The acceptance probability for a general Metropolis-Hastings algorithm

In a general setting, instead of generating X' from $q(\cdot|x)$ as is practised in the standard Metropolis-Hastings algorithm, X' can be defined in terms of a stochastic component $U \sim q(u|x)$ and a deterministic mapping g. As explained in connection with (4.108) and (4.109), in the move from X to X' the mapping is

$$(x', u') = g(x, u) = (g_1(x, u), g_2(x, u)) \tag{4.119}$$

and in the reverse move

$$(x, u) = g^{-1}(x', u') = \big(g_1^{-1}(x', u'), g_2^{-1}(x', u')\big). \tag{4.120}$$

The transition kernel in the move from X to X'

$$P(x, B) = \Pr(X' \in B|X = x) = \int I(x' \in B)p(x'|x)dx'$$

is now constructed in three steps. According to the Metropolis-Hastings protocol, a transition from x to x' requires first drawing u from $q(\cdot|x)$. Secondly, constructing $x' = g_1(x, u)$ and thirdly accepting it with probability $a(g_1(x, u)|x)$. Define

$$S(x) = \Pr(x' \text{ is rejected } |X = x).$$

Then

$$\Pr(X' \in B|X = x) = \int I(g_1(x, u) \in B)q(u|x)a(g_1(x, u)|x)du + S(x)I(x \in B), \tag{4.121}$$

where the second term in the right hand side specifies the probability of rejecting x' but the current state x is already in the set B. The left hand side of (4.117) takes the form

$$\begin{aligned} & \int_A \Pr(X' \in B|X = x)\pi(x)dx \\ = & \int\int I(x \in A, g_1(x, u) \in B)\pi(x)q(u|x)a(g_1(x, u)|x)dxdu \\ & + \int S(x)I(x \in A \cap B)\pi(x)dx. \end{aligned} \tag{4.122}$$

A transition from x' to x is accomplished by first drawing u' from $q(\cdot|x')$. Secondly, constructing $x = g_1(x', u')$ and thirdly accepting it with probability $a(g_1(x', u')|x')$.

The right hand side of (4.117) takes the form

$$\begin{aligned} & \int\int I(x' \in B, g_1(x', u') \in A)\pi(x')q(u'|x')a(g_1(x', u')|x')dx'du' \\ & + \int S(x')I(x' \in A \cap B)\pi(x')dx'. \end{aligned} \tag{4.123}$$

The second terms in (4.122) and (4.123) are equal and therefore a sufficient condition for (4.117) to hold is

$$
\begin{aligned}
&\int\int I(x \in A, g_1(x,u) \in B)\pi(x)q(u|x)a(g_1(x,u)|x)dxdu \\
&= \int\int I(x' \in B, g_1(x',u') \in A)\pi(x')q(u'|x')a(g_1(x',u')|x')dx'du'.
\end{aligned} \tag{4.124}
$$

The final step is to find a way to equalise the indicator functions of both sides of equation (4.124). This is achieved performing the change of variable

$$
(x,u) = g(x',u') = (g_1(x',u'), g_2(x',u')) \tag{4.125}
$$

and making use of

$$
g^{-1}(x',u') = g(x',u'). \tag{4.126}
$$

Substituting in the argument of the indicator function of the right hand side of (4.124), both indicator functions are equalised. It is at this point of the derivation that the constraint (4.110) or (4.125) becomes relevant (see also (4.111)). Using this transformation and setting $dx'du' = |J|dxdu$, the right hand side of (4.124) takes the form

$$
\iint I(g_1(x,u) \in B, x \in A)\pi(g_1(x,u))q(g_2(x,u)|g_1(x,u))a(x|g_1(x,u))|J|dxdu, \tag{4.127}
$$

where

$$
J = \det\begin{bmatrix} \frac{\partial g_1(x,u)}{\partial x} & \frac{\partial g_2(x,u)}{\partial x} \\ \frac{\partial g_1(x,u)}{\partial u} & \frac{\partial g_2(x,u)}{\partial u} \end{bmatrix}.
$$

Examination of the left hand side of (4.124) and of (4.127) shows that the reversibility condition (4.117) is satisfied if

$$
\pi(x)q(u|x)a(g_1(x,u)|x) = \pi(g_1(x,u))q(g_2(x,u)|g_1(x,u))a(x|g_1(x,u))|J|.
$$

The same argument leading to (4.106) indicates that a valid choice for $a(x'|x)$ is

$$
a(x'|x) = \min\left[1, \frac{\pi(x')q(u'|x')}{\pi(x)q(u|x)}|J|\right]. \tag{4.128}
$$

References