

An Observation About Subsection 6.6: On Average Training MSE Underestimates Validation MSE

The expected value of the validating mean squared error, taken over the distribution of training data y and validating data y_v is shown in equation (6.65) on page 285:

$$E_{y,y_v}(\text{MSE}_v) = \sigma^2 + \frac{1}{N} \sum_i \text{Var}(\hat{y}_{v,i}) + \text{bias}^2. \quad (6.65)$$

The expected value of the training mean squared error, taken over the distribution of training data y is shown in equation (6.66):

$$E_y(\text{MSE}_t) = \sigma^2 + \frac{1}{N} \sum_i \text{Var}(\hat{y}_i) + \text{bias}^2 - \frac{2}{N} \sum_i \text{Cov}(y_i, \hat{y}_i). \quad (6.66)$$

The predicted values in training and validating data are $\hat{y}_i = \hat{f}(x_i)$ and $\hat{y}_{v,i} = \hat{f}(x_{v,i})$, a function of known covariates x_i and $x_{v,i}$, respectively. These covariates are not necessarily the same. However with common covariates such as genetic markers, for a given number of these covariates, as the number of records increases the second terms after the equal sign of these expressions will tend to converge to the same value. Regardless of the number of records, if the covariates in the training and validating data take the same values, $\frac{1}{N} \sum_i \text{Var}(\hat{y}_{v,i}) = \frac{1}{N} \sum_i \text{Var}(\hat{y}_i)$. This is not made explicit in the book. Then it follows that

$$E_y(\text{MSE}_t) = E_{y,y_v}(\text{MSE}_v) - \frac{2}{N} \sum_i \text{Cov}(y_i, \hat{y}_i). \quad (6.67)$$

References