion. Furthermore, we see that for any totally ordered subcollection $\mathcal{T} \subseteq \mathcal{C}$ we clearly have $\bigcup_{\mathcal{A} \in \mathcal{T}} \mathcal{A} \in \mathcal{C}$ so Zorn's lemma does indeed apply. Moreover, in a general metric space, the collection $\mathcal{B}'_1$ could be uncountable. But in a separable metric space all pairwise disjoint collections of balls must be countable.

Next, for every $i \geq 2$ assume that $\mathcal{B}'_1 \subseteq \mathcal{B}_1, \ldots, \mathcal{B}'_{i-1} \subseteq \mathcal{B}_{i-1}$ have been defined. Then we let $\mathcal{B}'_i$ be a maximal pairwise disjoint subcollection of

$$\left\{ B \in \mathcal{B}_i \middle| B \cap B' = \emptyset, \text{ when } B' \in \bigcup_{j=1}^{i-1} \mathcal{B}'_j \right\}.$$

Another Zorn's lemma argument guarantees such a maximal collection exists. Now if $i \geq 1$ and $B \in \mathcal{B}_i$ we have $B \cap B' \neq \emptyset$ for some $B' \in \bigcup_{j=1}^{i} \mathcal{B}'_j$, otherwise this would contradict maximality of $\mathcal{B}'_i$. For such a pair B and B' we have

$$\operatorname{diam} B \leq R/2^{i-1} = 2R/2^i \leq 2 \operatorname{diam} B'$$

so that lemma 2.8 holds, and in particular $B \subseteq \widehat{B'}$. ■

In the following corollary we say that $A \subseteq X$ is *covered finely* by a collection $\mathcal{B}$ of balls, if for all $x \in A$ and all $\varepsilon > 0$ there is a ball $B \in \mathcal{B}$ with $x \in B$ and $\operatorname{diam} B < \varepsilon$.

Corollary 2.9. *Let $\mathcal{B}$ be as in lemma 2.8. Then if $A \subseteq X$ is covered finely by $\mathcal{B}$ and $\mathcal{B}' \subseteq \mathcal{B}$ is any pairwise disjoint subcollection of $\mathcal{B}$ satisfying lemma 2.8, then for each finite subcollection $\{B_1, \ldots, B_N\} \subseteq \mathcal{B}'$ we have*

$$A \setminus \bigcup_{i=1}^{N} B_i \subseteq \bigcup_{B \in \mathcal{B}' \setminus \{B_1, \ldots, B_N\}} \widehat{B}.$$

Proof. Let $x \in A \setminus \bigcup_{i=1}^{N} B_i$. Then since A is covered finely by $\mathcal{B}$ and $X \setminus \bigcup_{i=1}^{N} B_i$ is open, there is a $B \in \mathcal{B}$ with $B \cap \left(\bigcup_{i=1}^{N} B_i \right) = \emptyset$ and $x \in B$, and by lemma 2.8 there is a $B' \in \mathcal{B}'$ with $B' \cap B \neq \emptyset$ such that $B \subseteq \widehat{B'}$. This implies that $B' \cap B_i \neq \emptyset$ for all $i = 1, \ldots, N$ so $x \in \bigcup_{B' \in \mathcal{B}' \setminus \{B_1, \ldots, B_N\}} \widehat{B'}$. ■

Theorem 2.10. *Let μ be a Borel regular measure on X, $t \geq 0$ and $A_1 \subseteq A_2 \subseteq X$. Then if*

$$\Theta^{*n}(\mu, A_2, x) \geq t,$$

for all $x \in A_1$, then

$$t\mathcal{H}^m(A_1) \leq \mu(A_2)$$

Proof. If $\mu(A_2) = \infty$ or $t = 0$, the result is trivial, so assume that $\mu(A_2) < \infty$ and $t > 0$.

Let $\tau \in (0, t)$ be given, and assume that $\Theta^{*m}(\mu, A_2, x) > \tau$ for all $x \in A_1$. For any $\delta > 0$, define $\mathcal{B}_\delta$ by

$$\mathcal{B}_\delta = \left\{ B_r(x) \middle| x \in A_1, r \in (0, \delta/2), \frac{\mu(A_2 \cap B_r(x))}{\omega_m r^m} \geq \tau \right\}$$

This collection $\mathcal{B}_\delta$ covers A_1 finely, so there is a subcollection $\mathcal{B}' \subseteq \mathcal{B}_\delta$ such that lemma 2.8 holds. Then since $\mu(A_2 \cap B) > 0$ for each $B \in \mathcal{B}_\delta$ and since

$$\sum_{i=1}^{N} \mu(A_2 \cap B_i) = \mu\left(A_2 \cap \left(\bigcup_{i=1}^{N} B_i\right)\right) \leq \mu(A_2) < \infty$$

for all balls $B_1, \ldots, B_N \in \mathcal{B}'$, it follows that $\mathcal{B}'$ is a countable collection $\{B_{r_1}(x_1), B_{r_2}(x_2), \ldots\}$ and hence corollary 2.9 implies that

$$A_1 \setminus \bigcup_{i=1}^{N} B_{r_i}(x_i) \subseteq \bigcup_{i=N+1}^{\infty} B_{5r_i}(x_i)$$

for all $N \geq 1$.

Furthermore, by the construction of $\mathcal{B}_\delta$, we have that

$$\tau \sum_{i=1}^{\infty} \omega_m r_i^m \leq \sum_{i=1}^{\infty} \mu(A_2 \cap B_{r_i}(x_i)) = \mu\left(A_2 \cap \left(\bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right)\right) \leq \mu(A_2) < \infty.$$

Therefore

$$A_2 \subseteq \left(\bigcup_{i=1}^{N} B_{r_i}(x_i)\right) \cup \left(\bigcup_{i=N+1}^{\infty} B_{5r_i}(x_i)\right)$$

and hence by the definition of $\mathcal{H}^m_\delta$, we have

$$\mathcal{H}^m_{5\delta}(A_1) \leq \sum_{i=1}^{N} \omega_m r_i^m + 5^m \sum_{i=N+1}^{\infty} \omega_m r_i^m.$$

So by letting $N \to \infty$ we get that

$$\tau \mathcal{H}^m_{5\delta}(A_1) \leq \mu(A_2)$$

And then letting $\delta \to 0$, and then $\tau \to t$ the result follows. ∎

Theorem 2.11 (Upper density theorem)**.** *If μ is a Borel regular measure on X and if $A \subseteq X$ is μ-measurable with $\mu(A) < \infty$, then*

$$\Theta^{*m}(\mu, A, x) = 0$$

for $\mathcal{H}^m$-a.e. $x \in X \setminus A$.

Proof. For a given $t > 0$, define

$$S_t := \{x \in X \setminus A \mid \Theta^{*m}(\mu, A, x) \geq t\}$$

and let C be any closed subset of A. Then, since $X \setminus C$ is open, and $S_t \subseteq X \setminus A \subseteq X \setminus C$ we have

$$\Theta^{*m}(\mu, A \cap (X \setminus C), x) = \Theta^{*m}(\mu, A, x) \geq t$$

for all $x \in S_t$. Thus we can apply theorem 2.10 with $\mu \llcorner A, S_t$ and $X \setminus C$ in place of μ, A_1 and A_2 to obtain that $t\mathcal{H}^m(S_t) \leq \mu(A \setminus C)$ for each closed set $C \subseteq A$.

Now since μ is an outer measure on a metric space X, it has the Radon measure property that $\mu(A) = \sup_C\{\mu(C)\}$ where the supremum is taken over all closed subsets, C, of A. Therefore we can find a sequence $\{C_i\}_{i\in\mathbb{N}}$ such that $C_i \subseteq A$ are closed for all i, and such that $\mu(A \setminus C_i) \to 0$ as $i \to \infty$, which implies that $\mathcal{H}^m(S_t) = 0$. Finally letting $t \to \infty$ we conclude that

$$\mathcal{H}^m(\{x \in X \setminus A \mid \Theta^{*m}(\mu, A, x) > 0\}) = 0,$$

which finishes the proof. ■

2.3 The Area and Co-Area Formulae

Next we want to develop a generalized area formula similar to the one in the introduction. We shall need the following theorem, which we will not prove.

Theorem 2.12 (Rademacher's Theorem)**.** *If f is Lipschitz on $\mathbb{R}^n$, then f is differentiable $\mathcal{H}^n$-a.e. i.e. the gradient $\nabla f(x) = (D_1 f(x), \ldots, D_n f(x))$ exists and*

$$\lim_{y \to x} \frac{f(y) - f(x) - \nabla f(x)(y - x)}{|y - x|} = 0$$

for $\mathcal{H}^n$-a.e. $x \in \mathbb{R}^n$.

It is well known that for a linear map $L : \mathbb{R}^m \to \mathbb{R}^m$, and $A \subseteq \mathbb{R}^n$ we have that $\lambda_m(L(A)) = |\det L|\lambda_m(A)$. We want to develop a counterpart to this.

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, we define the Jacobian determinant, J_L of L as

$$J_L = \sqrt{\det(L^* \circ L)}$$

Now let $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, for $n \leq m$ be a Lipschitz mapping. By Rademacher's theorem, f is differentiable at λ_n-a.e. $x \in \mathbb{R}^n$. Hence the Jacobian matrix $Df(x)$ exists and can be expressed as a matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

at λ_n-a.e. $x \in \mathbb{R}^n$.

Now $Df(x)$ is a linear mapping whenever it exists, so we can similarly define the Jacobian determinant, J_f of f at a point x where f is differentiable as

$$J_f := J_{Df}$$

Lemma 2.13. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $n \leq m$ be a linear map. Then*

$$\mathcal{H}^n(LA) = J_L \mathcal{H}^n(A)$$

for all $A \subseteq \mathbb{R}^n$.

Proof. Since L is linear, $L(\mathbb{R}^n) \subseteq F$ for some n-dimensional subspace, $F \subseteq \mathbb{R}^m$. Therefore we can choose an orthogonal transformation $q : \mathbb{R}^m \to \mathbb{R}^m$ such that $q(F) = \mathbb{R}^n \times \{0\}$. We can further define a projection $p : \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^m \to \mathbb{R}^n$ such that $p(x, y) = x$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$. We then see that $p \circ q \circ L : \mathbb{R}^n \to \mathbb{R}^n$ and thus by the discussion at the beginning of the section, $\mathcal{H}^n((p \circ q \circ L)(A)) = |\det(p \circ q \circ L)| \mathcal{H}^n(A)$ for all $A \subseteq \mathbb{R}^n$. Using the adjoint, we get that $(p \circ q \circ L)^*(p \circ q \circ L) = L^* \circ q^* \circ (p^* \circ p) \circ q \circ L$. But $p^* \circ p$ is the identity on $\mathbb{R}^n \times \{0\}$, and q is orthogonal so $q^* \circ q$ is the identity on $\mathbb{R}^m$, hence $L^* \circ q^* \circ (p^* \circ p) \circ q \circ L = L^* \circ L$. Therefore $|\det(p \circ q \circ L)| = \sqrt{\det(L^* \circ L)} = J_L$. This finishes the proof. ■

Now, we can generalize this to Lipschitz functions, to get the following theorem, which can be proved by an approximation argument based on the linear case

Theorem 2.14 (Area formula)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and injective, then*

$$\mathcal{H}^n(f(A)) = \int_A J_f \, d\mathcal{H}^n.$$

for all $\mathcal{H}^n$-measurable $A \subseteq \mathbb{R}^n$.

Example 2.15. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz, and define $f : \mathbb{R}^m \to \mathbb{R}^{m+1}$ by $f(x) = (x, g(x))$. Then*

$$Df = \begin{pmatrix} I \\ \nabla g \end{pmatrix}$$

and hence $J_f^2 = 1 + |\nabla g|^2$. Given an open set $U \subseteq \mathbb{R}^m$, the graph, Γ, of g over U is given by

$$\Gamma = \{(x, g(x)) \mid x \in U\}$$

and therefore

$$\mathcal{H}^m(\Gamma) = \int_U \sqrt{1 + |\nabla g|^2} d\mathcal{H}^m(x).$$

This looks strikingly similar to (1.1)*, so we are going in the right direction.*

We state without proof the following generalisations of the area formula.

Theorem 2.16 (Area formula)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and $n \leq m$. If $A \subseteq \mathbb{R}^n$ is λ_n-measurable then*

$$\int_A J_f(x)\, d\lambda_n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(f^{-1}(\{y\} \cap A)\, d\mathcal{H}^n(y).$$

Note that $\mathcal{H}^0$ is the counting measure.

Theorem 2.17. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and $n \leq m$. If g is an λ_n-integrable function, then*

$$\int_{\mathbb{R}^n} g(x) J_f(x)\, d\lambda_n = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} g(x)\, d\mathcal{H}^n(y).$$

Next we briefly discuss the co-area formula, which is generalization of Fubini's theorem.

Theorem 2.18 (The co-area formula)**.** *Let $n \geq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then for every λ_m-measurable set $A \subseteq \mathbb{R}^n$ we have*

$$\int_A J_f(x)\, d\lambda_n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\lambda_m(y)$$

Theorem 2.19 (Change of variables)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, for $n \geq m$ be Lipschitz. Then for every λ_n integrable function $g : \mathbb{R}^n \to \mathbb{R}$, $g|_{f^{-1}(y)}$ is $\mathcal{H}^{n-m}$ integrable for λ_m-a.e. $y \in \mathbb{R}^m$ and*

$$\int_{\mathbb{R}^n} g(x) J_f(x)\, d\lambda_n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)} g\, d\mathcal{H}^{n-m} \right) d\lambda_m(y)$$

2.4 Sobolev Spaces

We shall briefly discuss the theory of Sobolev spaces, which we will used to prove the Allard Regularity theorem 5.8. See [5] for reference.

Definition 2.20. *Let $u \in L^1_{loc}(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, and let $\alpha \in \mathbb{N}^n$ be a multi-index. Then $v \in L^1_{loc}(\Omega)$ is the weak partial derivative of order α of u, written $D^\alpha u = v$ if*

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v\varphi \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$

Then we can define the Sobolev spaces as

Definition 2.21. *For all open sets $\Omega \subseteq \mathbb{R}^n$, define the Sobolev space by*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}.$$

For every $u \in W^{k,p}(\Omega)$ we define the Sobolev norm by

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \; dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, and otherwise

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_\Omega |D^\alpha u| \,,$$

where ess sup denotes the essential supremum, that is, for every measurable function $f : (X, \mu) \to \mathbb{R}$ where μ is some measure, the essential supremum of f is defined as the smallest number κ for which

$$\mu(f^{-1}(\kappa, \infty)) = \mu(\{x \in X \mid f(x) > \kappa\}) = 0.$$

If no such κ exists, we define the essential supremum to be ∞.

Theorem 2.22 (Poincaré inequality)**.** *Let $1 \leq p < \infty$ be given, and $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then there is some constant $C = C(p)$ for which*

$$\int_\Omega |u|^p \, dx \leq C \operatorname{diam}(\Omega)^p \int_\Omega |Du|^p \, dx$$

for every $u \in W_0^{1,p}(\Omega)$ (= the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm).

Proof. We prove the theorem for $u \in C_c^\infty(\Omega)$, and then since $W_0^{1,p}(\Omega)$ can be approximated by functions in $C_c^\infty(\Omega)$ (see [5] theorem 1.18), the theorem follows by an approximation argument.

So let $u \in C_c^\infty(\Omega)$, and $y \in \Omega$. Then letting $a_i := y_i - \operatorname{diam}(\Omega)$ and $b_i = y_i + \operatorname{diam}(\Omega)$ for all $i = 1, \ldots, n$, we have

$$\Omega \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Then as in the proof of [5] theorem 2.2, we get

$$\begin{aligned}
|u(x)| &\leq \int_{a_i}^{b_i} |Du(x_1, \ldots, t_i, \ldots, x_n)| \, dt_i \\
&\leq (2 \operatorname{diam}(\Omega))^{1-\frac{1}{p}} \left(\int_{a_i}^{b_i} |Du(x_1, \ldots, t_i, \ldots, x_n)|^p \, dt_i \right)^{\frac{1}{p}}
\end{aligned}$$

for all $i = 1, \ldots, n$. This allows us to use the Hölder inequality, and write

$$\begin{aligned}
\int_\Omega |u(x)|^p \, dx &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |u(x)|^p \, dx_1 \ldots dx_n \\
&\leq (2 \operatorname{diam}(\Omega))^{p-1} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{a_1}^{b_1} |Du(t_1, x_2, \ldots, x_n)|^p \, dt_1 dx_2 \ldots dx_n \\
&\leq (2 \operatorname{diam}(\Omega))^{p} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |Du(t_1, x_2, \ldots, x_n)|^p \, dt_1 \ldots dx_n \\
&= (2 \operatorname{diam}(\Omega))^{p} \int_\Omega |Du(x)|^p \, dx
\end{aligned}$$

which finishes the proof. ■

A fact that we shall use later in the Allard regularity theorem is the following Rellich-Kondrachov Compactness theorem, which we state without a proof. See [2] pp 167–168.

Theorem 2.23 (Rellich-Kondrachov compactness theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open. Then the Sobolov space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q < \frac{np}{n-p}$ if $1 \leq p < n$.*

Notice that an embedding is compact if and only if the identity operator is a compact operator. Thus the Rellich-Kondrachov compactness theorem implies that any uniformly bounded sequence in $W^{1,p}(\Omega)$ has a subsequence that converges in $L^p(\Omega)$.

Chapter 3

Countably m-rectifiable sets

3.1 Preliminaries

In the introduction we used standard surfaces in 3 dimensions. However we can expand the notion of minimal surfaces, to a broader class than those. First of all, we could consider manifolds as in the following definition. However, we can generalize even further, as we shall see in this chapter.

Definition 3.1. *Let $r \geq 1$, $m, n \in \mathbb{N}$ with $m < n$. Then $M \subseteq \mathbb{R}^n$ is called an m-dimensional C^ℓ submanifold of $\mathbb{R}^n$ if for each $y \in M$ there are open sets $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^n$ with $y \in W$, and a bijective C^ℓ map $\psi : V \to W$ with*

$$\psi(V) = W \cap M$$

and such that ψ is proper (i.e. if $K \subseteq W$ is compact, then $\psi^{-1}(K)$ is compact in V) and $D\psi(x)$ has rank m at each point $x \in V$.

Such an m-dimensional C^ℓ submanifold of $\mathbb{R}^n$ admits a local graphical representation. Indeed if $\psi : V \to W$ is the local representation for M as in the definition, $y_0 \in M \cap W$ and $x_0 = \psi^{-1}(y_0) \in V$, then by definition we have that rank $D\psi(x_0) = m$ hence there are indices $1 \leq k_1 < \cdots < k_n \leq n$ such that $\det(D_{k_i}\psi_j(x_0)) \neq 0$. Now if $k_i = i$ for all $i = 1, \ldots, m$, then letting $\tilde{\psi} = (\psi_1, \ldots, \psi_m)$, the inverse function theorem implies the existence of open sets $V_0, U \subseteq \mathbb{R}^m$ such that $x_0 \in V_0 \subseteq V$ and such that $\tilde{\psi}|_{V_0}$ is a C^ℓ diffeomorphism of V_0 onto U. We now see that $G := \psi \circ (\tilde{\psi}|_{V_0})^{-1} : U \to W$ has the form

$$G(x) = (x, u(x)), \quad x \in U$$

where $u : U \to \mathbb{R}^\ell$ is given by $u = (\psi_{m+1}, \ldots, \psi_n) \circ (\tilde{\psi}|_{V_0})^{-1}$, i.e. G is the graph map $x \mapsto (x, u(x))$ of u, namely $G(U) = \operatorname{graph} u = \psi(V_0) = M \cap W_0$.

This remains true without the assumption that $k_i = i$ for $i = 1, \dots, m$, because we can just compose with a permutation map such that the coordinates $x_{k_1}, \dots, x_{k_m}$ are permuted to the first m entries. Thus M is a C^ℓ submanifold of $\mathbb{R}^n$ if and only if M admits a local representation around each of its points as the graph of a C^ℓ function u.

We state without proof the following theorem

Theorem 3.2 (C^1 approximation theorem)**.** *If $f : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz. Then for each $\varepsilon > 0$ there exist a mapping $g \in C^1(\mathbb{R}^m)$ such that*

$$\lambda_m(\{x : f(x) \neq g(x)\} \cup \{x : \nabla f(x) \neq \nabla g(x)\}) < \varepsilon.$$

3.2 Countably m-rectifiable sets

Definition 3.3. *Let $m, n \in \mathbb{N}$ with $m \leq n$. A set $M \subseteq \mathbb{R}^n$ is called (countably) m-rectifiable if*

$$M \subseteq M_0 \cup \left(\bigcup_{j=1}^{\infty} F_j(\mathbb{R}^m) \right),$$

where $\mathcal{H}^n(M_0) = 0$ and $F_j : \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz mappings.

A countably m-rectifiable set is an extension of ordinary m-dimensional embedded C^1 submanifolds of $\mathbb{R}^n$ in the following sense

Lemma 3.4. *M is countably m-rectifiable if and only if $M \subseteq \bigcup_{i=0}^{\infty} N_j$ where $\mathcal{H}^m(N_0) = 0$ and where each N_i, $i \geq 1$ is an m-dimensional embedded C^1 submanifold of $\mathbb{R}^n$.*

Proof. First assume that N is an m-dimensional C^1 submanifold. Then using the discussion after definition 3.1 we see that N has a local graphical representation. This means that for each $x \in N$ there exists a radius $r_x > 0$ such that $B_{r_x}(x) \cap N = \psi(V)$ for a suitable open set $V \subseteq \mathbb{R}^m$ and C^1 map $\psi : V \to \mathbb{R}^n$. All such C^1 maps are automatically Lipschitz in each closed ball in V, so it is clear that M satisfies definition 3.3

the reverse implication is a consequence of the C^1 approximation theorem 3.2. Indeed, the theorem states that for each $j = 1, 2, \dots$ there exists C^1 functions $G_{1j}, G_{2j}, \dots : \mathbb{R}^m \to \mathbb{R}^n$ such that if F_j are Lipschitz functions as in definition 3.3 then $\mathcal{H}^m(\{x \mid F_j(x) \neq G_{ij}(x)\}) < 1/i$. Now define

$$Z_j := \mathbb{R}^n \setminus \left(\bigcup_{i=1}^{\infty} \{x \mid F_j(x) = G_{ij}(x)\} \right),$$

then $\mathcal{H}^m(Z_j) = 0$, and

$$\operatorname{graph} F_j \subseteq F_j(Z_j) \cup \left(\bigcup_{i=1}^{\infty} G_{ij}(\mathbb{R}^n) \right), \quad j = 1, 2, \dots .$$

Now the area formula (theorem 2.14) says that $\mathcal{H}^m(F_j(Z_j)) = 0$ because F_j is Lipschitz and $\mathcal{H}^m(Z_j) = 0$, so with $N_0 := \bigcup_{j=1}^{\infty} E_j$, we have $\mathcal{H}^m(N_0) = 0$, and so we have proved that

$$M \subseteq (M_0 \cup N_0) \cup \left(\bigcup_{i,j=1}^{\infty} G_{ij}(\mathbb{R}^n) \right).$$

Now by the area formula $\mathcal{H}^m(\{x \mid J_{G_{ij}} = 0\}) = 0$. So if the Jacobian of G_{ij} is non-zero at a point x, then there is an $r > 0$ such that $G_{ij}(\mathring{B}_r(x))$ is an m-dimensional C^1 submanifold of $\mathbb{R}^m$ (that is G_{ij} provides a local representation in a neighborhood of the point $y = G_{ij}(x)$). So $\bigcup_{ij} G_{ij}$ can be written as the union of a set of measure zero and countably many m-dimensional C^1 submanifolds of $\mathbb{R}^n$. ■

In particular, with the N_i from the theorem, we can define $M_0 = M \cap N_0$, and then

$$M_i = M \cap N_i \setminus \bigcup_{j=0}^{i-1} M_j$$

for $i \geq 1$. Hence we can write $M = \bigcup_i M_i$, where the sets M_i are all $\mathcal{H}^m$-measurable and pairwise disjoint.

Definition 3.5. *A set $P \subseteq \mathbb{R}^n$ is called purely m-unrectifiable if*

$$\mathcal{H}^m(P \cap R) = 0$$

for all m-rectifiable $R \subseteq \mathbb{R}^n$.

Theorem 3.6. *Let $A \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$ measurable such that $\mathcal{H}^m(A) < \infty$, with $m \in \mathbb{N}$. Then there exists $\mathcal{H}^m$-measurable sets P and R, such that P is purely m-unrectifiable and R is m-rectifiable, and such that*

$$A = R \cup P, \quad \text{and} \quad R \cap P = \emptyset$$

Proof. Let $M = \sup\{\mathcal{H}^m(R) \mid R \subseteq A, R \text{ is } m\text{-rectifiable}\}$. For each $i \in N$ choose an m-rectifiable set R_i such that

$$\mathcal{H}^m(R_i) > M - 1/i.$$

Letting $R = \bigcup_i R_i$ and $P = A \setminus R$ finishes the proof. ■

3.3 Approximate tangent spaces and characterisation of countably m-rectifiable sets

Next, we will describe approximate tangent spaces. These will help us characterise rectifiable sets, and give us some invaluable tools, when we begin to talk about varifolds in the next chapter, and in the theorems of the last chapter.

Definition 3.7. *Let M be an $\mathcal{H}^m$-measurable subset of $\mathbb{R}^n$ with $\mathcal{H}(M \cap K) < \infty$ for all compact sets K. Then we say that an m-dimensional vector subspace $V \subseteq \mathbb{R}^n$ is an **approximate tangent space** of M at $x \in \mathbb{R}^n$ if*

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(M)} f(y)\, d\mathcal{H}^m(y) = \int_V f(y)\, d\mathcal{H}^m(y)$$

for all $f \in C_c^0(\mathbb{R}^n)$, where $\eta_{x,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$ for $x, y \in \mathbb{R}^n$ and $\lambda > 0$.

Given some $f \in C_c^0(\mathbb{R}^n)$, $\mu := \int_M f(y)\, d\mathcal{H}^m(y)$ is a Radon measure on M. We say that μ has approximate tangent space V, if the above equality holds.

We see that if M has an approximate tangent space V, then it is unique, and we denote it by $T_x M$.

As we mentioned, we can characterise most rectifiable sets by their having approximate tangent spaces.

Theorem 3.8. *If M is $\mathcal{H}^m$-measurable and $\mathcal{H}^m(M \cap K) < \infty$ for all compact sets K, then M is countably m-rectifiable if and only if the approximate tangent space $T_x M$ exists for $\mathcal{H}^m$-a.e. $x \in M$.*

Proof. We will only prove " $\Rightarrow$ ". By lemma 3.4 and the remark afterwards, we can find a disjoint cover of $\mathcal{H}^m$-measurable sets $M = \bigcup_i M_i$ such that $\mathcal{H}^m(M_0) = 0$ and $M_i \subseteq N_i$, $i \geq 1$ where N_i are embedded C^1 submanifolds of dimension m.

Now with $r > 0$ and $f \in C_c^0(\mathbb{R}^n)$ where $f \equiv 0$ in $\mathbb{R}^n \setminus B_r(0)$ we have

$$\int_{\eta_{x,\lambda}(M)} f\, d\mathcal{H}^m = \int_{\eta_{x,\lambda}(N_i)} f\, d\mathcal{H}^m - \int_{\eta_{x,\lambda}(N_i \setminus M_i)} f\, d\mathcal{H}^m + \int_{\eta_{x,\lambda}(M \setminus M_i)} f\, d\mathcal{H}^m$$

for all $x \in M_i$, and since $x \in M_i \subseteq N_i$ and N_i is a C^1 submanifold

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(N_i)} f\, d\mathcal{H}^m = \int_{T_x N_i} f\, d\mathcal{H}^m.$$

Hence by the upper density theorem 2.11

$$\begin{aligned} \left| \int_{\eta_{x,\lambda}(M \setminus N_i)} f\, d\mathcal{H}^m \right| &\leq \sup |f| \mathcal{H}^m(B_r(0) \cap \eta_{x,\lambda}(M \setminus N_i)) \\ &= \sup |f| \lambda^{-m} \mathcal{H}^m(B_{\lambda r}(x) \cap M \setminus N_i) \to 0 \end{aligned}$$

for $\mathcal{H}^m$-a.e. $x \in M_i$. The upper density theorem also gives us that

$$\left| \int_{\eta_{x,\lambda}(N_i \setminus M_i)} f \, d\mathcal{H}^m \right| \to 0$$

for $\mathcal{H}^m$-a.e. $x \in M_i$. Hence we have shown that $T_x M$ exists and is equal to $T_x N_i$ for $\mathcal{H}^m$-a.e. $x \in M_i$. ■

We relax the conditions in the previous definition of approximate tangent space, and arrive at the following definition

Definition 3.9. *Let $M \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$-measurable and let θ be a positive $\mathcal{H}^m$-measurable function on M with $\int_{M\cap K} \theta \, d\mathcal{H}^m < \infty$ for all compact sets $K \subseteq R^n$. Then for each $x \in \mathbb{R}^n$ we say that an m-dimensional vector subspace $V \subseteq \mathbb{R}^n$ is an approximate tangent space of M with respect to the multiplicity θ if*

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(M)} f(y)\theta(x + \lambda y) \, d\mathcal{H}^m(y) = \theta(x) \int_{V_x} f(y) \, d\mathcal{H}^m(y)$$

for each $f \in C_c^0(\mathbb{R}^n)$. Again, if V exists, it is unique, and so we denote it by $T_x M$, which also agrees with our previous definition of the approximate tangent space in case $\mathcal{H}^m(M \cap K) < \infty$ for all compact $K \subseteq \mathbb{R}^n$ and $\theta \equiv 1$.

We can then characterise rectifiable m-varifolds anew. To do that, we will first need Lusin's theorem.

Theorem 3.10 (Lusin's Theorem)**.** *Let μ be a Borel regular outer measure on a metric space X, let $A \subseteq X$ be μ-measurable with $\mu(A) < \infty$ and let $f : A \to \mathbb{R}$ be μ-measurable. Then for each $\varepsilon > 0$ there is a closed set $C \subseteq A$, such that $\mu(A \setminus C) < \varepsilon$ and $f|_C$ is continuous*

Proof. For every $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ let

$$A_{ij} = f^{-1}\left[\frac{j-1}{i}, \frac{j}{i}\right)$$

and note that for a given i, A_{ij} are pairwise disjoint, when ranging over $j \in \mathbb{Z}$, and $\bigcup_{j\in\mathbb{Z}} A_{ij} = A$ for every $i \in \mathbb{N}$.

Now, since A is μ-measurable and μ is Borel regular, then $\mu \llcorner A$ is Borel regular as well, and since A is finite, $\mu \llcorner A$ is in fact a Radon measure. So for each $\varepsilon > 0$ there exist closed sets $C_{ij} \subseteq A_{ij}$ such that

$$\mu(A_{ij} \setminus C_{ij}) = (\mu \llcorner A)(A_{ij} \setminus C_{ij}) < 2^{-i-|j|-2}\varepsilon.$$

This implies that

$$\mu\left(A_{ij} \setminus \bigcup_{\ell\in\mathbb{Z}} C_{i\ell}\right) < 2^{-i-|j|-2}\varepsilon,$$

which in turn implies that

$$\mu\left(A \setminus \bigcup_{j\in\mathbb{Z}} C_{ij}\right) < 2^{-i}\varepsilon.$$

So for each $i \in \mathbb{N}$ we can find some corresponding integer $j(i)$ such that

$$\mu\left(A \setminus \bigcup_{|j|\leq j(i)} C_{ij}\right) < 2^{-i}\varepsilon.$$

Now, if we let $C = \left(\bigcap_{i\in\mathbb{N}}\left(\bigcup_{|j|\leq j(i)} C_{ij}\right)\right)$ (which is a closed set) then we see that $A \setminus C = \bigcup_{i\in\mathbb{N}}(A \setminus \bigcup_{|j|\leq j(i)} C_{ij})$ which implies that $\mu(A\setminus C) < \varepsilon$.

Now, for every $i \in \mathbb{N}$ we define $g_i : \bigcup_{|j|\leq j(i)} C_{ij} \to \mathbb{R}$ by $g_i(x) = \frac{j-1}{i}$ on C_{ij} where $|j| \leq j(i)$. But then since $C_{i1}, \dots, C_{ij(i)}$ are pairwise disjoint closed sets, g_i is continuous and its restriction to C is continuous as well for every $i \in \mathbb{N}$. Finally, by construction $0 \leq f(x) - g_i(x) \leq 1/i$ for every $x \in C$ and $i \in \mathbb{N}$, hence $g_i|_C$ converges uniformly to $f|_C$ on C and hence $f|_C$ is continuous. ■

Theorem 3.11. *If $M \subseteq \mathbb{R}^n$ is $\mathcal{H}^m$-measurable and θ is a positive $\mathcal{H}^m$-measurable function on M with $\int_{M\cap K} \theta \, d\mathcal{H}^m < \infty$ for all compact sets $K \subseteq \mathbb{R}^n$, then M is countably m-rectifiable if and only if M has an approximate tangent space T_xM with respect to θ for $\mathcal{H}^m$-a.e. $x \in M$.*

Proof. By Lusin's theorem 3.10, we can find an increasing sequence $\{M_i\}_{i\in\mathbb{N}}$ of compact subsets of M such that

$$\mathcal{H}^m\left(M \setminus \bigcup_{i=1}^{\infty} M_i\right) = 0$$

and such that $\theta|_{M_i}$ is continuous. This implies that $\theta|_{M_i}$ has a positive lower bound, and therefore $\mathcal{H}^m(M_i) < \infty$ for all $i \in \mathbb{N}$. So we can use theorem 3.8 to see that the sets M_i are countably m-rectifiable if and only if the approximate tangent space T_xM_i exists for $\mathcal{H}^m$-a.e. $x \in M_i$. This finishes the proof. ■

If M is countable m-rectifiable, it can be covered by countably many m-dimensional C^1 submanifolds (as in the remark after lemma 3.4), say M_i, $i = 1, 2, \dots$. The above proof then shows that for $\mathcal{H}^m$-a.e. $x \in M \cap M_i$, T_xM is equal to the usual tangent space of M_i.

This also implies that if $A \subseteq B \subseteq \mathbb{R}^m$ are $\mathcal{H}^m$-measurable with finite measure, then for $\mathcal{H}^m$-a.e. $x \in A$, T_xA exists if and only if T_xB exists. And if they exist, they are equal $\mathcal{H}^m$-a.e.

Corollary 3.12. *Let $P \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(P) < \infty$. Then P is purely m-unrectifiable if and only if the set of points $x \in P$ for which $T_x^m P$ exists has $\mathcal{H}^m$-measure zero.*

If $M \subseteq \mathbb{R}^n$ is countably m-rectifiable then given a Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}$, theorem 3.8 allows us to define the gradient $\nabla^M f$ at $\mathcal{H}^m$-a.e. $x \in M$ by

$$\nabla^M f(x) = \sum_{i=1}^{m} \partial_{v_i} f(x) v_i$$

where $(v_1, \ldots, v_m)$ is an orthonormal basis of $T_x^m M$ and $\partial_{v_i} f(x)$ denotes the directional derivatives of f in direction v_i.

By lemma 3.4 we can write $M = M_0 \cup \bigcup_{i=1}^{\infty} M_i$ for C^1 submanifolds M_i, $i > 0$. Indeed we can even ensure that the unions are pairwise disjoint, by letting $N_0 = M_0$, and then $N_j = M_j \setminus \bigcup_{i=0}^{j-1} M_i$. Then $N_i \subseteq M_i$ and

$$M = N_0 \sqcup \bigsqcup_{i=1}^{\infty} N_i.$$

This implies $\nabla^M f(x) = \nabla^{N_i} f(x)$ when $x \in N_i$ and $f|_{M_i}$ is differentiable at x, which, by Rademachers theorem holds for $\mathcal{H}^m$-a.e. $x \in M_i$. We can thus define the linear map $d^M f_x : T_x^m M \to \mathbb{R}$ by

$$d^M f_x(v) = \left\langle v, \nabla^M f(x) \right\rangle$$

for $v \in T_x^m M$, at all points where $T_x^m M$ and $\nabla^M f(x)$ exists. If now $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ is Lipschitz, we can similarly define the linear map $d^M f_x : T_x^m M \to \mathbb{R}^\ell$ by

$$d^M f_x(v) = \sum_{i=1}^{\ell} \left\langle v, \nabla^M f_j(x) \right\rangle e_j$$

where $(e_1, \ldots, e_\ell)$ is the standard basis for $\mathbb{R}^\ell$. If $\ell \geq m$, we define the Jacobian, $J_f^M(x)$ of f for $\mathcal{H}^m$-a.e. $x \in M$ by

$$J_f^M(x) = \sqrt{\det((d^M f_x)^* \circ (d^M f_x))}.$$

In this setting, the area formula also holds, namely

$$\int_A J_f^M(x) \, d\mathcal{H}^m = \int_{\mathbb{R}^\ell} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^m(y)$$

for every $\mathcal{H}^m$-measurable $A \subseteq M$. And similarly for $\ell < m$ we define

$$J_f^M(x) = \sqrt{\det((d^M f_x) \circ (d^M f_x)^*)}.$$

and the co-area formula holds, namely

$$\int_A J_f^M(x)\, d\mathcal{H}^m(x) = \int_{\mathbb{R}^\ell} \mathcal{H}^{m-\ell}(A \cap f^{-1}(y))\, d\mathcal{H}^\ell(y)$$

for every $\mathcal{H}^m$-measurable set A.

Chapter 4

Varifolds

We will now study varifolds, which can be thought of as a measure theoretic generalization of differentiable manifolds.

Definition 4.1. *Let $U \subseteq \mathbb{R}^n$ be open, and let M, M' be $\mathcal{H}^m$-measurable and countably m-rectifiable subsets of U and let $\theta : M \to [0, \infty)$, $\theta' : M' \to [0, \infty)$ be non-negative and locally $\mathcal{H}^m$-integrable in M, M' respectively (i.e. θ, θ' is integrable on every compact subset of M, M' respectively). We say that (M, θ) and (M', θ') are equivalent if*

$$\mathcal{H}^n((M \setminus M') \cup (M' \setminus M)) = 0$$

and $\theta = \theta'$ $\mathcal{H}^m$-a.e. in $M \cap M'$. Then a (countably) ***rectifiable*** m***-varifold*** *in U denoted by $V = V(M, \theta)$ is the equivalence class of the pair (M, θ) as above, and the pair is called a representative for V. If θ is integer-valued, $V(M, \theta)$ is called an integer (multiplicity rectifiable) m-varifold.*

For simplicity, we adopt the convention that $\theta \equiv 0$ on $\mathbb{R}^n \setminus M$.

To every rectifiable m-varifold $V = V(M, \theta)$ in an open set $U \subseteq \mathbb{R}^n$ we can associate a Radon measure μ_V, called the **weight measure** of V, given by

$$\mu_V(A) = \mathcal{H}^m \llcorner \theta(A) := \int_{A \cap M} \theta \, d\mathcal{H}^m$$

for every $\mathcal{H}^m$-measurable set $A \subseteq U$. The weight (or mass), $\mathbb{M}_V$ of V is then defined by

$$\mathbb{M}_V = \mu_V(U).$$

Furthermore, if $V = V(M, \theta)$ is a rectifiable m-varifold, and $x \in \mathbb{R}^n$ then we define

$$T_x V = T_x M$$

whenever $T_x M$ exists. Notice, that this definition is independent of the choice of representative of the rectifiable m-varifold. Lastly, for $V = V(M, \theta)$ we define

$$\operatorname{supp} V = \operatorname{supp} \mu_V .$$

4.1 First and second variation formulae

We want to study how the mass $\mathbb{M}_V$ of a varifold is affected when it is perturbed by a diffeomorphism, similar to the 3 dimensional. Ultimately we are seeking the varifolds which are critical under such perturbations, called stationary varifolds. It will turn out that we can relate the rate change of the mass of a varifold to the mean curvature. It will be these stationary varifolds that we can think of as "minimal surfaces".

Consider first, for simplicity sake, the case when M is an m-dimensional C^1 submanifold of $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be open such that $U \cap M \neq \emptyset$ and such that $\mathcal{H}^m(K \cap M) < \infty$ for every compact subset $K \subseteq U$. Now let $\{\phi_t\}_{t\in(-1,1)}$ be a 1-parameter family of diffeomorphisms with $\phi_t : U \to U$ with the following properties

$$\phi : (-1, 1) \times U \to U \text{ given by } \phi(t, x) = \phi_t(x) \text{ is } C^2, \tag{4.1}$$

$$\phi_0(x) = x, \text{ for all } x \in U, \text{ and} \tag{4.2}$$

$$\phi_t(x) = x, \text{ for all } x \in U \setminus K \text{ and } t \in (-1, 1), \tag{4.3}$$

for some compact subset $K \subseteq U$. Then we define the initial velocity and acceleration vectors for ϕ_t, $X = (X^1, \ldots, X^n) : U \to \mathbb{R}^n$ and $Z = (Z^1, \ldots, Z^n) : U \to \mathbb{R}^n$ respectively, by

$$X(x) = \frac{\partial \phi(t, x)}{\partial t}|_{t=0}, \quad \text{and} \quad Z(x) = \frac{\partial^2 \phi(t, x)}{\partial^2 t}|_{t=0}. \tag{4.4}$$

Then

$$\phi_t(x) = x + tX(x) + \frac{t^2}{2} Z(x) + O(t^3) \tag{4.5}$$

where $O(t^3) \in \mathbb{R}^n$ is such that for some $c > 0$, $|O(t^3)| \leq c|t^3|$ for all $t \in (-1, 1)$. Since $\phi_t(x) = x$ for all $x \in U \setminus K$ and $t \in (-1, 1)$ this means that $X(x)$ and $Z(x)$ are compactly supported on a subset of K.

With K as before, let $M_t = \phi_t(M \cap K)$. Then M_t is a 1-parameter family of manifolds such that $M_0 = M \cap K$ and M_t agrees with M outside some compact subset of U. The first and second variation of M are then defined respectively by

$$\frac{d}{dt}\mathcal{H}^m(M_t)|_{t=0}, \quad \text{and} \quad \frac{d^2}{dt^2}\mathcal{H}^m(M_t)|_{t=0}.$$

With K as before we can benefit from the area formula which yields

$$\mathcal{H}^m(M_t) = \mathcal{H}^m(\phi_t(M \cap K)) = \int_{M\cap K} J_{\psi_t}\, d\mathcal{H}^m$$

where $\psi_t = \phi_t|_{M\cap U}$. To compute the first and second variation, we can switch the order of integration and differentiation, and the computation is reduced to calculating

$$\frac{\partial}{\partial t}J_{\psi_t}|_{t=0}, \quad \text{and} \quad \frac{\partial^2}{\partial t^2}J_{\psi_t}|_{t=0}.$$

So let us do that short computation. We fix two orthonormal bases: $\tau_1, \dots, \tau_m$ of T_xM for $x \in M$ and $e_1, \dots, e_n$ of $\mathbb{R}^n$. Then with the induced linear map $d\psi_{t,x} : T_xM \to \mathbb{R}^n$ of ψ_t at $x \in M$ defined as

$$d\psi_{t,x}(\tau) = \partial_\tau \phi_t(x) = \partial_\tau \psi_t(x), \quad \tau \in M$$

we get by (4.5) that

$$d\psi_{t,x}(\tau) = \tau + t\partial_\tau X(x) + \frac{t^2}{2}\partial_\tau Z(x) + O(t^3).$$

We can express the matrix of the map $d\psi_{t,x}$ with respect to the basis $\tau_1, \dots, \tau_m$ of T_xM and $e_1, \dots, e_n$ of $\mathbb{R}^n$, and we get that the matrix has as i'th row

$$\partial_{\tau_i}\psi_t(x) = \tau_i + t\partial_{\tau_i}X + \frac{t^2}{2}\partial_{\tau_i}Z + O(t^3)$$

for $i = 1, \dots, m$. Hence, with respect to $\tau_1, \dots, \tau_m$, $(d\psi_{t,x})^* \circ (d\psi_{t,x})$ has matrix $(b_{ij})_{m\times m}$ where

$$\begin{aligned} b_{ij} &= \partial_{\tau_i}\psi_{t,x} \cdot \partial_{\tau_j}\psi_{t,x} \\ &= \delta_{ij} + t(\langle \tau_i, \partial_{\tau_j}X\rangle + \langle \tau_j, \partial_{\tau_i}X\rangle) \\ &\qquad + t^2\left(\frac{1}{2}(\langle \tau_i, \partial_{\tau_j}Z\rangle + \langle \tau_j, \partial_{\tau_i}Z\rangle) + \langle \partial_{\tau_i}X, \partial_{\tau_j}X\rangle\right) + O(t^3). \end{aligned}$$

By a straightforward computation, we see that for symmetric square matrices $I = (\delta_{ij})$, $A = (A_{ij})$ and

$B = (B_{ij})$ we have

$$\det(I + tA + t^2 B) = 1 + t \operatorname{Tr} A + t^2 \operatorname{Tr} B + \frac{1}{2} t^2 ((\operatorname{Tr} A)^2 - \operatorname{Tr} A^2) + O(t^3).$$

Letting

$$\begin{aligned} A_{ij} &= \langle \tau_i, \partial_{\tau_j} X \rangle + \langle \tau_j, \partial_{\tau_i} X \rangle = A_{ji}, \quad \text{and} \\ B_{ij} &= \frac{1}{2} (\langle \tau_i, \partial_{\tau_j} Z \rangle + \langle \tau_j, \partial_{\tau_i} Z \rangle) + \langle \partial_{\tau_i} X, \partial_{\tau_j} X \rangle, \end{aligned}$$

we get that

$$\begin{aligned} J^2_{\psi_t}(x) &= \det((d\psi_{t,x})^* \circ (d\psi_{t,x}() = \det(b_{ij}) \\ &= 1 + 2t \sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X \rangle + t^2 \sum_{i=1}^m \left(\langle \tau_i, \partial_{\tau_i} Z \rangle + |\partial_{\tau_i} X|^2 \right) + 2t^2 \left(\sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X \rangle \right)^2 \\ &\qquad - \frac{t^2}{2} \sum_{i,j=1}^m \left(\langle \tau_i, \partial_{\tau_j} X \rangle + \langle \tau_j, \partial_{\tau_i} X \rangle \right)^2 + O(t^3) \\ &= 1 + 2t \operatorname{div}_M X + t^2 \operatorname{div}_M Z + t^2 \sum_{i=1}^m |\partial_{\tau_i} X|^2 + 2t^2 (\operatorname{div}_M X)^2 \\ &\qquad - t^2 \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X \rangle^2 - t^2 \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X \rangle \langle \tau_j, \partial_{\tau_i} X \rangle + O(t^3) \\ &= 1 + 2t \operatorname{div}_M X \\ &\qquad + t^2 \left(\operatorname{div}_M Z + 2(\operatorname{div}_M X)^2 + \sum_{i=1}^m \left| (\partial_{\tau_i} X)^\perp \right|^2 - \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X \rangle \langle \tau_j, \partial_{\tau_i} X \rangle \right) \\ &\qquad + O(t^3) \end{aligned}$$

where

$$(\partial_{\tau_i} X)^\perp = \partial_{\tau_i} X - \sum_{j=1}^m \langle \tau_j, \partial_{\tau_i} X \rangle \tau_j$$

is the normal part of $\partial_{\tau_i} X$, and $\operatorname{div}_M X$ is the divergence of X at $x \in M$ with respect to M defined as

$$\operatorname{div}_M X = \sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X \rangle$$

Finally, using the Taylor expansion, we see that

$$\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + O(x^3)$$

and thus

$$
\begin{aligned}
J_{\psi_t}(x) &= 1 + t \operatorname{div}_M X \\
&\qquad + \frac{t^2}{2}\left(\operatorname{div}_M Z + 2(\operatorname{div}_M X)^2 + \sum_{i=1}^{m} \left|(\partial_{\tau_i} X)^{\perp}\right|^2 - \sum_{i,j=1}^{m} \left\langle \tau_i, \partial_{\tau_j} X\right\rangle \left\langle \tau_j, \partial_{\tau_i} X\right\rangle\right) \\
&\qquad - \frac{t^2}{8}(2 \operatorname{div}_M X)^2 + O(t^3) \\
&= 1 + t \operatorname{div}_M X \\
&\qquad + \frac{t^2}{2}\left(\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^{m} \left|(\partial_{\tau_i} X)^{\perp}\right|^2 - \sum_{i,j=1}^{m} \left\langle \tau_i, \partial_{\tau_j} X\right\rangle \left\langle \tau_j, \partial_{\tau_i} X\right\rangle\right) \\
&\qquad + O(t^3).
\end{aligned}
$$

Hence we can finally say that

$$\frac{\partial}{\partial t} J_{\psi_t}|_{t=0} = \operatorname{div}_M X,$$

and thus, by the area formula, we get the first variation formula

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}^m(M_t)|_{t=0} &= \int_{M \cap K} \frac{\partial}{\partial t} J_{\psi_t}|_{t=0}\, d\mathcal{H}^m \\
&= \int_{M \cap K} \operatorname{div}_M X \, d\mathcal{H}^m \\
&= \int_{M} \operatorname{div}_M X \, d\mathcal{H}^m
\end{aligned}
$$

where the last inequality is due to the fact that X has support in a subset of K. This is very similar to the area functional in the 3 dimensional case.

Furthermore, we get the second variation formula, which is not as pretty

$$
\begin{aligned}
&\frac{d^2}{dt^2}\mathcal{H}^m(M_t)|_{t=0} \\
&\quad = \int_M \left(\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^{m} \left|(\partial_{\tau_i} X)^{\perp}\right|^2 - \sum_{i,j=1}^{m} \left\langle \tau_i, \partial_{\tau_j} X\right\rangle \left\langle \tau_j, \partial_{\tau_i} X\right\rangle\right) d\mathcal{H}^m.
\end{aligned}
$$

Definition 4.2. *With ϕ_t, and K as in (4.2), we say that an m-dimensional C^1 submanifold $M \subseteq \mathbb{R}^n$ is stationary in an open set $U \subseteq \mathbb{R}^n$ if $\mathcal{H}^m(M \cap C) < \infty$ for all compact subsets $C \subseteq U$, if*

$$\frac{d}{dt}\mathcal{H}^m(M_t)|_{t=0} = 0$$

For $M_t = \phi_t(M \cap K)$.

By the above discussion, M is stationary in U if and only if

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^m = 0$$

for every C^1 map $X : U \to \mathbb{R}^n$ with compact support in U.

Let $m < n$. When $M \subseteq \mathbb{R}^n$ is an m-dimensional C^2 submanifold, $U \subseteq \mathbb{R}^n$ is open such that $\overline{U} \cap M$ is compact, and H is the mean curvature of M, then M is stationary in U if and only if $H \equiv 0$ in $M \cap U$.

We will try and generalize the first variation formula to rectifiable m-varifolds, along with a definition of a stationary varifold, and a treatment of mean curvature.

So let $U \subseteq \mathbb{R}^n$ be an open set, and let $V(M, \theta)$ be a rectifiable m-varifold in U. Suppose for simplicity that $\theta(x) \geq 1$ for $\mathcal{H}^m$-a.e. $x \in M$. We do this to avoid discussion of approximate tangent spaces, and Jacobians with respect to θ.

Let $N \geq n$, $U' \subseteq \mathbb{R}^N$ be open and $f : U \to U'$ be Lipschitz. Recall that we defined J_f^M by

$$J_f^M(x) = \sqrt{\det((d^M f_x)^* \circ (d^M f_x))}$$

then we note that if (M, θ) and $(\tilde{M}, \tilde{\theta})$ are two representatives for the same rectifiable m-varifold, V, then for $\mathcal{H}^m$-a.e. $x \in M \cap \tilde{M}$, $J_f^M(x) = J_f^{\tilde{M}}(x)$ and we will thus denote it by J_f^V.

By the general area formula theorem 2.14 we get that for every nonnegative $\mathcal{H}^m$-measurable mapping g on M, and every $\mathcal{H}^m$-measurable $A \subseteq M$

$$\int_A g J_f^E \, d\mathcal{H}^m = \int_{f(M)} \sum_{x \in A \cap f^{-1}(y)} g(x) \, d\mathcal{H}^m = \int_{f(M)} \left(\int_{A \cap f^{-1}(y)} g \, d\mathcal{H}^0 \right) d\mathcal{H}^m.$$

We clearly see that $f(M)$ is an m-rectifiable subset of U'. Furthermore, if we assume that $f : U \to U'$ is proper (i.e. $f^{-1}(K) \subseteq U$ is compact for every compact $K \subseteq U'$), we can then define θ' on U' by

$$\theta'(y) = \sum_{x \in M \cap f^{-1}(y)} \theta(x) = \int_{M \cap f^{-1}(y)} \theta \, d\mathcal{H}^0$$

and then define the image varifold $f_\# V$ by

$$f_\# V = V(f(M), \theta').$$

Indeed, this is well defined, since

$$\int_K \theta' \, d\mathcal{H}^m = \int_{f(M)\cap K} \theta' \, d\mathcal{H}^m = \int_{M\cap f^{-1}(K)} \theta J_f^M \, d\mathcal{H}^m$$

for every compact $K \subseteq U'$, so θ' is locally $\mathcal{H}^m$-integrable in U'. Therefore $f_\# V$ is a rectifiable m-varifold in U' with multiplicity θ'. Furthermore

$$\mathbb{M}_{f_\# V} = \int_{f(M)} \theta' \, d\mathcal{H}^m = \int_M F_f^M \, d\mathcal{H}^m$$

which leads us to defining the first variation of V. So let $\{\phi_t\}$ be a 1-parameter family of diffeomorphisms $\phi_t : U \to U$ as in (4.2). We let $V \llcorner K = V(M \cap K, \theta|_K)$ when $K \subseteq U$ is the compact set from (4.2). Then

$$\mathbb{M}_{\phi_{t\#}(V \llcorner K)} = \int_{M\cap K} J_{\phi_t}^M \theta \, d\mathcal{H}^m$$

and we can thus compute the first variation as in the C^1 case and get that

$$\frac{d}{dt}\mathbb{M}_{\phi_{t\#}(V \llcorner K)}|_{t=0} = \int_M \operatorname{div}_M X \, d\mu_V$$

where X is as in (4.4), and $\operatorname{div}_M X$ is the divergence of X with respect to M, given by

$$\operatorname{div}_M X = \sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X(x) \rangle$$

with $\tau_1, \dots, \tau_m$ an orthonormal basis of $T_x M$. We can thus define, exactly as in the case with C^1-submanifolds, the following.

Definition 4.3. *A rectifiable m-varifold $V = V(M, \theta)$ is **stationary** in an open set $U \subseteq \mathbb{R}^n$ if*

$$\int_M \operatorname{div}_M X \, d\mu_V = 0$$

for all C^1 mappings $X : U \to \mathbb{R}^n$ with compact support in U.

And we also generalize the notion of mean curvature as follows.

Definition 4.4. *Let $V = V(M, \theta)$ be a rectifiable m-varifold in an open set $U \subseteq \mathbb{R}^n$. Suppose $H : M \cap U \to \mathbb{R}^n$ is locally μ_V-integrable. Then we say that $V(E, \theta)$ has **generalized mean curvature** H in U, if*

$$\int_U \operatorname{div}_M X \, d\mu_V = -\int_U X \cdot H \, d\mu_V$$

for all C^1 mappings $X : U \to \mathbb{R}^n$ with compact support in U.

Hence a rectifiable m-varifold $V = V(M, \theta)$ is stationary in an open set $U \subseteq \mathbb{R}^n$ if and only if it has generalized mean curvature 0 in U.

4.2 Monotonicity Formulae

In [1] the author discusses the monotonicity formulae, specifically for integer rectifiable varifolds. We shall describe them here a little more generally.

Given various restrictions on the generalized mean curvature of a rectifiable m-varifold $V = V(M, \theta)$, one can show various results on the monotonicity of the function $\rho \mapsto \rho^{-m}\mu(B_\rho(\xi))$ or functions closely related to this. If the generalized mean curvature is zero, one can show that this above function is increasing in ρ and hence that the density

$$\Theta^m(\mu_V, \xi) = \lim_{\rho \to 0} \frac{\mu_V(B_\rho(\xi))}{\omega_m \rho^m}$$

exists and is real for every $\xi \in U$.

We will instead assume that the generalized mean curvature is merely bounded, and arrive at the next result.

First though, if $U \subseteq \mathbb{R}^n$ is open, and $V = V(M, \theta)$ is a rectifiable m-varifold in U, then for every differentiable $g : U \to \mathbb{R}$, we denote by $\nabla^\perp g(x)$ the projection of ∇g onto $(T_x M)^\perp$ for the $\mathcal{H}^m$-a.e. $x \in M$ where it is defined.

Theorem 4.5 (Monotonicity Formula). *Let $V = V(M, \theta)$ be a rectifiable m-varifold in some open set $U \subseteq \mathbb{R}^n$ with generalized mean curvature H. Assume that the generalized mean curvature H of V is bounded (i.e. there is a constant Λ such that $|H| \leq \Lambda$). Let $x \in U$, then for every $0 < \sigma < \rho < \operatorname{dist}(\xi, \partial U)$ we have*

$$\frac{\mu_V(B_\rho(\xi))}{\rho^m} - \frac{\mu_V(B_\rho(\xi))}{\sigma^m} = \int_{B_\rho(\xi)} \frac{H}{m}(x - \xi)\left(\frac{1}{\max\{r, \sigma\}^m} - \frac{1}{\rho^m}\right) d\mu_V + \int_{B_\rho(\xi) \setminus B_\sigma(\xi)} \frac{|\nabla^\perp r|^2}{r^m} d\mu_V$$

Where $r = r(x) = |x - \xi|$. In particular, the map $\rho \mapsto e^{\|H\|_\infty \rho}\rho^{-m}\mu_V(B_\rho(\xi))$ is monotone increasing.

Proof. By translating we can assume that $\xi = 0$, and thus $r(x) = |x|$. Let $\gamma \in C_c^1([0, 1))$ with $\gamma \equiv 1$ in some neighborhood of 0. For every $s \in (0, \partial U)$ we define a vector field X_s by $X_s = \gamma\left(\frac{|x|}{s}\right) x$. Then $X_s \in C_c^1(U)$ and we can therefore write

$$\int \operatorname{div}_{T_x M} X_s \, d\mu_V = -\int H \cdot X_s \, d\mu_V. \tag{4.6}$$

Now, $T_x M$ sits inside $\mathbb{R}^n$, and it has an orthonormal basis $e_1, \ldots, e_k$ which can be extended to be an orthonormal basis of $\mathbb{R}^n$. We can then compute

$$\begin{aligned}
\operatorname{div}_{T_xM} X_s &= m\gamma\left(\frac{r}{s}\right) + \sum_{i=1}^{m} e_i x \gamma'\left(\frac{r}{s}\right)\frac{x\cdot e_i}{|x|s} \\
&= m\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\sum_{i=1}^{m}\left(\frac{x\cdot e_i}{|x|}\right)^2 \\
&= m\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\left(1-\sum_{i=m+1}^{n}\left(\frac{x\cdot e_i}{|x|}\right)^2\right) \\
&= m\gamma\left(\frac{r}{s}\right) + \frac{r}{s}\gamma'\left(\frac{r}{s}\right)\left(1-|\nabla^{\perp} r|^2\right) \qquad (4.7)
\end{aligned}$$

We can then insert (4.7) in to (4.6), divide by s^{m+1} and then integrate between σ and ρ, which yields

$$\begin{aligned}
&\int_\sigma^\rho \int_{\mathbb{R}^n} \frac{m}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) d\mu_V\, ds + \int_\sigma^\rho \int_{\mathbb{R}^n} \frac{|x|}{s^{m+2}}\gamma'\left(\frac{|x|}{s}\right)\left(1-|\nabla^\perp r|^2\right) d\mu_V\, ds \\
&\quad = -\int_\sigma^\rho \int_{\mathbb{R}^n} \frac{H\cdot x}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) d\mu_V\, ds
\end{aligned}$$

Using Fubini, we can switch the order of integration, and distribute the integrals out, which gives

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) + \frac{|x|}{s^{m+2}}\gamma'\left(\frac{|x|}{s}\right) ds\, d\mu_V \\
&= \int_{\mathbb{R}^n} |\nabla^\perp r|^2 \int_\sigma^\rho \frac{|x|}{s^{m+2}}\gamma'\left(\frac{|x|}{s}\right) ds\, d\mu_V - \int_{\mathbb{R}^n} H\cdot x \int_\sigma^\rho \frac{1}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) ds\, d\mu_V \qquad (4.8)
\end{aligned}$$

By the product rule, it is easily seen that

$$-\int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) + \frac{|x|}{s^{m+2}}\gamma'\left(\frac{|x|}{s}\right) ds = \rho^{-m}\gamma\left(\frac{|x|}{\rho}\right) - \sigma^{-m}\gamma\left(\frac{|x|}{\sigma}\right)$$

and we can thus rewrite (4.8) as

$$\begin{aligned}
&\rho^{-m}\int \gamma\left(\frac{|x|}{\rho}\right) d\mu_V(x) - \sigma^{-m}\int\gamma\left(\frac{|x|}{\sigma}\right) d\mu_V(x) - \int_{\mathbb{R}^n} H\cdot x\int_\sigma^\rho s^{-m-1}\gamma\left(\frac{|x|}{s}\right) ds\, d\mu_V(x) \\
&= \int_{\mathbb{R}^n} |\nabla^\perp r|^2\left[\rho^{-m}\gamma\left(\frac{|x|}{\rho}\right) - \sigma^{-m}\gamma\left(\frac{|x|}{\sigma}\right) + \int_\sigma^\rho \frac{m}{s^{m+1}}\gamma\left(\frac{|x|}{s}\right) ds\right] d\mu_V(x). \qquad (4.9)
\end{aligned}$$

Replacing γ with a sequence of mollifiers $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \to \mathbb{1}_{(-1,1)}$ from below as $n\to\infty$, we can use the dominated convergence theorem to see that we are allowed to insert $\gamma = \mathbb{1}_{(0,1)}$ in (4.9), and get

$$\int_\sigma^\rho \frac{m}{s^{m+1}}\mathbb{1}_{(0,1)}\left(\frac{|x|}{s}\right) ds = \mathbb{1}_{B_\rho}(x)\int_{\max\{|x|,\sigma\}}^\rho \frac{m}{s^{m+1}}\, ds = \left(\frac{1}{\max\{|x|,\sigma\}^m} - \frac{1}{\rho^m}\right)\mathbb{1}_{B_\rho}(x).$$

which finishes the main statement of the theorem.

For the last part of the theorem, define $f(\rho) := \rho^{-m}\mu_V(B_\rho)$. The main result of this proof then gives us the trivial bound

$$\frac{f(\rho)-f(\sigma)}{\rho-\sigma} \geq -\frac{\|H\|_\infty}{m}\int_{B_\rho} |x|\frac{\max\{|x|,\sigma\}^{-m}-\rho^{-m}}{\rho-\sigma}\,d\mu_V(x) \geq -\frac{\|H\|_\infty}{m}\mu_V(B_\rho)\rho\frac{\sigma^{-m}-\rho^{-m}}{\rho-\sigma}.$$

Since $\rho \mapsto \rho^{-m}$ is convex, we can conclude, by setting $\rho = \sigma+\varepsilon$ for some given $\varepsilon > 0$, that

$$\frac{f(\sigma+\varepsilon)-f(\sigma)}{\varepsilon} \geq -\mu_V(B_\rho)\|H\|_\infty \frac{(\sigma+\varepsilon)}{\sigma^{m+1}} = -\|H\|_\infty f(\sigma+\varepsilon)\frac{(\sigma+\varepsilon)^{m+1}}{\sigma^{m+1}}. \tag{4.10}$$

Finally, let ψ_δ be some smooth non-negative mollifier. We can then take the convolutions of both sides of (4.10) as functions of σ, and then, letting $\varepsilon \to 0$ we can conclude that $(f*\psi_\delta)' + \|H\|_\infty(f*\psi_\delta) \geq 0$. This implies that the function $g_\delta(\rho) := e^{\|H\|_\infty \rho} f*\psi_\delta(\rho)$ is monotone increasing. Finally letting $\delta \to 0$ from above, gives us that $\rho \mapsto e^{\|H\|_\infty \rho}\rho^{-m}\mu_V(B_\rho)$ is monotone increasing as wanted. ■

As a consequence of this theorem, we see that for a rectifiable varifold $V = V(M,\theta)$, the density $\Theta^n(\mu_V,\xi)$ is an upper semi-continuous function on U and coincides with θ μ_V-a.e. This follows from the monotonicity of $e^{\|H\|_\infty \rho}\rho^{-m}\mu_V(B_\rho(x))$ and the fact that $\mu_V = \Theta^n$ μ_V-a.e. (a consequence of the Radon-Nikodym theorem).

One can generalize the above theorem a little further, and instead assume that the generalized mean curvature has an L^p condition. One then arrives at a similar conclusion. Again, it becomes evident that $\Theta^n(\mu_V,\xi)$ is an upper semi-continuous function on U. We will not prove this here, but the result will be used later.

Chapter 5

Allard's regularity theorem

We will now move towards the main theorem of this thesis. The main theorem, and, in fact, the whole chapter, focuses not only on stationary varifolds, but varifolds with an L^p condition on the generalized mean curvature (See (†) for the specific L^p condition).

In the following chapter, we assume that $U \subseteq \mathbb{R}^n$ is open, $0 \in U$, $V = V(M, \theta)$ is a rectifiable m-varifold in U, and that V has generalized mean curvature H in U as defined in definition 4.4. We also assume that $\rho > 0$ is such that $B_\rho(0) \subseteq U$ and continue to assume that $m \leq n$.

5.1 Tilt-Excess Decay Lemma

Definition 5.1. *If $B_\sigma(\xi) \subseteq U$, and T is an m-dimensional subspace of $\mathbb{R}^n$, we define the **tilt-excess**, $E(\xi, \sigma, T)$ relative to the rectifiable m-varifold V by*

$$E(\xi, \sigma, T) = \sigma^{-m} \int_{B_\sigma(\xi)} |p_{T_x M} - p_T|^2 \, d\mu_V$$

In the above, $p_{T_x M}$ and p_T specify the orthogonal projections of U onto the respective spaces $T_x M$ and T.

For some given subspace T, we see that the tilt-excess $E(\xi, \sigma, T)$ measures the mean square deviation of T away from the approximate tangent space $T_x M$, in a small area around ξ.

We note that if $T = \mathbb{R}^m \times \{0\}$, and (e_{ij}), (ε_{ij}) denote the $n \times n$ matrices of $p_{T_x m}$ and p_T respectively, then since these matrices are idempotent and have trace $= m$ we get that

$$\begin{aligned}|p_{T_xM} - p_T|^2 &= 2\sum_{i,j}(e_{ij}^2 + \varepsilon_{ij}^2 - 2e_{ij}\varepsilon_{ij}) \\ &= 2(m - \sum_{j=1}^{m} e_{jj}) \\ &= 2\sum_{j=m+1}^{n} e_{jj} \\ &= 2\sum_{j=1}^{n-m} |\nabla^M x_{m+j}|^2\end{aligned}$$

hence in this case

$$E(\xi,\sigma,T) = 2\sigma^{-m}\int_{B_\sigma(\xi)}\sum_{i=1}^{k}\left|\nabla^M x_{m+i}\right|^2\, d\mu_V \tag{5.1}$$

Lemma 5.2. *If $B_\rho(\xi) \subseteq U$, then for any m-dimensional subspace $T \subseteq \mathbb{R}^n$*

$$E(\xi,\rho/2,T) \leq C\rho^{-m}\int_{B_\rho(\xi)}\left(\frac{\operatorname{dist}(x-\xi,T)}{\rho}\right) d\mu_V + C\rho^{2-m}\int_{B_\rho(\xi)}|H|^2\, d\mu_V,$$

for some constant $C = C(m)$.

Proof. We can assume without loss of generality that $\xi = 0$ and that $T = \mathbb{R}^m \times \{0\}$. Recalling definition 4.4 of the generalized mean curvature, the idea of this proof is to find some suitable X to put in that formula that will yield the wanted result. Given $x = (x_1,\ldots,x_n) \in U$, define $x' = (0,\ldots,0,x_{m+1},\ldots,x_n)$, then our suitable choice will be

$$X = \zeta^2(x)x'$$

where $\zeta \in C_c^1(U)$, with $\zeta \geq 0$, is to be found below. Now, if (e_{ij}) represents the matrix of the projection p_{T_xM} relative to the standard normal basis for $\mathbb{R}^n$, we have

$$\operatorname{div}_M(x') = \sum_{i=m+1}^{n} e_{ii}$$

for μ_V-a.e. $x \in M$ by the definition of div_M. We then denote by (ε_{ij}) the matrix of the projection $p_{\mathbb{R}^m}$, and see that $\varepsilon_{ij} = 0$ if $i > m$. Then remembering that projections are idempotent, and noting that the trace of (e_{ij}) equals n, we can write

$$\operatorname{div}_M(x') = \sum_{i=m+1}^{n} e_{ii} = (n - \sum_{i=1}^{m} e_{ii}) = \sum_{i=m+1}^{n}\sum_{j=1}^{n}(e_{ij} - \varepsilon_{ij})^2 = \frac{1}{2}|p_{T_xM} - p_{\mathbb{R}^n}|^2.$$

for μ_V-a.e. $x \in M$. This gives us the connection to the tilt-excess. So using the definition of the generalized mean curvature from definition 4.4 we get that

$$
\begin{aligned}
\int \operatorname{div}_M(x')\zeta^2\, d\mu_V &= -\int 2\zeta x' \operatorname{div}_M(x')\nabla^M \zeta\, d\mu_V - \int \zeta^2 H x'\, d\mu_V \\
&= -\int 2\zeta \sum_{i=m+1}^{n} \sum_{j=1}^{n} x_i(e_{ij} - \varepsilon_{ij})\nabla^M \zeta\, d\mu_V - \int \zeta^2 H x'\, d\mu_V \\
&\leq \left| \int -2\zeta \sum_{i=m+1}^{n} \sum_{j=1}^{n} x_i(e_{ij} - \varepsilon_{ij})\nabla^M \zeta - \zeta^2 H x'\, d\mu_V \right| \\
&\leq \int \left| 2\zeta \sum_{i=m+1}^{n} \sum_{j=1}^{n} x_i(e_{ij} - \varepsilon_{ij})\nabla^M \zeta - \zeta^2 H x' \right| d\mu_V \\
&\leq \int 2\zeta |x'| \sqrt{\operatorname{div}_M(x')} |\nabla^M \zeta| + \zeta^2 |H||x'| d\mu_V \\
&\leq \int \frac{1}{2} \operatorname{div}_M(x')\zeta^2 + 2|x'|^2 |\nabla^M \zeta|^2 + \zeta^2 |H||x'| d\mu_V
\end{aligned}
$$

where we used, that $\zeta \geq 0$ (when we find it), the triangle inequality, and Young's inequality. We continue, with a bound which could be made tighter, but which suffices for our use

$$
\int \operatorname{div}_M(x')\zeta^2\, d\mu_V \leq 4 \int |x'|^2 |\nabla^M \zeta|^2 + \zeta^2 |H||x'| d\mu_V
$$

Now choosing some ζ satisfying that $\zeta \equiv 1$ on $B_{\rho/2}(0)$, $\zeta \equiv 0$ outside $B_\rho(0)$ and $|\nabla^M \zeta| \leq 3/\rho$ and then noting that $|x'||H| = (\rho^{-1}|x'|)(|H|\rho) \leq \frac{1}{2}\rho^{-2}|x'|^2 + \frac{1}{2}(|H|\rho)^2$ the lemma follows, since with these choices, $\int \operatorname{div}_M(x')\zeta^2\, d\mu_V = E(\xi, \rho/2, T)$. ■

Given some β assume that $\rho^{-m}\mu_V(B_\rho(\xi)) \leq \beta$, We can then apply Hölder to estimate the term $\int_{B_\rho(\xi)} |H|^2\, d\mu_V$ in the above lemma and obtain

$$
\rho^{2-m} \int_{B_\rho(\xi)} |H|^2\, d\mu_V \leq C \left(\rho^{p-m} \int_{B_\rho(\xi)} |H|^p\, d\mu_V \right)^{1/p}
$$

for all $p > 2$, and some $C = C(m, p, \beta)$. We can then alter the result of the lemma, to say

$$
E(\xi, \rho/2, T) \leq C\rho^{-m} \int_{B_\rho(\xi)} \left(\frac{\operatorname{dist}(x - \xi, T)}{\rho} \right) d\mu_V + C \left(\rho^{p-m} \int_{B_\rho(\xi)} |H|^p\, d\mu_V \right)^{1/p}
$$

for $p \geq 2$.

For the rest of this chapter, we are going to assume, that for some $\delta \in (0, 1/2)$ that we specify below, and $\mu_V = \mathcal{H}^m \llcorner \theta$, that

1. $\theta \geq 1$ μ_V-a.e.

2. $0 \in \operatorname{supp} V$

3. $B_\rho(0) \subseteq V$

4. $\frac{\mu_V(B_\rho(0))}{\omega_m \rho^m} \leq 1 + \delta$

5. $\left(\rho^{p-m} \int_{B_\rho(0)} |H|^p \, d\mu_V\right)^{1/p} \leq \delta$

We denote these assumptions by a (†). It can be shown that for $\delta \leq \delta_0(m, \ell, p) \in (0, 1/4]$, subject to (†), that

$$\left.\begin{aligned} \frac{1}{2} \leq 1 - C\delta \leq \frac{\mu_V(B_\sigma(y))}{\omega_m \sigma^m} \leq 1 + C\delta \leq 2 \\ 0 < \sigma \leq (1-\delta)\rho \end{aligned}\right\} \tag{5.2}$$

for $y \in \operatorname{supp} V \cap B_{\delta\rho}(0)$ and C dependent on m, ℓ and p.

We can then prove that locally, $\operatorname{supp} \mu_V \cap B_{\delta\rho}(0)$ is approximately affine in nature.

Lemma 5.3 (Affine Approximation Lemma)**.** *If $\delta \in (0, 1/4]$ and (†) holds, then for each $\xi \in \operatorname{supp} V \cap B_{\delta\rho}(0)$ and $\sigma \in (0, 2\delta\rho]$ there is an m-dimensional subspace $T = T(\xi, \sigma)$ such that*

$$\frac{E(\xi, \sigma/2, T)^{1/2}}{C} \leq \sigma^{-1} \sup\{\operatorname{dist}(x, \xi + T) \mid x \in \operatorname{supp} V \cap B_\sigma(\xi)\} \leq C\delta^{1/(2m+2)}$$

for some $C = C(m, \ell, p) > 0$.

Proof. By scaling and translating we can assume that $\sigma = 1$ and $\xi = 0$, so then since $\delta \leq 1/4$ we have $(1-\delta)\rho/(\delta\rho) \geq 3$, and (5.2) holds for any $\sigma \leq 2$, and any $y \in \operatorname{supp} V \cap B_1(0)$.

By the Monotonicity formula (theorem 4.5) together with (†) we get that

$$\int_{B_2(y)} |p_{(T_x M)^\perp}(x-y)|^2 \leq \int_{B_2(y)} |p_{(T_x M)^\perp}(x-y)|^2 |x-y|^{-m-2} \, d\mu_V \leq C\delta \tag{5.3}$$

for all $y \in \operatorname{supp} V \cap B_1(0)$ and some $C = C(m, \ell, p)$. Now let $\alpha \in (0, 1)$ be given, and recall that if K is compact and $\eta > 0$, then any maximal pairwise disjoint family $B_{\eta/2}(y_i)$ with $y_i \in K$ has the property that $K \subseteq \bigcup_i B_\eta(y_i)$. Taking $\eta = \delta^\alpha$, we can use this, and get pairwise disjoint balls $B_{\delta^\alpha/2}(y_1), \ldots, B_{\delta^\alpha/2}(y_N)$ with $y_i \in \operatorname{supp} V \cap B_1(0)$, such that

$$\operatorname{supp} V \cap B_1(0) \subseteq \bigcup_{i=1}^{N} B_{\delta^\alpha}(y_i). \tag{5.4}$$

Using (5.2) with $\sigma = \delta^\alpha \rho$ we get

$$\frac{\delta^{\alpha m}}{C} \leq \mu_V(B_{\delta^\alpha/2}(y_i)) \leq C\delta^{\alpha m}$$

for all $i = 1, \dots, N$ and some $C = C(m, \ell, p)$. This implies that

$$\frac{N\delta^{\alpha n}}{C} \le \sum_{i=1}^N \mu_V(B_{\delta^\alpha/2}(y_i)) = \mu_V\left(\bigcup_{i=1}^N B_{\delta^\alpha/2}(y_i)\right) \le C\mu_V(B_2(0)) \le 2C.$$

Therefore $N \le C\delta^{-\alpha m}$. So using (5.3) with $y = y_j$ while noting that $B_2(y_i) \supseteq B_1(0)$ for each i, we get that

$$\int_{B_1(0)} \sum_{i=1}^N |p_{T_xM^\perp}(x - y_i)|^2 \, d\mu_V \le NC\delta \le C\delta^{1-\alpha m}.$$

So for any $k \ge 1$ we can write

$$\sum_{i=1}^N |p_{T_xM^\perp}(x - y_i)|^2 \le Ck\delta^{1-\alpha m}, \tag{5.5}$$

except maybe on a set $x \in B_1(0) \cap \operatorname{supp} V$ which has μ_V-measure $1/k$. By (5.2) we see that $\mu_V(B_{\delta^\alpha}(0)) \ge C^{-1}\delta^{\alpha m}$, which implies that we can, by taking $k = C\delta^{-\alpha m}$, ensure that (5.5) holds for some $x_0 \in \operatorname{supp} V \cap B_{\delta^\alpha}(0)$. This shows that there is an $x_0 \in \operatorname{supp} V \cap B_{\delta^\alpha}(0)$ such that

$$\sum_{i=1}^N |p_{T_{x_0}M^\perp}(x_0 - y_i)|^2 \le C\delta^{1-2\alpha m}$$

and therefore also that

$$|p_{T_{x_0}M^\perp}(x_0 - y_i)| \le C\delta^{1/2-\alpha m}$$

for all $i = 1, \dots, N$. Furthermore since $x_0 \in B_{\delta^\alpha}(0)$, then $|x_0| \le \delta^\alpha$ and hence

$$|p_{T_{x_0}M^\perp} y_i| \le C(\delta^{1/2-\alpha m} + \delta^\alpha)$$

for all $i = 1, \dots, N$.

Finishing off, we select $\alpha = \frac{1}{2m+2}$, because then $1/2 - \alpha m = \alpha$, which yields that all points $y_1, \dots, y_N$ are in the $C\delta^{1/(2m+2)}$ neighborhood of the subspace $T_0 = T_{x_0}M$, and thus (5.4) gives us that

$$\operatorname{dist}(y, T_0) \le C\delta^{1/(2m+2)}$$

for all $y \in \operatorname{supp} V \cap B_1(0)$. Thus the second inequality in the theorem is shown by taking $T = T_0 = T_{x_0}M$, and the first inequality follows from our discussion on using the Hölder inequality after lemma 5.2. ■

We can then use this lemma to prove the following Lipschitz approximation lemma, which states that in a small disc, the weight measure for any varifold V can be approximated by a well-behaved Lipschitz mapping up to some error given by the tilt-excess.

Lemma 5.4 (Lipschitz Approximation Lemma)**.** *For every $L \in (0, 1]$ there is a $\beta = \beta(m, \ell, p) \in (0, \frac{1}{4}]$ such that if $\delta \in (0, (\beta L)^{2m+2}]$, if (†) holds, if the subspaces $T(\sigma, \xi)$ are as in the affine approximation* *lemma 5.3, if* $\sigma_0 = \delta\rho$ *and if we furthermore assume (without loss of generality) that* $T(2\sigma_0, 0) = \mathbb{R}^m \times \{0\}$, *then there is a Lipschitz mapping* $f : B^n_{\sigma_0/2}(0) \to \mathbb{R}^\ell$, *such that* $\operatorname{Lip} f \leq L$, $\sup |f| \leq C\delta^{1/(2m+2)}$ *and such that*

$$\mu_V\left(B_{\sigma_0/2}(0) \cap (\operatorname{supp}\mu_V \setminus \operatorname{graph} f)\right) + \mathcal{H}^m\left(B_{\sigma_0/2}(0) \cap (\operatorname{graph} f \setminus \operatorname{supp}\mu_V)\right)$$
$$\leq CL^{-2} \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T(2\sigma_0,0)}|^2 \, d\mu_V$$

for some $C = C(m, \ell, p)$.

Proof. First, lets assume that $\delta \in (0, \frac{1}{4}]$ is arbitrary. Then the affine approximation lemma 5.3 tells us that

$$\sigma_0^{-m} \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V(x) \leq C\delta^{1/(m+1)} \tag{5.6}$$

for some $C = C(m, \ell, p)$. Let, for the moment $\beta \in (0, \frac{1}{4}]$ be arbitrary and define

$$G := \{y \in \operatorname{supp}\mu_V \cap B_{3\sigma_0/4}(0) \mid \sup_{\sigma \in (0,\sigma_0/2]} \sigma^{-m} \int_{B_{\sigma/2}(y)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V \leq \beta^2 L^2\}.$$

Then if $y \in \operatorname{supp}\mu_V \cap B_{3\sigma_0/4}(0) \setminus G$ then there exists some $\sigma \in (0, \sigma_0/2]$ for which

$$\beta^2 L^2 \sigma^m \leq \int_{B_{\sigma/2}(y)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V. \tag{5.7}$$

Now, The 5-times covering lemma tells us that there exists a family of disjoint balls $\{B_{\sigma_i}(y_i)\}_{i \in \mathbb{N}}$ with $\sigma = \sigma_i \in (0, \sigma_0/2]$ and $y = y_i \in \operatorname{supp}\mu_V \cap B_{\sigma_0}(0) \setminus G$ such that (5.7) holds and such that

$$\operatorname{supp}\mu_V \cap B_{3\sigma_0/4}(0) \setminus G \subseteq \bigcup_i B_{5\sigma_i}(y_i).$$

So using (5.7) with $\sigma = \sigma_i$ and $y = y_i$ and summing over i we get

$$\begin{aligned}
\beta^2 L^2 \mu_V(B_{3\sigma_0/4}(0) \setminus G) &\leq \beta^2 L^2 \sum_i \mu_V(B_{5\sigma_i}(y_i)) \\
&\leq \beta^2 L^2 \, C \sum_i \sigma_i^n \\
&\leq C \int_{\bigcup_i B_{\sigma_i/2}(y_i)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V \\
&\leq C \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V
\end{aligned}$$

Dividing through with $(\beta L)^2$ we thus obtain

$$\mu_V(B_{3\sigma_0/4}(0) \setminus G) \leq C\beta^{-2} L^{-2} \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V.$$

Next, we want to show that G is entirely contained in the graph of some Lipschitz mapping. To that end, let $y_1, y_2 \in G$ be arbitrary, define $\sigma := |y_1 - y_2|$, and observe that then $\sigma \leq \sigma_0$. Furthermore we see that by the definition of G and $T(y_1, \sigma)$ the following two inequalities holds

$$\sigma^{-m} \int_{B_{\sigma/2}(y_1)} |p_{T_x M} - p_{T_0}|^2 \, d\mu_V \leq \beta^2 L^2$$
$$\sigma^{-m} \int_{B_{\sigma/2}(y_1)} |p_{T_x M} - p_{T(y_1,\sigma)}|^2 \, d\mu_V \leq \delta^{1/(m+1)}$$

and so since $|p_{T_0} - p_{T(y_1,\sigma)}|^2 \leq 2|p_{T_x M} - p_{T_0}|^2 + 2|p_{T_x M} - p_{T(y_1,\sigma)}|^2$, we get that

$$\begin{aligned} |p_{T_0} - p_{T(y_1,\sigma)}|^2 &= C\sigma^{-m} \int_{B_{\sigma/2}(y_1)} |p_{T_0} - p_{T(y_1,\sigma)}|^2 \, d\mu_V \\ &\leq C\sigma^{-m} \int_{B_{\sigma/2}(y_1)} 2|p_{T_x M} - p_{T_0}|^2 + 2|p_{T_x M} - p_{T(y_1,\sigma)}|^2 \, d\mu_V \\ &\leq C(\beta^2 L^2 + \delta^{1/(m+1)}), \end{aligned}$$

from which we obtain that

$$|p_{T_0} - p_{T(y_1,\sigma)}| \leq C(\beta L + \delta^{1/(2m+2)}). \tag{5.8}$$

The affine approximation lemma now gives us the following inequality

$$|p_{T(y_1,\sigma)^\perp}(y_1 - y_2)| = \operatorname{dist}(y_2, y_1 + T(y_1,\sigma)) \leq C\delta^{1/(2m+2)}\sigma$$

which, with (5.8), implies that

$$\begin{aligned} \operatorname{dist}(y_2, y_1 + T_0) &= |p_{T_0^\perp}(y_1 - y_2)| \\ &= |(p_{T(y_1,\sigma)} + (p_{T_0^\perp} - p_{T(y_1,\sigma)^\perp}))(y_1 - y_2)| \\ &\leq \operatorname{dist}(y_2, y_1 + T(y_1,\sigma)) + |p_{T_0} - p_{T(y_1,\sigma)}|\sigma \\ &= C(\beta L + \delta^{1/(2m+2)})\sigma \end{aligned}$$

where we used that $\sigma := |y_1 - y_2|$. We will then restrict $\delta \leq (\beta L)^{2m+2}$ as in the assumptions of the theorem, which enables us to write

$$\operatorname{dist}(y_2, y_1 + T_0) \leq C(\beta L + \delta^{1/(2m+2)})\sigma \leq C\beta L.$$

This gives us that

$$|p_{T_0^\perp}(y_2 - y_1)| \leq C\beta L|y_1 - y_2| \leq C\beta L(|p_{T_0^\perp}(y_1 - y_2)| + |p_{T_0}(y_1 - y_2)|)$$

Restricting $C\beta \leq \frac{1}{2}$ we then have that $|p_{T_0^\perp}(y_1) - p_{T_0^\perp}(y_2)| \leq 2C\beta L|p_{T_0}(y_1) - p_{T_0}(y_2)|$, and since y_1 and y_2 were chosen arbitrarily in G this shows that G is contained in the graph of a Lipschitz mapping with corresponding Lipschitz constant $\leq L$, if, indeed, we choose $\beta = \beta(m, \ell, p)$ such that $C\beta \leq \frac{1}{2}$, along with the assumptions (†) with $\delta \in (0, (\beta L)^{2m+2}]$.

Employing the Lipschitz extension theorem we see that there exists some Lipschitz mapping $f = (f_1, \ldots, f_\ell) : \mathbb{R}^m \to \mathbb{R}^\ell$ with $\operatorname{Lip} f \leq C\beta L$ $(\leq L)$ such that $G \subseteq \operatorname{graph} f$, and

$$\mu_V(B_{3\sigma_0/4}(0) \setminus G) \leq CL^{-2} \int_{B_{\sigma_0}} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V. \tag{5.9}$$

Furthermore, we have by the affine approximation lemma that for all x such that $(x, f(x)) \in G$, then $|f_i(x)| \leq C\delta^{1/(2m+2)}$ for all $i = 1, \ldots, \ell$. Thus replacing f_i with

$$\tilde{f}_i = \max\{\min\{f_i, C\delta^{1/(2m+2)}\}, -C\delta^{1/(2m+2)}\}$$

for all $i = 1, \ldots, \ell$, we can assume that $\sup|f| \leq C\delta^{1/(2m+2)}$, and it now only remains to show that

$$\mathcal{H}^m(B_{\sigma_0/2}(0) \cap (\operatorname{graph} f \setminus \operatorname{supp} \mu_V)) \leq CL^{-2} \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V. \tag{5.10}$$

To do this, let $\eta \in (\operatorname{graph} f \setminus \operatorname{supp} \mu_V) \cap B_{\sigma_0/2}(0)$ be arbitrary and choose $\sigma \in (0, \sigma_0/4]$ such that $B_\sigma(\eta) \cap \operatorname{supp} \mu_V = \emptyset$ and $B_{2\sigma}(\eta) \cap \operatorname{supp} \mu_V \neq \emptyset$. We can assure such an η exists, since $\eta \in B_{\sigma_0/2}$ and $0 \in \operatorname{supp} \mu_V$. Then this, along with the monotonicity formula (theorem 4.5) tells us that

$$\begin{aligned}
\mu_V(B_{3\sigma}(\eta)) &= \mu_V(B_{3\sigma}(\eta)) - \mu_V(B_\sigma(\eta)) \\
&\leq C\sigma^m \int_{B_{3\sigma}(\eta) \setminus B_\sigma(\eta)} |x - \eta|^{-m} \left| p_{(T_xM)^\perp}\left(\frac{x - \eta}{|x - \eta|}\right)\right|^2 d\mu_V + C\delta\sigma^m.
\end{aligned}$$

Since $\operatorname{supp} \mu_V \cap B_{2\sigma}(\eta) \neq \emptyset$ we can use (5.2) to see that $\frac{\mu_V(B_{3\sigma}(\eta))}{\omega_m \sigma^m} \geq 1/2$, which, together with the above inequality implies that for some suitable $\delta = \delta(m, \ell, p)$,

$$
\begin{aligned}
\sigma^m &\leq C \int_{B_\sigma(\eta)} \left| p_{(T_x M)^\perp} \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V \\
&= C \int_{B_\sigma(\eta)} \left| ((\mathrm{id}(x) - p_{T_0}) + (p_{T_0} - p_{T_x M})) \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V \\
&\leq 2C \left(\int_{B_\sigma(\eta)} \left| p_{T_0^\perp} \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V + \int_{B_\sigma(\eta)} \left| (p_{T_x M} - p_{T_0}) \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V \right) \\
&\leq C \Big(\int_{B_\sigma(\eta) \cap F} \left| p_{T_0^\perp} \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V + \int_{B_\sigma(\eta) \setminus F} 1 \, d\mu_V + \\
&\qquad \int_{B_\sigma(\eta)} \left| (p_{T_x M} - p_{T_0}) \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V \Big) \\
&= C \Big(\int_{B_\sigma(\eta) \cap F} \left| p_{T_0^\perp} \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V + \mu_V(B_\sigma(\eta) \setminus F) + \\
&\qquad \int_{B_\sigma(\eta)} \left| (p_{T_x M} - p_{T_0}) \left(\frac{x-\eta}{\sigma} \right) \right|^2 d\mu_V \Big)
\end{aligned}
$$

where we used that $p_{T^\perp}(x) = \mathrm{id}(x) - p_T(x)$ for any subspace $T \subseteq \mathbb{R}^n$, and that $(x-\eta)/\sigma \leq 1$ for all $x \in B_\sigma(\eta)$, hence the projection is less than 1. Now, since $\mathrm{Lip}\, f \leq \beta L$, we have for all $x, y \in \operatorname{graph} f \cap B_\sigma(\eta)$

$$
\left| p_{T_0^\perp} \left(\frac{x-y}{\sigma} \right) \right| \leq C\beta,
$$

and using (5.2) to see that $\frac{\mu_V(B_\sigma(\eta))}{\omega_m \sigma^m} \leq 2$, we get that

$$
\sigma^m \leq C \left(\beta L \sigma^m + \mu_V(B_\sigma(\eta) \setminus \operatorname{graph} f) + \int_{B_\sigma(\eta)} |p_{T_x M} - p_{T_0}|^2 \, d\mu_V \right). \tag{5.11}
$$

Choosing $\beta \in (0, \min\{\frac{1}{2C}, 1/4\}]$, we assure that $C\beta \leq 1/2$ which yields by rearranging (5.11)

$$
\sigma^m \leq C \left(\mu_V(B_\sigma(\eta) \setminus \operatorname{graph} g) + \int_{B_\sigma(\eta)} |p_{T_x M} - p_{T_0}|^2 \, d\mu_V \right)
$$

Now, the collection of balls $B_\sigma(\eta)$ cover $B_{\sigma_0/2}(0) \cap \operatorname{graph} f \setminus \operatorname{supp} \mu_V$ by definition, so by the 5-times covering lemma 2.8 there exists some pairwise disjoint collection of balls $B_{\sigma_i}(\eta_i)$ such that for all i and $\bigcup_i B_{5\sigma_i}(\eta_i) \supseteq B_{\sigma_0/2}(0) \cap \operatorname{graph} f \setminus \operatorname{supp} \mu_V$ we have

$$
\sigma_i^m \leq C \left(\mu_V(B_{\sigma_i}(\eta_i) \setminus \operatorname{graph} f) + \int_{B_{\sigma_i}(\eta_i)} |p_{T_x M} - p_{T_0}|^2 \, d\mu_V \right). \tag{5.12}
$$

Since f is Lipschitz with $\mathrm{Lip}\, f \leq 1$, we have that $\mathcal{H}^m(B_{5\sigma_i}(\eta_i) \cap \operatorname{graph} f) \leq C\sigma_i^m$ for all i and therefore

(5.12) shows (5.10) by

$$
\begin{aligned}
\mathcal{H}^m(B_{\sigma_0/2}(0) \cap \operatorname{graph} f \setminus \mu_V) &\leq \mathcal{H}^m\left(F \cap \bigcup_i B_{5\sigma_i}(\eta_i)\right) \\
&\leq \sum_i \mathcal{H}^m(\operatorname{graph} f \cap B_{5\sigma_i}(\eta_i)) \\
&\leq C \sum_i \left(\mu_V(B_{\sigma_i}(\eta_i) \setminus \operatorname{graph} f) + \int_{B_{\sigma_i}(\eta_i)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V\right) \\
&\leq C\left(\mu_V(\bigcup_i B_{\sigma_i}(\eta_i) \setminus \operatorname{graph} f) + \int_{\bigcup_i B_{\sigma_i}(\eta_i)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V\right) \\
&\leq C\left(\mu_V(B_{3\sigma_0/4}(0) \setminus \operatorname{graph} f) + \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V\right) \\
&\overset{(5.9)}{\leq} C \int_{B_{\sigma_0}(0)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V
\end{aligned}
$$

which finishes the proof. ■

The following corollary shows that given some further restrictions on the regularity of V, that in fact within some small ball, the support of V is the graph of a Lipschitz mapping.

Corollary 5.5. *Assuming the notation and assumptions of the Lipschitz approximation lemma 5.4, then $\beta = \beta(m, \ell, p)$ can be chosen such that if*

$$
\sup_{\substack{\xi \in \operatorname{supp} \mu_V \cap B_{\sigma_0/2}(0), \\ \sigma \in (0, \sigma_0/2]}} \sigma^{-m} \int_{B_\sigma(\xi)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V \leq (\beta L)^2
$$

then for some Lipschitz mapping $f : B^m_{\sigma_0/4}(0) \to \mathbb{R}^\ell$ with $\operatorname{Lip} f \leq L$ and $\sup |f| \leq C\delta^{1/(2m+2)}\sigma_0$ we have

$$
\operatorname{supp} \mu_V \cap B_{\sigma_0/4}(0) = \operatorname{graph} f \cap B_{\sigma_0/4}(0)
$$

Proof. We see that if

$$
\sup_{\substack{\xi \in B_{\sigma_0/2}(0), \\ \sigma \leq \sigma_0/2}} \int_{B_\sigma(\xi)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V \leq (\beta L)^2
$$

then the set G in the proof of the Lipschitz approximation lemma by definition includes all of $\operatorname{supp} \mu_V \cap B_{\sigma_0/2}(0)$, hence $\operatorname{supp} \mu_V \cap B_{\sigma_0/2}(0) \subseteq \operatorname{graph} f$ where f has the properties stated in the corollary. Moreover, if $\eta \in \operatorname{graph} f \cap B_{\sigma_0/4}(0) \setminus \operatorname{supp} \mu_V$ then (5.11) from the proof of the Lipschitz approximation lemma gives

$$
\sigma^m \leq C \int_{B_\sigma(\eta)} |p_{T_xM} - p_{T_0}|^2 \, d\mu_V \leq C(\beta L)^2 \sigma^m \leq C\beta^2 \sigma^m
$$

for some $\sigma \in (0, \sigma_0/4]$. However this is impossible with a $\beta = \beta(m, \ell, p)$ satisfying $C\beta^2 \leq 1$, hence with such a β we have $\operatorname{graph} f \cap B_{\sigma_0/4}(0) \setminus \operatorname{supp} \mu_V = \emptyset$. ■

We are now ready to prove the following lemma which will be crucial for the Allard regularity theorem. It shows that some weakly differentiable functions can be approximated well by harmonic functions within some small ball.

Recall that $\mathring{B}_1^m(0)$ is the open unit ball in $\mathbb{R}^m$, and that for some open set $\Omega \subseteq \mathbb{R}^n$ a function $u : \Omega \to \mathbb{R}$ is said to be harmonic if

$$\nabla^2 u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Lemma 5.6 (Harmonic Approximation lemma)**.** *Let $B := \mathring{B}_1^m(0)$. Then for all $\varepsilon > 0$ there is a $\delta = \delta(m, \varepsilon) > 0$ such that for all $f \in W^{1,2}(B)$ for which*

$$\int_B |\nabla f|^2 \leq 1$$
$$\left| \int_B \nabla f \cdot \nabla \varphi \, d\lambda^m \right| \leq \delta \sup |\nabla \varphi|, \qquad \forall \varphi \in C_c^\infty(B)$$

there is a harmonic function u on B such that $\int_B |\nabla u|^2 \leq 1$ and

$$\int_B (u - f)^2 \, d\lambda_m \leq \varepsilon.$$

Proof. Assume for the sake of contradiction that the lemma is false. Then there is some $\varepsilon > 0$ and some sequence $\{f_k\}_{k \in K} \subseteq W^{1,2}(B)$, for some index set $K \subseteq \mathbb{N}$, for which

$$\int_B |\nabla f_k|^2 \leq 1 \tag{5.13}$$
$$\left| \int_B \nabla f_k \cdot \nabla \varphi \, d\lambda^m \right| \leq k^{-1} \sup |\nabla \varphi| \tag{5.14}$$

but where

$$\int_B |u - f_k|^2 > \varepsilon, \tag{5.15}$$

for every harmonic functions u on B with $\int_B |\nabla u|^2 \leq 1$.

Define

$$\tau_k = \frac{\int_B f_k \, d\lambda_m}{\omega_m},$$

then the Poincaré inequality (theorem 2.22) says that

$$\int_B |f_k - \tau_k|^2 \leq C \int_B |\nabla f_k|^2 \leq C$$

and then the Rellich-Kondrachov compactness theorem 2.23, or rather the remark afterwards, says that there exists some subsequence $K' \subseteq K$, and some $w \in W^{1,2}(B)$ with $\int_B |\nabla w|^2 \leq 1$ such that $\|(f_{k'} -$

$\tau_{k'}) - w\|_{L^2(B)} \to 0$, and such that $\nabla f_{k'} \to \nabla w$ weakly in $L^2(B)$ for $k' \in K$.

Now, (5.14) and the weak convergence of $\nabla f_{k'}$ to ∇w implies that

$$\int_B \nabla w \nabla \varphi \, d\lambda_m = \lim_k \int_B \nabla f_k \nabla \varphi \, d\lambda_m = 0$$

for all $\varphi \in C_c^\infty(B)$. This means that w is harmonic in B and

$$\int_B |(f_{k'} - \tau_{k'}) - w|^2 \to 0.$$

But then $w + \tau_{k'}$ is harmonic, which contradicts (5.15). ■

As a short aside, which we shall use later, we see that if u is harmonic on $B_\sigma(0)$, and ℓ is the affine approximation of u, that is $\ell(x) = u(0) + x \cdot \nabla u(0)$, then

$$\left.\begin{aligned} |\ell(0)| = |u(0)| &\le C\sigma^{-m/2}\|u\|_{L^2(B_\sigma(0))} \\ |\nabla \ell| = |\nabla u(0)| &\le C\sigma^{-m/2}\|\nabla u\|_{L^2(B_\sigma(0))} \\ \sup_{B_{\eta\sigma}(0)} |u - l| \le (\eta\sigma)^2 \sup_{B_{\eta\sigma}(0)} |D^2 u| &\le (\eta\sigma)^2 \sup_{B_{\sigma/2}(0)} |D^2 u| \le C\eta^2\sigma^{1-m/2}\|\nabla u\|_{L^2(B_\sigma(0))} \end{aligned}\right\} \tag{5.16}$$

for $\eta \in (0, 1/4]$ and some $C = C(m)$. We will not prove this.

In the following, let $\ell = n - m$, and let

$$E_*(\xi, \sigma, T) := \max\left\{ E(\xi, \sigma, T), \delta^{-1}\left(\sigma^{p-m}\int_{B_\sigma(\xi)} |H|^p \, d\mu_V\right)^{2/p}\right\}$$

where δ is as in (†).

Then the following theorem says, that if we know that the tilt-excess E_* behaves nicely on a small disc, we can bound the behavior of the tilt-excess on larger discs.

Theorem 5.7 (Tilt-excess Decay Theorem)**.** *There are constants $\eta = \eta(m, \ell, p), \delta_0 = \delta_0(m, \ell, p) \in (0, 1/4]$, such that if (†) holds for some $\delta \in (0, \delta_0]$, $\rho \in (0, \delta\rho/2]$, $\xi \in \operatorname{supp} \mu_V \cap B_{\delta\rho}(0)$ and T is an m-dimensional subspace of $\mathbb{R}^n$, then*

$$E_*(\xi, m\sigma, S) \le \eta^{2(1-m/p)} E_*(\xi, \sigma, T)$$

for some m-dimensional subspace $S \subseteq \mathbb{R}^n$.

Proof. We can by translation and rotation assume that $\xi = 0$ and that $T = \mathbb{R}^m \times \{0\}$. Using the Affine approximation lemma 5.3 we see that the subspace $\tilde{T} = T(0, 2\sigma)$ (In the notation of the affine approximation lemma), has the property that

$$E(0,\sigma,\tilde{T}) \leq C\delta^{1/(m+1)}, \tag{5.17}$$

for some $C = C(m,\ell,p)$. Hence we can assume the same inequality for T instead, that is, we can assume

$$E(0,\sigma,T) \leq C\delta^{1/(m+1)} \tag{5.18}$$

because if this was not the case, we could just prove the lemma for $\tilde{T}$ which would imply the lemma for T.

Now, the affine approximation lemma 5.3 implies that

$$\sup_{x\in \text{supp }\ \mu_V \cap B_\sigma(0)} \operatorname{dist}(x,\tilde{T}) \leq C\delta^{1/(2m+2}$$

which together with (5.18) implies that $|p_T - p_{\tilde{T}}| \leq C\delta^{1/(2m+2)}$, and hence

$$\sup_{x\in \text{supp }\ \mu_V \cap B_\sigma(0)} \operatorname{dist}(x,T) \leq C\delta^{1/(2m+2}$$

which can also be written as

$$\sup_{B_\sigma(0)\cap \operatorname{supp}\mu_V} \sum_{i=1}^{\ell} |x_{m+i}| \leq C\delta^{1/(2m+2)}\sigma, \tag{5.19}$$

where $\ell = n-m$. We can then apply the Lipschitz approximation lemma 5.4 with $L = 1$, to obtain a Lipschitz mapping $f = (f_1,\dots,f_\ell) : B^m_\sigma(0) \to \mathbb{R}^\ell$ where $\operatorname{Lip} f \leq 1$, $\sup|f| \leq C\delta^{1/(2m+2)}\sigma$ and

$$\mu_V(\operatorname{supp}\mu_V \cap B_\sigma(0) \setminus \operatorname{graph} f) + \mathcal{H}^m(\operatorname{graph} f \cap B_\sigma(0) \setminus \operatorname{supp}\mu_V) \leq CE_0\sigma^m \tag{5.20}$$

where $E_0 := E_*(0,\sigma,T)$. Restricting, we can assume that $C\delta^{1/(2m+2)} \leq \frac{1}{4}$ which, with (5.19), implies that

$$(B^m_{\sigma/2}(0)\times\mathbb{R}^\ell) \cap \operatorname{supp}\mu_V \subseteq B^m_{\sigma/2}(0)\times B^\ell_{\sigma_4}(0). \tag{5.21}$$

We now want to prove that f_i is well-approximated by a harmonic function for all $i = 1,\dots,\ell$. To that end, let $X = \varphi e_{m+i}$ for $\varphi \in C^1_c(\mathring{B}_\sigma(0))$, where e_j are the standard basis vectors of $\mathbb{R}^n$. Then we note that by the definition of the generalized mean curvature H of V with this given X, we get that

$$\int_M \nabla^M_{m+i}\varphi\, d\mu_V = -\int_M e_{m+i}\cdot H\varphi\, d\mu_V, \qquad \varphi \in C^1_c(\mathring{B}_\sigma(0)) \tag{5.22}$$

for all $i = 1, \dots, \ell$, where $\nabla^M_{m+i} = p_{T_xM}(e_{m+i}) \cdot \nabla^M = (\nabla^M x_{m+i})\nabla^M$. That is,

$$\int_M (\nabla^M x_{m+i})\nabla^M \varphi \, d\mu_V = -\int_M e_{m+i} \cdot H\varphi \, d\mu_V, \qquad \varphi \in C^1_c(\mathring{B}_\sigma(0)).$$

Now, for $x = (x_1, \dots, x_m) \in \mathbb{R}^n$, define $\tilde{f}_i(x_1, \dots, x_n) := f_i(x_1, \dots, x_m)$. Then all $x \in M_1 = M \cap \operatorname{graph} f$ has the form $x = (x_1, \dots, x_m, f_1(x), \dots, f_\ell(x))$, hence $x_{m+i} = \tilde{f}_i(x)$ on M_1. Thus by the definition of ∇^M we have

$$\nabla^M x_{m+i} = \nabla^M \tilde{f}_i(x) \tag{5.23}$$

for μ_V-a.e. $x \in M_1$. Hence (5.22) can be rewritten as

$$\int_{M_1} (\nabla^M \tilde{f}_i) \cdot (\nabla^M \varphi) \, d\mu_V = -\int_{M \setminus M_1} (\nabla^M x_{m+i}) \cdot (\nabla^M \varphi) \, d\mu_V - \int_M e_{m+i} \cdot H\varphi \, d\mu_V, \tag{5.24}$$

Then, using Hölder's inequality with (5.2) we get

$$\begin{aligned}
\int_{B_\sigma(0)} |H| \, d\mu_V &= \int_{B_\sigma(0)} |H \cdot 1| \, d\mu_V \\
&\leq \left(\int_{B_\sigma(0)} |H|^p \, d\mu_V \right)^{1/p} \left(\int_{B_\sigma(0)} 1 \, d\mu_V \right)^{1-1/p} \\
&= \left(\int_{B_\sigma(0)} |H|^p \, d\mu_V \right)^{1/p} (\mu(B_\sigma(0)))^{1-1/p} \\
&= \delta^{1/2} \sigma^{m/p-1} \sqrt{\delta^{-1} \left(\sigma^{p-m} \int_{B_\sigma(0)} |H|^p \, d\mu_V \right)^{2/p}} (\mu(B_\sigma(0)))^{1-1/p} \\
&\overset{(5.2)}{\leq} \delta^{1/2} \sigma^{m/p-1} E_0^{1/2} (2\omega_m \sigma^m)^{1-1/p} \\
&\leq \delta^{1/2} \sigma^{m/p-1} E_0^{1/2} C \sigma^{m-m/p} \\
&= C \delta^{1/2} \sigma^{m-1} E_0^{1/2},
\end{aligned}$$

which implies, with (5.20) and (5.24), that for every smooth φ with $\operatorname{supp} \varphi \subseteq \mathring{B}_\sigma(0)$, we have

$$\begin{aligned}
\left| \sigma^{-m} \int_{M_1} (\nabla^M \tilde{f}_i) \cdot \nabla^M \varphi \, d\mu_V \right| &\leq C(\sigma^{-1} \sup |\varphi| \delta^{1/2} E_0^{1/2} + \sup |\nabla \varphi| E_0) \\
&\leq C \sup |\nabla \varphi| (\delta^{1/2} E_0^{1/2} + E_0),
\end{aligned} \tag{5.25}$$

and using (5.23) and (5.1) we see that

$$\sigma^{-m} \int_{M_1 \cap B_\sigma(0)} |\nabla^M \tilde{f}_i|^2 \, d\mu_V \leq E_0. \tag{5.26}$$

Doing the same trick as above with f and $\tilde{f}$, we can, for every $\varphi_0 \in C_c^1(B^m_{\sigma/2}(0))$ associate a new function $\tilde{\varphi}_0(x_1, \ldots, x_n) = \varphi_0(x_1, \ldots, x_m)$ where $\operatorname{supp} \tilde{\varphi}_0 = \operatorname{supp} \varphi_0 \times \mathbb{R}^\ell \subseteq B^m_{\sigma/2}(0) \times \mathbb{R}^\ell$. Furthermore by (5.21) there is some function $h \in C_c^1(B_\sigma(0))$ for which $h \equiv 1$ in some neighborhood around $\operatorname{supp} \mu_V \cap \operatorname{supp} \tilde{\varphi}_0$. Thus we can replace φ in the above with $\tilde{\varphi}_0 h$, or, since $h \equiv 1$ on some neighborhood of $\operatorname{supp} \mu_V \cap \operatorname{supp} \tilde{\varphi}_0$, we can instead replace φ with $\tilde{\varphi}_0$. So lets do that, and note, that since $p_T(\nabla \tilde{\varphi}_0) = \nabla \tilde{\varphi}_0$ and $p_T(\nabla \tilde{f}_i) = \nabla \tilde{f}_i$ we have

$$\begin{aligned}
\nabla^M \tilde{f}_i \cdot \nabla^M \tilde{\varphi}_0 &= p_{T_x M}(\nabla \tilde{f}_i) \cdot \nabla \tilde{\varphi}_0 \\
&= \nabla \tilde{f}_i \cdot \nabla \tilde{\varphi}_0 - p_{(T_x M)^\perp}(\nabla \tilde{f}_i) \cdot \nabla \tilde{\varphi}_0 \\
&= \nabla \tilde{f}_i \cdot \nabla \tilde{\varphi}_0 - (p_T \circ p_{(T_x M)^\perp} \circ p_T)(\nabla \tilde{f}_i) \cdot \nabla \tilde{\varphi}_0 \qquad (5.27)
\end{aligned}$$

where we used that $p_{S^\perp} = \operatorname{id}(x) - p_S$ for all subspaces $S \subseteq \mathbb{R}^n$.

It can be shown that the operator norm and the inner product norm are equivalent, in fact $\|p_S\| \leq |p_S|$ (see [6]), which shows that

$$\begin{aligned}
\|p_T \circ p_{(T_x M)^\perp} \circ p_T\| &= \|(p_T - p_{T_x M}) \circ p_{(T_x M)^\perp} \circ (p_T - p_{T_x M})\| \qquad (5.28) \\
&\leq \|p_T - p_{T_x M}\|^2 \qquad (5.29) \\
&\leq |p_T - p_{T_x M}|^2 \qquad (5.30)
\end{aligned}$$

so (5.27) implies that

$$|\nabla^M \tilde{f}_i \cdot \nabla^M \tilde{\varphi}_0 - \nabla \tilde{f}_i \cdot \nabla \tilde{\varphi}_0| \leq |p_T - p_{T_x M}|^2 \sup |\nabla \tilde{\varphi}_0|. \tag{5.31}$$

which, together with (5.25) implies that

$$\left| \sigma^{-m} \int_{M_1} \nabla \tilde{f}_i \cdot \nabla \tilde{\varphi}_0 \, d\mu_V \right| \leq C \sup |\nabla \varphi_0| (\delta^{1/2} E_0^{1/2} + E_0). \tag{5.32}$$

We can validly replace φ_0 with f_i in (5.27) and (5.31) and use (5.26) to see that also

$$\sigma^{-m} \int_{M_1 \cap B_\sigma(0)} |\nabla \tilde{f}_i|^2 \, d\mu_V \leq C E_0. \tag{5.33}$$

Using the area formula (theorem 2.14) this implies that

$$\left|\sigma^{-m}\int_{B_\sigma(0)} \nabla f_i \cdot \nabla\varphi_0 \theta\circ F\, J_F\, d\lambda_m\right| \le C\delta^{1/2}E_0^{1/2}\sup|\nabla\varphi_0|,$$

and

$$\sigma^{-m}\int_{B_\sigma(0)} |\nabla f_i|^2\theta\circ F\, J_F\, d\lambda_m \le CE_0 \tag{5.34}$$

where θ is the multiplicity of the varifold $V = V(M,\theta)$, $F : \mathbb{R}^m \to \mathbb{R}^n$ is the graph map of f, i.e. $F(x) = (x, f(x))$ and where J_F is the Jacobian of F i.e.

$$J_F(x) = \sqrt{\det(D_iF(x)\cdot D_jF(x))} = \sqrt{\det(\delta_{ij} + D_if(x)\cdot D_jf(x))}.$$

We note that $1 \le J_F \le 1 + C|\nabla f|^2$ on $B_\sigma(0)$ and $1 \le \theta \le 1 + C\delta$ which implies that

$$\begin{aligned}\left|\sigma^{-m}\int_{B_\sigma(0)} \nabla f_i\cdot\nabla\varphi_0\, d\lambda_m\right| &\le C\left(\delta^{1/2}E_0^{1/2} + \delta\sigma^{-m}\int_{B_{\sigma/2}(0)}|\nabla f_i|\, d\lambda_m\right)\sup|\nabla\varphi_0| \\ &\le C\delta^{1/2}E_0^{1/2}\sup|\nabla\varphi_0| \end{aligned} \tag{5.35}$$

and

$$\sigma^{-m}\int_{B_\sigma(0)} |\nabla f_i|^2\, d\lambda_m \le CE_0. \tag{5.36}$$

We can now use the Harmonic approximation lemma 5.6 along with (5.35) and (5.36) but replacing f with $(CE_0)^{-1/2}f_i$ to see that for any $\varepsilon \in (0,1)$ there is some $\delta_0 = \delta_0(m)$ such that if the assumptions in the Harmonic approximation lemma 5.6 holds with $\delta \le \delta_0$, then there exists ℓ harmonic functions $u_1,\ldots,u_\ell$ on $B_{\sigma/2}(0)$ such that

$$\sigma^{-m}\int_{B_{\sigma/2}(0)} |Du|^2\, d\lambda_m \le CE_0, \tag{5.37}$$

and

$$\sigma^{-m-2}\int_{B_{\sigma/2}(0)} |f-u|^2\, d\lambda_m \le \varepsilon E_0. \tag{5.38}$$

Since $\sup|f| \le C\delta^{1/(2m+2)}\sigma$ we see that $|u(x)| \le |u(x) - f(x)| + C\delta^{1/(2m+2)}$ and hence

$$\int_{B_{\sigma/2}(0)} |u|^2\, d\lambda_m \le 2\int_{B_{\sigma/2}(0)} |u-f|^2\, d\lambda_m + C\delta^{1/(m+1)}E_0.$$

Furthermore, (5.16) and (5.37) and (5.38) implies that

$$\left.\begin{aligned} \sigma^{-1}|u(0)| &\leq C(\varepsilon^{1/2}E_0^{1/2} + \delta^{1/(2m+2)}) \leq C\delta^{1/(2m+2)} \\ |Du(0)| &\leq CE_0^{1/2}. \end{aligned}\right\} \tag{5.39}$$

We now define $L(x) = (L_1(x), \ldots, L_\ell(x))$ such that the i'th entry is the affine approximation to u_i, that is $L_i(x) = u_i(x) + x \cdot \nabla u_i(0)$ for $i = 1, \ldots, \ell$. So then, using (5.16) with $\eta \in (0, 1/4)$ we get

$$\begin{aligned} (\eta\sigma)^{-m-2} \int_{B_{\eta\sigma}(0)} |f - L|^2 \, d\lambda_m &\leq 2(\eta\sigma)^{-m-2} \int_{B_{\eta\sigma}(0)} (|f - u|^2 + |u - L|^2) \, d\lambda_m \\ &\leq 2\eta^{-m-2}\varepsilon E_0 + 2\omega_m(\eta\sigma)^{-2} \sup_{B_{\eta\sigma}(0)} |u - L|^2 \\ &\leq 2\eta^{-m-2}\varepsilon E_0 + C\eta^2\sigma^{-m} \int_{B_\sigma(0)} |Du|^2 \, d\lambda_m \\ &\leq 2\eta^{-m-2}\varepsilon E_0 + C\eta^2 E_0. \end{aligned} \tag{5.40}$$

Now, define S as the m-dimensional subspace $S = \operatorname{graph}(L - L(0)) = (x, L(x) - L(0))$ and let $\tau = (0, L(0))$. We then note that $|\tau| \leq C\delta^{1/(2m+2)}\sigma$ by the above. Thus $\operatorname{dist}(x, \tau + S) \leq |f(x') - L(x')|$ for all $x = (x', f(x')) \in B_{\eta\sigma}(\tau) \cap \operatorname{graph} f$ hence (5.40) implies that

$$(\eta\sigma)^{-m-2} \int_{B_{\eta\sigma}(\tau) \cap \operatorname{graph} f} \operatorname{dist}(x - \tau, S)^2 \, d\mathcal{H}^m \leq C\eta^{-m-2}\varepsilon E_0 + C\eta^2 E_0.$$

and then (5.39) and the first paragraph of this proof shows that

$$(\eta\sigma)^{-m-2} \int_{B_{\eta\sigma}(\tau)} \operatorname{dist}(x - \tau, S)^2 \, d\mu_V \leq C\eta^{-m-2}\varepsilon E_0 + C\delta^{1/(m+1)} E_0 + C\eta^2 E_0,$$

where we used that $\theta(\xi) \leq 1 + C\delta \leq 2$ in $B_\sigma(0)$. We can then use Hölder on the tilt-excess and get that

$$E(\tau, \frac{\eta\sigma}{2}, S) \leq C\eta^{-m-2}\varepsilon E_0 + C\delta^{1/(m+1)} E_0 + C(\eta^2 + \delta) E_0. \tag{5.41}$$

and hence using that $|\tau| \leq C\delta^{1/(2m+2)}\sigma$ yields that whenever $C\delta^{1/(2m+2)} < \eta/4$ we have $B_{\eta\sigma/4}(0) \subseteq B_{\eta\sigma/2}(\tau)$ whenever $\delta = \delta(m, \ell, p)$ is small enough. and then (5.41) shows that

$$E(0, \frac{\eta\sigma}{4}, S) \leq C\eta^{-m-2} \left(\varepsilon + \delta^{1/(2m+2)}\right) E_0 + C(\eta^2 + \delta) E_0. \tag{5.42}$$

Finally, select C from (5.42), choose $\eta = \eta(m, \ell, p)$ such that $C\eta^2 \leq \frac{1}{4}\left(\frac{\eta}{4}\right)^{2(1-m/p)}$, then choose $\varepsilon = \varepsilon(m, \ell, p)$ such that $C\eta^{-m-2}\varepsilon \leq \frac{1}{4}\left(\frac{\eta}{4}\right)^{2(1-m/p)}$, and finally, choose $\delta_0 = \delta_0(m, \ell, p)$ small enough such that $B_{\eta\sigma/4}(0) \subseteq B_{\eta\sigma/2}(\tau)$ as above, and such that the harmonic approximation above holds true with the just chosen ε and such that $C\eta^{-m-2}\delta^{1/(2m+2)} \leq \frac{1}{4}\left(\frac{\eta}{4}\right)^{2(1-m/p)}$, and then select some $\delta \leq \delta_0$.

Coming to the conclusion, (5.42) implies that

$$E(0, \frac{\eta\sigma}{4}, S) \leq \left(\frac{\eta}{4}\right)^{2(1-m/p)} E_0, \tag{5.43}$$

and then, since

$$\left(\left(\frac{\eta\sigma}{4}\right)^{p-m} \int_{B_{\eta\sigma/4}(0)} |H|^p \, d\mu_V\right)^{1/p} \leq \left(\frac{\eta}{4}\right)^{1-m/p} \left((\sigma)^{p-m} \int_{B_{\eta\sigma/4}(0)} |H|^p \, d\mu_V\right)^{1/p}$$

we can now finally conclude, since $B_{\eta\sigma/4}(0) \subseteq B_\sigma(0)$, that

$$E_*(0, \frac{\eta\sigma}{4}, S) \leq \left(\frac{\eta}{4}\right)^{2(1-m/p)} E_*(0, \sigma, T)$$

which completes the proof with $\eta/4$ in place of η. ■

Notice that we trivially have, for any S that satisfies the statement in the Tilt-excess decay theorem,

$$(\eta\sigma)^{-m} \int_{B_{\eta\sigma}(\xi)} |p_{T_xM} - p_T|^2 \, d\mu_V \leq \eta^{-m} E(\xi, \sigma, T),$$

and the Tilt-excess decay theorem 5.7 implies that

$$(\eta\sigma)^{-m} \int_{B_{\eta\sigma}(\xi)} |p_{T_xM} - p_S|^2 \, d\mu_V \leq E_*(\xi, \sigma, T),$$

So since $|p_S - p_T|^2 \leq 2|p_{T_xM} - p_T| + 2|p_{T_xM} - p_S|^2$, and since $\mu_V(B_{\eta\sigma}(\xi)) \geq \frac{1}{2}(\omega\eta\sigma)^m$ we have

$$|p_S - p_T|^2 \leq C\eta^{-m} E_*(\xi, \sigma, T) \tag{5.44}$$

5.2 Allard's Regularity Theorem

Recall that the Hölder space $C^{k,\alpha}(U, \mathbb{R}^N)$ for $k \in \mathbb{N}$, $\alpha \in (0, 1]$ and some open set $U \subseteq \mathbb{R}^M$, is the space of functions $f \in C^k(U, \mathbb{R}^N)$ for which $|D^k f(x) - D^k(y)| \leq C(|x - y|)^\alpha$.

We are now ready to prove the main theorem of this thesis. It says that under some regularity conditions, then inside some small ball the support of a varifold V is a Hölder continuous m-dimensional manifold.

Theorem 5.8 (Allard's Regularity Theorem). *For any $p > m$ there is some $\delta_0 = \delta_0(m, \ell, p), \gamma = \gamma(m, \ell, p) \in (0, 1)$ such that if (†) holds with $\delta \leq \delta_0$ then there exists some linear isometry $q : \mathbb{R}^n \to \mathbb{R}$ and some $u = (u_1, \ldots, u_\ell) \in C^{1,1-m/p}(B^m_{\gamma\rho}(0), \mathbb{R}^\ell)$ such that $Du(0) = 0$, $\operatorname{supp} V \cap B_{\gamma\rho}(0) = q(\operatorname{graph} u) \cap B_{\gamma\rho}(0)$*

and such that

$$\rho^{-1}\sup|u| + \sup|Du| + \rho^{1-m/p}\sup_{x,y\in B^m_{\gamma\rho}(0),x\neq y}\frac{|Du(x)-Du(y)|}{|x-y|^{1-m/p}} \leq C\delta^{1/(2m+2)}$$

for some $C = C(m,\ell,p)$

Proof. We will prove a slight improvement of the theorem, that is, for every $\gamma\in(0,1)$ there is some $\delta = \delta(\gamma,m,\ell,p)\in(0,1)$ such that (†) implies the theorems conclusion.

So let $C = C(m,\ell,p) > 0$ be arbitrary. Take some $\xi\in B_{\delta\rho/2}(0)\cap\operatorname{supp}V$ and $\sigma\in(0,\delta\rho/2]$ and some m-dimensional subspace $S_0\subseteq\mathbb{R}^n$. We know from the Tilt-Excess Decay theorem 5.7 that there exists $\delta = \delta(m,\ell,p)$ and $\eta = \eta(m,\ell,p)$ such that (†) implies the existence of some m-dimensional subspace $S_1\subseteq\mathbb{R}^n$ for which

$$E_*(\xi,\eta\sigma,S_1)\leq\eta^{2(1-m/p)}E_*(\xi,\sigma,S_0).$$

In fact, we can do this inductively, and see that for $\xi\in\operatorname{supp}V\cap B_{\delta\rho/2}(0)$ and $\sigma_0 = \delta\rho/2$ there exists some sequence $S_1,S_2,\dots\subseteq\mathbb{R}^n$ of m-dimensional subspaces such that

$$E_*(\xi,\eta^i\sigma_0,S_i)\leq\eta^{2(1-m/p)}E_*(\xi,\eta^{i-1}\rho/2,S_{i-1})\leq\eta^{2i(1-m/p)}E_*(\xi,\sigma_0,S_0) \tag{5.45}$$

for all $i = 1,2,\dots$.

Now, with $T_0 = T(0,2\sigma_0)$ the Affine Approximation lemma 5.3 implies that $E(0,\sigma_0,T_0)\leq C\delta^{1/(m+1)}$ and for all $\xi\in\operatorname{supp}V\cap B_{\sigma_0/2}(0)$ that $E(\xi,\sigma_0/2,T_0)\leq 2^mC\delta^{1/(m+1)}$ with the same C. So taking $S_0 = T_0$ in (5.45) we get that

$$E_*(\xi,\eta^i\sigma_0,S_i)\leq\eta^{2(1-m/p)}E_*(\xi,\eta^{i-1}\sigma_0/2,S_{i-1})\leq\eta^{2i(1-m/p)}E_0$$

for every $\xi\in\operatorname{supp}V\cap B_{\sigma_0/2}(0)$ where $E_0 = E_*(0,\sigma_0,T_0)$. We can then employ (5.44) to see that

$$|p_{S_i}-p_{S_{i-1}}|^2\leq CE_*(\xi,\eta^{i-1}\sigma_0,S_{i-1})\leq C\eta^{2i(1-m/p)}E_*(\xi,\sigma_0,S_0), \tag{5.46}$$

for all $i\geq 1$. Let us now consider $|p_{S_\ell}-p_{S_i}|$, which by (5.46) can be bounded as follows

$$\begin{aligned}|p_{S_\ell}-p_{S_i}|^2 &= \left|\sum_{j=i+1}^{\ell}p_j-p_{j-1}\right|^2\\ &\leq\sum_{j=i+1}^{\ell}|p_j-p_{j-1}|^2\\ &\leq C\eta^{2i(1-m/p)}E_0\end{aligned} \tag{5.47}$$

for every $\ell \geq i \geq 0$. This implies that for every ξ there is some $S(\xi)$ realised as $S(\xi) = \lim_{\ell\to\infty} S_\ell$ such that

$$|p_{S(\xi)} - p_{S_i}|^2 \leq C\eta^{2i(1-m/p)} E_0 \tag{5.48}$$

and nota that with $i = 0$ we get

$$|p_{S(\xi)} - p_{T_0}|^2 \leq CE_0. \tag{5.49}$$

Now, that was for $\sigma_0 = \delta\rho/2$. So if $\sigma \in (0, \sigma_0/2]$ is arbitrary, we can find some $i \geq 0$ for which $\eta^i\sigma_0/2 < \sigma \leq \eta^{i-1}\sigma_0/2$, and then (5.45) and (5.48) implies that

$$E_*(\xi, \sigma, S(\xi)) \leq C\left(\frac{\sigma}{\sigma_0}\right)^{2(1-m/p)} E_0 \tag{5.50}$$

for all $\xi \in \operatorname{supp} V \cap B_{\sigma_0/2}(0)$. In addition, (5.49) and (5.50) imply that

$$E_*(\xi, \sigma, T_0) \leq CE_0 \leq C\delta^{1/(2m+2)}. \tag{5.51}$$

We can assume without loss of generality that $T_0 = \mathbb{R}^m \times \{0\}$, and then corollary 5.5 and (5.51) implies that if $L_0 \in (0, 1/4]$ and if $\delta \leq \delta_0 L_0^{2m+2}$ for some small enough $\delta_0 = \delta_0(m, \ell, p)$ then there exists some Lipschitz mapping $f : B^m_{\sigma_0/2}(0) \to \mathbb{R}^\ell$ with $\operatorname{Lip} f \leq L_0$ such that

$$\operatorname{supp} V \cap B_{\sigma_0/4}(0) = \operatorname{graph} f \cap B_{\sigma_0/4}(0). \tag{5.52}$$

Now, define $F = \operatorname{graph} f$ and let $\xi = (\xi', f(\xi')) \in \operatorname{graph} f$. Then (5.50) and (5.52) implies that

$$\lim_{\sigma\to 0^+} \sigma^{-m} \int_{B_\sigma(\xi)\cap F} |p_{T_x F} - p_{S(\xi)}|^2 \, d\mathcal{H}^m = 0$$

for $\mathcal{H}^m$-a.e. $\xi \in F \cap B_{\sigma_0/2}(0)$. But by definition, this is exactly what it means for $S(\xi)$ to be the tangent space of F at all such ξ, that is $S(\xi) = p_{T_\xi F}$. Hence (5.50) can be rewritten as

$$\sigma^{-m} \int_{B_\sigma(\xi)\cap F} |p_{T_x F} - p_{T_\xi F}|^2 \, d\mathcal{H}^m \leq C\left(\frac{\sigma}{\sigma_0}\right)^{2(1-m/p)} E_0. \tag{5.53}$$

Now, we note that $p_{T_\xi F} : \mathbb{R}^n \to T_\xi F$ is given by

$$p_{T_\xi F}(v) = \sum_{i=1}^m (\tau_i \cdot v)\tau_i$$

where $\tau_1, \ldots, \tau_m$ is an orthonormal basis for $T_\xi F$. We can then use the Gram-Schmidt orthogonalization process, starting with $(e_i, D_i f(\xi'))$ for $i = 1, \ldots, m$ as the basis for $T_\xi F$, which shows that $p_{T_\xi F}$ has a corresponding matrix of the form

$$P_\xi = \begin{pmatrix} I_{m\times m} & Df(\xi') \\ (Df(\xi'))^t & O_{\ell\times\ell} \end{pmatrix} + \mathcal{F}(Df(\xi))$$

where $I_{m\times m}$ is the identity matrix, $O_{\ell\times\ell}$ is the zero-matrix, and where $\mathcal{F}:\mathbb{R}^m\times\mathbb{R}^\ell\to\mathbb{R}$ is a real analytical function with $\mathcal{F}(0)=0$, $D_p\mathcal{F}(0)=0$ and therefore $|\mathcal{F}(p_1)-\mathcal{F}(p_2)|\leq C(m,\ell)(|p_1|+|p_2|)|p_1-p_2|$ whenever $|p_1|,|p_2|\leq 1$. So choosing L_0 small enough, this allows us to see that

$$|Df(x')-Df(\xi')|^2\leq|p_{T_xF}-p_{T_\xi F}|^2\leq 3|Df(x')-Df(\xi')|^2$$

and then (5.53) implies that

$$\sigma^{-m}\int_{B^m_\sigma(\xi')}|Df(x)-Df(\xi)|^2\,d\lambda_m(x)\leq C\left(\frac{\sigma}{\sigma_0}\right)^{2(1-m/p)}E_0$$

whenever $\sigma\in(0,\sigma_0/4)$. Now for μ_V-a.e. $x_1,x_2\in\operatorname{supp}V\cap B_{\sigma_0/8}(0)$ we can use this inequality with $\sigma=|x_1-x_2|$ and with $\xi=x_1,x_2$ respectively. Then since $|Df(x_1)-Df(x_2)|^2\leq 2|Df(x)-Df(x_1)|^2+2|Df(x)-Df(x_2)|^2$ for every $x\in B^m_\sigma(x_1)\cap B^m_\sigma(x_2)$ and since $B^m_\sigma(x_1)\cap B^m_\sigma(x_2)\supseteq B^m_{\sigma/2}(\frac{x_1+x_2}{2})$ we see that

$$|Df(x_1)-Df(x_2)|\leq C\left(\frac{|x_1-x_2|}{\sigma_0}\right)^{1-m/p}E_0^{1/2}$$

for λ_m-a.e. $x_1,x_2\in B^m_{\sigma_0/4}(0)$. Finally, we see that since f is Lipschitz, then the above inequality shows that $f\in C^{1,1-m/p}$ for every $x_1,x_2\in B^m_{\sigma_0/4}(0)$. So choosing δ small enough to satisfy the restrictions above, the theorem follows when selecting $u=f$ and $\gamma=\delta/4$. ■

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