Curs 2 – variabile aleatoare

$$P_B(A) = \frac{P(A\cap B)}{P(B)}$$

[redacted]

$P(A\cap B) = P(A)\cdot P(B)$ - evenimente independente

17/20

$P(A_1 \cup A_2)$ în loc de $P(A)$ – linia 2

Temă

Schema lui Poisson
Schema multinominală
Schema geometrică
Exemple. (minim 3 > 1 fiecare)

Curs 3

X, Y - independente

$P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j)$

$X\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\quad Y\begin{pmatrix} -2 & 1 & 5 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

$p_1 = P(X=0) = \frac{1}{2}$ $\quad q_1 = P(Y=-2) = \frac{1}{4}$

$p_2 = P(X=1) = \frac{1}{2}$ $\quad q_2 = P(Y=1) = \frac{1}{2}$

$p_1 + p_2 = 1$ $\quad q_3 = P(Y=5) = \frac{1}{4}$

$X \setminus Y$	-2	1	5	p_i
0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
q_j	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\boxed{1}$

$p_{11} = P(X=0, Y=-2) \overset{\text{indep}}{=}$
$= P(X=0) \cdot P(Y=-2) = p_1 \cdot q_1 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

$p_{12} = P(X=0, Y=1) = p_1 \cdot q_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$p_{13} = P(X=0, Y=5) = p_1 \, q_3 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$p_{21} = P(X=1, Y=-2) = p_2 \cdot q_1 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

$p_{22} = \frac{1}{4}$ $\quad p_{23} = \frac{1}{8}$

$$X \cdot Y \begin{pmatrix} x_i \cdot y_j \\ p_{ij} \end{pmatrix}$$

$$\boxed{\begin{matrix} X+Y \\ X/Y \end{matrix} \Big\} \text{temă}}$$

X, Y - independente $\Rightarrow p_{ij} = p_i \cdot q_j$

$$XY \begin{pmatrix} 0\cdot(-2) & 0\cdot 1 & 0\cdot 5 & 1\cdot(-2) & 1\cdot 1 & 1\cdot 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$XY \begin{pmatrix} -2 & 0 & 1 & 5 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$

$p_1 = P(X \cdot Y = -2) = P(X = 1, Y = -2) = \frac{1}{8}$.

$p_2 = P(XY = 0) = P(X = 0, Y = 2) +$
$+ P(X = 0, Y = 1) + P(X = 0, Y = 5)$
$= \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2} = P(X = 0)$

$p_3 = P(XY = 1) = P(X = 1, Y = 1) = \frac{1}{4}$

$p_4 = P(XY = 5) = P(X = 1, Y = 5) = \frac{1}{8}$

$$XY \begin{pmatrix} -2 & 0 & 1 & 5 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

$\frac{1}{8} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1$ ✓

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$$X+Y\begin{pmatrix} x_i + y_j \\ r_{ij} \end{pmatrix}$$

$$X+Y\begin{pmatrix} -2 & -1 & 1 & 2 & 5 & 6 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{pmatrix}$$

$Q_1 = P(X+Y=1) = P(X=0,\ Y=-2) = \frac{1}{8}$ - - - -

Functia de repartitie

$$Y\begin{pmatrix} -2 & 1 & 5 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$F_Y(x) = P(Y < x) = \begin{cases} 0 & ;\ x \le 2 \\ \frac{1}{4} & ;\ -2 < x \le 1 \\ \frac{1}{4}+\frac{1}{2} & ;\ 1 < x \le 5 \\ \frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1 & ;\ x > 5 \end{cases}$$

$\boxed{\text{Temă}}$ → dem relatii slide 18/19 - curs 3

Functia de repartitie a unui vect. aleator:

$F: \mathbb{R}^2 \to \mathbb{R},\ F(x,y) = P(X < x,\ Y < y),\ (x,y) \in \mathbb{R}^2$

Variabile aleatoare de tip continuu:

$F(x) = \int_{-\infty}^{x} \rho(t)\,dt;$

ρ = densitatea de probabilitate $\boxed{\rho = R_0}$

<u>Obs</u>: În aplicații

$\rho(x)$ este densitatea de prob.

$$\rho(x) \geq 0$$

$$\int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{2}, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \\ \frac{x}{4}, & 2 < x \leq 4 \\ 1, & x > 4 \end{cases}$$

$l_s(0) = l_d(0) = 0$

$l_s(1) = \frac{1}{2} = l_d(1)$

$l_s(2) = \frac{1}{2} = l_d(2)$

$l_s(4) = l_d(4) = 1$

F cont pe $\mathbb{R}$

$F'(x) = \rho(x)$

$$f(x)=\begin{cases} 0 & , x\le 0 \\ \frac{1}{2} & , 0<x\le 1 \\ 0 & , 1<x\le 2 \\ \frac{1}{4} & , 2<x\le 4 \\ 0 & , x>4 \end{cases}$$

$$f(x)=\begin{cases} \frac{1}{2} , & 0<x\le 1 \\ \frac{1}{4} , & 2<x\le 4 \\ 0 , & \text{în rest.} \end{cases}$$

$$P(X>3)=P(X\geq 3)-P(X=3)$$

$$P(X\geq 3)=1-F(\cancel{x})=1-F(3)=1-\frac{3}{4}=\frac{1}{4}$$

$$P(X=3)=F(3+0)-F(3)=$$

$$=\lim_{x\searrow 3} F(x)-F(3)=F(3)-F(3)=0$$

f cont în $x=3$

$f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases} k\cdot \sin x , & x\in[0,\pi] \\ 0, \text{ în rest} , & x\notin[0,\pi] \end{cases}$$

$k=?$ a.î. f - dens. de prob.

a) $f(x)\geq 0$ $(\forall)\, x\in\mathbb{R}$.

$x\in[0,\pi]$, $f(x)\geq 0 \Leftrightarrow k\cdot\sin x\geq 0$

$\sin x\geq 0$, $x\in[0,\pi]$

$\boxed{k\geq 0}$

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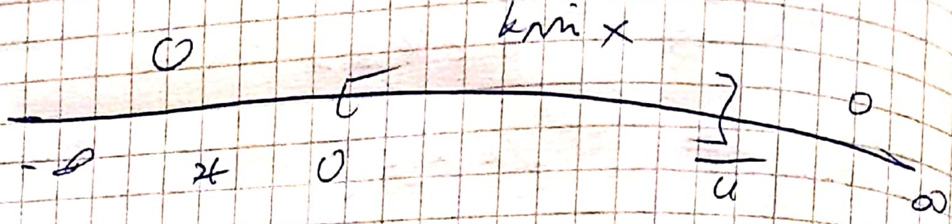
a)



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\pi} k\sin x\,dx + \int_{\pi}^{\infty} 0\,dx = k\int_{0}^{\pi} \sin x\,dx = 2k.$$

$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{2}\sin x, & x \in [0, \pi] \\ 0, & \text{în rest} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

I. $\boxed{x<0}$

$$F(x) = P(X<x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

II $0 \leq x < \pi$

$$F(x) = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{0} + \int_0^x dt = \int_0^x \frac{1}{2}\sin t\,dt$$

$$= \frac{1}{2}(-\cos t)\Big|_0^x = \frac{1}{2}(1-\cos x)$$

III $x \geq \pi$

$$F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^{\pi} f(t)\,dt + \int_{\pi}^{x} f(t)\,dt$$

$$= 1$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(1-\cos x), & 0 \leq x < \pi \\ 1, & x > \pi \end{cases}$$

$$P\left(0 \leq X \leq \frac{\pi}{4}\right) = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

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PROP

Fie vectorul aleator de tip continuu $U=(X,Y)$ având fct. de repartiție $F=F(x,y)$ și densit. de probabilitate $f=f(x,y)$. Atunci:

a) $f(x,y)\geq 0$, $(\forall)(x,y)\in\mathbb{R}^2$

b) $f(x,y)=\dfrac{\partial^2 F}{\partial x\partial y}$

c) $$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy=1$$

d) $$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

(f_X, f_Y densitățile v.a. X, Y)

Exemplu (slide 28)

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\underbrace{\int_{-\infty}^{1} f(x,y)\,dy}_{0}+\int_1^3 f(x,y)\,dy$$

$$+\underbrace{\int_3^{\infty} f(x,y)\,dy}_{0})=$$

$$=\int_1^3 \frac{1}{\cdot}(x+y+\cdot)\ldots$$

$$= \frac{1}{20}\left(xy + \frac{y^2}{2} + 2y\right)\Big|_1^3 = \frac{2x+8}{20}$$

$$f_X(x) = \begin{cases} \frac{2x+8}{20}, & x \in [0,2] \\ 0, & \text{în rest} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \underbrace{\int_{-\infty}^{0}}_{0} + \underbrace{\int_0^2}_{\Downarrow} + \underbrace{\int_2^{\infty}}_{0}$$

$$\int_0^2 f(x,y)\,dx = \frac{1}{20}\left(\frac{x^2}{2}, xy + 2x\right)\Big|_0^2$$

$$= \frac{6+2y}{20}$$

$$f_Y(y) = \begin{cases} \frac{6+2y}{20}, & y \in [1,3] \\ 0, & \text{în rest.} \end{cases}$$

b) Aflăm funcția de repartiție a var x, y

$$\boxed{q(1-P(B)) = 1 - P(A) - P(B) + p\,P(B)}$$

Curs 5

Caracteristici numerice ale var. aleatoare:

- medie (valoarea medie)

v.a. discrete

Ex. $X\begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$$M(X) = -1\cdot\frac{1}{4} + 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$M(X) = \sum x_i p_i$$

v.a. continue

$$M(X) = \int_{-\infty}^{\infty} x f(x)\,dx \quad (\text{integrala este convergentă})$$

Propr. dem.

1) $M(aX+b) = a\,M(X) + b$, $(\forall)\ a, b \in \mathbb{R}$

ex:

a) $X\begin{pmatrix} x_i \\ p_i \end{pmatrix}$ $\quad p_i = P(X = x_i)$

$$\sum p_i = 1$$

$aX + b\begin{pmatrix} ax_i + b \\ p_i \end{pmatrix}$

$$M(aX+b) = \sum (ax_i + b)p_i = \sum (ax_i p_i + b p_i)$$

$$= a\underbrace{\sum x_i p_i}_{M(X)} + b\underbrace{\sum p_i}_{1} = aM(X) + b$$

b) $X\begin{pmatrix} x \\ f(x) \end{pmatrix}$ $x \in \mathbb{R}$.

$$M(aX+b) = \int_{-\infty}^{\infty} (ax+b)\cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} [ax\,f(x) + b f(x)]\,dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{M(X)} + b\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= aM(X) + b$$

2) $M(X+Y) = M(X) + M(Y)$

$X\begin{pmatrix} x_i \\ p_i \end{pmatrix}$

$Y\begin{pmatrix} y_j \\ q_j \end{pmatrix}$

dem:

$X+Y\begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}$

$$p_{ij} = P(X = x_i, Y = y_j)$$

$$\sum_i \sum_j p_{ij} = 1, \quad p_i = \sum_j p_{ij}$$

$$q_j = \sum_i p_{ij}.$$

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$$M(X+Y) = \sum_i \sum_j (x_i + y_j)\, p_{ij} = \sum_i \sum_j (x_i p_{ij} + y_j p_{ij})$$

$$= \sum_i \sum_j x_i p_{ij} + \sum_i \sum_j y_j p_{ij}$$

$$= \sum_i \Big(x_i \underbrace{\sum_j p_{ij}}_{p_i} \Big) + \sum_j y_j \underbrace{\sum_i p_{ij}}_{q_j}$$

$$= \sum_i x_i p_i + \sum_j y_j q_j = M(X) + M(Y)$$

3) $M(X \cdot Y) = M(X) \cdot M(Y)$, dacă X, Y sunt v.a. independente

dem:

$X \cdot Y \begin{pmatrix} x_i \cdot y_j \\ p_{ij} \end{pmatrix}$

X, Y indep. $\Rightarrow p_{ij} = p_i \cdot q_j$

$$M(X \cdot Y) = \sum_i \sum_j x_i y_j p_i q_j$$

$$= \Big(\sum_i x_i p_i \Big) \Big(\sum_j y_j q_j \Big)$$

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- Dispersia (varianța) $\Delta^2(X) = M[(X - M(X))^2]$

$\sigma(X) = \sqrt{\Delta^2(X)}$ - abatere pătratică medie.

$$X\begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$M(X) = -1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$Y\begin{pmatrix} -1000 & 0 & 1000 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$M(Y) = -1000\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 1000\cdot\frac{1}{4} = 0$$

Propr

1) $\Delta^2(X) = M(X^2) - M^2(X)$

dem:

$$\Delta^2(X) = M[(X - M(X))^2]$$
$$= M(X^2 - 2X\underbrace{M(X)}_{\in\mathbb{R}} + \underbrace{M^2(X)}_{\in\mathbb{R}})$$
$$= M(X^2) - M(\underbrace{2XM(X)}_{\text{const}}) + M(M^2(X))$$
$$= M(X^2) - 2M(X)\cdot M(X) + M^2(X)$$
$$= M(X^2) - M^2(X)$$

2) $\Delta^2(aX + b) = a^2\Delta^2(X)$ disp. ct = 0 $\Delta^2(b) = 0$

$$\Delta^2(aX + b) = M[(aX + b)^2] - M^2(aX + b)$$
$$= M(a^2X^2 + 2abX + b^2) - [aM(X) + b]^2$$

$= a^2 M(X^2) + 2ab\, M(X) + b^2 - a^2 M^2(X) -$
$- 2ab\, M(X) - b^2$

$$= a^2 [\underbrace{M(X^2) - M^2(X)}_{D^2(X)}] = a^2 D^2(X)$$

3) dacă v.a. X, Y - indep

$$D^2(X+Y) = D^2(X) + D^2(Y)$$

$$D^2(X+Y) = M[(X+Y)^2] - M^2(X+Y)$$

$$= M(X^2) + \cancel{2M(XY)} + M(Y^2) - M^2(X) - \cancel{2M(X)M(Y)} - M^2(Y)$$

$$= M(X^2) - M^2(X) + M(Y^2) - M^2(Y)$$

$$= D^2(X) + D^2(Y)$$

Ex: $X\begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$M(X^2) =$

Obs

discret.

$X\begin{pmatrix} x_i \\ p_i \end{pmatrix}$

$M(X^2) = \sum x_i^2 p_i$

continuu

$X\begin{pmatrix} x \\ f(x) \end{pmatrix}$

$M(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$

$M(X^2) = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}$

$D^2(X) = M(X^2) - M^2(X)$

$= \frac{2}{3}$

$Y \begin{pmatrix} -1000 & 0 & 1000 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

$M(Y^2) = (-1000)^2 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1000^2 \cdot \frac{1}{4} = \frac{1000^2}{2}$

$D^2(Y) = \frac{1000^2}{2}$

<u>Inegalitatea lui Cebîşev</u> $(\forall)\ \varepsilon > 0$

$P(|X - M(X)| < \varepsilon) \geq 1 - \frac{D^2(X)}{\varepsilon^2}$

$F_X(x) = P(X < x)$

<u>Ex.</u> $X \begin{pmatrix} -1 & 0 & 1 \\ p_1 & p_2 & p_3 \end{pmatrix}$ Det p_1, p_2, p_3 a.î. $M(X) = 0$
$M(X^2) = 0{,}9$

$p_1 + p_2 + p_3 = 1$

$-1p_1 + 0 \cdot p_2 + 1p_3 = 0$ $M(X)$

$+1p_1 + p_3 = 0.9$ $M(X^2)$

$\Rightarrow \begin{cases} p_1 + p_2 + p_3 = 1 \\ -p_1 + p_3 = 0 \\ p_1 + p_3 = 0.9 \end{cases} \Rightarrow 2p_3 = 0.9 \Rightarrow p_3 = \frac{0.9}{2}$ $\Rightarrow p_2 = 0.1$

$-p_1 = -\frac{0.9}{2} \Rightarrow p_1 = \frac{0.9}{2}$

$$X \begin{pmatrix} -1 & 0 & 1 \\ 0.45 & 0.1 & 0.45 \end{pmatrix}$$

Ex: $k = ? \in \mathbb{R}$ a.î. $f(x) = \begin{cases} k \cdot x, & x \in [1,3] \\ 0, & \text{în rest} \end{cases}$

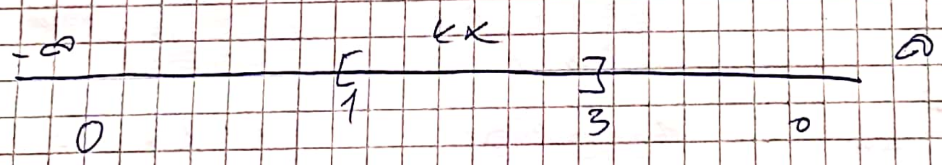
să fie densitate de prob. a unei v.a. X. Să se det apoi media, dispersia și abaterea pătratică

$$f(x) = \begin{cases} kx, & x \in [1,3] \\ 0, & \text{în rest} \end{cases}$$

$f(x) \geq 0, (\forall) x \in \mathbb{R}$

$k \cdot x \geq 0, \ x \in [1,3] \Rightarrow k \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow$$



$$\Leftrightarrow \underbrace{\int_{-\infty}^{1} 0\,dx}_{0} + \int_{1}^{3} kx\,dx + \underbrace{\int_{3}^{\infty} 0\,dx}_{0}$$

$$= \int_{1}^{3} kx\,dx = k\int_{1}^{3} x\,dx = k\,\frac{x^2}{2}\Big|_{1}^{3} =$$

$$= k\left(\frac{3^2}{2} - \frac{1^2}{2}\right) = k\left(\frac{9}{2} - \frac{1}{2}\right) = 4k \Rightarrow 4k = 1$$

$$\Rightarrow k = \frac{1}{4} \geq 0 \ (A)$$

$\rightarrow$) $f(x) = \begin{cases} \frac{x}{4}, & x \in [1, 3] \\ 0, & \text{în rest.} \end{cases}$

$$M(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{1} 0\, dx + \int_{1}^{3} \frac{x^2}{4}\, dx + \int_{3}^{\infty} 0\, dx$$

$$= \frac{1}{4} \int_{1}^{3} x^2\, dx = \frac{1}{4} \left. \frac{x^3}{3} \right|_{1}^{3} = \frac{1}{4}\left(\frac{27}{3} - \frac{1}{3} \right)$$

$$= \frac{1}{4} \cdot \frac{26}{3} = \frac{26}{12}$$

$$= \frac{13}{6}$$

$$D^2(X) = M(X^2) - M^2(X)$$

$$M(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_{1}^{3} \frac{x^3}{4}\, dx$$

$$= \frac{1}{4}\left(\left. \frac{x^4}{4} \right|_{1}^{3} \right)$$

$$= \frac{1}{4}\left(\frac{3^4 - 1}{4} \right) = \frac{81-1}{16}$$

$$= \frac{80}{16} = 5$$

$$D^2(X) = 5 - \left(\frac{13}{6}\right)^2 = \text{---}$$

- Momente

1) moment inițial de ordin k

$$m_k = M(X^k) = \{$$

2) $\mu_k = M[\ldots$

$$\mu_1 = M(X - M(x))) = M(x) - M(x) = 0.$$

$$D^2(x) = M[(X - M(x))^2] = \mu_2$$

3) momentul inițial de ordin (k, l)

$$m_{kl} = M(X^k Y^l)$$

- Corelație / covarianță

$$C(X, Y) = M[(X - M(x))(Y - M(Y))]$$

$$= M[XY - X \cdot M(Y) - YM(X) + M(X)M(Y)]$$

$$= M(XY) - M(X) \cdot M(Y) - M(Y)M(X) + M(X)M(Y)$$

$$\Rightarrow C(X, Y) = M(XY) - M(X)M(Y)$$

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Se num. coeficient / raport de corelație

$$r(X,Y)=\frac{C(X,Y)}{\sqrt{D^2(X)}\sqrt{D^2(Y)}}$$

Prop: X, Y indep. $\Rightarrow r(X,Y)=0$.

$|r(X,Y)|\leq 1$

$r(X,Y)=1 \Leftrightarrow Y=aX+b,\ a>0,$

$r(X,Y)=-1 \Leftrightarrow Y=aX+b,\ a<0.$

exemplu slide 20/20.

$M(X)=-1\cdot 9/24+0\cdot 8/24+1\cdot 7/24$

$=\frac{-9}{24}+\frac{7}{24}=\frac{-2}{24}=-\frac{1}{12}$

$M(Y)=\cancel{-\frac{6}{24}}+\frac{8}{24}+\cancel{\frac{6}{24}}=\frac{1}{3}.$

$M(X^2)=\frac{2}{3}$

$M(Y^2)=\frac{13}{12}$

$D^2(X)=M(X^2)-M^2(X)=\frac{95}{144}$

$D^2(Y)=M(Y^2)-M^2(Y)=\frac{35}{36}.$

$M(XY)=$

$$XY\begin{pmatrix}-2 & -1 & 0 & 1 & 2\\ p_1 & p_2 & p_3 & p_4 & p_5\end{pmatrix}$$

$P_1 = P(X Y = -2) = \frac{1}{24}$.

$P_2 = P(X \cdot Y = 1) = P(X = 1, Y = -1) + P(X = -1, Y = 1)$

$P_3 = (P(X \cdot Y = 0) = \frac{1}{24} + \frac{1}{12} = \frac{3}{24} = \frac{1}{8}$

$P_4 (P(X \cdot Y = 1) =$

$P_5 (P(X \cdot Y = 2)) =$

Seminar

Aplicații la formula probabilității totale

① Trei producători trimit spre vânzare același tip de produs.

Primul producător are 3% produse defecte.

Al doilea 2% — " —

Al treilea 1% — " —

Se știe că al doilea producător trimite

$\boxed{\text{Curs 6}}$

Alte caracteristici numerice de var. aleator.

S.n. $\boxed{\text{mediana}}$ v.a. X caracteristica notată $\boxed{m_e}$ care satisface urm. cond.

$$P(X \geq m_e) \geq \frac{1}{2} \leq P(X \leq m_e) \quad (1)$$

OBS $F_X(x) = P(X < x)$

$$P(X \geq m_e) = P(\overline{X < m_e}) = 1 - P(X < m_e) =$$
$$= 1 - F(m_e)$$

$P(X \leq m_e) = F(m_e + 0)$ (lim la dr)

$$(1) \iff \underbrace{1 - F(m_e) \geq \frac{1}{2}} \leq F(m_e + 0)$$

$$\left.\begin{array}{l} 1 - F(m_e) \geq \frac{1}{2} \\ F(m_e) = \frac{1}{2} \end{array}\right\} \Rightarrow F(m_e) \leq \frac{1}{2} \leq F(m_e + 0)$$

Obs Dacă F este cont. $\Rightarrow F(m_e) = \frac{1}{2}$

S.n. valoarea (modală) sau modul a v.a. X orice punct de maxim local al distribuției, respectiv densității v.a.

S.n. asimetrie sau coef. de asimetrie

$$S = \frac{\mu_3}{\sigma^3}$$

$$\sigma = \sqrt{D^2(X)}, \quad \mu_3 = M[(X - M(X))^3]$$

exces $e = \frac{\mu_4}{\sigma^4} - 3$

Repartiții clasice

1) Repartiția binomială

V.a. de tip discret $X : \Omega \to \mathbb{R}$ urmează legea binomială dacă $X \begin{pmatrix} k \\ P(n,k) \end{pmatrix}_{k=\overline{0,n}}$ unde

$P(n,k) = C_n^k p^k q^{n-k}$; $p + q = 1$

PROP: Dacă X urmează legea binomială de param. n și $p \Rightarrow M(X) = n \cdot p$

$D^2(X) = n \cdot p \cdot q$

Dem $M(X) = \sum_{k=1}^{n} k \cdot P(n,k)$

$$= \sum_{k=1}^{n} k \cdot C_n^k p^k q^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^{n-k} q^k$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \cdot p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)![(n-1)-(k-1)]!} \cdot p \cdot p^{k-1} q^{(n-1)-(k-1)}$$

$$= n \cdot p \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} p^{k-1} q^{(n-k)-(k-1)}$$

$$= n \cdot p \cdot \sum_{i=0}^{n} \frac{(n-1)!}{i!(n-1-i)!} \cdot p^i q^{(n-1)-i} = n \cdot p (p+q)^{n-1} = n \cdot p$$

obs

$$(p\cdot t+q)^n=\sum_{k=0}^{n} C_n^k (p\cdot t)^k \cdot q^{n-k}$$

$$=\sum_{k=0}^{n} C_n^k \cdot p^k \cdot t^k \cdot q^{n-k} \; //$$

$$n(pt+q)^{n-1}\cdot p=\sum_{k=0}^{n} C_n^k \cdot p^k \cdot k\cdot t^{k-1}\cdot q^{n-k}$$

$$n(pt+q)^{n-1}\cdot p=\sum_{k=1}^{n} k\cdot C_n^k p^k \cdot q^{n-k}\cdot t^{k-1}$$

$$\boxed{t=1} \Rightarrow n(p+q)^{n-1}\cdot p=\sum_{k=1}^{n} k\cdot P(n,k)$$

$$n\cdot p=\sum_{k=1}^{n} k\cdot P(n,k) \Rightarrow n\cdot p=M(X)$$

Temă: $D^2(X)=n\cdot p\cdot q$

$$M(X^2)=\sum_{k=1}^{n} k^2\cdot p(n,k)$$

2) Repartiția hipergeometrică

(-> schema bilei nerevenite)

$N = a + b$.

Spunem că v.a $X : \Omega \to \mathbb{R}$ de tip discret urmează legea hipergeom. dacă are distribuția

$$X \begin{pmatrix} k \\ P(n,k) \end{pmatrix} \qquad P(n,k) = \frac{C_a^k \cdot C_b^{n-k}}{C_{a+b}^n}, \ n \le a.$$

Dacă X este o var. aleatoare ce urmează legea hipergeom atunci $M(X) = n \cdot p$

$$D^2(X) = n \cdot p \cdot q \cdot \frac{a+b-n}{a+b-1}$$

$$p = \frac{a}{a+b}$$

$$q = \frac{b}{a+b}$$

ex: Fie un lot de 200 de produse din care 13 % nu se încadrează în limitele de funcționare admise.

Să se determine legea de repartiție a v.a. ~~care repr. nr. de aparate din cele 10 care~~ ~~fără bine.~~

Alegând rand. 10 produse.

nu se încadrează ...

Calc. dispersie și medie

$a = 26$

$b = 174$

$n = 10$

$$X \begin{pmatrix} k \\ P(10,k) \end{pmatrix}_{k=\overline{0,10}} \quad n=10$$

$$P(10,k) = \frac{C_{26}^{k} \cdot C_{174}^{10-k}}{C_{200}^{10}}$$

$$M(X) = n \cdot p = n \cdot \frac{a}{a+b} = 10 \cdot \frac{26}{200} = \frac{2,6}{2} = 1,3$$

$$D^2(X) = 10 \cdot \frac{26}{200} \cdot \frac{174}{200} \cdot \frac{190}{199}$$

3) Repartiția POISSON

(legea evenimentelor rare)

Def. Spunem că v.a. de tip discret $X : \Omega \to \mathbb{R}$ urm. legea POISSON dacă

$$X \begin{pmatrix} k \\ P_k(\lambda) \end{pmatrix}_{k \geq 0}$$

$$P_k(\lambda) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}, \; \lambda > 0$$

obs

n - nr. probabilităților → mare
p - prob → mică

$\lambda = n \cdot p$

PROP Dacă X urmează legea POISSON

$$M(X) = \lambda$$
$$D^2(X) = \lambda$$

Dem $M(X) = \sum_{k \geq 0} k \, P_\lambda(k) = \sum_{k \geq 0} k \cdot \frac{\lambda^k}{k!} e^{-\lambda}$

$= \sum_{k \geq 1} \frac{\lambda^k}{(k-1)!} \cdot e^{-\lambda} = \lambda \cdot e^{-\lambda} \sum_{k \geq 1} \frac{\lambda^{k-1}}{(k-1)!}$

Mac Laurin

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} + f''(0) + \dots$

$f(x) = e^x$ $\quad f(0) = 1$
$f'(x) = e^x$ $\quad f'(0) = 1$
$f''(x) = e^x$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n \geq 0} \frac{x^n}{n!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{i \geq 0} \frac{\lambda^i}{i!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

4) <u>Repartiție uniformă</u>

<u>Def</u> Spunem că v.a de tip continuu $X : \Omega \to \mathbb{R}$ urmează repartiție uniformă pe $[a, b]$ dacă
$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{în rest.} \end{cases}$$

<u>Obs</u>:

(1) $f(x) \geq 0, (\forall) x \in \mathbb{R}$

$x \in [a, b), \frac{1}{b-a} > 0.$

(2) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\underbrace{\int_{-\infty}^{a} 0}_{0} + \int_a^b f(x)\,dx = \frac{1}{b-a}\,dx + \underbrace{\int_b^{\infty} 0\,dx}_{0} = 1$$

$$\to \frac{1}{b-a} x \Big|_a^b = 1 \to \frac{b-a}{b-a} = 1 \quad A.$$

PROP Dacă v.a. X urmează legea uniformă pe $[a,b]$, atunci

$$M(X) = \frac{a+b}{2}$$

$$D^2(X) = \frac{(b-a)^2}{12}$$

Dem $M(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_{a}^{b} x \frac{1}{b-a}\,dx + \int_{b}^{\infty} 0\,dx$

$$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{b+a}{2}.$$

5) Repartiția normală

→ are un rol fundamental în teoria probabilităților și în statistică stând la baza metodelor de prelucrare a datelor.

V.a de tip continuu $X : \Omega \to \mathbb{R}$ urm. legea normală de param m, σ^2 (notăm $N(m, \sigma^2)$)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}},\ x \in \mathbb{R}$$

Fct de repartiție

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}}\,dt.$$

<u>Prop</u>

Dacă X urmează $N(m, \sigma^2)$ atunci

$M(X) = m, \quad D^2(X) = \sigma^2$

Dem

$$M(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

$x = t\cdot\sigma + m \Rightarrow dx = \sigma\,dt$

$$M(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} (t\sigma + m) \cdot e^{-\frac{t^2\sigma^2}{2\sigma^2}} \cdot \sigma\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (t\sigma + m) \cdot e^{-\frac{t^2}{2}}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} t\,\sigma\, e^{-\frac{t^2}{2}}\,dt + m \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt \right]$$

$f(t) = t \cdot e^{-\frac{t^2}{2}}$

$f(-t) = -t \cdot e^{-\frac{t^2}{2}} \Rightarrow f$ fct impară

Scanned with CamScanner

$$\Rightarrow \int_{-\infty}^{\infty} \varphi(t)\, dt = 0 \quad = \boxed{\frac{m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt.}$$

Integrala lui POISSON

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}$$

$$\Rightarrow M(X) = m$$

$$D^2(X) = M(X^2) - M^2(X)$$

(temă)

⊕ Legea normală redusă $N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt = \phi(x)$$

Funcția lui LAPLACE

(! tabel)

$$\phi(x)+\phi(-x)=1$$

$$\phi(-x)=\int_{-\infty}^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

$t=-u$

$dt=-du$

$t=-\infty \Rightarrow u=\infty$

$t=-x \Rightarrow u=x$

$$\phi(-x)=+\int_{\infty}^{x}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{u^2}{2}}(-du)$$

$$\Rightarrow \phi(-x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-\frac{t^2}{2}}dt$$

$$\phi(x)+\phi(-x)=$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt+\int_{x}^{\infty}e^{-\frac{t^2}{2}}dt\right)$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt=\int_{-\infty}^{\infty}f(t)dt=1$$

$P(a<X<b)$, $X\sim\mathcal{N}(m,\sigma^2)$

$$=\int_{a}^{b}f(x)dx$$

$$=\int_{a}^{b}\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-m)^2}{2\sigma^2}}dx.$$

$x=t\sigma+m$; $dx=\sigma dt$

$x = a \Rightarrow a = t\sigma + m \Rightarrow t = \frac{a-m}{\sigma}$

$x = b \Rightarrow b = t\sigma + m \Rightarrow t = \frac{b-m}{\sigma}$

$$P(a < X < b) = \int_{z=\frac{a-m}{\sigma}}^{\beta=\frac{b-m}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} \sigma\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\beta} e^{-\frac{t^2}{2}} dt$$

OBS

$\Rightarrow$ Dacă v.a. X urmează $N(m, \sigma^2)$

$Y = \frac{X-m}{\sigma}$ urm. $N(0,1)$