

An overview of partial variances and covariances

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Consider the joint distribution

$$(x, y, z) \sim N \left(\begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{pmatrix} \right). \quad (1)$$

The interpretation of the dispersion parameters in (1) is that σ_x^2 is the marginal variance of x ; that is,

$$\sigma_x^2 = E(x^2) - (E(x))^2$$

where expectations are taken with respect to $p(x) = \int \int p(x, y, z) dy dz$. Similarly,

$$\sigma_{xy} = E(xy) - E(x)E(y)$$

where $E(xy)$ is taken with respect to $p(x, y) = \int p(x, y|z) p(z) dz$.

We now consider the conditional distribution of x and y given z . From normal theory,

$$(x, y|z) \sim N(E(x, y|z), \text{Var}(x, y|z))$$

where

$$E(x, y|z) = \begin{pmatrix} \mu_x + \frac{\sigma_{xz}}{\sigma_z^2} (z - \mu_z) \\ \mu_y + \frac{\sigma_{yz}}{\sigma_z^2} (z - \mu_z) \end{pmatrix}, \quad (2)$$

$$\begin{aligned} \text{Var}(x, y|z) &= \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} - \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} \frac{1}{\sigma_z^2} \begin{pmatrix} \sigma_{xz} & \sigma_{yz} \end{pmatrix} \\ &= \begin{pmatrix} \sigma_x^2 (1 - \rho_{xz}^2) & \sigma_x \sigma_y (\rho_{xy} - \rho_{xz} \rho_{yz}) \\ \sigma_x \sigma_y (\rho_{xy} - \rho_{xz} \rho_{yz}) & \sigma_y^2 (1 - \rho_{yz}^2) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{x.z}^2 & \sigma_{xy.z} \\ \sigma_{xy.z} & \sigma_{y.z}^2 \end{pmatrix} \end{aligned} \quad (3)$$

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so that

$$\rho_{xy.z} = \frac{\sigma_{xy.z}}{\sigma_{x.z}\sigma_{y.z}} = \frac{\rho_{xy} - \rho_{xz}\rho_{yz}}{\sqrt{(1-\rho_{xz}^2)}\sqrt{(1-\rho_{yz}^2)}}, \quad (4a)$$

$$b_{xy.z} = \frac{\sigma_{xy.z}}{\sigma_{y.z}^2} = \frac{\sigma_x}{\sigma_y} \frac{\rho_{xy} - \rho_{xz}\rho_{yz}}{(1-\rho_{yz}^2)}. \quad (4b)$$

Expressions (4) are partial coefficients.

Let in general for arbitrary vectors (x_1, x_2)

$$Var(x_1, x_2) = \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad (5)$$

denote their variance structure and let the inverse, known as the precision matrix, be

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}.$$

The block Σ^{11} is obtained using the inverse of a partitioned matrix, which is

$$\Sigma^{11} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} = (\Sigma_{1.2})^{-1} \quad (6)$$

and therefore

$$\Sigma_{1.2} = Var(x_1|x_2) = (\Sigma^{11})^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \quad (7)$$

which retrieves (3) for the particular example of 3 variables (x, y, z) . In this particular case, $\Sigma_{11} = Var(x, y)$, $\Sigma_{12} = \sigma_{xz}$, $\Sigma_{22} = \sigma_z^2$ and $(\Sigma^{11})^{-1} = Var(x, y|z)$.

In general, for an n -variate system, in the partition (5), let Σ_{ij} be the 2×2 matrix representing the covariance structure between the two scalar variables x_i and x_j , with Σ denoting the covariance structure of the n -variate system. Let Σ^{ij} be the 2×2 block of matrix Σ^{-1} and

$$(\Sigma^{ij})^{-1} = \begin{pmatrix} Q_{ii} & Q_{ij} \\ Q_{ji} & Q_{jj} \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} Q_{jj} & -Q_{ij} \\ -Q_{ji} & Q_{ii} \end{pmatrix}$$

with $\Delta = Q_{ii}Q_{jj} - Q_{ij}Q_{ji}$. Then the partial correlation between x_i and x_j , given all the remaining $n - 2$ variates is

$$Corr(x_i, x_j|rest) = -\frac{Q_{ij}}{\sqrt{Q_{ii}Q_{jj}}} \quad (8)$$

where *rest* denotes the remaining $n - 2$ variates.

To interpret the i th diagonal element of Σ^{-1} , we use the notation

$$\Sigma = \begin{pmatrix} \Sigma_{ii} & \Sigma_{i,-i} \\ \Sigma_{-i,i} & \Sigma_{-i,-i} \end{pmatrix}$$

and

$$\Sigma^{-1} = \begin{pmatrix} \Sigma^{ii} & \Sigma^{i,-i} \\ \Sigma^{-i,i} & \Sigma^{-i,-i} \end{pmatrix}$$

where Σ_{ii} and Σ^{ii} are scalars, $\Sigma_{i,-i}$ and $\Sigma^{i,-i}$ are row vectors of size $(n-1)$, $\Sigma_{-i,-i}$ and $\Sigma^{-i,-i}$ are $(n-1) \times (n-1)$ matrices. Then

$$\Sigma^{ii} = (\Sigma_{ii} - \Sigma_{1,-i} \Sigma_{-i,-i}^{-1} \Sigma_{-i,i})^{-1} = (\Sigma_{ii,-i})^{-1}$$

and

$$\Sigma_{ii,-i} = \text{Var}(x_i | x_{-i}) = (\Sigma^{ii})^{-1}$$

the inverse of the i th diagonal element of Σ^{-1} .

References