Equations

Double pendulum

The lagrangian

$$\mathscr{L} = T - U$$
$$T = \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2(\dot{\theta}_2^2 l_2^2 + \dot{\theta}_1^2 l_2^2 + 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\cos(\theta_1 - \theta_2))$$
$$U = g\Big[(m_1 + m_2)l_1\cos(\theta_1) + m_2 l_2\cos\theta_2\Big]$$

so

$$\mathscr{L} = \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2(\dot{\theta}_2^2 l_2^2 + \dot{\theta}_1^2 l_2^2 + 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\cos(\theta_1 - \theta_2)) - g\Big[(m_1 + m_2)l_1\cos(\theta_1) + m_2 l_2\cos\theta_2\Big]$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}} = \frac{\partial L}{\partial q}$$

We take the derivatives with $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$

First, θ_1. Lets take the two derivatives:

$$\frac{\partial\ell}{\partial\theta_1} = g(m_1 + m_2)l\sin(\theta_1) - m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$
$$\frac{\partial\ell}{\partial\dot{\theta}_1} = l_1^2 m_1\dot{\theta}_1 + m_2 l_2^2\dot{\theta}_1 + m_2\dot{\theta}_2 l_1 l_2\cos(\theta_1 - \theta_2)$$

We can calculate

$$\frac{d}{dt}\frac{\partial\ell}{\partial\dot{\theta}_1} = (m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} + m_2 l_1 l_2(\ddot{\theta}_2\cos(\theta_1 - \theta_2)) - \dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

We now have both sides for the Lagrange differential equation.

We also have the derivatives with θ_2:

$$\frac{\partial\ell}{\partial\theta_2} = gm_2 l_2\sin\theta_2 + m_2\theta_2 l_2^2 + m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$
$$\frac{\partial\ell}{\partial\dot{\theta}_2} = m_2 l_2^2 + m_2\dot{\theta}_1 l_1 l_2\cos(\theta_1 - \theta_2)$$
$$\frac{d}{dt}\frac{\partial\ell}{\partial\dot{\theta}_2} = m_2 l_2^2\ddot{\theta}_2 + l_1 l_2 m_2\Big[\dot{\theta}_1(-\sin(\theta_1 - \theta_2))(\dot{\theta}_1 - \dot{\theta}_2) + \cos(\theta_1 - \theta_2)\ddot{\theta}_1\Big]$$

We now have the two lagrange equations of motion:

For θ_1:

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} + m_2 l_1 l_2(\ddot{\theta}_2\cos(\theta_1 - \theta_2)) - \dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) = g(m_1 + m_2)l\sin(\theta_1) - m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$

Lets collect terms to make this a nice separable equation:

$$\boxed{(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta}_1 + (m_2 l_1 l_2\cos(\theta_1 - \theta_2))\ddot{\theta}_2 = m_1 l_1 l_2\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) + g(m_1 + m_2)l\sin(\theta_1) - m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)}$$

For θ_2:

$$m_2 l_2^2 \ddot{\theta}_2 + l_1 l_2 m_2 \left[\dot{\theta}_1(-\sin(\theta_1 - \theta_2))(\dot{\theta}_1 - \dot{\theta}_2) + \cos(\theta_1 - \theta_2)\ddot{\theta}_1\right] = g m_2 l_2 \sin\theta_2 + m_2 \theta_2 l_2^2 + m_2$$

We can rearrange this to be nicer:

$$(m_2 l_1 l_2 \cos(\theta_1 - \theta_2))\ddot{\theta}_1 + (m_2 l_2)\ddot{\theta}_2 = (m_2 l_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2))\dot{\theta}_1 + g m_2 l_2 \sin\theta_2 + m_2 \theta_2 l_2^2 + m_2$$

We need to solve the equations

$$\alpha\ddot{\theta}_1 + \beta\ddot{\theta}_2 = \gamma$$
$$\delta\ddot{\theta}_1 + \epsilon\ddot{\theta}_2 = \phi$$

So we have the matrix operation

$$\begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \begin{vmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{vmatrix} = \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$