Orthogonality in Rn

Orthonormal sets

A set of vectors in $\mathbb{R}^n$ is called an #orthogonal_set if all pairs of distinct vectors in the set are orthogonal, ie

$$\vec{v}_i \cdot \vec{v}_j = 0 \, \forall i \neq j, \, i, j \in Z$$

An orthogonal set of unit vectors is called an orthonormalset. That is

$$\vec{v}_i \cdot \vec{v}_j = \begin{Bmatrix} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{Bmatrix}$$

For example:

$$\left\{ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} m\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{pmatrix} \right\}$$

We can see that all pairs are orthonormal by computing the dot product between all of them.

We get an orthonormal set by making all of these unit vectors, ie

$$\vec{V}_1 = \frac{v_1}{|v_1|}, \text{ etc}$$

This doesn't change the angle of any vector, so they are still orthonormal.

Why do we care about orthonormal sets?
Suppose $\vec{v}_1, \ldots, \vec{v}_k$ is an orthogonal set and all $\vec{v}_i$ are nonzero, and consider the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

For any $j = 1, \ldots, k$ we can compute

$$\begin{aligned} 0 = \vec{v}_j \cdot 0 = c_1(\vec{v}_j \cdot v_1) + \cdots + c_j(\vec{v}_j \cdot \vec{v}_j) \\ = \vec{v}_j \cdot \vec{v}_j \end{aligned}$$

Since $\vec{v}_j$ is nonzero, we have $c_j = 0$. Since our choice of j was arbitrary, we know that $c_1 = c_2 = \cdots = c_k$. All of the constants are zero, so this set is linearly independent.

Theorem: if $\left\{\vec{v}_1, \ldots, \vec{v}_k\right\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then they are linearly independent.

We can use this to compute the constants.

$$\vec{w} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

taking dot products,

$$\begin{aligned} \vec{v}_j \cdot \vec{w} &= c_1(\vec{v}_j \cdot \vec{v}_1) + \cdots + c_k(\vec{v}_j + \vec{v}_k) \\ &= c_j(\vec{v}_j \cdot \vec{v}_j) \end{aligned}$$

so $c_j = \frac{\vec{v}_j \cdot \vec{w}}{\vec{v}_j \cdot \vec{v}_j}$

For example:
take two vectors in $\mathbb{R}^2$ which are linearly independent and orthogonal

$$V = \left\{\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$W = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a linear combination of $\vec{v}_1, \vec{v}_2$

$$\vec{w} = \underbrace{\left(\frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1}_{\text{proj}_{\vec{v}_1}(\vec{w})} + \underbrace{\left(\frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2}_{\text{proj}_{\vec{v}_2}(\vec{w})} = \frac{2}{8}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{-3}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

* Check this! *

Orthogonal transformations

Interesting examples:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This second one is a "permutation" if the identity matrix, ie just simple row operations have generated it.

Orthogonal matrices are #angle-preserving and #length-preserving.

we can write out these properties as such:

$$\begin{aligned} &(Q\vec{x}) \cdot (Q\vec{y}) \\ &= (Q\vec{x})^T (Q\vec{y} \\ &= \vec{x}^T Q^T Q\vec{y} \\ &= \vec{x}^T y \end{aligned}$$

Properties of orthogonal matrices

Let Q be an orthogonal matrix.

- The rows of Q also (like the columns) form an orthonormal set
- Q^{-1} is also an orthogonal matrix
- $\det Q = 1 \text{ or } -1$
- if λ is an eigenvalue of Q, then $|\lambda|$ = 1
- The product of orthogonal matrices is orthogonal.

See previous: Similarity and Diagonalization