1 Equality of Sets

Theorems:

Proof. $L_1 \times (L_2 \cup L_3) \equiv (L_1 \times L_2) \cup (L_1 \times L_3)$

The proof will be divided into two parts. the first will show that:
$L_1 \times (L_2 \cup L_3) \subseteq (L_1 \times L_2) \cup (L_1 \times L_3)$

And the second that:
$(L_1 \times L_2) \cup (L_1 \times L_3) \subseteq L_1 \times (L_2 \cup L_3)$

Proof:
let $w \in L_1 \times (L_2 \cup L_3)$

Concatenation definition:
$\implies \exists y, z \in \Sigma^* \mid w = yz \ and \ (y \in L_1) \wedge (z \in (L_2 \cup L_3))$
Union definition:
$\implies \exists y, z \in \Sigma^* \mid w = yz \ and \ (y \in L_1) \wedge (z \in L_2 \vee z \in L_3)$
Distribution definition:
$\implies \exists y, z \in \Sigma^* \mid w = yz \ and \ (y \in L_1 \wedge z \in L_2) \vee (y \in L_1 \wedge z \in L_3)$
Concatenation definition:
$\implies w \in L_1 \times L_2 \vee w \in L_1 \times L_3$
Union definition:
$\implies w \in (L_1 \times L_2) \cup (L_1 \times L_3))$
Subset definition:
$\implies L_1 \times (L_2 \cup L_3) \subseteq (L_1 \times L_2) \cup (L_1 \times L_3)$

Let $w \in (L_1 \times L_2) \cup (L_1 \times L_3)$

Union definition:
$\implies w \in (L_1 \times L_2) \vee w \in (L_1 \times L_3)$
Concatenation definition:
$\implies \exists y, z \in \Sigma^* \mid w = yz \ and \ (y \in L_1 \wedge z \in L_2) \vee (y \in L_1 \wedge z \in L_3)$
Union definition:
$\implies \exists y, z \in \Sigma^* \mid w = yz \ and \ (y \in L_1 \wedge z \in (L_2 \cup L_3))$
Concatenation definition:
$\implies w \in L_1 \times (L_2 \cup L_3)$
Subset definition:
$\implies (L_1 \times L_2) \cup (L_1 \times L_3) \subseteq (L_1 \times (L_2 \cup L_3))$
Group equality definition:
$\implies L_1 \times (L_2 \cup L_3) \equiv (L_1 \times L_2) \cup (L_1 \times L_3)$

$\square$

Proof. $L_1^* \cup L_2^*(L_1 \cap L_2)^* \subseteq (L_1 \cup L_2)^*$

let $w \in L_1^* \cup L_2^*(L_1 \cap L_2)^*$
let $t \in (L_1 \cup L_2)^*$

Definition of union:
$\Longrightarrow w \in L_1^* \vee w \in L_2^*(L_1 \cap L_2)^*$
Differenciating possible circumstances:
Assume: $w \in L_1^*$

It would be sufficient for such circumstances to prove that:
$w \in (L_1 \cup L_2)^*$

Proof:
$w \in L_1^*$
Definition of Kleene Star:
$\Longrightarrow w = w_1 \circ w_2 \circ \ldots \circ w_n \mid w_{1,2,\ldots,n} \in L_1$
Definition of Union:
$\Longrightarrow w_{1,2,\ldots,n} \in L_1 \cup L_2$
Definition of Kleene Star:
$\Longrightarrow w_1 \circ w_2 \circ \ldots \circ w_n \in (L_1 \cup L_2)^*$
Definition of w:
$\Longrightarrow w \in (L_1 \cup L_2)^*$

Assume: $w \in L_2^*(L_1 \cap L_2)^*$

It would be sufficient for such circumstances to prove that:
$w \in (L_1 \cup L_2)^*$
Definition of Concatenation:
$\Longrightarrow \exists y \in L_2^* \exists z \in (L_1 \cap L_2)^* \mid w = yz$
Definition of Kleene Star:
$\Longrightarrow z = z_1 \circ z_2 \circ \ldots \circ z_n \mid z_{1,2,\ldots,n} \in (L_1 \cap L_2)$
Definition of Intersection:
$\Longrightarrow z_{1,2,\ldots,n} \in L_2$
Definition of Kleene Star:
$\Longrightarrow z \in L_2^*$
Definition of z:
$\Longrightarrow \exists y \in L_2^* \exists z \in L_2^* \mid w = yz$
$\Longrightarrow \exists y, z \in L_2^* \mid w = yz$
Definition of Kleene Star:
$\Longrightarrow w \in L_2^*$
Definition of Kleene Star:
$\Longrightarrow w = w_1 \circ w_2 \circ \ldots \circ w_n | w_{1,2,\ldots,n} \in L_2$
Definition of Union:
$w_{1,2,\ldots,n} \in L_1 \cup L_2$
Definition of Kleene Star:

$\implies w_1 \circ w_2 \circ \ldots \circ w_n \in (L_1 \cup L_2)^*$
Definition of w:
$\implies w \in (L_1 \cup L_2)^*$

Proved for all possible circumstances.

$\square$