Equations

Double pendulum

The lagrangian

$$\mathscr{L} = T - U$$
 $T = rac{1}{2} m_1 \dot{ heta}_1^2 l_1^2 + rac{1}{2} m_2 (\dot{ heta}_2^2 l_2^2 + \dot{ heta}_1^2 l_2^2 + 2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \cos(heta_1 - heta_2))$ $U = g igg[(m_1 + m_2) l_1 \cos(heta_1) + m_2 l_2 \cos{ heta}_2 igg]$

so

$$\mathscr{L} = rac{1}{2} m_1 \dot{ heta}_1^2 l_1^2 + rac{1}{2} m_2 (\dot{ heta}_2^2 l_2^2 + \dot{ heta}_1^2 l_2^2 + 2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \cos(heta_1 - heta_2)) - g igg[(m_1 + m_2) l_1 \cos(heta_1) + m_2 l_2 \cos{ heta_2} igg]$$

We have the differential equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{\partial}q} = \frac{\partial L}{\partial q}$$

We take the derivatives with $heta_1, heta_2, \dot{ heta}_1, \dot{ heta}_2$

First, θ_1 . Lets take the two derivatives:

$$egin{aligned} rac{\partial \ell}{\partial heta_1} &= g(m_1+m_2)l\sin(heta_1) - m_2\dot{ heta_1}\dot{ heta_2}l_1l_2\sin(heta_1- heta_2) \ rac{\partial \ell}{\partial \dot{ heta}_1} &= l_1^2m_1\dot{ heta_1} + m_2l_2^2\dot{ heta_1} + m_2\dot{ heta_2}l_1l_2\cos(heta_1- heta_2) \end{aligned}$$

We can calculate

$$rac{d}{dt}rac{\partial \ell}{\partial \dot{ heta}_1} = (m_1l_1^2+m_2l_2^2)\ddot{ heta} + m_2l_1l_2(\ddot{ heta}_2\cos(heta_1- heta_2)) - \dot{ heta}_2\sin(heta_1- heta_2)(\dot{ heta}_1-\dot{ heta}_2)$$

We now have both sides for the Lagrange differential equation.

We also have the derivatives with θ_2 :

$$\begin{split} \frac{\partial \ell}{\partial \theta_2} &= g m_2 l_2 \sin \theta_2 + m_2 \theta_2 l_2^2 + m_2 \dot{\theta}_1 \dot{\theta}_2 l_1 l_2 \sin(\theta_1 - \theta_2) \\ \frac{\partial \ell}{\partial \dot{\theta}_2} &= m_2 l_2^2 + m_2 \dot{\theta}_1 l_1 l_2 \cos(\theta_1 - \theta_2) \\ \\ \frac{d}{dt} \frac{\partial \ell}{\partial \dot{\theta}_2} &= m_2 l_2^2 \ddot{\theta}_2 + l_1 l_2 m_2 \bigg[\dot{\theta}_1 (-\sin(\theta_1 - \theta_2)) (\dot{\theta}_1 - \dot{\theta}_2) + \cos(\theta_1 - \theta_2) \ddot{\theta}_1 \bigg] \end{split}$$

We now have the two lagrange equations of motion:

For θ_1 :

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{ heta} + m_2 l_1 l_2 (\ddot{ heta}_2 \cos(heta_1 - heta_2)) - \dot{ heta}_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2) = g(m_1 + m_2) l \sin(heta_1) - m_2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \sin(heta_1 - heta_2)$$

Lets collect terms to make this a nice separable equation:

$$\left| (m_1 l_1^2 + m_2 l_2^2) \ddot{ heta}_1 + (m_2 l_1 l_2 \cos(heta_1 - heta_2)) \ddot{ heta}_2 = m_1 l_1 l_2 \dot{ heta}_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2) + g(m_1 + m_2) l \sin(heta_1) - m_2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2) + g(m_1 + m_2) l \sin(heta_1) - m_2 \dot{ heta}_1 \dot{ heta}_2 l_1 l_2 \sin(heta_1 - heta_2) (\dot{ heta}_1 - \dot{ heta}_2) (\dot{ heta}_2 - \dot{ heta}_2) (\dot{$$

For θ_2 :

$$m_2 l_2^2 \ddot{ heta}_2 + l_1 l_2 m_2 igg[\dot{ heta}_1 (-\sin(heta_1 - heta_2)) (\dot{ heta}_1 - \dot{ heta}_2) + \cos(heta_1 - heta_2) \ddot{ heta}_1 igg] = g m_2 l_2 \sin heta_2 + m_2 heta_2 l_2^2 + m_2$$

We can rearrange this to be nicer:

$$(m_2l_1l_2\cos(heta_1- heta_2))\ddot{ heta}_1+(m_2l_2)\ddot{ heta}_2=(m_2l_1l_2\sin(heta_1- heta_2)(\dot{ heta}_1-\dot{ heta}_2))\dot{ heta}_1+gm_2l_2\sin heta_2+m_2 heta_2l_2^2+m_2$$

We need to solve the equations

$$lpha\ddot{ heta}_1 + eta\ddot{ heta}_2 = \gamma \ \delta\ddot{ heta}_1 + \epsilon\ddot{ heta}_2 = \phi$$

So we have the matrix operation

$$\begin{bmatrix} \alpha & \beta \\ \delta & \epsilon \end{bmatrix} \begin{array}{c} \ddot{\theta_1} \\ \ddot{\theta_2} \end{array} = \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$