Equations

Double pendulum

The Lagrangian $\mathscr L$ is given by

$$\mathscr{L} = T - U$$
 $T = rac{1}{2} m_1 \dot{ heta}_1^2 l_1^2 + rac{1}{2} m_2 (\dot{ heta}_2^2 l_2^2 + \dot{ heta}_1^2 l_2^2 + 2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \cos(heta_1 - heta_2))$ $U = g igg[(m_1 + m_2) l_1 \cos(heta_1) + m_2 l_2 \cos{ heta_2} igg]$

so

$$\mathscr{L} = rac{1}{2} m_1 \dot{ heta}_1^2 l_1^2 + rac{1}{2} m_2 (\dot{ heta}_2^2 l_2^2 + \dot{ heta}_1^2 l_2^2 + 2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \cos(heta_1 - heta_2)) - g igg[(m_1 + m_2) l_1 \cos(heta_1) + m_2 l_2 \cos{ heta_2} igg]$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{\partial}q} = \frac{\partial L}{\partial q}$$

We take the derivatives with $heta_1, heta_2, \dot{ heta}_1, \dot{ heta}_2$

First, θ_1 . Lets take the two derivatives:

$$\begin{split} \frac{\partial \ell}{\partial \theta_1} &= g(m_1+m_2)l_1\sin(\theta_1) - m_2\dot{\theta_1}\dot{\theta_2}l_1l_2\sin(\theta_1-\theta_2) \\ \frac{\partial \ell}{\partial \dot{\theta}_1} &= l_1^2m_1\dot{\theta_1} + m_2l_2^2\dot{\theta_1} + m_2\dot{\theta_2}l_1l_2\cos(\theta_1-\theta_2) \\ \\ \frac{d}{dt}\frac{\partial \ell}{\partial \dot{\theta}_1} &= (m_1l_1^2 + m_2l_2^2)\ddot{\theta}_1 + m_2l_1l_2(\ddot{\theta}_2\cos(\theta_1-\theta_2)) - m_1l_1l_2\sin(\theta_1-\theta_2)(\dot{\theta_1}-\dot{\theta_2})\dot{\theta}_2 \end{split}$$

We have the LaGrange equation of motion for θ_1 (rearranging to have the accelerations on the left and others on the right):

$$(m_1 l_1^2 + m_2 l_2^2) \ddot{ heta}_1 + m_2 l_1 l_2 \cos(heta_1 - heta_2) \ddot{ heta}_2 = m_1 l_1 l_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2) \dot{ heta}_2 + g(m_1 + m_2) l_1 \sin(heta)_1 - m_2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \sin(heta_1 - heta_2)$$

We also have the derivatives with θ_2 :

$$egin{aligned} rac{\partial\mathscr{L}}{\partial heta_2} &= gm_2l_2\sin(heta_2) + m_2\dot{ heta}_1\dot{ heta}_2l_1l_2\sin(heta_1- heta_2) \ &rac{\partial\mathscr{L}}{\partial\dot{ heta}_2} &= m_2\dot{ heta}_2l_2^2 + m_2\dot{ heta}_1l_1l_2\cos(heta_1- heta_2) \ &rac{d}{dt}rac{\partial\mathscr{L}}{\partial heta_2} &= m_2l_2^2\ddot{ heta}_2 + m_2l_1l_2\cos(heta_1- heta_2)\ddot{ heta}_1 - m_2\dot{ heta}_1l_1l_2\sin(heta_1- heta_2)(\dot{ heta}_1-\dot{ heta}_2) \end{aligned}$$

Which gets the equation of motion

For θ_2 :

$$m_2 l_1 l_2 \cos(heta_1 - heta_2) \ddot{ heta}_1 + m_2 l_2^2 \ddot{ heta}_2 = m_2 \dot{ heta}_1 l_1 l_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2)$$

We need to solve the equations

$$lpha\ddot{ heta}_1 + eta\ddot{ heta}_2 = \gamma$$
 $\delta\ddot{ heta}_1 + \epsilon\ddot{ heta}_2 = \phi$

$$\begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \begin{bmatrix} \ddot{\theta_1} \\ \ddot{\theta_2} \end{bmatrix} = \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$

Single Pendulum

The Lagrangian $\mathscr L$ is given by

$$\mathscr{L} = T - U$$

$$T = \frac{1}{2}m\omega^2$$

$$U = L(1-\cos\theta)mg$$

so

$$\mathscr{L} = rac{1}{2} m \omega^2 - L m (1 - \cos heta) g$$

First, Lets take the two derivatives:

$$egin{aligned} rac{\partial \ell}{\partial heta_1} &= -L m g \sin heta \ rac{\partial \ell}{\partial \dot{ heta}} &= m \omega \end{aligned}$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{\partial}q} = \frac{\partial L}{\partial q}$$

We now have the Lagrange equation of motion:

$$m\ddot{ heta} = -Lmg\sin{ heta} \ \ddot{ heta} = -Lg\sin{ heta}$$