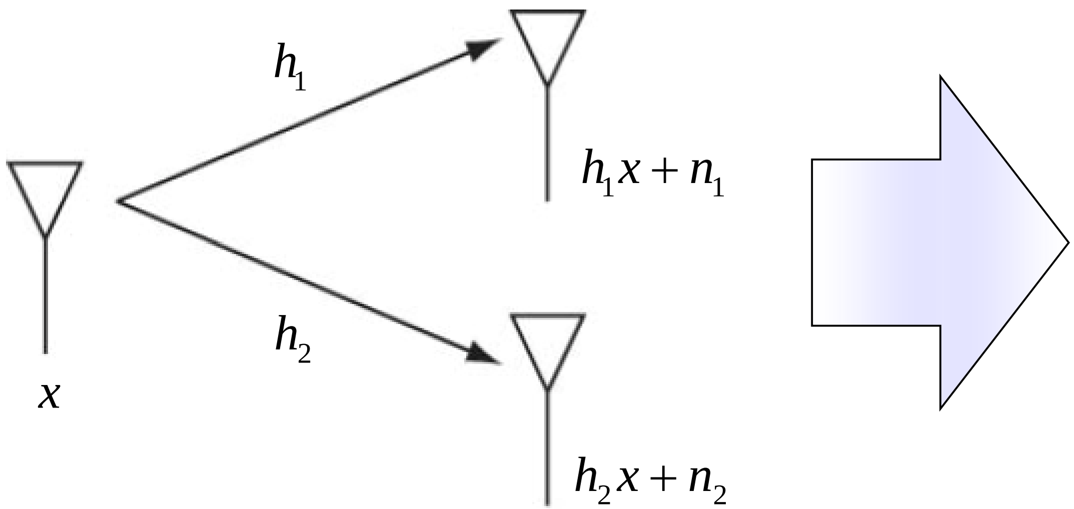
# Note 3. MIMO Systems – Diversity, Multiplexing, Capacity

# Receive Diversity in SIMO Systems

- SIMO (Single-Input Multiple-Output)
  - 단일 입력에 대해 여러 개의 출력을 가지는 시스템
  - 단일 송신 안테나 → 다중 수신 안테나




$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

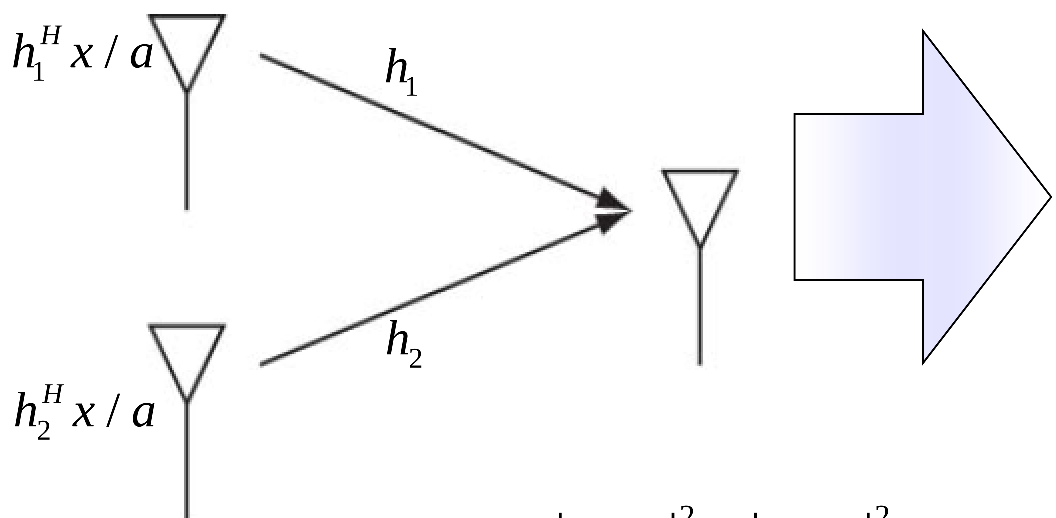
  - Maximal-Ratio Combining (MRC)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 x + n_1 \\ h_2 x + n_2 \end{bmatrix} \Rightarrow \begin{bmatrix} h_1^H y_1 \\ h_2^H y_2 \end{bmatrix} = \begin{bmatrix} h_1^H h_1 x + h_1^H n_1 \\ h_2^H h_2 x + h_2^H n_2 \end{bmatrix} \Rightarrow h_1^H y_1 + h_2^H y_2$$

    - 수신 신호의 결합 이후 SNR 값이 최대가 될 수 있도록 Combining
    - The resulting SNR: $\sum_{k=1}^{N} SNR_k$

# Transmit Diversity in MISO Systems (1/2)

- Maximal-Ratio Transmission (MRT)
  - MISO 시스템에서 Transmit Diversity를 달성하기 위한 송신 방법 중 하나
  - SIMO의 MRC와 동일한 원리 → 동일 성능
    - 차이점) 채널 정보를 송신단에서 알고 있어야 함.
  - 송신 Power를 더 쓰지 않도록 각 송신 안테나 신호에 대한 Power Normalization이 필요




$$y = (h_1 h_1^H x + h_2 h_2^H x) / a + n$$

$$a = \|\mathbf{h}\| : \text{ Power Normalization factor}$$

$$\mathbf{h} = [h_1, h_2]$$

$$Transmit\ Power: \left|\frac{h_1^H x}{a}\right|^2 + \left|\frac{h_2^H x}{a}\right|^2 = \frac{|h_1|^2 |x|^2}{|h_1|^2 + |h_2|^2} + \frac{|h_2|^2 |x|^2}{|h_1|^2 + |h_2|^2} = |x|^2$$

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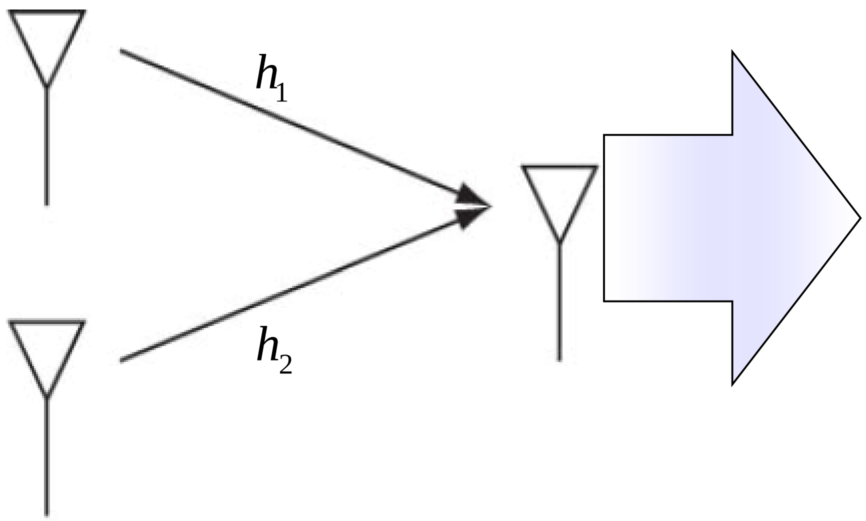
# Transmit Diversity in MISO Systems (2/2)

- Transmit Antenna Selection
  - MISO 시스템에서 Transmit Diversity를 달성하기 위한 송신 방법 중 하나
  - SIMO의 SC와 동일한 원리 → 동일한 성능
  - SISO 환경과 동일한 신호 파워로 전송

$$\begin{cases} x \ \text{if} \ |h_1| > |h_2| \\ 0 \ \text{if} \ |h_1| < |h_2| \end{cases}$$

$$\begin{cases} 0 \ \text{if} \ |h_1| > |h_2| \\ x \ \text{if} \ |h_1| < |h_2| \end{cases}$$




$$y = \begin{cases} h_1 x + noise \ \text{if} \ |h_1| > |h_2| \\ h_2 x + noise \ \text{if} \ |h_1| < |h_2| \end{cases}$$

$$Transmit\ Power: \ |x|^2$$

# Alamouti STBC in MISO Systems (1/5)

- MRT (Maximal-Ratio Transmission) / Antenna Selection
  - MISO System을 위한 Diversity 달성 기법
  - 송신단에서 채널 정보를 알아야만 사용 가능한 기술들
    - MRT: 정확한 채널 값
    - Antenna Selection: 어떤 안테나의 채널이 더 좋은지

  → 수신단에서부터 송신단으로의 피드백이 기본적으로 필요

  - 채널 정보를 모르고 MISO 시스템에서 Diversity를 달성할 수 있는 방법은 없는가??

- Space-time block coding: 시공간 블록 부호
  - 안테나와 시간 자원을 사용하여 Diversity를 얻기 위한 블록 부호
  - 동일한 방법을 주파수 단에도 적용할 수 있음

    → Space-frequency block coding

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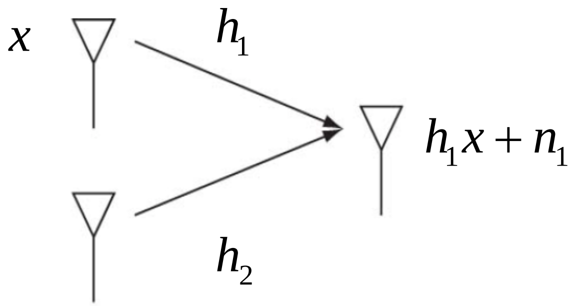
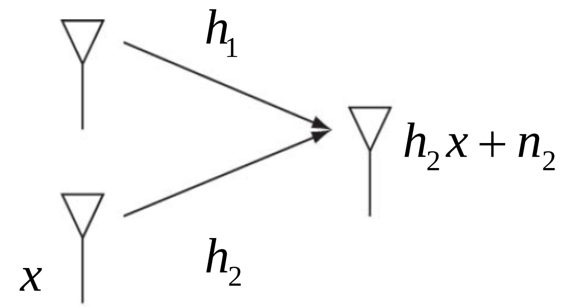
# Alamouti STBC in MISO Systems (2/5)

- Simplest space-time block coding




**Time 1**



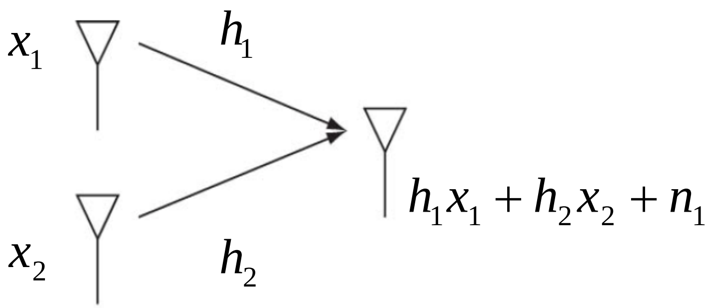

**Time 2**

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

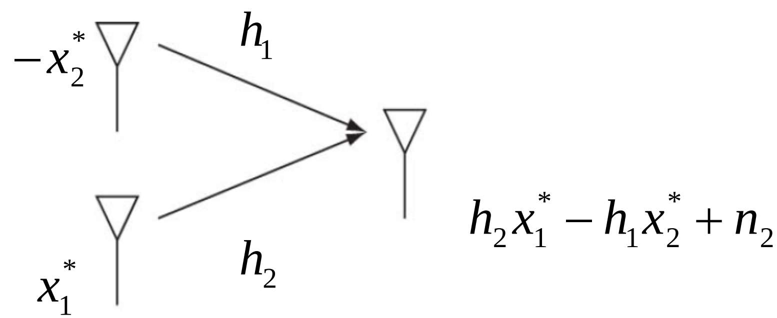
- 각 시간마다 한 TX Antenna만을 사용하여 Diversity를 확보하는 방법
  - 본 기법: 0.5 심볼 전송 / 시간, 채널 정보 필요 X (MISO)
  - MRT: 1 심볼 전송 / 시간, 채널 정보 필요 O (MISO)
  - MRC: 1 심볼 전송 / 시간, 채널 정보 필요 X (SIMO)

# Alamouti STBC in MISO Systems (3/5)

- Alamouti STBC in MISO systems (2x1 MISO systems)
  - The most well-known space-time block codes




**Time 1**




**Time 2**

- 기본 송수신 벡터 모델

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1x_1 + h_2x_2 \\ h_2x_1^* - h_2x_2^* \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1x_1 + h_2x_2 \\ h_2^*x_1 - h_1^*x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix} \quad \rightarrow \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

# Alamouti STBC in MISO Systems (4/5)

- Alamouti STBC – Equivalent System Model

$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

- Detection for the 1st TX symbol

$$h_1^* y_1 + h_2 y_2^* = |h_1|^2 x_1 + h_1^* h_2 x_2 + |h_2|^2 x_1 - h_2 h_1^* x_2 + noise$$

$$= |h_1|^2 x_1 + |h_2|^2 x_1 + noise$$

- Detection for the 2nd TX symbol

$$h_2^* y_1 - h_1 y_2^* = h_2^* h_1 x_1 + |h_2|^2 x_2 - h_1 h_2^* x_1 + |h_1|^2 x_2 + noise$$

$$= |h_1|^2 x_2 + |h_2|^2 x_2 + noise$$

# Alamouti STBC in MISO Systems (5/5)

- Alamouti STBC in MISO Systems (2x1)
  - 두 개의 신호를 두 번의 시간 동안 전송함
  - 두 전송 신호에 대해 두개의 채널(Diversity)을 보장하며, 또한 상호간 간섭 없는 신호를 추출할 수 있는 간단한 복호과정을 가짐
    - Alamouti STBC: 1 심볼 전송 / 시간, 채널 정보 필요 X (MISO 2x1)
    - Simple STBC: 0.5 심볼 전송 / 시간, 채널 정보 필요 X (MISO)
    - MRT: 1 심볼 전송 / 시간, 채널 정보 필요 O (MISO)
    - MRC: 1 심볼 전송 / 시간, 채널 정보 필요 X (SIMO)
- 상기 기법들은 2x1 MISO / 1x2 SIMO를 가정할 때 모두 2개의 독립적인 채널을 통한 Diversity 획득
- **Alamouti STBC의 제약 사항**
  - **MISO 시스템 중에서는 2x1 환경에만 적용 가능**
  - **두 전송 시간동안 채널이 변하지 않아야 최대 성능 보장.**
    - 채널이 변화할 경우 간섭 성분이 남을 수 있음

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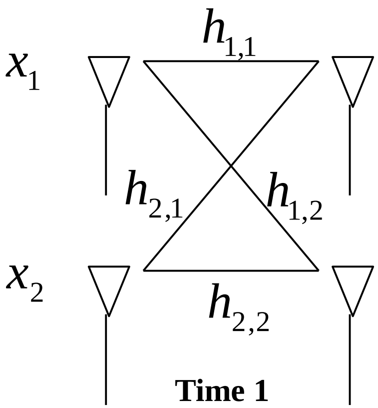
# Diversity in MIMO Systems

- Total diversity in MIMO Systems
  - Total available diversity: # of available independent channels
  - MISO with *N* TX antennas: *N* at maximum
  - SIMO with *M* RX antennas: *M* at maximum
  - MIMO with *N* TX and *M* RX antennas: *MN* at maximum

- How to achieve the diversity in MIMO systems?
  - Antenna Selection could be applied for MIMO systems
    - Antenna Selection → Channel is transformed into 1 TX and *M* RX antennas
    - Still needs the information about which TX antenna has the best channel.
  - Also, MRT can be applied for MIMO systems
  - Also, space-time block codes can be applied for MIMO systems

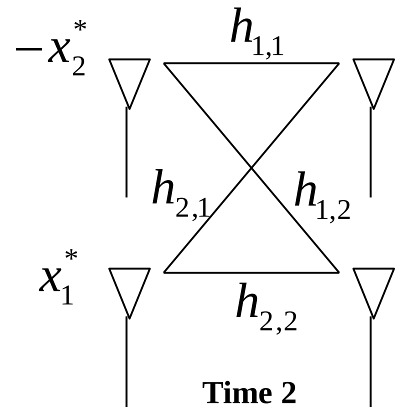
# Alamouti STBC in MIMO Systems (1/3)

- Alamouti STBC for MIMO Systems (2x2)
  - Same approach to the case of MISO systems




$h_{1,1}x_1 + h_{1,2}x_2 + n_{1,1}$

$h_{2,1}x_1 + h_{2,2}x_2 + n_{1,2}$




$-h_{1,1}x_2^* + h_{1,2}x_1^* + n_{2,1}$

$-h_{2,1}x_2^* + h_{2,2}x_1 + n_{2,2}$

  - 기본 송수신 벡터 모델

$$\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_{1,1} \\ n_{1,2} \end{bmatrix}$$

**Received signal vector at Time 1**

$$\begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + \begin{bmatrix} n_{2,1} \\ n_{2,2} \end{bmatrix}$$

**Received signal vector at Time 2**

# Alamouti STBC in MIMO Systems (2/3)

- Alamouti STBC – Equivalent system model

$$\begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_{1,1} \\ n_{1,2} \end{bmatrix} \qquad \begin{bmatrix} y_{2,1} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + \begin{bmatrix} n_{2,1} \\ n_{2,2} \end{bmatrix}$$

**Received signal vector at Time 1** **Received signal vector at Time 2**

- Detection for the 1st TX symbol

$$\begin{bmatrix} h_{11}^* & h_{21}^* \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} + \begin{bmatrix} h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} y_{2,1}^* \\ y_{2,2}^* \end{bmatrix} = (|h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2) x_1 + noise$$

- Detection for the 2nd TX symbol

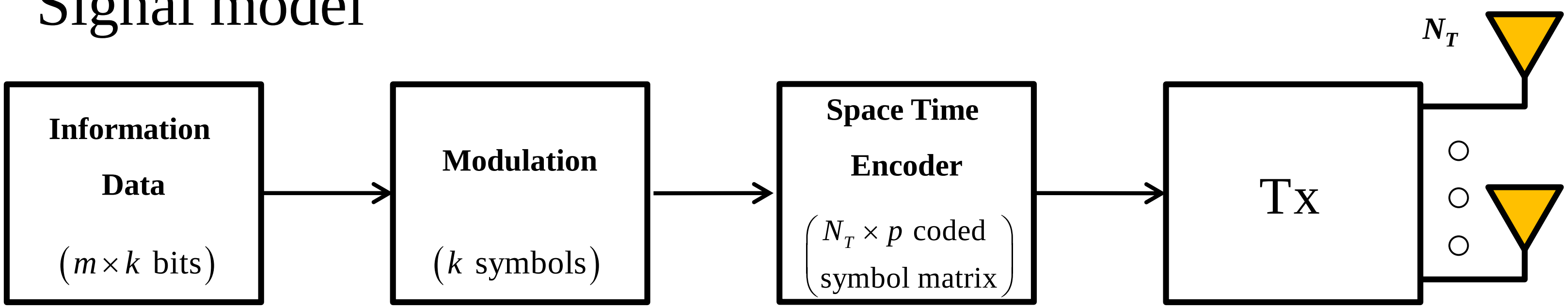
$$\begin{bmatrix} h_{12}^* & h_{22}^* \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix} - \begin{bmatrix} h_{11} & h_{21} \end{bmatrix} \begin{bmatrix} y_{2,1}^* \\ y_{2,2}^* \end{bmatrix} = (|h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2) x_2 + noise$$

# Alamouti STBC in MIMO Systems (3/3)

- Alamouti STBC in MIMO Systems (2x2)
  - 두 개의 신호를 두 번의 시간 동안 전송함
  - 두 전송 신호에 대해 4개의 채널(Diversity)을 보장하며, 또한 상호간 간섭 없는 신호를 추출할 수 있는 간단한 복호과정을 가짐
  - Orthogonal STBC
    - Alamouti STBC 2x2: 1 심볼 전송 / 시간, 채널 정보 필요 X, 4 Diversity (MIMO 2x2)
    - Alamouti STBC 2x1: 1 심볼 전송 / 시간, 채널 정보 필요 X, 2 Diversity (MISO 2x1)
    - Simple STBC: 0.5 심볼 전송 / 시간, 채널 정보 필요 X, 2 Diversity (MISO 2x1)
    - MRT: 1 심볼 전송 / 시간, 채널 정보 필요 O, 2 Diversity (MISO 2x1)
    - MRC: 1 심볼 전송 / 시간, 채널 정보 필요 X, 2 Diversity (SIMO 2x1)
- **Alamouti STBC의 제약 사항**
  - **MIMO 시스템 중에서는 2x2 환경에만 적용 가능**
  - **두 전송 시간동안 채널이 변하지 않아야 최대 성능 보장.**
    - 채널이 변화할 경우 간섭 성분이 남을 수 있음

# Space Time Block Codes (1/4)

- Signal model




  - $m$=$\log_2 M$ ($M$: modulation order)
  - $k$: # of transmitted symbols
  - $p$: ST coded symbol duration
- Spatial rate: $r=k/p$
- Effective data rate: $R_{\text{eff}}=km/p$ bps/Hz

- STBC based on orthogonal design

$$\mathbf{X}\mathbf{X}^H = c\left(|x_1|^2 + |x_1|^2 + \cdots + |x_k|^2\right)\mathbf{I}_{N_T},\ \ \mathbf{x}_i\mathbf{x}_j = 0,\ \text{for } i \neq j$$

  - $\mathbf{X}$: $N_T \times p$ ST coded matrix

# Space Time Block Codes (2/3)

- Orthogonal STBC (OSTBC)
  - Full diversity gain, maximum spatial rate ($r$) =1
  - Low decoding complexity
  - Real orthogonal design
    - For system using real symbol (ex. PAM), $r = 1$

| $N_T$=3, $k$=4, $p$=4, $r$=1 | $N_T$=4, $k$=4, $p$=4, $r$=1 |
|---|---|
| $\mathbf{X}_3 = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ x_3 & -x_4 & x_1 & x_2 \end{bmatrix}$ | $\mathbf{X}_4 = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{bmatrix}$ |

  - Complex orthogonal designs
    - For system using complex symbol (ex. QAM), $r$<1 ($r$=1 only for $\boxed{N_T=2}$)

**Alamouti code**

| $N_T$=3, $k$=4, $p$=8, $r$=1/2 |
|---|
| $\mathbf{X}_3 = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & x_1^* & -x_2^* & -x_3^* & -x_4^* \\ x_2 & x_1 & x_4 & -x_3 & x_2^* & x_1^* & x_4^* & -x_3^* \\ x_3 & -x_4 & x_1 & x_2 & x_3^* & -x_4^* & x_1^* & x_2^* \end{bmatrix}$ |

# Space Time Block Codes (3/3)

- Quasi-orthogonal STBC (QOSTBC)
  - Fixed spatial rate = 1
  - Diversity order loss → Orthogonality is no longer maintained.
  - ABBA code (4x1 system)
    - Diversity order = 2, Spatial rate = 1
    - Code structure

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}, \mathbf{B} = \begin{bmatrix} s_3 & s_4 \\ -s_4^* & s_3^* \end{bmatrix}$$

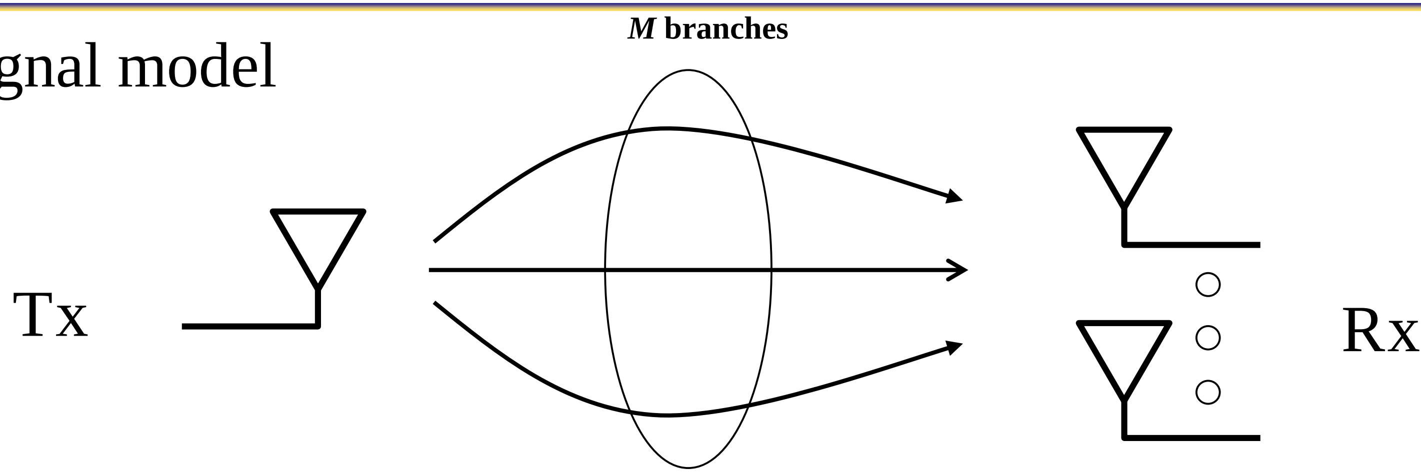
  - Jafarkhani code (4x1 system)
    - Diversity order = 2, Spatial rate = 1
    - Code structure

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^* & \mathbf{A}^* \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}, \mathbf{B} = \begin{bmatrix} s_3 & s_4 \\ -s_4^* & s_3^* \end{bmatrix}$$

  - OSTBC / QOSTBC
    - OSTBC → full diversity gain, not full rate $\left(N_T \geq 3\right)$
    - QOSTBC → not full diversity gain, full rate

# Diversity gain analysis (1/3)

- Signal model




  - Assume $M$ identical independent Rayleigh fading links between Tx and Rx
  - Signal model is

$$y_i = \sqrt{\frac{E_s}{M}} h_i s + n_i, \quad i = 1, \ldots, M$$

$y_i$ : received signal on the $i^{\text{th}}$ diversity branch

$h_i$ : channel corresponding to the $i^{\text{th}}$ diversity branch

$s$ : transmitted symbol

$n_i$ : AWGN noise with variance $N_0$

# Diversity gain analysis (2/3)

- Effect of diversity on SER performance
  - Assume perfect channel knowledge at the Rx and MRC

$$z = \sum_{i=1}^{M} h_i^* y_i = \sqrt{\frac{E_s}{M}} \sum_{i=1}^{M} |h_i|^2 s + \sum_{i=1}^{M} h_i^* n_i$$

  - The average received SNR $\eta$ is given by

$$\eta = \frac{E_s \left( \sum_{i=1}^{M} |h_i|^2 \right)^2}{M N_0 \sum_{i=1}^{M} |h_i|^2} = \frac{1}{M} \sum_{i=1}^{M} |h_i|^2 \rho \quad where \;\; \rho = E_s / N_0$$

  - Assuming ML detection, the probability of symbol error is

$$P_e \approx \overline{N}_e Q\left( \sqrt{\frac{\eta d_{\min}^2}{2}} \right)$$

$\overline{N}_e$ : the number of nearest neighbors

$d_{\min}$ : minimum distance of separation

  - Using the Chernoff bound, $Q(x) \le e^{-x^2/2}$

$$P_e \le \overline{N}_e e^{-\left( \sum_{i=1}^{M} |h_i|^2 \right) \frac{\rho d_{\min}^2}{4M}}$$

# Diversity gain analysis (3/3)

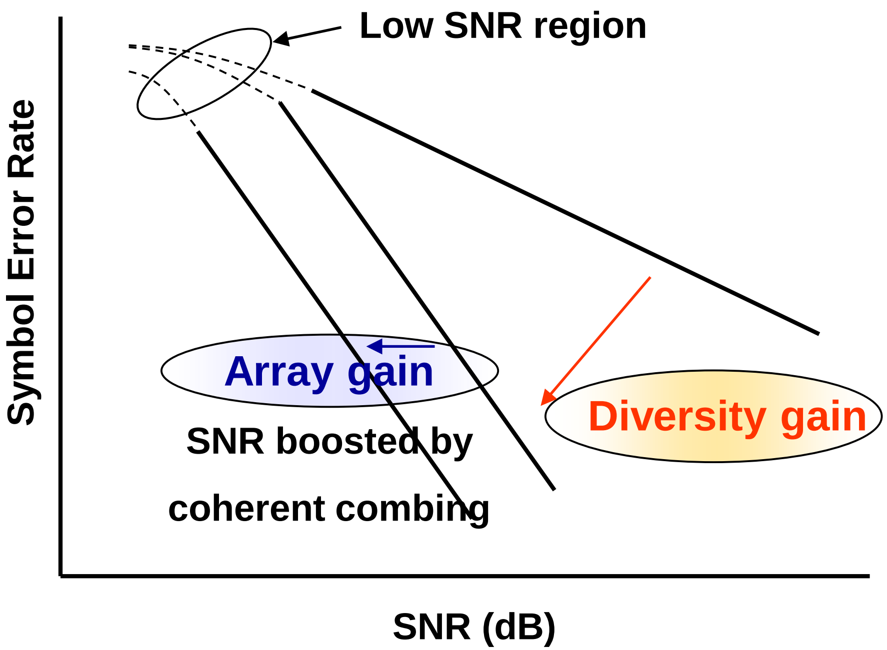
- Exponential function decomposition $e^{-\left(\sum_{i=1}^{M}|h_i|^2\right)\frac{\rho d_{\min}^2}{4M}} = \prod_{i=1}^{M} e^{-|h_i|^2\frac{\rho d_{\min}^2}{4M}}$
- Moment generating function of chi-square r.v. $|h_i|^2$
  - Sum of squares of two indep. Gaussian r.v., → degree of freedoms:2, weight 0.5
  - MGF: $M_{|h_i|^2}(a) \overset{\textbf{(a)}}{=} E[e^{-a|h_i|^2}] \overset{\textbf{(b)}}{=} \frac{1}{1+a}$
    - (a): Definition of MGF for any r.v.
    - (b): MGF for chi-square r.v.,
- Average symbol error prob.: $\bar{P}_e \le E\left[\bar{N}_e e^{-\left(\sum_{i=1}^{M}|h_i|^2\right)\frac{\rho d_{\min}^2}{4M}}\right] = \bar{N}_e \prod_{i=1}^{M} \frac{1}{1+\frac{\rho d_{\min}^2}{4M}}$
- Hign SNR, $(\rho \gg 1)$

$$\bar{P}_e \le \bar{N}_e \left(\frac{\rho d_{\min}^2}{4M}\right)^{-M}$$

**<u>diversity order</u>**

# Diversity Order (1/2)

- Diversity 효과 vs. SNR 증대 효과
  - Diversity 효과: Fading Channel의 Averaging 효과
  - SNR 증대 효과: 신호의 증폭 효과
    - $x_1 + x_2 + x_3 + \cdots + x_N \approx N \cdot \mathrm{E}[x]$ → $\mathrm{E}[x]$를 $N$번 더한 것과 같음
    - $x_1 + x_1 + x_1 + \cdots + x_1 = N \cdot x_1$ → $x_1$을 $N$번 더한 것과 같음




$$\bar{P}_e \leq \bar{N}_e \left( \frac{\rho d_{\min}^2}{4M} \right)^{-M}$$

$$\approx \frac{c}{(\gamma_c \rho)^M}$$

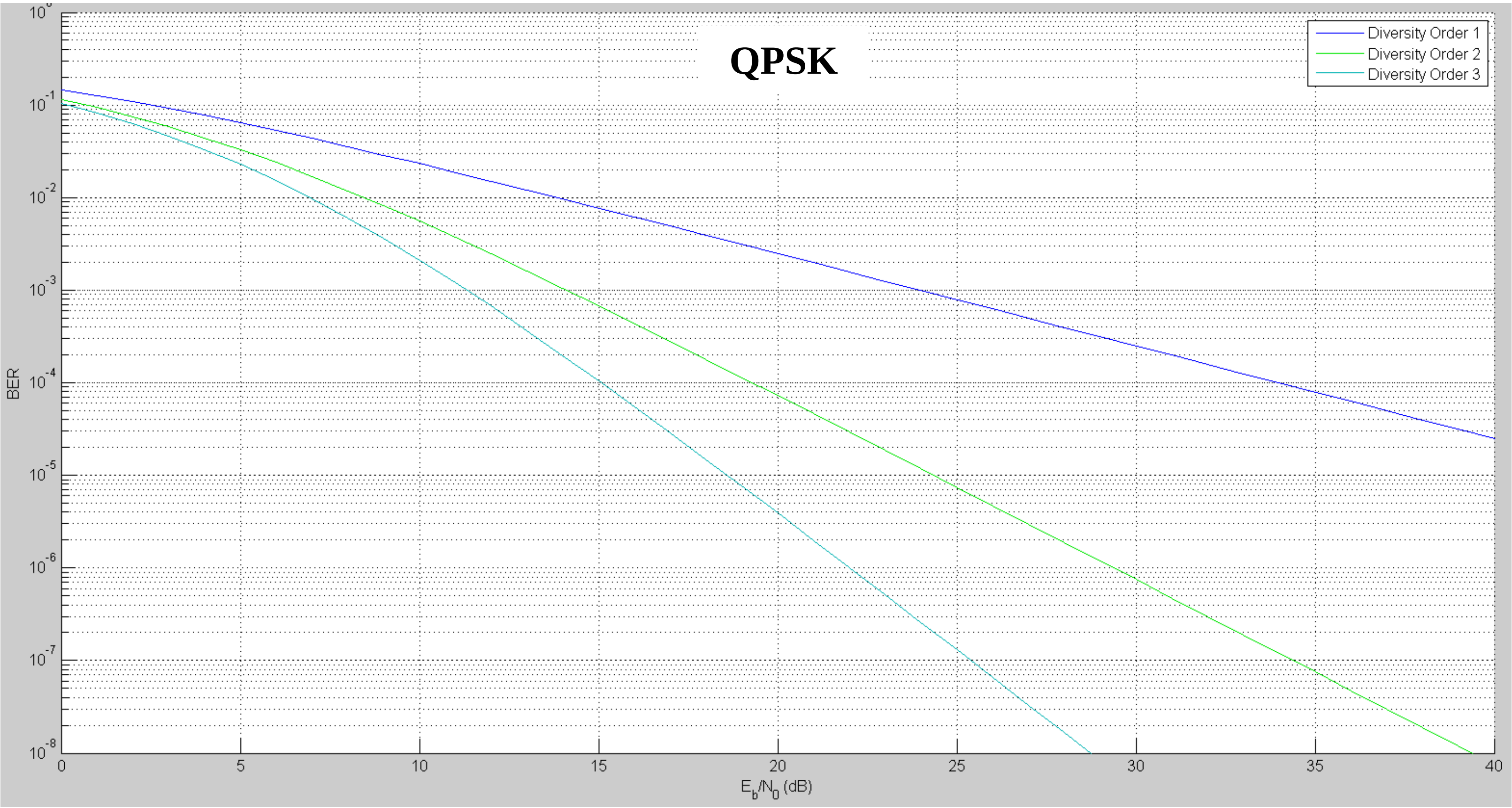
$c$ : scailing constant

$\gamma_c$ : coding (array) gain

$M$ : diversity gain

# Diversity Order (2/2)

- Diversity order (Rayleigh fading assumption)
  - # of independent fading channels that the symbol was experienced
  - Can be checked by decreasing rate of BER in the high SNR region



# Spatial Multiplexing (1/2)

- Up to now, the diversity schemes were introduced.
- However, instead of diversity(reliability), MIMO systems can increase data rates as well.

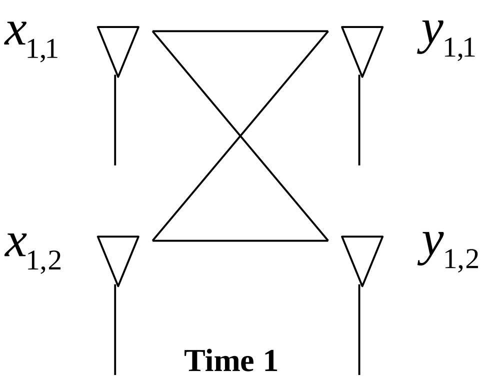
- Multiple input multiple output (MIMO) systems
  1. Increase ***data rates*** through ***multiplexing***

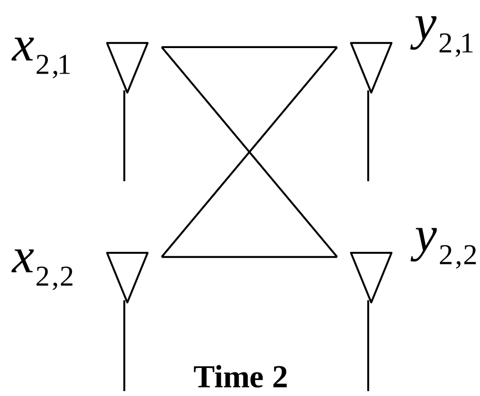
     $\Rightarrow$ Independent signaling paths that can be used to send independent data
  2. Improve ***performance*** through ***diversity*** $\Rightarrow$ Diversity gain

- How to increase data rate? → Spatial multiplexing

# Spatial Multiplexing (2/2)

- Spatially multiplexed MIMO systems (2x2 example)




$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x}_1 + \mathbf{n}_1 \qquad \mathbf{y}_2 = \mathbf{H}_2\mathbf{x}_2 + \mathbf{n}_2$$

- MIMO System with *N* TX & *M* RX antennas
  - 매 전송시간마다 *N*개의 독립적인 송신 심볼이 전송될 수 있음.
  - 하지만, Diversity(MISO/SIMO) 기법들과는 다르게, 이제 각 심볼들은 서로에게 영향을 주는 형태로 들어오며, Alamouti 기법처럼 간단하게 분리할 수도 없음
  - ML (Maximum-likelihood), MMSE (Minimum mean-square-error), ZF 등의 검출 기술이 필요

# To do list

- 1x1, 1x2 (SIMO), 2x1 (MISO), 2x2 (MIMO) QPSK BER Simulation
  - *N*x*M*: TX Antennas – *N*, RX Antennas – *M*
- SNR = 1 / $\sigma^2$ ($E_s/N_o$)
- 5 dB step ( From 0dB ~20dB)
- MRC, EGC, SC for SIMO, MRT for MISO
- Independent Rayleigh Fading Channels for MISO/SIMO environment
  - 모든 안테나간 채널이 독립적인 i.i.d. complex Gaussian R.V.,
  - Alamouti의 경우, 두 시간동안은 채널이 변하지 않고 다음 두 시간에 변하는 경우. 다른 기법의 경우 매 시간마다 채널 변화
- 1x1, 1x2 MRC, 1x2 EGC, 1x2 SC, 2x1 MRT, 2x1 Alamouti, 2x2 Alamouti 등 총 7개 기법에 대한 BER 도출

# Thank You!