An overview of partial variances and covariances

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Consider the joint distribution

$$(x, y, z) \sim N\left(\left(\begin{array}{c}\mu_x\\ \mu_y\\ \mu_z\end{array}\right), \left(\begin{array}{ccc}\sigma_x^2 & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2\end{array}\right)\right). \tag{1}$$

The interpretation of the dispersion parameters in (1) is that σ_x^2 is the marginal variance of x; that is,

$$\sigma_x^2 = E\left(x^2\right) - \left(E\left(x\right)\right)^2$$

where expectations are taken with respect to $p\left(x\right) = \int\int p\left(x, y, z\right) dydz$. Similarly,

$$\sigma_{xy} = E\left(xy\right) - E\left(x\right) E\left(y\right)$$

where $E\left(xy\right)$ is taken with respect to $p\left(x, y\right) = \int p\left(x, y|z\right) p\left(z\right) dz$.

We now consider the conditional distribution of x and y given z. From normal theory,

$$(x, y|z) \sim N\left(E\left(x, y|z\right), Var\left(x, y|z\right)\right),$$

where

$$\begin{aligned}
E\left(x, y|z\right) &= \left(\begin{array}{c}\mu_x + \frac{\sigma_{xz}}{\sigma_z^2}\left(z - \mu_z\right)\\ \mu_y + \frac{\sigma_{yz}}{\sigma_z^2}\left(z - \mu_z\right)\end{array}\right), && (2)\\
Var\left(x, y|z\right) &= \left(\begin{array}{cc}\sigma_x^2 & \sigma_{xy}\\ \sigma_{xy} & \sigma_y^2\end{array}\right) - \left(\begin{array}{c}\sigma_{xz}\\ \sigma_{yz}\end{array}\right)\frac{1}{\sigma_z^2}\left(\begin{array}{cc}\sigma_{xz} & \sigma_{yz}\end{array}\right)\\
&= \left(\begin{array}{cc}\sigma_x^2\left(1 - \rho_{xz}^2\right) & \sigma_x\sigma_y\left(\rho_{xy} - \rho_{xz}\rho_{yz}\right)\\ \sigma_x\sigma_y\left(\rho_{xy} - \rho_{xz}\rho_{yz}\right) & \sigma_y^2\left(1 - \rho_{yz}^2\right)\end{array}\right)\\
&= \left(\begin{array}{cc}\sigma_{x.z}^2 & \sigma_{xy.z}\\ \sigma_{xy.z} & \sigma_{y.z}^2\end{array}\right), && (3)
\end{aligned}$$

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so that

$$\rho_{xy.z} = \frac{\sigma_{xy.z}}{\sigma_{x.z}\sigma_{y.z}} = \frac{\rho_{xy} - \rho_{xz}\rho_{yz}}{\sqrt{(1-\rho_{xz}^2)}\sqrt{(1-\rho_{yz}^2)}}, \tag{4a}$$

$$b_{xy.z} = \frac{\sigma_{xy.z}}{\sigma_{y.z}^2} = \frac{\sigma_x}{\sigma_y}\frac{\rho_{xy} - \rho_{xz}\rho_{yz}}{(1-\rho_{yz}^2)}. \tag{4b}$$

Expressions (4) are partial coefficients.

Let in general for arbitrary vectors (x_1, x_2)

$$Var\,(x_1, x_2) = \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{5}$$

denote their variance structure and let the inverse, known as the precision matrix, be

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}.$$

The block Σ^{11} is obtained using the inverse of a partitioned matrix, which is

$$\Sigma^{11} = \left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1} = (\Sigma_{1.2})^{-1} \tag{6}$$

and therefore

$$\Sigma_{1.2} = Var\,(x_1|x_2) = \left(\Sigma^{11}\right)^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{7}$$

which retrieves (3) for the particular example of 3 variables (x, y, z). In this particular case, $\Sigma_{11} = Var\,(x, y)$, $\Sigma_{12} = \sigma_{xz}$, $\Sigma_{22} = \sigma_z^2$ and $(\Sigma^{11})^{-1} = Var\,(x, y|z)$.

In general, for an $n-$variate system, in the partition (5), let Σ_{ij} be the 2×2 matrix representing the covariance structure between the two scalar variables x_i and x_j, with Σ denoting the covariance structure of the $n-$variate system. Let Σ^{ij} be the 2×2 block of matrix Σ^{-1} and

$$\left(\Sigma^{ij}\right)^{-1} = \begin{pmatrix} Q_{ii} & Q_{ij} \\ Q_{ji} & Q_{jj} \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} Q_{jj} & -Q_{ij} \\ -Q_{ji} & Q_{ii} \end{pmatrix}$$

with $\Delta = Q_{ii}Q_{jj} - Q_{ij}Q_{ji}$. Then the partial correlation between x_i and x_j, given all the remaining $n-2$ variates is

$$Corr(x_i, x_j|rest) = -\frac{Q_{ij}}{\sqrt{Q_{ii}Q_{jj}}} \tag{8}$$

where $rest$ denotes the remaining $n-2$ variates.

To interpret the ith diagonal element of Σ^{-1}, we use the notation

$$\Sigma = \begin{pmatrix} \Sigma_{ii} & \Sigma_{i,-i} \\ \Sigma_{-i,i} & \Sigma_{-i,-i} \end{pmatrix}$$

and

$$\Sigma^{-1} = \begin{pmatrix} \Sigma^{ii} & \Sigma^{i,-i} \\ \Sigma^{-i,i} & \Sigma^{-i,-i} \end{pmatrix}$$

where Σ_{ii} and Σ^{ii} are scalars, $\Sigma_{i,-i}$ and $\Sigma^{i,-i}$ are row vectors of size $(n-1)$, $\Sigma_{-i,-i}$ and $\Sigma^{-i,-i}$ are $(n-1)\times(n-1)$ matrices. Then

$$\Sigma^{ii} = \left(\Sigma_{ii} - \Sigma_{1,-i}\Sigma^{-1}_{-i,-i}\Sigma_{-i,i}\right)^{-1} = \left(\Sigma_{ii.-i}\right)^{-1}$$

and

$$\Sigma_{ii.-i} = Var\left(x_i|x_{-i}\right) = \left(\Sigma^{ii}\right)^{-1}$$

the inverse of the ith diagonal element of Σ^{-1}.

References