Equations

Double pendulum

The Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L} = T - U$$

$$T = \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2(\dot{\theta}_2^2 l_2^2 + \dot{\theta}_1^2 l_2^2 + 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2 \cos(\theta_1 - \theta_2))$$

$$U = g\Big[(m_1 + m_2)l_1\cos(\theta_1) + m_2 l_2\cos\theta_2\Big]$$

so

$$\mathscr{L} = \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2(\dot{\theta}_2^2 l_2^2 + \dot{\theta}_1^2 l_2^2 + 2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\cos(\theta_1 - \theta_2)) - g\Big[(m_1 + m_2)l_1\cos(\theta_1) + m_2 l_2\cos\theta_2\Big]$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\dot{\partial q}} = \frac{\partial L}{\partial q}$$

We take the derivatives with $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$

First, θ_1. Lets take the two derivatives:

$$\frac{\partial\ell}{\partial\theta_1} = g(m_1 + m_2)l_1\sin(\theta_1) - m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$

$$\frac{\partial\ell}{\partial\dot{\theta}_1} = l_1^2 m_1\dot{\theta}_1 + m_2 l_2^2\dot{\theta}_1 + m_2\dot{\theta}_2 l_1 l_2\cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\frac{\partial\ell}{\partial\dot{\theta}_1} = (m_1 l_1^2 + m_2 l_2^2)\ddot{\theta}_1 + m_2 l_1 l_2(\ddot{\theta}_2\cos(\theta_1 - \theta_2)) - m_1 l_1 l_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2$$

We have the LaGrange equation of motion for θ_1 (rearranging to have the accelerations on the left and others on the right):

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta_1} + m_2 l_1 l_2\cos(\theta_1 - \theta_2)\ddot{\theta}_2 = m_1 l_1 l_2\sin(\theta_1 - \theta_2)(\dot{\theta_1} - \dot{\theta}_2)\dot{\theta}_2 + g(m_1 + m_2)l_1\sin(\theta)_1 - m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$

We also have the derivatives with θ_2:

$$\frac{\partial\mathscr{L}}{\partial\theta_2} = gm_2 l_2\sin(\theta_2) + m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_2\sin(\theta_1 - \theta_2)$$

$$\frac{\partial\mathscr{L}}{\partial\dot{\theta}_2} = m_2\dot{\theta}_2 l_2^2 + m_2\dot{\theta}_1 l_1 l_2\cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\dot{\partial\theta}_2} = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2\dot{\theta}_1 l_1 l_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

Which gets the equation of motion

For θ_2:

$$m_2 l_1 l_2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 = m_2\dot{\theta}_1 l_1 l_2\sin(\theta_1 - \theta_2)(\theta_1 \dot{-} \dot{\theta}_2)$$

We need to solve the equations

$$\alpha\ddot{\theta}_1 + \beta\ddot{\theta}_2 = \gamma$$

$$\delta\ddot{\theta}_1 + \epsilon\ddot{\theta}_2 = \phi$$

So we have the matrix operation

$$\begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$

Single Pendulum

The Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L} = T - U$$

$$\begin{aligned} T &= \frac{1}{2} m\omega^2 \\ U &= L(1 - \cos\theta)mg \end{aligned}$$

so

$$\mathscr{L} = \frac{1}{2} m\omega^2 - Lm(1 - \cos\theta)g$$

First, Lets take the two derivatives:

$$\begin{aligned} \frac{\partial \ell}{\partial \theta_1} &= -Lmg\sin\theta \\ \frac{\partial \ell}{\partial \dot{\theta}} &= m\omega \end{aligned}$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\dot{\partial} q} = \frac{\partial L}{\partial q}$$

We now have the Lagrange equation of motion:

$$\begin{aligned} m\ddot{\theta} &= -Lmg\sin\theta \\ \ddot{\theta} &= -Lg\sin\theta \end{aligned}$$