Problem Set II

Linear Algebra
July 9, 2023

ALON FILLER[1]

1 :

Problem 1.1. Find the canonical row echelon form of the matrix:

$$\begin{pmatrix} 7 & 7 & 4 & 28 & 21 \\ 3 & 3 & 1 & 12 & 9 \\ 6 & 5 & 2 & 20 & 18 \\ 2 & 3 & 1 & 12 & 6 \\ 4 & 6 & 4 & 24 & 12 \end{pmatrix}$$

Solution.

$$\xRightarrow{\text{swap } R_1 \text{ with } R_5}$$

$$\begin{pmatrix} 4 & 6 & 4 & 24 & 12 \\ 3 & 3 & 1 & 12 & 9 \\ 6 & 5 & 2 & 20 & 18 \\ 2 & 3 & 1 & 12 & 6 \\ 7 & 7 & 4 & 28 & 21 \end{pmatrix}$$

$$\xRightarrow{\text{Divide the new } R_1 \text{ by } R_{1,1}}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 3 & 3 & 1 & 12 & 9 \\ 6 & 5 & 2 & 20 & 18 \\ 2 & 3 & 1 & 12 & 6 \\ 7 & 7 & 4 & 28 & 21 \end{pmatrix}$$

$$\xRightarrow{\text{From each row } R_{i>1} \text{ subtract } R_i * R_1}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & -1.5 & -2 & -6 & 0 \\ 0 & -4 & -4 & -16 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -3.5 & -3 & -14 & 0 \end{pmatrix}$$

[1] Sorer

$$\xRightarrow{\text{swap } R_2 \text{ with } R_3}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & -4 & -4 & -16 & 0 \\ 0 & -1.5 & -2 & -6 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -3.5 & -3 & -14 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{Divide the new } R_2 \text{ by } R_{2,2}}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & -1.5 & -2 & -6 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -3.5 & -3 & -14 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{From each row } R_{i>2} \text{ subtract } R_i * R_2}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{Multiply } R_3 \text{ by -2}}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{From each row } R_{i>3} \text{ subtract } R_i * R_3}$$

$$\begin{pmatrix} 1 & 1.5 & 1 & 6 & 3 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{From each row } R_{i<2} \text{ subtract } R_{i,2} * R_2}$$

$$\begin{pmatrix} 1 & 0 & -0.5 & 0 & 3 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xRightarrow{\text{From each row } R_{i<3} \text{ subtract } R_{i,3} * R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\blacksquare$

Problem 1.2. Solve the System of Equations above $\mathbb{R}$:

$$\begin{aligned} 2x + 7y + 13z &= 33 \\ 2x + 4y + 7z &= 15 \\ 1x + 2y + 4z &= 7 \end{aligned}$$

Solution.

$$\xRightarrow{\text{Display as Matrix}}$$

$$\left(\begin{array}{ccc|c} 2 & 7 & 13 & 33 \\ 2 & 4 & 7 & 15 \\ 1 & 2 & 4 & 7 \end{array}\right)$$

$$\xRightarrow{R_2 = R_2 - R_1}$$

$$\left(\begin{array}{ccc|c} 2 & 7 & 13 & 33 \\ 0 & -3 & -6 & -18 \\ 1 & 2 & 4 & 7 \end{array}\right)$$

$$\xRightarrow{R_3 = 2 * R_3 - R_1}$$

$$\left(\begin{array}{ccc|c} 2 & 7 & 13 & 33 \\ 0 & -3 & -6 & -18 \\ 0 & -3 & -5 & -19 \end{array}\right)$$

$$\xRightarrow{R_3 = R_3 - R_2}$$

$$\left(\begin{array}{ccc|c} 2 & 7 & 13 & 33 \\ 0 & -3 & -6 & -18 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = 3 * R_1 + 7 * R_2}$$

$$\left(\begin{array}{ccc|c} 6 & 0 & -3 & -27 \\ 0 & -3 & -6 & -18 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 + 3 * R_3}$$

$$\left(\begin{array}{ccc|c} 6 & 0 & 0 & -30 \\ 0 & -3 & -6 & -18 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 + 6 * R_3}$$

$$\left(\begin{array}{ccc|c} 6 & 0 & 0 & -30 \\ 0 & -3 & 0 & -24 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1/6, R_2 = R_2/(-3)}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{\text{Display as System of Equations}}$$

$$\begin{array}{lcl} 2x + 7y + 13z & = & 33 \\ -3y + -6z & = & -18 \\ 1z & = & -1 \end{array}$$

$$\xrightarrow{\text{z = -1}}$$

$$\begin{array}{lcl} 2x + 7y + 13z & = & 33 \\ y & = & 8 \\ z & = & -1 \end{array}$$

$$\xrightarrow{\text{y = 8, z = -1}}$$

$$\begin{array}{lcl} x & = & -5 \\ y & = & 8 \\ z & = & -1 \end{array}$$

■

Problem 1.3. Given the following System of Equations, determine for which a values the System has one solution, no solutions or an infinite amount of solutions:

$$\begin{aligned} x + y + az &= 1 \\ x + ay + z &= 1 \\ ax + y + z &= 1 \end{aligned}$$

Solution.

$\xRightarrow{\text{Display as Matrix}}$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \end{array}\right)$$

$\xRightarrow{R_2 = R_2 \text{ - } R_1}$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ a & 1 & 1 & 1 \end{array}\right)$$

$\xRightarrow{R_3 = R_3 \text{ - a * } R_1}$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & 1-a \end{array}\right)$$

$\xRightarrow{R_3 = R_3 + R_2}$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & -(a-1)(a+2) & 1-a \end{array}\right)$$

$\xRightarrow{\text{Testing Edge Cases:}}$

$$\begin{cases} a = 1 \Rightarrow -(1-1)(1+2) = 1-1 \Rightarrow 0 = 0 \Rightarrow x, y, z \in \mathbb{R} \\ a = 2 \Rightarrow -(-2-1)(-2+2) = 1-(-2) \Rightarrow 0 = 3 \Rightarrow x, y, z \notin \mathbb{R} \end{cases} \tag{1}$$

Now, assuming $a \neq$ 1,-2 let us find general solutions

$$\xrightarrow{R_1 = R_1 - \frac{1}{a-1}R_2}$$

$$\begin{pmatrix} 1 & 0 & a+1 & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & -a^2-a+2 & 1-a \end{pmatrix}.$$

$$\xrightarrow{R_1 = R_1 - \frac{a+1}{-a^2-a+2}R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1-\frac{1-a^2}{-a^2-a+2} \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & -a^2-a+2 & 1-a \end{pmatrix}.$$

$$\xrightarrow{R_2 = R_2 - \frac{1-a}{-a^2-a+2}R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{a+2} \\ 0 & a-1 & 0 & -\frac{(1-a)^2}{-a^2-a+2} \\ 0 & 0 & -a^2-a+2 & 1-a \end{pmatrix}.$$

$$\xrightarrow{\text{Display as System of Equations}}$$

$$\begin{aligned} x &= \tfrac{1}{a+2} \\ (a-1)y &= -\tfrac{(1-a)^2}{-a^2-a+2} \\ (-a^2-a+2)z &= 1-a \end{aligned}$$

$$\xrightarrow{\text{(2): /:(a - 1)}}$$

$$\begin{aligned} x &= \tfrac{1}{a+2} \\ y &= -\tfrac{(1-a)}{-a^2-a+2} \\ (-a^2-a+2)z &= 1-a \end{aligned}$$

$$\xrightarrow{\text{(3): /:(}-a^2\text{ - a + 2), (2): finding roots}}$$

$$\begin{aligned} x &= \tfrac{1}{a+2} \\ y &= \tfrac{1}{a+2} \\ z &= \tfrac{1}{a+2} \end{aligned}$$

$$\xrightarrow{\text{z = -y}}$$

$$\begin{aligned} x &= \tfrac{1}{a+2} \\ y &= -\tfrac{1}{a+2} \\ z &= -y \end{aligned}$$

And hence the solution is:

$$\begin{pmatrix} \frac{1}{a+2} \\ \frac{1}{a+2} \\ \frac{1}{a+2} \end{pmatrix}$$

$\blacksquare$